Logarithmic Hochschild homology and cohomology

Leonardi, F.

Citation

Leonardi, F. (2025, June 3). *Logarithmic Hochschild homology and cohomology*. Retrieved from https://hdl.handle.net/1887/4247737

Logarithmic Hochschild homology and cohomology

Proefschrift

ter verkrijging van
de graad van doctor aan de Universiteit Leiden,
op gezag van rector magnificus prof. dr. ir. H. Bijl,
volgens besluit van het college voor promoties
te verdedigen op dinsdag 3 juni 2025
klokke 11.30 uur

door

Francesca Leonardi
geboren te Rome, Italië
in 1997

Promotor:

Prof.dr. R.M. van Luijk

Copromotor:

Dr. M. Hablicsek

Promotiecommissie:

Prof.dr.ir. G.L.A. Derks

Prof.dr. J.B. Vonk

Prof.dr. T.L. Kelly (Queen Mary University of London)

Prof.dr. B. Toën (CNRS, Université de Toulouse)

Dr. L.S. Herr (Virginia Polytechnic Institute and State University)

Contents

Introduction

In this thesis, we study Hochschild homology and cohomology, which have nice expressions and properties for smooth and proper schemes and separated Deligne-Mumford stacks. The goal of this work is to adapt it to such spaces when they are obtained as compactifications of non-proper ones recording the boundaries, in the language of logarithmic geometry.

For simplicity, this introduction covers the main results present in this work stated for schemes. However, most of them apply to Deligne-Mumford stacks as well. They will be expressed in this more general version in the body of this thesis.

Logarithmic schemes

A classical way to encode the boundaries of non-proper algebraic schemes is via logarithmic algebraic geometry, which is the formalism that we work with. In this framework, the classical notion of smoothness is replaced by the one of logarithmic smoothness.

Logarithmic schemes are schemes with a logarithmic structure, i.e. a sheaf of monoids which is a submonoid of the structure sheaf (regarded with the multiplication) containing the units. Examples of logarithmically smooth logarithmic schemes include (proper) toric varieties, equipped by additional structure determined by the dense torus, and pairs made of a smooth (and proper) variety and a s.n.c. divisor (defining the logarithmic structure). Moreover, logarithmically smooth spaces typically include those with “mild” singularities (nodal) when equipped with an appropriate logarithmic structure. In fact, for such singularities, one may picture logarithmic smoothness as the smoothness of the strata of the logarithmic structure.

Logarithmic schemes naturally arise from compactifications to the sense of [Nag56], which associates smooth varieties with pairs of a proper variety and a (possibly empty) s.n.c. divisor, such that the original variety is embedded in the proper one with the divisor as complement. Such compactifications are, a priori, not unique. Think, for instance, of the affine plane $\mathbb{A}^2$ admitting both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ as compactifications. The pairs $(\mathbb{P}^2, \mathbb{P}^1)$ and $(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \cup \mathbb{P}^1)$ give two different logarithmic schemes, both associated with $\mathbb{A}^2$ in the above sense.

Hochschild homology and cohomology

Hochschild homology and cohomology are tools of crucial importance in commutative and noncommutative algebra and algebraic geometry. The geometric approach to the subject is to define them through derived complexes (the Hochschild homology and cohomology complexes), whose global sections give the Hochschild homology and cohomology vector spaces.

Definition 1. Let X be a smooth algebraic variety over $S = \operatorname{Spec} k$. Denote by $\Delta : X \to X \times_S X$ the diagonal morphism. The Hochschild homology (resp. cohomology) derived complex is

$$\mathbf{L}\Delta^*\mathbf{R}\Delta_*\mathcal{O}_X \quad (\text{resp. } \Delta^!\mathbf{R}\Delta_*\mathcal{O}_X).$$

The Hochschild homology (resp. cohomology) of X is the graded k-vector space $\mathbf{R}\Gamma(X, \mathbf{L}\Delta^*\mathbf{R}\Delta_*\mathcal{O}_X)$ (resp. $\mathbf{R}\Gamma(X, \Delta^!\mathbf{R}\Delta_*\mathcal{O}_X)$).

They have a nice "sheafy" expression for smooth varieties in terms of the cotangent and tangent bundles, and when the varieties are moreover proper the Hochschild homology and cohomology spaces have good properties. This is the content of the HKR theorem ([HKR62], [AC12]). A direct consequence is that the dimensions of the Hochschild homology spaces for smooth and proper varieties give information on its Hodge diamond. The low-degree Hochschild cohomology spaces play a role in deformation theory, encoding their first-order deformations and their obstructions ([Ger64], [Tod09]). The Hochschild homology and cohomology spaces are preserved under derived equivalence ([Cal03]) and may be defined for abstract derived categories and abelian categories ([LVdB05], [Kel21]). To conclude, Hochschild homology and cohomology may be defined for noncommutative schemes. In the noncommutative framework, the homology spaces play the role of Hodge cohomology in the commutative setup. A closely related notion is the one of cyclic homology, which may be interpreted as the noncommutative counterpart of the de Rham cohomology ([Lod98]).

Logarithmic Hochschild homology and cohomology

This thesis may be resumed as an extended affirmative answer to the following question.

Question 2. Are there Hochschild-like invariants for logarithmic schemes, coinciding with the usual ones for schemes with the trivial logarithmic structure and with an HKR-like decomposition in terms of the logarithmic cotangent and tangent bundles?

The main difficulty that one encounters trying to define similar endofunctors which are logarithmically compatible is that there are many attempts to define logarithmic analogous of the derived category of coherent sheaves ([MS80, Yok95], [TV18], [Vai17]) which appear to be convenient for some purposes but less appropriate for others. The author has no reason to believe that any of these definitions should prevail over the others. Therefore, our techniques avoid the use of such a derived category. In particular, the derived pullback and pushforward for morphisms of logarithmic schemes are just those for the underlying scheme and do not extend to functors recording any logarithmic information. To make similar endofunctors to those in Definition 1 encode the logarithmic data, we alter the diagonal morphism, replacing Δ by i defined by the diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \times X \arrow[d] \\
& \Theta_X \arrow[r] & \Theta_{X \times X}
\end{tikzcd}, \qquad (1)$$

where Θ_X and $\Theta_{X\times X}$ are the Artin fans of X and $X \times X$ respectively and B is the pullback in the category of logarithmic algebraic stacks. This alteration of the diagonal, which we call strict or logarithmic diagonal, does not modify the logarithmic structure.

Artin fans

In the above construction, the Artin fans of X and $X \times X$ appear. They are logarithmic algebraic stacks encoding the logarithmic structure of the logarithmic scheme they are associated with, i.e. such that the morphism $X \to \Theta_X$ is strict. For a toric variety V with dense torus $\mathbb{G}_m^k \simeq T \subset V$, the Artin fan is simply given by the quotient stack $[V/T]$. Such quotients are called Artin cones. Geometrically, it coincides with the fan.

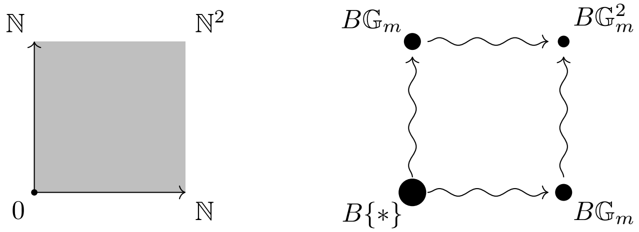

Figure 0.1: The fan of the affine plane $\mathbb{A}^2$ as toric variety, and its Artin fan.

Logarithmically smooth logarithmic schemes admit étale covers by toric varieties. the Artin fan is obtained by Artin cones glueing étale locally. A more extensive definition and treatment of the topic will be given in Section 2.3.

Working with Artin fans can be complicated, as the construction is not functorial. A famous example of the failure of this property is given by the Whitney umbrella [ACM+15, Subsection 5.4.1]. To make sure that the bottom arrow in Diagram (1) exists and makes the diagram commute, we prove the following result.

Theorem A (Corollary 2.3.30)**.** Let X and Y be two logarithmically smooth quasicompact logarithmic schemes. Then the Artin fan of the product $X \times Y$ is isomorphic to the product of the Artin fans of the factors:

$$\Theta_{X\times Y} \xrightarrow{\sim} \Theta_X \times \Theta_Y.$$

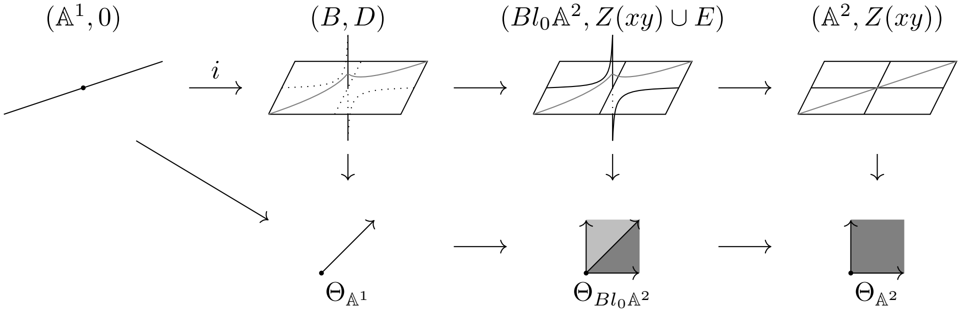

Figure 0.2: Construction of the logarithmic diagonal of the logarithmic scheme given by the pair $(\mathbb{A}^1, 0)$. For this example, it can be obtained by two subsequent pullbacks, first factoring the diagonal of the Artin fan through the subdivision of the fan. The target of the strict diagonal obtained like so is the open $B = Bl_0\mathbb{A}^2 \setminus Z(xy)$ with logarithmic structure induced by the exceptional divisor minus the points where it intersects the axis $D = E\setminus(E\cap Z(xy))$. Here we are representing the Artin fans via the fans as toric varieties.

This result arises as a corollary of the more general Theorem 2.3.29. Theorem A allows us to define the strict diagonal.

A formality result

After constructing the strict diagonal via Diagram (1), one can define the Hochschild homology and cohomology endofunctors as self-intersections along the strict diagonal.

Definition 3 (definition 3.1.7)**.** Let X be a logarithmically smooth, weakly logarithmically separated (definition 2.3.24), quasicompact logarithmic scheme. The logarithmic Hochschild homology (resp. cohomology) endofunctor of the derived cateogry of unbounded coherent sheaves of X is

$$\mathbf{L}i^*\mathbf{R}i_* : \mathscr{D}(X) \to \mathscr{D}(X) \quad (\text{resp. } i^!\mathbf{R}i_* : \mathscr{D}(X) \to \mathscr{D}(X)).$$

The logarithmic Hochschild homology (resp. cohomology) of X is

$$\mathrm{HH}^{\ell}_{\bullet}(X) = \mathbf{R}\Gamma(X, \mathbf{L}i^*\mathbf{R}i_*\mathscr{O}_X) \quad (\text{resp. } \mathrm{HH}^{\bullet}_{\ell}(X) = \mathbf{R}\Gamma(X, i^!\mathbf{R}i_*\mathscr{O}_X)).$$

The reason why $B := X \times X \times_{\Theta_{X\times X}} \Theta_X \simeq X \times_{\Theta_X} X$ is a suitable choice for the space where to regard the diagonal morphism is given by the following list of properties:

- the morphism $i : X \to B$ is strict: this means that it induces an isomorphism between the logarithmic structures;
- the morphism $B \to X \times X$ is logarithmically étale: this implies that the two spaces have the isomorphic logarithmic cotangent complexes; for easy

examples, one may think of the morphism to be a logarithmic blow-up (see Figure 0.2);

- the morphism $B \simeq X \times_{\Theta_X} X \to X \times_{\mathscr{L}} X$ is a open immersion, where $\mathscr{L}$ is the logarithmic algebraic stack $\mathscr{T}or$ from [Ols03], classifying (fs) logarithmic structures;
- the morphism i is a l.c.i. closed embedding: this allows us to use the techniques of [AC12].

With the aid of the properties listed above, we are able to prove the following statement. For brevity, we formulate it for the logarithmic Hochschild homology, but we point out that it has a cohomological version, that the reader can find in this thesis.

Theorem B (Corollary 3.2.6, Theorem 3.2.7)**.** Let X be a quasicompact, weakly logarithmically separated, logarithmically smooth logarithmic scheme.

- The endofunctor $\mathbf{L}i^*\mathbf{R}i_*$ is formal: there exists an isomorphism of dg endofunctors $\mathscr{D}(X) \to \mathscr{D}(X)$

  $$\mathbf{L}i^*\mathbf{R}i_*(-) \xrightarrow{\sim} (-) \otimes \operatorname{Sym}(\Omega_X^{1,\log}[1]) \simeq (-) \otimes \bigwedge^{\bullet} \Omega_X^{1,\log}[\bullet].$$

  Here, Sym is the derived symmetric algebra.
- The logarithmic Hochschild homology of $\mathscr{O}_X$ can be computed in terms of the logarithmic cotangent bundle:

  $$\mathrm{HH}_\bullet^\ell(X) = \bigoplus_{q-p=\bullet} H^p(X, \Omega_X^{q,\log}).$$

Theorem B has some nice consequences. First, there is a natural transformation

$$\mathbf{L}\Delta^*\mathbf{R}\Delta_*(-) \to \mathbf{L}i^*\mathbf{R}i_*(-)$$

induced by the unit of the adjunction along the morphism $B \to X \times X$, giving rise to a long exact sequence imitating an Eilenberg–Steenrod axiom for smooth pairs (X, D), relating the Hochschild homology spaces of X and D, and the logarithmic Hochschild homology of the logarithmic scheme associated with the pair.

Another implication of Theorem B is the invariance of the endofunctor under pullback along logarithmically étale morphisms. In particular, the pullback along logarithmic blow-ups induces an isomorphism of the logarithmic Hochschild homology vector spaces. This fact suggests that there should be a relation between the logarithmic derived category of a logarithmic scheme and the one of its logarithmic blow-up. In the non-logarithmic case, we know by [BO95] that, for the blow-up $Bl_{(0,0)}\mathbb{P}^2 \to \mathbb{P}^2$ with exceptional divisor $j : E \to Bl_{(0,0)}\mathbb{P}^2$, we have

$$\mathscr{D}^b(Bl_{(0,0)}\mathbb{P}^2) = \langle j_*\mathscr{O}_E(-1), \mathscr{D}^b(\mathbb{P}^2)\rangle.$$

Supposing that the logarithmic derived category of the blow-up is subject to a similar decomposition, the component associated with the exceptional divisor should have trivial logarithmic Hochschild homology.

Finally, there is a logarithmic Künneth formula, stating that the logarithmic Hochschild homology graded vector space of a finite product of logarithmic schemes is isomorphic to the tensor products of the logarithmic Hochschild homology graded vector spaces of the factors

$$\mathrm{HH}^{\ell}_{\bullet}(X \times Y) \simeq \mathrm{HH}^{\ell}_{\bullet}(X) \otimes \mathrm{HH}^{\ell}_{\bullet}(Y).$$

Moreover, the approach to logarithmic Hochschild homology and cohomology via the strict diagonal morphism $i : X \to B$ allows us to revisit the construction using the perspective of the loop space [BZN12, TV07, BM93], naturally giving as logarithmic loop space of logarithmically flat logarithmic schemes the derived algebraic stack

$$L_X^{\log} := X \times_B^{\mathbf{R}} X.$$

The logarithmic loop space allows us to mimic the classical theory defining cyclic, negative and periodic homology theories in a compatible way with the logarithmic Hochschild homology.

We point out that an alternative definition of logarithmic Hochschild homology has been formulated in [BLPØ23], via a topological approach. The two definitions may be seen to coincide in the logarithmically smooth case using the HKR theorem, and are conjecturally equivalent in broader generality. However, the geometric approach followed in this thesis allows us to prove some properties typically appearing in the field, like some of those mentioned above.

An invariance result

One of the properties of ordinary Hochschild homology and cohomology that is not easy to state in its logarithmic version is the derived invariance, proved for instance in [Cal03]. The reason behind this is once again the lack of a broadly accepted logarithmic version of the derived category. For the simple reason that the logarithmic Hochschild homology and cohomology spaces differ from the ordinary ones, we cannot expect the invariance to hold whenever the underlying schemes are derived equivalent.

We describe a condition, that we call strict logarithmic derived equivalence, under which invariance of logarithmic Hochschild homology and cohomology hold.

Definition 4 (definition 4.3.1)**.** Let X and Y be logarithmic schemes, whose underlying schemes are smooth and proper. We say that X and Y are strictly logarithmically derived equivalent if there are

- an Artin fan $\mathscr{A}$ and strict morphisms $\Theta_X \to \mathscr{A}$ and $\Theta_Y \to \mathscr{A}$;

- a smooth and proper logarithmic scheme Z and a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\
X \arrow[dr] & & Y \arrow[dl] \\
 & \mathscr{A} &
\end{tikzcd}$$

  where p_X and p_Y are strict and proper;

- an equivalence of type

$$\mathbf{R}p_{Y,*}(\mathbf{L}p_X^*(-) \otimes \mathscr{P}) : \mathscr{D}^b(X) \to \mathscr{D}^b(Y)$$

  for some $\mathscr{P} \in \mathscr{D}^b(Z)$.

Notice that every strict logarithmic derived equivalence is also a derived equivalence. This suggests the insufficiency of this condition for a comprehensive notion of logarithmic derived equivalence. However, we expect that any good definition of logarithmic derived equivalence should be satisfied by strict logarithmic derived equivalences.

We prove the invariance result under this condition.

Theorem C (theorem 4.3.6)**.** *If X and Y are strictly logarithmically derived equivalent, separated, weakly logarithmically separated and logarithmically flat logarithmic schemes, then their logarithmic Hochschild homology and cohomology spaces are isomorphic.*

Imitating the strategy of [Cal03], we define a logarithmic version of the logarithmic Serre functor relating logarithmic Hochschild homology and cohomology and we show that strict logarithmic derived equivalences conjugate them.

A formality result for orbifolds

The HKR theorem may be refined for orbifolds. An orbifold is a quotient stack obtained by the action of a finite group G on a scheme X. In particular, it is a Deligne-Mumford stack. We stress that, unlike in other references, we require them to be global quotients, instead of just local ones.

It is proved in [ACH19] that the Hochschild homology endofunctor for an orbifold $[X/G]$ may be decomposed as

$$\bigoplus_{g\in G} \mathbf{L}\Delta_g^*\mathbf{R}\Delta_* \simeq \bigoplus_{g\in G} \mathbf{R}q_* \left(\mathbf{L}p^*(-) \otimes \operatorname{Sym}((\Omega_X^1)_g[1])\right)$$

where $\Delta_g : X \to X \times X$ is the twisted diagonal and q and p are the pullbacks of Δ and Δ_g respectively. After taking the global sections, the above expression provides the HKR decomposition for orbifolds.

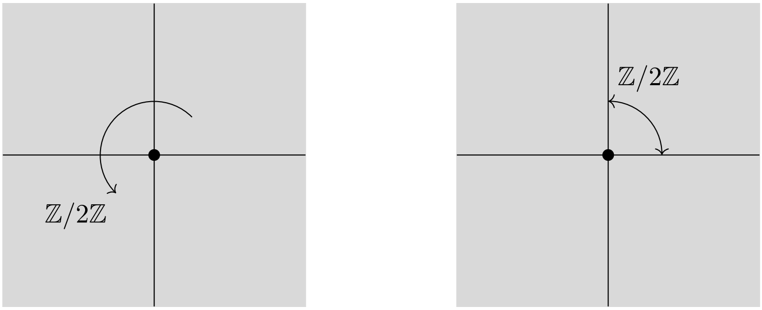

Figure 0.3: On the left, the firm action of $\mathbb{Z}/2\mathbb{Z}$ on the logarithmic scheme associated with $(\mathbb{A}^2, Z(xy))$, rotating the plane of 180°; this action maps each axis onto itself by reflecting it. On the right, the non-firm action of $\mathbb{Z}/2\mathbb{Z}$ on the same logarithmic scheme given by the reflection across the line $x = y$, swapping the two axes.

In order to prove a similar statement for logarithmic Hochschild homology, we need to understand the intersection functors $\mathbf{L}i_g^*\mathbf{R}i_*$. This problem is hard to tackle with our techniques in the very general case. The reason behind this difficulty is the fact that the action of G on X induces an action of $G \times G$ on $X \times X$, but this does not always lift to the strict diagonal $B = X \times_{\Theta_X} X$. For this reason, to produce a result along the lines of [ACH19], we restrict to firm actions, i.e. actions that induce the trivial action on the Artin fan.

For such actions, we prove the following theorem.

Theorem D (Corollary 5.1.16, Corollary 5.1.17)**.** Let X be a quasicompact, weakly logarithmically separated, logarithmically smooth logarithmic scheme and G a finite group acting firmly on X (definition 5.1.3).

- Denoting the strict diagonal for the orbifold $[X/G]$ by
$$\iota : [X/G] \to [X/G] \times_{\Theta_{[X/G]}} [X/G],$$
we have the decomposition of the Hochschild homology endofunctor
$$\mathbf{L}\iota^*\mathbf{R}\iota_*(-) \simeq \bigoplus_{g\in G} \mathbf{L}i_g^*\mathbf{R}i_*(-)$$
where $i_g : X \to X \times_{\Theta_X} X$ denotes the strict twisted diagonal induced by $x \mapsto (x, g.x)$;

- There are natural isomorphisms of endofunctors
$$\mathbf{L}i_g^*\mathbf{R}i_*(-) \simeq \mathbf{R}q_* \left(\mathbf{L}p^*(-) \otimes \operatorname{Sym}\left(\left(\Omega_X^{1,\log}\right)_g [1] \right) \right);$$

- There are isomorphisms of vector spaces

$$\mathrm{HH}^{\ell}_{\bullet}\left([X/G]\right) = \left(\bigoplus_{g\in G} \mathrm{HH}^{\ell}_{n}\left(X^{g}_{\log}\right)\right)^{G} = \left(\bigoplus_{g\in G}\bigoplus_{q-p=n} H^{p}\left(X^{g}_{\log}, \Omega^{q,\log}_{X^{g}_{\log}}\right)\right)^{G}.$$

Structure of the thesis

This thesis is articulated in five chapters. In Chapter 1, we provide an overview of classical facts and methods about Hochschild homology and cohomology, including their definitions for different structures, the HKR theorem, some invariance results, their role in Hodge theory and the on of the cohomology in deformation theory, with a focus on the approach used in [AC12, ACH19] to reprove the HKR theorem. In Chapter 2, we establish the background on logarithmic geometry, starting with the classical approach of [Kat89, Ogu18], then revisiting it is the more modern lexicon of [Ols03, Ols05] and eventually introducing Artin fans and their role in logarithmic geometry and proving Theorem A. In Chapter 3, we give the definition of logarithmic Hochschild homology and cohomology and prove Theorem B, we prove some properties relying on the HKR theorem, we explain what these results mean for smooth pairs, we exhibit some examples of computation of logarithmic Hochschild homology, we define the logarithmic loop space and describe how it fits in the picture and conjecture over the Duflo isomorphism for the logarithmic Hochschild cohomology. In Chapter 4, we introduce the logarithmic Serre functor and use Fourier–Mukai theory to relate logarithmic Hochschild homology and cohomology and we prove Theorem C. In Chapter 5, we investigate intersection theory along l.c.i. morphisms between logarithmic schemes which are not smooth in general (but typically the l.c.i. morphism in question is strict and between schemes which are logarithmically smooth), we use it to prove Theorem D and we conduct an explicit computation for the orbifold obtained by the shuffle action of the symmetric group, displaying a HKR-like decomposition for this example too.

Conventions and notations

Unless differently stated, all schemes and algebraic stacks are meant over $S = \operatorname{Spec} k$ where k is an algebraically closed field of characteristic 0. Similarly, the symbols $\times, \otimes, \mathrm{Hom}$ stand for $\times_S, \otimes_k, \mathrm{Hom}_k$ respectively. By variety, we mean a separated scheme of finite type over S. The properties for schemes or algebraic stacks like flatness, smoothness, separateness are meant over S.

The fiber product of logarithmic schemes is by default meant in the category of fs logarithmic schemes, unless stated otherwise or clear from the context. We denote by $\times, \ulcorner$ the fiber product in the category of logarithmic schemes, by $\times^{\ell}, \ulcorner\ell$ the fiber product in the category of fs logarithmic schemes and by $\times^{\not\ell}, \ulcorner\not\ell$ the fiber product when the two coincide. Throughout the thesis, the stack classifying logarithmic structures will be denoted by $\mathscr{L}og$ and the stack following the classical literature, whilst the stack denoted by $\mathscr{L}$ is the one classifying fs logarithmic structures denoted by $\mathscr{T}or$ in [Ols03]. From chapter 3, all logarithmic schemes are assumed to be fs logarithmic schemes.

Unless differently stated or clear from the context, all functors are derived. The symbols Sym, $\times$, $\otimes$, Hom, f^*, $f_*, \dots$ denote $\mathbf{L}$Sym, $\times^{\mathbf{L}}$, $\otimes^{\mathbf{L}}$, $\mathbf{R}$Hom, $\mathbf{L}f^*$, $\mathbf{R}f_*, \dots$ respectively.

CHAPTER 1

Hochschild homology and cohomology

This chapter is meant to provide background on Hochschild homology and cohomology. In Section 1.1, we give the original definition given for associative algebras and recall the Hochschild–Kostant–Rosenberg isomorphism (from now on, HKR isomorphism) originally proved in [HKR62]. In Section 1.2, we provide the geometric definition (for schemes and stacks), and suitable categories (abelian, dg). In Section 1.3, we list some well-known properties of Hochschild homology and cohomology, some of which we will later prove in the analogous logarithmic theoretical version. In Section 1.4, we revisit the HKR theorem the approach used by Arinkin, Căldăraru and Hablicsek in [AC12, ACH19].

1.1 Definition of Hochschild homology and cohomology

Let k be a field. We denote by $\otimes = \otimes_k$ the tensor product over k.

Definition 1.1.1. Let A be an associative k-algebra with unit. The Hochschild homology chain complex (elsewhere called bar complex) of A over k is

$$C_*(A) = \cdots \to A^{\otimes n+1} \xrightarrow{d_n} A^{\otimes n} \to \cdots \to A \otimes A \xrightarrow{d_1} A \to 0$$

where the differential $d_n : A^{\otimes n+1} \to A^{\otimes n}$ is given by $d_n = \sum_{i=0}^{n} (-1)^i \delta_n^i$ with δ_n^i is extended k-linearly by

$$\delta_n^i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{if } i \neq n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n \end{cases} .$$

The Hochschild homology of A is the homology of the Hochschild homology chain complex

$$\mathrm{HH}_n(A) = H_n(C_*(A)).$$

Example 1.1.2. The differential $d_1 : A \otimes A \to A$ is defined as

$$a \otimes b \mapsto ab - ba = [a, b].$$

The degree-zero Hochschild homology space is

$$\mathrm{HH}_0(A) = \mathrm{coker}(d_1) = A/[A, A]$$

and is isomorphic to A if and only if A is commutative.

Definition 1.1.3. Let A be an associative k-algebra with unit. The Hochschild cohomology complex of A over k is

$$C^*(A) = 0 \to A \xrightarrow{d^1} \mathrm{Hom}_k(A, A) \to \cdots \to \mathrm{Hom}_k(A^{\otimes n}, A) \xrightarrow{d^{n+1}} \mathrm{Hom}_k(A^{\otimes n+1}, A) \to \ldots$$

where $d^n = \sum_{i=1}^{n} (-1)^i \gamma_i^n$ with γ_i^n being the k-linear extension of

$$\gamma_i^n(f)(a_1 \otimes \cdots \otimes a_n) = \begin{cases} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) & \text{if } i \neq n \\ f(a_n a_1 \otimes a_1 \otimes \cdots \otimes a_{n-1}) & \text{if } i = n \end{cases}.$$

The Hochschild cohomology of A is the cohomology of the Hochschild cohomology cochain complex

$$\mathrm{HH}^n(A) = H^n(C^*(A)).$$

Example 1.1.4. The first differential d^1 is defined as

$$a \mapsto [a, -].$$

Thus $\mathrm{HH}^0(A) = \ker(d^1) = Z(A)$, denoting the center of A, and is isomorphic to A is and only in A is commutative.

1.1.1 Hochschild homology and cohomology with coefficients

Let A be an associative k-algebra with unit and M a A-bimodule. Hochschild homology and cohomology may be defined with coefficients in M.

Definition 1.1.5. The Hochschild homology complex of A with coefficients in M is

$$C_*(A, M) = \cdots \to M \otimes A^{\otimes n} \xrightarrow{d_n} M \otimes A^{\otimes n-1} \to \cdots \to M \otimes A \xrightarrow{d_1} M \to 0$$

where the differential $d_n : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ is given by $d_n = \sum_{i=0}^{n} (-1)^i \delta_n^i$ with δ_n^i is extended k-linearly by

$$\delta_n^i(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} m a_1 \otimes a_2 \otimes \cdots \otimes a_n & \text{if } i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{if } i \neq 0, n \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n \end{cases}.$$

The Hochschild homology of A with coefficients in M is the homology of the Hochschild homology chain complex

$$\mathrm{HH}_n(A, M) = H_n(C_*(A, M)).$$

Definition 1.1.6. The Hochschild cohomology complex of A with coefficients in M is

$$\begin{aligned} C^*(A, M) = 0 \to M \xrightarrow{d^1} \mathrm{Hom}_k(A, M) \to \dots \\ \dots \to \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{d^n} \mathrm{Hom}_k(A^{\otimes n+1}, M) \to \dots \end{aligned}$$

where $d^n = \sum_{i=0}^{n}(-1)^i \gamma_i^n$ with γ_i^n being the k-linear extension of

$$\gamma_i^n(f)(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 f(a_1 \otimes \cdots \otimes a_n) & \text{if } i = 0 \\ f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) & \text{if } i \neq 0, n \\ f(a_0 \otimes \cdots \otimes a_{n-1}) a_n & \text{if } i = n \end{cases} .$$

The Hochschild cohomology of A with coefficients in M is the cohomology of the Hochschild cohomology cochain complex

$$\mathrm{HH}^n(A, M) = H^n(C^*(A, M)).$$

Remark 1.1.7. The Hochschild homology and cohomology of A are, according to these more general definitions, the Hochschild homology and cohomology of A with coefficients in A:

$$\mathrm{HH}_\bullet(A) = \mathrm{HH}_\bullet(A, A) \quad \text{and} \quad \mathrm{HH}^\bullet(A) = \mathrm{HH}^\bullet(A, A).$$

1.2 Equivalent definitions and generalizations

The first result that we recall is proved by Cartan and Eilenberg in [CE56] and gives an equivalent definition of Hochschild homology and cohomology in terms of the Tor and Ext functors. Let k be a field, A an associative k-algebra with unit and M a A-bimodule. We denote $\otimes = \otimes_k$.

Definition 1.2.1. The enveloping algebra of A is $A^e := A^{op} \otimes A$.

Lemma 1.2.2 ([CE56])**.** The category of right A^e-modules $\mathrm{Mod}(A^e)$ is equivalent to the category of A-bimodules.

Proof. Let M be a A-bimodule. One may consider it as a right A^e-module considering the same underlying abelian group with multiplication with elements of A^e extended k-linearly from by $m \cdot (a \otimes b) = bma$. Vice versa, if N is a A^e-module, one can set the left (resp. right) multiplication times an element $a \in A$ to be given by $a \cdot m = m(1 \otimes a)$ (resp. $m \cdot a = m(a \otimes 1)$). □

Theorem 1.2.3 ([CE56])**.** For A and M as above,

$$\mathrm{HH}_\bullet(A, M) = \mathrm{Tor}_\bullet^{A^e}(A, M) \quad \text{and} \quad \mathrm{HH}^\bullet(A, M) = \mathrm{Ext}_{A^e}^\bullet(A, M)$$

1.2.1 Hochschild homology and cohomology for schemes

We now recall the geometric definition of Hochschild homology and cohomology for schemes. Let $S = \operatorname{Spec} k$ and X be a scheme over S. Denote by $\Delta : X \to X \times_S X$ the diagonal morphism (into the derived fiber product).

Definition 1.2.4. Let $\mathcal{F} \in \mathcal{D}(X)$. The Hochschild homology of X with coefficients in $\mathcal{F}$ is

$$\mathrm{HH}_n(X, \mathcal{F}) = \mathrm{Tor}_n^{\mathcal{O}_{X \times_S X}}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{F}).$$

The Hochschild cohomology of X with coefficients in $\mathcal{F}$ is

$$\mathrm{HH}^n(X, \mathcal{F}) = \mathrm{Ext}^n_{\mathcal{O}_{X \times_S X}}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{F}).$$

The Hochschild homology and cohomology of X are respectively the Hochschild homology and cohomology of X with coefficients in the structure sheaf $\mathcal{O}_X$ and we denote them by

$$\mathrm{HH}_\bullet(X) = \mathrm{HH}_\bullet(X, \mathcal{O}_X) \quad \text{and} \quad \mathrm{HH}^\bullet(X) = \mathrm{HH}^\bullet(X, \mathcal{O}_X).$$

Remark 1.2.5. It follows by Theorem 1.2.3 that the Hochschild homology and cohomology of a commutative ring are the Hochschild homology and cohomology of its spectrum (with coefficients in the structure sheaf). The Hochschild homology and cohomology of a commutative ring with coefficients in a finitely generated module (seen as a bimodule) are the Hochschild homology and cohomology of the spectrum with coefficients in the module seen as a coherent sheaf.

Equivalently, Hochschild homology and cohomology may be defined via derived self-intersections. With this approach, the Hochschild homology and cohomology spaces are realised as global sections of the Hochschild homology and cohomology complexes. This is the definition we will more often refer to in this thesis. In Section 1.4, we will give an outline of [AC12] and [ACH19], which constitute the main advantage of this approach for us.

Definition 1.2.6. The Hochschild homology endofunctor is given

$$\Delta^* \Delta_* : \mathcal{D}(X) \to \mathcal{D}(X).$$

The Hochschild cohomology endofunctor is given

$$\Delta^! \Delta_* : \mathcal{D}(X) \to \mathcal{D}(X).$$

Let $\mathcal{F} \in \mathcal{D}(X)$. The Hochschild homology and cohomology spaces of X with coefficients in $\mathcal{F}$ is

$$\mathrm{HH}_n(X, \mathcal{F}) = R^n\Gamma(X, \Delta^* \Delta_* \mathcal{F}) \quad \text{and} \quad \mathrm{HH}^n(X, \mathcal{F}) = R^n\Gamma(X, \Delta^! \Delta_* \mathcal{F})$$

respectively. The Hochschild homology and cohomology spaces of X are the Hochschild homology and cohomology spaces with coefficients in $\mathcal{O}_X$

$$\mathrm{HH}_n(X) = \mathrm{HH}_n(X, \mathcal{O}_X) \quad \text{and} \quad \mathrm{HH}^n(X) = \mathrm{HH}^n(X, \mathcal{O}_X).$$

Lastly, we recall the equivalent definition used by Kuznetsov in [Kuz09]. This will be useful in Chapter 4 for the proof of the invariance. We only give it for the space because of the use that we will make of it later.

Definition 1.2.7. Let X be a scheme over S. The Hochschild cohomology and homology spaces are respectively

$$\mathrm{HH}^\bullet(X) = \mathrm{Hom}^\bullet_{X\times_S X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$$

and

$$\mathrm{HH}_\bullet(X) = \mathbb{H}^\bullet(X \times_S X, \Delta_*\mathcal{O}_X \otimes \Delta_*\mathcal{O}_X)$$

respectively, where $\mathrm{Hom}^\bullet$ is graded by the degree of the morphisms.

1.2.2 Hochschild homology and cohomology for categories

In this subsection, we recall the definitions of Hochschild homology and cohomology for dg-categories and their relation to the Hochschild homology and cohomology of schemes. Although we will not conduct any investigation in this direction, we provide this background to give a flavour of other possible points of view on the topic and motivate the problem we will address in Chapter 4. More exhaustive discussions can be found in [Kel06] and [LVdB05].

We start with the definition of Hochschild homology and cohomology for k-linear categories.

Definition 1.2.8 ([Mit72])**.** Let k be a field and $\mathcal{C}$ a k-linear category. The Hochschild chain complex of $\mathcal{C}$ is $C_*(\mathcal{C})$ with

$$C_n(\mathcal{C}) = \bigsqcup_{X_0,\dots,X_n\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X_n, X_0) \otimes \mathrm{Hom}_{\mathcal{C}}(X_{n-1}, X_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(X_0, X_1)$$

and differentials $d_n = \sum_{i=0}^n (-1)^i \delta_n^i : C_n(\mathcal{C}) \to C_{n-1}(\mathcal{C})$ where

$$\delta_n^i(f_n \otimes f_{n-1} \otimes \cdots \otimes f_0) = \begin{cases} f_n \otimes \cdots \otimes f_i f_{i-1} \otimes \cdots \otimes f_0 & \text{if } i \neq 0 \\ f_0 f_n \otimes f_{n-1} \otimes \cdots \otimes f_1 & \text{if } i = 0 \end{cases}$$

The Hochschild homology of $\mathcal{C}$ is the homology of the Hochschild chain complex of $\mathcal{C}$.

Definition 1.2.9 ([Mit72])**.** Let k be a field and $\mathcal{C}$ a k-linear category. The Hochschild cochain complex of $\mathcal{C}$ is $C^*(\mathcal{C})$ with

$$C^n(\mathcal{C}) = \bigsqcup_{X_0,\dots,X_n\in\mathcal{C}} \mathrm{Hom}_k(\mathrm{Hom}_{\mathcal{C}}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(X_{n-1}, X_n), \mathrm{Hom}_{\mathcal{C}}(X_0, X_n))$$

and differentials as in Definition 1.1.3. The Hochschild cohomology of $\mathcal{C}$ is the homology of the Hochschild cochain complex of $\mathcal{C}$.

Example 1.2.10 ([McC94])**.** Let A be an associative k-algebra with unit. Consider A as a category with one object. The Hochschild chain complex of A as algebra coincides with the Hochschild chain complex as k-linear category. Moreover, the functor $A \to \operatorname{Proj} A$ where $\operatorname{Proj} A$ denotes the category of finitely generated projective A-modules induces a quasi-isomorphism $C_*(A) \to C_*(\operatorname{Proj} A)$.

We give the analogous definitions for dg-categories.

Definition 1.2.11. Let k be a field and $\mathcal{C}$ a dg-category over k. Define the double complex having component with index (p, q)

$$C_{p,q}(\mathcal{C}) = \bigsqcup_{\substack{X_0,\dots,X_q \in \mathcal{C} \\ i_0+\cdots+i_q=p}} \operatorname{Hom}_{\mathcal{C}}^{i_q}(X_q, X_0) \otimes \operatorname{Hom}_{\mathcal{C}}^{i_{q-1}}(X_{q-1}, X_q) \otimes \cdots \otimes \operatorname{Hom}_{\mathcal{C}}^{i_0}(X_0, X_1),$$

with the vertical differentials being those naturally given by the dg-structure and horizontal ones $d_{p,q} = \sum_{i=0}^{q} (-1)^i \delta^i_{p,q} : C_{p,q}(\mathcal{C}) \to C_{p,q-1}(\mathcal{C})$ where

$$\delta^i_{p,q}(f_q \otimes f_{q-1} \otimes \cdots \otimes f_0) = \begin{cases} f_q \otimes \cdots \otimes f_i f_{i-1} \otimes \cdots \otimes f_0 & \text{if } i \neq 0 \\ (-1)^{(q+r)} f_0 f_q \otimes f_{q-1} \otimes \cdots \otimes f_1 & \text{if } i = 0 \end{cases}$$

with $r = i_0(i_1 + \cdots + i_q)$. The Hochschild homology complex of $\mathcal{C}$ is the sum total complex

$$C_*(\mathcal{C}) = \operatorname{Tot}^{\oplus}_*(C_{\bullet,\bullet}(\mathcal{C})) = \bigoplus_{p+q=*} C_{p,q}(\mathcal{C})$$

The Hochschild homology of $\mathcal{C}$ is the homology of the Hochschild chain complex of $\mathcal{C}$.

Definition 1.2.12. Let k be a field and $\mathcal{C}$ a small cofibrant dg-category over k. Define the double complex having component with index (p, q)

$$C^{p,q}(\mathcal{C}) = \prod_{X_0,\dots,X_q \in \mathcal{C}} \operatorname{Hom}_k^p(\operatorname{Hom}_{\mathcal{C}}(X_0, X_1) \otimes \cdots \otimes \operatorname{Hom}_{\mathcal{C}}(X_{q-1}, X_q), \operatorname{Hom}_{\mathcal{C}}(X_0, X_q))$$

with the vertical differentials being those naturally given by the dg-structure and horizontal ones given as in Definition 1.1.3. with $r = i_0(i_1 + \cdots + i_q)$. The Hochschild homology complex of $\mathcal{C}$ is the product total complex

$$C_*(\mathcal{C}) = \operatorname{Tot}^{\Pi}_*(C^{\bullet,\bullet}(\mathcal{C})) = \prod_{p+q=*} C^{p,q}(\mathcal{C})$$

The Hochschild homology of $\mathcal{C}$ is the homology of the Hochschild chain complex of $\mathcal{C}$.

We will not dive into the technicalities of this definition and the techniques involved in this approach. However, these definitions allow us to state an important result for the spirit of our further investigation.

Theorem 1.2.13 ([Kel06], [LVdB05])**.** Let X be a smooth and proper variety. Then

$$\mathrm{HH}_\bullet(X) = \mathrm{HH}_\bullet(\mathcal{D}^b(X)) \quad \text{and} \quad \mathrm{HH}^\bullet(X) = \mathrm{HH}^\bullet(\mathcal{D}^b(X))$$

where $\mathcal{D}^b(X)$ denotes the bounded derived category of coherent sheaves on X.

1.3 Properties of Hochschild homology and cohomology

In this section, we recall some of the properties of Hochschild homology and cohomology in the different setups. These features constitute the motivation for the study of these tools.

1.3.1 Functoriality

The constructions of Hochschild homology performed earlier in this chapter are functorial. However, the Hochschild cohomology is generally not functorial, but in some setting that we will outline upon the interest of our investigation. Let us begin by elaborating on this for the Hochschild homology for algebras and schemes.

Lemma 1.3.1. Let A and B be associative k-algebras with unit and $f : A \to B$ a morphism. There is an induced morphism of graded algebras

$$\mathrm{HH}_\bullet(f) : \mathrm{HH}_\bullet(A) \to \mathrm{HH}_\bullet(B).$$

Let M and N be A-bimodules and $g : M \to N$ a morphism. There are induced morphisms of k-vector spaces

$$\mathrm{HH}_\bullet(g) : \mathrm{HH}_\bullet(A, M) \to \mathrm{HH}_\bullet(A, N) \quad \text{and} \quad \mathrm{HH}^\bullet(g) : \mathrm{HH}^\bullet(A, M) \to \mathrm{HH}^\bullet(A, N).$$

Proof. The morphism $A \to B$ induces a morphism of chain complexes

$$\begin{tikzcd}
\dots \arrow[r] & A^{\otimes n+1} \arrow[r] \arrow[d, "f^{\otimes n+1}"] & A^{\otimes n} \arrow[r] \arrow[d, "f^{\otimes n}"] & \dots \arrow[r] & A \otimes A \arrow[r] \arrow[d, "f \otimes f"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\dots \arrow[r] & B^{\otimes n+1} \arrow[r] & B^{\otimes n} \arrow[r] & \dots \arrow[r] & B \otimes B \arrow[r] & B \arrow[r] & 0
\end{tikzcd}.$$

Then $\mathrm{HH}_\bullet(f) = H_\bullet(f^{\otimes *})$. The morphism $g : M \to N$ induces a morphism of chain complexes

$$\begin{tikzcd}
\dots \arrow[r] & M \otimes A^{\otimes n} \arrow[r] \arrow[d, "g \otimes \mathrm{Id}"] & M \otimes A^{\otimes n-1} \arrow[r] \arrow[d, "g \otimes \mathrm{Id}"] & \dots \arrow[r] & M \arrow[r] \arrow[d, "g"] & 0 \\
\dots \arrow[r] & N \otimes A^{\otimes n} \arrow[r] & N \otimes A^{\otimes n-1} \arrow[r] & \dots \arrow[r] & N \arrow[r] & 0
\end{tikzcd}.$$

and of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "g"] & \dots \arrow[r] & \mathrm{Hom}(A^{\otimes n}, M) \arrow[r] \arrow[d, "g \circ -"] & \mathrm{Hom}(A^{\otimes n+1}, M) \arrow[r] \arrow[d, "g \circ -"] & \dots \\
0 \arrow[r] & N \arrow[r] & \dots \arrow[r] & \mathrm{Hom}(A^{\otimes n}, N) \arrow[r] & \mathrm{Hom}(A^{\otimes n+1}, M) \arrow[r] & \dots
\end{tikzcd}$$

Then $\mathrm{HH}_\bullet(g) = H_\bullet(g \otimes \mathrm{Id})$ and $\mathrm{HH}^\bullet(g) = H^\bullet(g \circ -)$. $\square$

Remark 1.3.2. Functoriality fails for the Hochschild cohomology spaces of algebras. In the non-commutative setting, counterexamples are easy to produce by taking an algebra with trivial center and a commutative subalgebra, onto which the algebra projects. In a similar fashion, take $B = \mathrm{Mat}_n(k)$ the k-algebra of square matrices of dimension $n \times n$. We have that $Z(B) = \{c \cdot \mathrm{Id}_n\}_{c \in k} \simeq k$. Let $A \subset B$ be the commutative subalgebra of diagonal matrices $A = \{(c_{ij})_{i,j=1,\dots,n} | c_{ij} = 0 \text{ if } \ i \neq j\} \simeq k^n$. Factor the identity of A as

$$A \hookrightarrow B \twoheadrightarrow A.$$

Since $\mathrm{HH}^0(A) = A \simeq k^n$, and $\mathrm{HH}^0(B) \simeq k^n$, any induced morphism factors as

$$k^n \to k \to k^n,$$

which never gives the identity.

Functionality for Hochschild homology of schemes is better expressed in terms of the endofunctors.

Lemma 1.3.3. Let X, Y be two schemes over k, with Y quasi-separated over k with quasi-separated diagonal $\Delta_Y : Y \to Y \times_{\mathrm{Spec}\, k} Y$. Let $f : X \to Y$ be a morphism. Then there are natural transformations of functors of bounded derived coherent sheaves

$$\Delta_Y^* \Delta_{Y,*} f_* \to f_* \Delta_X^* \Delta_{X,*} \quad \text{and} \quad f^* \Delta_Y^* \Delta_{Y,*} \to \Delta_X^* \Delta_{X,*} f^*.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\Delta_X"'] & & \\
& X \times_Y X \arrow[r, "g"] \arrow[d, "\delta"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\
& X \times X \arrow[r, "f \times f"] & Y \times Y
\end{tikzcd} . \tag{1.1}$$

We compose

$$\Delta_Y^* \Delta_{Y,*} f_* \simeq \Delta_Y^* (f \times f)_* \Delta_{X,*} \to g_* \delta^* \Delta_{X,*} \to g_* h_* h^* \delta^* \Delta_{X,*} \simeq f_* \Delta_X^* \Delta_{X,*}$$

where the second transformation is given by [Sta24, 02N7] and the third one is given by the counit of the adjunction (h^*, h_*). Similarly when have

$$f^* \Delta_Y^* \Delta_{Y,*} \simeq \Delta_X^* (f \times f)^* \Delta_{Y,*} \to \Delta_X^* \delta_* g^* \to \Delta_X^* \delta_* h_* h^* g^* \simeq \Delta_X^* \Delta_{X,*} f^*.$$

□

Remark 1.3.4. Many properties of the natural transformations in the statement of Lemma 1.3.3 depend on the properties of h (in particular, of the counit $1 \to h_* h^*$), which are typically induced by properties of the map f. We will see an example below.

A consequence of the lemma is the following invariance result with respect to étale morphisms of schemes.

Corollary 1.3.5. Let $f : X \to Y$ be an étale morphism of schemes smooth and proper schemes. The morphism

$$f^* \Delta_Y^* \Delta_{Y,*} \mathscr{F} \to \Delta_X^* \Delta_{X,*} f^* \mathscr{F}$$

from Lemma 1.3.3 induces an isomorphism of graded algebras between the Hochschild homology spaces of X and Y.

Proof. Let $\mathscr{F} \in \mathscr{D}^b(Y)$ and consider the morphism

$$f^* \Delta_Y^* \Delta_{Y,*} \mathscr{F} \to \Delta_X^* \Delta_{X,*} f^* \mathscr{F}.$$

When f étale, $f \times f$ is étale as well. We use the notation of Diagram 1.1. The morphism g is étale, being the pullback of an étale morphism. The morphism h is étale since both f and g are. Thus $1 \to h_* h^*$ is an equivalence. The morphism of derived coherent sheaves is thereby given by a composition of isomorphisms, and so is an isomorphism. Computing its derived global sections

$$R\Gamma(X, f^* \Delta_Y^* \Delta_{Y,*} \mathscr{F}) \xrightarrow{\sim} R\Gamma(X, \Delta_X^* \Delta_{X,*} f^* \mathscr{F}).$$

Because f is étale, by [Sta24, 03YU],

$$R\Gamma(X, f^* \Delta_Y^* \Delta_{Y,*} \mathscr{F}) \simeq R\Gamma(Y, \Delta_Y^* \Delta_{Y,*} \mathscr{F}).$$

Take $\mathscr{F} = \mathscr{O}_X$. We obtain the isomorphism

$$\mathrm{HH}_\bullet(Y) \simeq \mathrm{HH}_\bullet(X).$$

□

Remark 1.3.6. For $f : X \to Y$ as in the statement of Lemma 1.3.3 which is moreover unramified, there is a similar equivalence between the Hochschild cohomology endofunctors

$$\Delta_X^! \Delta_{X,*} f^* \simeq f^* \Delta_Y^! \Delta_{Y,*},$$

implying

$$\mathrm{HH}^\bullet(X) \simeq \mathrm{HH}^\bullet(Y).$$

1.3.2 The HKR isomorphism and the role of Hochschild co/homology in Hodge theory

The most interesting property of these spaces for our investigation is the HKR isomorphism. It was originally proved by Hochschild, Kostant and Rosenberg for algebras over rings and has been generalized for schemes and stacks later. We recall the statements of the consecutive versions.

Theorem 1.3.7 (HKR for algebras, [HKR62])**.** Let k be a field and A a finitely presented and smooth commutative k-algebra with unit. Then there is an isomorphism of graded k-algebras

$$\wedge^{\bullet}\Omega_k(A) \simeq \mathrm{HH}_{\bullet}(A),$$

where the left-hand side denotes the complex with zero differentials $\wedge^{\bullet+1}\Omega_k(A) \to \wedge^{\bullet}\Omega_k(A)$ and an isomorphism of graded k-modules

$$\wedge^{\bullet}\mathrm{Der}_k(A,A) \simeq \mathrm{HH}^{\bullet}(A),$$

where the left-hand side is the complex with zero differentials

$$\wedge^{\bullet}\mathrm{Der}_k(A,A) \to \wedge^{\bullet+1}\mathrm{Der}_k(A,A).$$

This statement has more and more general scheme-theoretical versions. We recall the statement as in [Cal05], recovered as a corollary of a theorem on the Hochschild homology complex. The reader might want to refer to [Kon03], [Swa96] and [Yek01] for an overview of the history towards this result.

Theorem 1.3.8 ([Cal05])**.** Let k be an algebraically closed field of characteristic 0. Let X be a smooth, quasi-projective variety of dimension n, and let $\Delta : X \to X \times_{\operatorname{Spec} k} X$ be the diagonal embedding. Then there exists a quasi-isomorphism

$$I : \Delta^*\Delta_*\mathcal{O}_X \to \bigoplus_{i=0}^{\dim X} \bigwedge^{i} \Omega^1_X[i]$$

where the right-hand side denotes the complex with $\bigwedge^i \Omega^1_X$ in degree $-i$ and zero differentials.

Corollary 1.3.9 (HKR for schemes, [Cal05])**.** The quasi-isomorphism I induces isomorphisms of graded vector spaces

$$\mathrm{HH}_{\bullet}(X) \xrightarrow{\sim} \bigoplus_{q-p=\bullet} H^p(X,\Omega^q_X);$$

$$\mathrm{HH}^{\bullet}(X) \xrightarrow{\sim} \bigoplus_{p+q=\bullet} H^p(X,\wedge^q T_X).$$

The HKR theorem has its main application in Hodge theory. The result formally states the informal fact that the Hochschild homology spaces compute the diagonal parts of the Hodge diamond and, for Calabi-Yau varieties, the Hochschild cohomology spaces compute the horizontal parts of the Hodge diamond. In particular, they give an exhaustive description of Hodge diamonds for smooth and projective surfaces and Calabi-Yau threefolds.

Corollary 1.3.10.

1. Let X be a connected, smooth and proper curve or surface. The dimensions of its Hochschild homology spaces determine the Hodge diamond of X.

2. Let X be a connected, smooth and proper variety of dimension 3 or 4. Additionally, suppose that the canonical bundle of X is trivial. Then the dimensions of its Hochschild homology and cohomology spaces determine the Hodge diamond of X.

Proof. Preliminary, recall that by Hodge symmetry $h^{p,q} = h^{q,p}$, and by Serre duality $h^{p,q} = h^{d-p,d-q}$, where $d = \dim X$. Let X be a curve. Its Hodge diamond is determined by its genus $g = h^{1,0} = h^{0,1}$. By Corollary 1.3.9, $h^{0,1} = \dim \mathrm{HH}_1(X)$. For surfaces, the Hodge diamond is of the shape

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2}. \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

By the symmetries of the Hodge diamond, it suffices to compute $h^{0,0}, h^{0,1}, h^{0,2}, h^{1,1}$. Because X is connected, $h^{0,0} = 1$. By Corollary 1.3.9 and the symmetry properties

$$\begin{aligned} \dim \mathrm{HH}_2(X) &= h^{0,2}; \\ \dim \mathrm{HH}_1(X) &= h^{0,1} + h^{1,2} = h^{0,1} + h^{2,1} = 2h^{0,1}; \\ \dim \mathrm{HH}_0(X) &= h^{0,0} + h^{1,1} + h^{2,2} = 2h^{0,0} + h^{1,1} = 2 + h^{1,1}. \end{aligned}$$

Let us consider the case of threefolds with trivial canonical bundle. The Hodge diamond is

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & \underline{h^{1,2}} & & \underline{h^{0,3}} \\ & h^{2,0} & & \underline{h^{1,1}} & & \underline{h^{0,2}} & \\ & & h^{1,0} & & \underline{h^{0,1}} & & \\ & & & \underline{h^{0,0}} & & & \end{array}$$

with $h^{0,0} = h^{3,3} = 1$ because of connectedness, and $h^{0,3} = h^{3,0}$ by Hodge symmetry. The Hodge diamond is determined by the underlined entries. In a similar fashion as above,

$$\begin{aligned} \dim \mathrm{HH}_0(X) &= 2h^{0,0} + 2h^{1,1} = 2 + 2h^{1,1}; \\ \dim \mathrm{HH}_1(X) &= 2h^{0,1} + h^{1,2}; \\ \dim \mathrm{HH}_2(X) &= 2h^{0,2}; \\ \dim \mathrm{HH}_3(X) &= h^{0,3}. \end{aligned}$$

In order to determine $h^{0,1}$ and $h^{1,2}$, we observe that Corollary 1.3.9 and Serre duality together give

$$\begin{aligned} \dim \mathrm{HH}^1(X) &= \dim H^1(X, \mathcal{O}_X) + \dim H^0(X, T_X) \\ &= \dim H^1(X, \mathcal{O}_X) + \dim H^3(X, \Omega^1_X) = h^{1,0} + h^{3,1} \end{aligned}$$

Using the symmetry properties recalled above, we obtain that $\dim \mathrm{HH}^1(X) = h^{0,1} + h^{0,2}$. This, together with the previous data, determines the Hodge numbers. □

1.3.3 Invariance theorems for the Hochschild homology and cohomology

One of the features (or, more precisely, a class of features) making Hochschild homology and cohomology interesting data is constituted by their invariance properties. We will see below that these spaces respect notions of equivalences which are suitable for the setting (algebraic, geometric, derived).

Proposition 1.3.11 ([Wei94])**.** Let A and B associative k-algebras with unit which are Morita equivalent. Then

$$\mathrm{HH}_\bullet(A) \simeq \mathrm{HH}_\bullet(B) \quad \text{and} \quad \mathrm{HH}^\bullet(A) \simeq \mathrm{HH}^\bullet(B).$$

If M is a A-bimodule and N is a B-bimodule corresponding under Morita equivalence, then

$$\mathrm{HH}_\bullet(A, M) \simeq \mathrm{HH}_\bullet(B, N) \quad \text{and} \quad \mathrm{HH}^\bullet(A, M) \simeq \mathrm{HH}^\bullet(B, N).$$

Proposition 1.3.12 ([Kel06])**.** Let A and B quasi-isomorphic differential graded k-algebras. Then

$$\mathrm{HH}_\bullet(A) \simeq \mathrm{HH}_\bullet(B) \quad \text{and} \quad \mathrm{HH}^\bullet(A) \simeq \mathrm{HH}^\bullet(B).$$

Proposition 1.3.13 ([Kel21])**.** Let $\mathscr{D}$ and $\mathscr{D}'$ be two derived equivalent derived categories. Then their Hochschild homology and cohomology groups are isomorphic.

This statement can somehow be refined to a stronger one, making the Hochschild homology spaces additive invariants.

Proposition 1.3.14 ([Tab05])**.** Let $\mathscr{D}$ be a derived category with a semiorthogonal decomposition

$$\mathscr{D} = \langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle.$$

Then $\mathrm{HH}_\bullet(\mathscr{D}) = \bigoplus_{i=1}^n \mathrm{HH}_\bullet(\mathscr{A}_i)$.

A straightforward consequence of Proposition 1.3.13, together with Theorem 1.2.13, provides the geometric invariance result.

Corollary 1.3.15. Let X and Y two derived equivalent schemes smooth and proper. Then

$$\mathrm{HH}_\bullet(X) \simeq \mathrm{HH}_\bullet(Y) \quad \text{and} \quad \mathrm{HH}^\bullet(X) \simeq \mathrm{HH}^\bullet(Y).$$

This is a straightforward consequence of the results mentioned above and is consistent with the intention of generalising the theory from schemes to derived categories. However, the statement of the last corollary has been known for longer than the more advanced technology employed here. A proof for Fourier–Mukai equivalences appears in [Cal03]. This will be relevant for our treatment of the analogous logarithmic-theoretical statement proved in this thesis.

1.3.4 Hochschild cohomology and deformation theory

This subsection is devoted to the interpretation of low-degree Hochschild cohomology vector spaces, investigated in [Ger64].

Let us first consider an associative k-algebra with unit A and a A-bimodule M. Following the same reasoning as in Example 1.1.4, the degree-zero Hochschild cohomology vector spaces is

$$\mathrm{HH}^0(A, M) = \ker \begin{pmatrix} M & \xrightarrow{d_1} & \mathrm{Hom}(A, M) \\ m & \mapsto & [m, -] \end{pmatrix} = \{m \in M | ma = am \ \forall a \in A\} = M^A.$$

For the degree-one part, let us notice that every element of $m \in M$ naturally induces a derivation $l_m(-) = [m, -]$, which is exactly the image of m via the Hochschild differential d_1. We call derivations of this type inner derivations. Let us now look at

$$\begin{aligned} d_2 : \mathrm{Hom}(A, M) &\to \mathrm{Hom}(A \otimes A, M) \\ f &\mapsto (a \otimes b \mapsto af(b) - f(ab) + f(a)b). \end{aligned}$$

An element in $\mathrm{Hom}(A, M)$ is annihilated d_2 if and only if it satisfies Leibniz's rule. Therefore

$$\mathrm{HH}^1(A, M) = \mathrm{Der}(A, M)/\{\text{inner derivations}\}.$$

A more involved discussion concerns $\mathrm{HH}^2(A)$ and $\mathrm{HH}^3(A)$, whose scheme-theoretical extension is the subject of the study in [Tod09] and allows explicit computations.

Definition 1.3.16. A first-order deformation of A is an associative product $*$ on $A[\varepsilon]/(\varepsilon^2)$ such that $a * b = ab \mod \varepsilon$. We say that two first-order deformations $(A[\varepsilon]/(\varepsilon^2), *)$ and $(A[\varepsilon]/(\varepsilon^2), *')$ of A are equivalent if there is an isomorphism of algebras between them that is the identity on A, i.e.

$$\begin{tikzcd} (A[\varepsilon]/(\varepsilon^2), *) \arrow[rr, "\sim"] \arrow[dr] & & (A[\varepsilon]/(\varepsilon^2), *') \arrow[dl] \\ & A & \end{tikzcd}.$$

The space of k-linear deformations of A is the quotient set of the set of deformations modulo equivalence and is denoted by $\mathcal{D}ef_k(A)$.

Deformations of A are parameterized by morphisms of algebras $\mu : A \otimes A \to A$ such that

$$a\mu(b \otimes c) + \mu(a \otimes (bc)) = \mu(a \otimes b)c + \mu((ab) \otimes c).$$

Such a morphism defines an associative product $*$ by setting $a * b = ab + \mu(a \otimes b)$ which gives a deformation. Vice versa, associativity of $*$ guarantees that $\mu(a \otimes b) := (a * b - ab)|_{\varepsilon=1}$ satisfies the condition above. Rearranging the equation, we obtain

$$a\mu(b \otimes c) - \mu(a \otimes b)c + \mu(a \otimes (bc)) - \mu((ab) \otimes c) = 0,$$

which is exactly the Hochschild differential computed at μ equated to zero. Therefore such morphisms $\mu : A \otimes A \to A$ are exactly the Hochschild 2-cocycles.

Suppose now that $*$ and $*'$ define equivalent deformations of A, let μ and μ' be the associated 2-cocycles and $\varphi : (A[\varepsilon]/(\varepsilon^2), *) \to (A[\varepsilon]/(\varepsilon^2), *')$ a compatible isomorphism. The compatibility implies that $\varphi(a_0) = a_0 + \lambda(a_0)\varepsilon$, for some k-linear morphism $\lambda : A \to A$. Moreover, λ must satisfy

$$\mu(a \otimes b) + \lambda(ab) = \mu'(a \otimes b) + a\lambda(b) + \lambda(a)b$$

since φ is an isomorphism of algebras. Similarly to the previous discussion, such morphisms parameterize the equivalences of deformations. Rearranging the equation above, we obtain

$$\mu(a \otimes b) - \mu'(a \otimes b) = a\lambda(b) - \lambda(ab) + \lambda(a)b = d(\lambda)(a \otimes b).$$

Therefore, equivalences between deformations of an algebra correspond to its Hochschild 2-coboundaries.

This proves that $\mathrm{HH}^2(A) \simeq \mathscr{D}ef_k(A)$ as sets.

To conclude this discussion, we give a deformation-theoretical interpretation for $\mathrm{HH}^3(A)$. To this purpose, let us define second-order deformations.

Definition 1.3.17. A second-order deformation of A is an associative product $*$ on $A[\varepsilon]/(\varepsilon^3)$ such that $a * b = ab \mod \varepsilon$. We say that a second-order deformation $(A[\varepsilon]/(\varepsilon^3), *)$ is an extension of a given first-order deformation $(A[\varepsilon]/(\varepsilon^2), \overline{*})$ if the projection $(A[\varepsilon]/(\varepsilon^3), *) \to (A[\varepsilon]/(\varepsilon^2), \overline{*})$ is a morphism of algebras.

In other words, a second-order deformation is uniquely determined by a morphism

$$\mu : A \otimes A \to A[\varepsilon]$$

given by $\mu = \mu_1\varepsilon + \mu_2\varepsilon^2$ with $\mu_1, \mu_2 : A \otimes A \to A$, satisfying the relation

$$\mu(a \otimes b)c + \mu(ab \otimes c) + \mu(\mu(a \otimes b) \otimes c) = a\mu(b \otimes c) + \mu(a \otimes bc) + \mu(a \otimes \mu(b \otimes c))$$

modulo ε^3, which is obtained by requiring associativity. Equivalently,

$$d^3(\mu)(a \otimes b \otimes c) = \mu(\mu(a \otimes b) \otimes c) - \mu(a \otimes \mu(b \otimes c)) \mod \varepsilon^3.$$

Expanding the right-hand side, one obtains

$$\mu_1(\mu_1(a \otimes b) \otimes c)\varepsilon^2 - \mu_1(a \otimes \mu_1(b \otimes c))\varepsilon^2 \mod \varepsilon^3.$$

We use the following technical result.

Lemma 1.3.18 ([Ger64])**.** The linear map $\nu_2 : A \otimes A \otimes A \to A$ given by

$$\nu_2(a \otimes b \otimes c) = \mu_1(\mu_1(a \otimes b) \otimes c) - \mu_1(a \otimes \mu_1(b \otimes c))$$

is a Hochschild 3-cocyle.

Given a first order deformation $(A[\varepsilon]/(\varepsilon^2), *)$, with corresponding $\mu : A \to A[\varepsilon]$ given by $\mu(a\otimes b) = ab + \mu_1(a\otimes b)\varepsilon$, an extraction of it is determined by a morphism $\mu_2 : A \otimes A \to A$ satisfying

$$\mu_2(ab\otimes c)+\mu_1(\mu_1(a\otimes b, c))+\mu_2(a\otimes b)c = \mu_2(a\otimes bc)+\mu_1(a\otimes\mu_1(b\otimes c))+a\mu_2(b\otimes c),$$

which is equivalent to

$$d^3(\mu_2)(a \otimes b \otimes c) = \mu_1(\mu_1(a \otimes b, c)) - \mu_1(a \otimes \mu_1(b \otimes c)) = \nu_2(a \otimes b \otimes c).$$

This proves that $\mathrm{HH}^3(A)$ is the obstruction space to extending first-order deformations of A.

The deformation-theoretical meaning of the degree-two Hochschild cohomology space of a scheme is given in terms of deformations of its category of its coherent sheaves. We will not define deformations of categories, as we will never explicitly reproduce the technicalities on which this discussion relies. However, the motivation behind some discussion in this thesis originates from this treatment, therefore we mention it. First, we recall the following basic result about deformations of schemes.

Proposition 1.3.19 ([Har10])**.** *Let X be a smooth variety over an algebraically closed field k. The set of first-order deformations of X is in 1-to-1 correspondence with $H^1(X, T_X)$.*

Suppose now that $\mathrm{char}(k) = 0$, additionally to the assumptions of the proposition. Corollary 1.3.9 allows us to identify the set of first-order deformations of X with a direct summand of $\mathrm{HH}^2(X)$.

The comprehensive description of $\mathrm{HH}^2(X)$ is given in [Tod09] and interprets deformations of the abelian category of coherent sheaves over X by distinguishing three types of deformations. We give here a summary of the mentioned portion of the paper.

Corollary 1.3.9 gives the decomposition

$$\mathrm{HH}^2(X) = H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \wedge^2 T_X).$$

Each of the direct summands naturally gives a deformation of $\mathrm{Coh}(X)$. In particular, we now explain that each of the direct summands can be interpreted in the following ways:

- $H^2(X, \mathcal{O}_X)$ gives gerby deformations;
- $H^1(X, T_X)$ gives commutative deformations;
- $H^0(X, \wedge^2 T_X)$ gives non-commutative deformations.

First, let us consider a class $\alpha \in H^2(X, \mathcal{O}_X)$. We can interpret α as a Čech 2-cocycles $\{\alpha_{i_0i_1i_2} \in \mathcal{O}_X(U_{i_0i_1i_2})\}_{1\le i_0,i_1,i_2\le N}$ on a fixed affine cover $\mathcal{U} = \{U_i\}_1^N$. One can define α-twisted modules to be pairs $\mathcal{F} = (\{\mathcal{F}_i\}_1^N, \{\phi_{i_0i_1}\}_{1\le i_0,i_1\le N})$ where $\phi_{i_0i_1} : \mathcal{F}_{i_0}|_{U_{i_0i_1}} \xrightarrow{\sim} \mathcal{F}_{i_1}|_{U_{i_0i_1}}$ as left $\mathcal{O}_X|_{U_{i_0i_1}}$-modules and the gluing maps satisfy the α-twisted cocycle condition $\phi_{i_2 i_0} \circ \phi_{i_1 i_2} \circ \phi_{i_0 i_1} = \alpha_{i_0 i_1 i_2} \cdot \mathrm{Id}_{\mathcal{F}_{i_0}}$. The notion of quasi-coherent (resp. coherent) sheaf on the patches induces a notion of quasi-coherent (resp. coherent) sheaf. A class $\alpha \in H^2(X, \mathcal{O}_X)$ deforms $\mathrm{Coh}(X)$ by taking it to the α-twisted coherent sheaves (which are the datum of coherent sheaves on the patches of a Čech satisfying the cocyle condition above).

Classes in $H^1(X, T_X)$ and $H^0(X, \wedge^2 T_X)$ deform coherent sheaves by deforming $\mathcal{O}_X$ and then by twisting coherent sheaves by the deformed structure sheaf. Let us consider a pair $(\beta, \gamma) \in H^1(X, T_X) \oplus H^2(X, \wedge^2 T_X)$. Consider β as a Čech 1-cocycle $\{\beta_{i_0 i_1}\}_{1 \leq i_0 i_1 \leq N}$ with $\beta_{i_0 i_1} : \mathcal{O}_{U_{i_0 i_1}} \to \mathcal{O}_{U_{i_0 i_1}}$ a derivation. Define the morphism

$$\begin{aligned} \delta(\beta_{i_0 i_1}) : \mathcal{O}_X \oplus \mathcal{C}^0(\mathcal{U}, \mathcal{O}_X) &\to \mathcal{C}^1(\mathcal{U}, \mathcal{O}_X) \\ (a, \{b_i\}_1^N) &\mapsto \beta_{i_0 i_1}(a) + \delta(\{b_i\}_1^N) \end{aligned}$$

where δ denotes the Čech differential. The deformed structure sheaf $\mathcal{O}_X^{(\beta,\gamma)}$ has underlying abelian group the kernel of $\delta(\beta_{i_0 i_1})$. To define the product on the deformed structure sheaf, consider $\gamma \in H^0(X, \wedge^2 T_X)$. This can be seen as a biderivation $\gamma : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$. Such γ naturally defines a product $*_\gamma$ on $\mathcal{O}_X \oplus \mathcal{C}^0(\mathcal{U}, \mathcal{O}_X)$ as

$$(a, \{b_i\}) *_\gamma (c, \{d_i\}) := (ac, ad_i + cb_i + \gamma(a, c)).$$

Typically, this product is not commutative, motivating the name of this class of deformations.

Remark 1.3.20. A class $\beta \in H^1(X, T_X) \subset H^1(X, T_X) \oplus H^2(X, \wedge^2 T_X)$ defines both a deformation $\mathfrak{X}$ of X as a scheme as well as a deformation $\mathcal{O}_X^{(\beta,0)}$ of its structure sheaf. Following the construction exposed in the proof of 1.3.19 contained in [Har10], it is easy to see that $\pi_* \mathcal{O}_{\mathfrak{X}} \simeq \mathcal{O}_X^{(\beta,0)}$, where $\pi_* : \mathfrak{X} \to X$. In this sense, the commutative deformations coincide with the geometric deformations.

1.4 The HKR theorem via derived self-intersections

In this last section, we present an alternative approach to the HKR theorem, which follows the work of Arinkin, Căldăraru and Hablicsek presented in [AC12] and [ACH19]. We assume all schemes to be defined over $\operatorname{Spec} k$ where k is an algebraically closed field of characteristic 0. While the previous sections in this chapter were meant to introduce the general topic and motivate its investigation, the exposition that follows aims to provide the background on the techniques that we will use in the core of this thesis (in particular chapters 3 and 5).

1.4.1 A criterion on formality of l.c.i. closed immersions

We present the main result of [AC12] and give an idea of the proof. Let us first introduce the setting. In what follows, for a morphism $f : X \to Y$, $\mathbb{L}_f \in \mathcal{D}(X)$ denotes the derived cotangent complex.

Let Y be a smooth scheme over S and X a smooth closed subscheme of Y. Denote by $i : X \hookrightarrow Y$ the closed immersion. The morphism i gives rise to the triangle of complexes

$$i^*\mathbb{L}_{Y/S} \to \mathbb{L}_{X/S} \to \mathbb{L}_i \xrightarrow{+1} .$$

Since X and Y are both smooth S-schemes, their cotangent complexes are concentrated in degree 0 and are $\mathbb{L}_{X/S} = \Omega^1_{X/S}[0]$ and $\mathbb{L}_{Y/S} = \Omega^1_{Y/S}[0]$ respectively. This implies that i is a local complete intersection (from now on, l.c.i.) as its cotangent complex is non-zero at most in degrees 0 and -1. Moreover, the description of the relative cotangent complex is explicitly given by

$$\mathbb{L}^0_i = \operatorname{coker}(i^*\Omega^1_{Y/S} \to \Omega^1_{X/S}) = 0 \qquad \mathbb{L}^{-1}_i = \ker(i^*\Omega^1_{Y/S} \to \Omega^1_{X/S}) = N^\vee_{X/Y}$$

where $N_{X/Y}$ denotes the normal sheaf to X in Y, which is a vector bundle (i.e. locally free) since i is smooth.

We now introduce the notion of formality, which will appear in the statement of the main result recalled in this section.

Definition 1.4.1. A derived coherent sheaf is formal if it is isomorphic to $\mathrm{Sym}(\mathcal{E}^\vee)$ for some $\mathcal{E} \in \mathcal{D}^{>0}(X)$ (a complex concentrated in strictly positive degree).

Example 1.4.2. Let $\mathcal{E} = E[-1]$ for some vector bundle E on X, then

$$\begin{aligned}\mathrm{Sym}(\mathcal{E}^\vee) &= \bigoplus_{k\in\mathbb{N}} \bigwedge^k E^\vee[k] \\ &= \cdots \to \underbrace{E^\vee \wedge \cdots \wedge E^\vee}_{k \text{ times}} \to \cdots \to E^\vee \wedge E^\vee \to E^\vee \to \mathcal{O}_X \to 0\end{aligned}$$

with the zero-differentials.

Let us recall the definition of first infinitesimal thickening, that will appear in the statement of Theorem 1.4.5.

Definition 1.4.3. Let $i : X \hookrightarrow Y$ be a closed immersion of schemes and denote by $\mathcal{I}$ the ideal cutting out X in Y. The first infinitesimal neighbourhood (elsewhere called first-order thickening) of X into Y is the closed subscheme X' cut out by $\mathcal{I}^2$.

Remark 1.4.4. The first infinitesimal neighbourhood is a nilpotent extension of the subscheme. In particular, they have the same underlying topological space and the $(X')^{\mathrm{red}} = X$ if X is reduced.

We are now able to state the generalized HKR theorem.

Theorem 1.4.5 ([AC12])**.** Let $i : X \hookrightarrow Y$ be a l.c.i. closed immersion. Then $i^*i_*\mathcal{O}_X$ is formal if and only if the normal bundle extends to the first infinitesimal neighbourhood of X in Y. In this case

$$i^*i_*\mathcal{O}_X \simeq \mathrm{Sym}(N^\vee_{X/Y}[1]).$$

Dually, one obtains that $i^! i_* \mathcal{O}_X \simeq \mathrm{Sym}(N_{X/Y}[-1])$ under the same condition.

Remark 1.4.6. It follows by the projection formula [Sta24, 01E6] that the condition above implies the equivalence of endofunctors

$$i^* i_*(-) \simeq - \otimes \mathrm{Sym}(N^\vee_{X/Y}[1]) : \mathcal{D}(X) \to \mathcal{D}(X).$$

The construction of the isomorphism is given explicitly in the paper. A relevant fact about the proof is that the isomorphism depends on the extension of $N_{X/Y}$ to X' and is generally non-functorial. The functoriality of this construction has been investigated in [Hua20].

The HKR theorem is recovered via the following consequence about the degeneration of a spectral sequence.

Corollary 1.4.7. Let $i : X \hookrightarrow Y$ be a l.c.i. closed immersion such that the normal bundle extends to the first infinitesimal neighbourhood of X in Y. Then the spectral sequences

$$E_2^{pq} = H^p(X, \wedge^q N_{X/Y}) \Rightarrow \mathrm{Hom}^{p+q}_{\mathcal{D}(X)}(i^* i_* \mathcal{O}_X, \mathcal{O}_X) = \mathrm{Ext}^{p+q}_Y(i_* \mathcal{O}_X, i_* \mathcal{O}_X)$$

and

$$E^2_{pq} = H_p(X, \wedge^q N^\vee_{X/Y}) \Rightarrow \mathrm{Tor}^Y_{p+q}(i_* \mathcal{O}_X, i_* \mathcal{O}_X).$$

degenerate on the second page, giving the decompositions

$$\mathrm{Ext}^n_Y(i_* \mathcal{O}_X, i_* \mathcal{O}_X) = \bigoplus_{p+q=n} H^p(X, \wedge^q N_{X/Y})$$

and

$$\mathrm{Tor}^Y_n(i_* \mathcal{O}_X, i_* \mathcal{O}_X) = \bigoplus_{q-p=n} H^p(X, \wedge^q N^\vee_{X/Y}).$$

Proof. The spectral sequences appearing in the statement are an example of those studied in [Wei94, Section 5.7]. The degeneration follows by the fact that the differentials of the self-intersection complex in 1.4.5 are the zero maps (see Example 1.4.2). ☐

Theorem 1.3.9 may be recovered by taking the diagonal morphism

$$\Delta : X \to X \times X,$$

which is closed for separated schemes (in particular, for varieties), and is a l.c.i. if X is smooth as discussed at the beginning of this section. The lifting property is easy to verify. Let us compute the cotangent bundle $N^\vee_{X/X\times X}$. By definition

$$0 \to N^\vee_{X/X\times X} \to \Delta^* \Omega^1_{X\times X} \to \Omega^1_X \to 0.$$

Because $\Delta^* \Omega^1_{X\times X} \simeq \Omega^1_X \oplus \Omega^1_X$, we have that $N^\vee_{X/X\times X} \simeq \Omega^1_X$. Dually $N_{X/X\times X} \simeq T_X$. The exact sequence above splits. In particular, the cotangent bundle extends to every intermediate scheme between X and $X \times X$. Plugging these expressions for the tangent and cotangent sheaves in the expressions in Corollary 1.4.7, we recover the desired result.

Remark 1.4.8. The derived self-intersection

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

The complex $\Delta^* \Delta_* \mathcal{O}_X$ is equivalent to $p_* \mathcal{O}_W$. The derived scheme W is the loop space of X and is denoted by LX. By its universal property, the projections onto X split.

1.4.2 The HKR theorem for orbifolds

The HKR decomposition can be further demystified when the space that one is studying is an orbifold. We start by introducing the definition of such object. The definition that we will give is much more restrictive that others that can be found in the literature. However, we would rather restrict from the beginning to the case for which we have more insight for our investigation.

Definition 1.4.9. An orbifold is a global quotient Deligne-Mumford stack. Explicitly, it is a quotient $[X/G]$ where X is a variety and G is a finite group with a free action as an algebraic group on X.

Remark 1.4.10. The functor of points of $[X/G]$ is given as follows. For T a scheme over k

$$[X/G](T) := \left\{ \begin{tikzcd} P \arrow[r, "f"] \arrow[d, "p"] & X \\ T & \end{tikzcd} \quad : \quad \begin{array}{c} p \text{ is a } G-\text{torsor} \\ f \text{ is } G-\text{equivariant} \end{array} \right\}.$$

A G-torsor is a morphism p which admits a covering $\{T_i\}_i$ of T such that

$$\begin{tikzcd} T_i \times_T P \arrow[rr, "\sim"] \arrow[dr, "p|_{T_i \times_T P}"'] & & T_i \times G \arrow[dl, "\pi_i"] \\ & T_i & \end{tikzcd} .$$

We briefly denote such objects by P. For a morphism $\phi : T \to T'$ of schemes over k, the morphisms between objects $P \in [X/G](T)$ and $P' \in [X/G](T')$ above ϕ are G-equivariant morphisms $\overline{\phi} : P \to P'$ such that

$$\begin{tikzcd} P \arrow[r, "\overline{\phi}"] \arrow[d, "\pi"] & P' \arrow[d, "\pi'"] \\ T \arrow[r, "\phi"] & T' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[rr, "\overline{\phi}"] \arrow[dr, "f"'] & & P' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commute.

In this framework, the HKR theorem follows from the following property. We state it in a less exhaustive form than in the paper where it appears, but still sufficient to serve our purpose. This is a more general version of Theorem 1.4.5 for the smooth case.

Theorem 1.4.11 ([ACH19]). Let S be a smooth variety and X and Y be smooth subvarieties of S and consider the pullback diagram of derived schemes:

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "i"] \\ Y \arrow[r, "j"] & S \end{tikzcd} .$$

Denote by

$$E = \frac{T_S|_W}{T_X|_W + T_Y|_W}$$

the excess bundle. The following conditions are equivalent:

1. there is an equivalence of endofunctors for $\mathcal{D}(X)$ to $\mathcal{D}(Y)$
$$j^* i_*(-) \simeq q_*(p^*(-) \otimes \mathrm{Sym}(E^\vee[1]));$$
2. The exact sequence
$$0 \to T_X|_W + T_Y|_W \to T_S|_W \to E \to 0$$
splits.

Theorem 1.4.5 is obtained by taking $i = j$ and Corollary 1.4.7 may be recovered by setting $i = j = \Delta : X \to X \times X$.

For the discussion about orbifolds, it is crucial to be able to treat intersections which are not self-intersections.

Corollary 1.4.12 (HKR for orbifolds, [ACH19]). Let X be a smooth variety and G a finite group acting on it. Denote $\overline{\Delta} : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ the diagonal and $p, q : L\mathfrak{X} \to \mathfrak{X}$ the projections form the loop space onto $\mathfrak{X}$. There are natural isomorphisms of derived functors

$$\overline{\Delta}^* \overline{\Delta}_*(-) \simeq q_* p^*(-) \simeq (-) \otimes q_* \mathcal{O}_{L\mathfrak{X}}.$$

Moreover $q_* \mathcal{O}_{L\mathfrak{X}} \in \mathcal{D}(\mathfrak{X})$ is represented by the object

$$\bigoplus_{g \in G} i_{g,*} \mathrm{Sym}((\Omega^1_X)_g[1]) \in \mathcal{D}(X)$$

where $i_g : X^g \to X$ is the pullback of the twisted diagonal $\Delta_g : X \to X \times X$ mapping $x \mapsto (x, g.x)$ in the pullback diagram in the category of schemes

$$\begin{tikzcd} X^g \arrow[r, "i_g"] \arrow[d, "i"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta_g"] & X \times X \end{tikzcd}$$

and $(\Omega^1_X)_g$ is the sheaf of g-coinvariant differential 1-forms. In particular

1. $\overline{\Delta}^{*}\overline{\Delta}_{*}\mathcal{O}_{\mathfrak{X}} \simeq \bigoplus_{g\in G} i_{g,*}\mathrm{Sym}((\Omega^1_X)_g[1])$;

2. $\mathrm{HH}_{\bullet}(\mathfrak{X}) = \left(\bigoplus_{g\in G}\bigoplus_{q-p=\bullet} H^p(X^g, \wedge^q(\Omega^1_X)_g)\right)^G$;

3. $\mathrm{HH}^{\bullet}(\mathfrak{X}) = \left(\bigoplus_{g\in G}\bigoplus_{q+p=\bullet} H^{p-\mathrm{codim}_X(X^g)}(X^g, \wedge^q T^g_X \otimes \omega_{i_g})\right)_G$,

where we denote by T^g_X the g-invariant subsheaf of the tangent bundle and ω_{i_g} is the relative canonical bundle.

The proof of the corollary is given by the computation of the excess bundle E which yields to $(T_X)_g$ (which is isomorphic to T^g_X) and exhibiting an explicit splitting of the exact sequence in Theorem 1.4.11 given by the averaging map:

$$\begin{array}{ccc} (T_X)_g & \to & T_X \\ t & \mapsto & \dfrac{1}{\mathrm{ord}(g)}\displaystyle\sum_{i=1}^{\mathrm{ord}(g)} g^i t. \end{array}$$

CHAPTER 2

Logarithmic geometry and Artin fans

In this chapter, we provide the background of logarithmic geometry that we will use in this thesis. In the first section, we recall some classical definitions and properties in logarithmic geometry. In the second one, we present the stack of logarithmic structures $\mathcal{L}og$ introduced by Olsson in [Ols03] and we use it to revisit some content of presented earlier in the chapter. In the last one, we remind of the Artin fans and prove a crucial result for the treatment of logarithmic Hochschild homology and cohomology in Chapter 3, already present in [HHL23].

2.1 Logarithmic structures

We begin by giving some classical definitions in the framework of logarithmic geometry. In this section, we follow [Kat89, Ogu18, Tem23].

2.1.1 Monoids

The local picture of a logarithmic structure is given by a monoid.

Definition 2.1.1. A monoid $(M, +_M, 0_M)$ is a set M with an associative binary operation $+ : M \times M \to M$ with a neutral element 0 (i.e. $m + 0 = 0 + m = m$ for every $m \in M$). For us all monoids are commutative. We denote by $M^\times$ the group of units of the monoid (i.e. elements $m \in M$ for which there is $m' \in M$ such that $m + m' = 0$). We denote $\overline{M} = M/M^\times$ and call it the ghost monoid or the characteristic monoid of M. We say that a monoid is sharp if $M = \overline{M}$.

Definition 2.1.2. A morphism of monoids $f : (M, +_M, 0_M) \to (N, +_N, 0_N)$ is a function $f : M \to N$ such that $f(m +_M m') = f(m) +_N f(m')$ and $f(0_M) = 0_N$.

Example 2.1.3. For $k \geq 0$, the set $\mathbb{N}^k$ with the usual sum component by component is a sharp monoid.

Example 2.1.4. For $k > 0$, the set $\mathbb{Z}^k$ with the usual sum component by component is a monoid which is never sharp.

Despite the examples looking very basic, a priori monoids can be wild. To avoid odd phenomena that may occur in the broad generality, we restrict to the study of fine and saturated monoids, whose definitions we recall below.

Definition 2.1.5. A monoid M is finitely generated if there exists a surjection $\mathbb{N}^k \to M$ for some $k \geq 0$. A monoid is integral if $m + n = m' + n$ implies that $m = m'$. A monoid is fine if it is finitely generated and integral.

To give the definition of saturated monoid, we introduce the notion of the Grothendieck group of a monoid.

Construction 2.1.6. Let M be a monoid. We define $M^{gp} := M \times M/ \sim$ where $(m, n) \sim (m', n')$ if $m+n'+l = m'+n+l$ for some $l \in M$. The set M^{gp} inherits the monoid structure from M. Moreover, every element has an inverse with respect to the induced operation (explicitly, $[(m, n)] + [(n, m)] = [(0, 0)] = 0$) and therefore it is a group. We call M^{gp} the Grothendieck group of M and we denote by M^{int} the image of M in M^{gp} via the homomorphism $m \mapsto [(m, 0)]$. The monoid M is integral if and only if $M \simeq M^{int}$.

Definition 2.1.7. An integral monoid M is saturated if for every $m \in M^{gp}$ such that $am \in M$ for some $a \in \mathbb{N}$ it is true that $m \in M$.

Example 2.1.8. Let $M = \mathbb{N}^k$ for some $k > 0$. Its Grothendieck group is $M^{gp} = \mathbb{Z}^k$. It is easy to see that M is saturated.

Example 2.1.9. Fix an integer $x \geq 2$. The set $M = \{y \in \mathbb{N} | y = 0 \,\text{or}\, y \geq x\}$ is a monoid with the usual sum. We have that $[(x + 1, x)] \notin M$, while $x + 1 \in M$. Thus M is not saturated.

Definition 2.1.10. A fs monoid is a fine and saturated monoid. A toric monoid is a sharp fs monoid.

Fact 2.1.11. A sharp fs monoid M is equivalent to the datum of a lattice M^{gp} and a strictly convex rational polyhedral cone $\sigma \subset M^{gp} \otimes \mathbb{R}$ such that $\sigma \cap M^{gp} = M$. This provides the construction as an affine toric variety of $\operatorname{Spec} k[M]$. This statement is fundamental to thinking of logarithmic schemes as an extension of the notion of toric variety.

2.1.2 Logarithmic structures on schemes

The notions introduced in the previous subsection will be used in the current ones to define the geometric tools that we will be using. In logarithmic geometry, the default topology we will work with is the étale topology. Unless we state otherwise, we will use this convention.

Definition 2.1.12. Let X be a scheme. A pre-logarithmic structure on X is a sheaf of monoids $\mathcal{M}$ together with a morphism of monoids $\alpha : \mathcal{M} \to (\mathcal{O}_X, \cdot)$. Denote the category of pre-logarithmic structures on X by $\mathbf{PreLog}(X)$ A logarithmic structure is a pre-logarithmic structure such that α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$.
A logarithmic scheme is a tuple $(X, \mathcal{M}, \alpha)$ where X is a scheme and $(\mathcal{M}, \alpha)$ is a logarithmic structure.
A morphism of logarithmic schemes $f : (X, \mathcal{M}_X, \alpha_X) \to (Y, \mathcal{M}_Y, \alpha_Y)$ is the data of a morphism between the underlying schemes $f : X \to Y$ and a morphism of monoids $f^\flat : \mathcal{M}_Y \to f_*\mathcal{M}_X$.
When the context will not require it, we will shorten the notation to $(X, \mathcal{M})$ or just X.

Example 2.1.13. The monoid $\mathcal{O}_X^\times$ with the natural inclusion gives the trivial logarithmic structure. This allows us to consider any scheme as a trivial logarithmic scheme and any morphism of schemes as a morphism of logarithmic schemes, giving a fully faithful embedding of the category of schemes into the category of logarithmic schemes.

Example 2.1.14. By definition, a normal crossing divisor is a divisor D which étale locally admits étale morphisms to $\mathbb{A}^n$ such that D is the preimage of $V(x_1 \cdot \cdots \cdot x_r)$ for some $r = 1, \ldots, n$ (r does not need to pe the same for every étale chart). Let $D \subset X$ be a normal crossing divisor. Denote $j : U := X \setminus D \hookrightarrow X$. Define

$$\mathcal{M} = j_*\mathcal{O}_U^\times \cap \mathcal{O}_X.$$

This monoid naturally embeds in $\mathcal{O}_X$ and therefore gives a logarithmic structure on X.

Example 2.1.15. Let $X = \mathbb{A}^1 = \operatorname{Spec} k[t]$ and $D = \{0\} = \operatorname{Spec} k[t]/(t)$ defining a logarithmic structure as in 2.1.14. The monoid is defined as

$$\mathcal{M} = j_*\mathcal{O}^\times_{\mathbb{A}^1 \setminus \{0\}} \cap \mathcal{O}_{\mathbb{A}^1}.$$

Away from the origin, the monoid is trivial (i.e. coincides with the group of units): if $U \subset \mathbb{A}^1$ and $0 \notin U$, $j_*\mathcal{O}^\times_{\mathbb{A}^1 \setminus \{0\}}(U) \simeq \mathcal{O}^\times_{\mathbb{A}^1}(U) \simeq k^\times$. At the origin, the regular functions ct^n for every $c \in k$ and $n \in \mathbb{N}$ lie in the monoid because they only vanish at 0, and are invertible away from it. The monoid is therefore generated by the regular function t. We have that

$$\overline{\mathcal{M}} = \mathcal{M}/\mathcal{M}^\times = \mathbb{N}_0$$

where $\mathbb{N}_0$ denotes the skyscraper sheaf at the origin. The affine line is a toric variety and the logarithmic structure described here is exactly the logarithmic structure of X as toric variety, which is induced by the complement of the dense torus. In particular, it is fine and saturated.

To define morphisms of logarithmic schemes, we need to introduce the logarithmic structure associated with a pre-logarithmic structure. Let $\mathcal{M}$ be a pre-logarithmic structure. Let $\mathcal{M}^a$ be the pushout of the diagram

$$\begin{tikzcd} \alpha^{-1}(\mathcal{O}_X^\times) \arrow[r] \arrow[d] & \mathcal{O}_X^\times \\ \mathcal{M} & \end{tikzcd} .$$

This construction gives a functor $(-)^a : \mathbf{PreLog}(X) \to \mathbf{Log}(X)$ which is the left adjoint to the fully faithful functor $E : \mathbf{Log}(X) \to \mathbf{PreLog}(X)$ which takes logarithmic structures to themselves seen as pre-logarithmic structures.

Lemma 2.1.16 ([Ogu18]). Let $f : X \to Y$ be a morphism of schemes. The pushforward $f_* : \mathbf{Log}(X) \to \mathbf{Log}(Y)$ has a left adjoint

$$f_{\log}^* := (-)^a \circ f^{-1} : \mathbf{Log}(Y) \to \mathbf{Log}(X).$$

Remark 2.1.17. The morphism of monoids $f^\flat : \mathcal{M}_Y \to f_*\mathcal{M}_X$ introduced as part of the data giving a logarithmic scheme in Definition 2.1.12 corresponds via the adjunction to a morphism $f_{\log}^*\mathcal{M}_Y \to \mathcal{M}_X$. Given this correspondence, $f^\flat$ will denote the above morphism between sheaves of monoids on the source space X.

Denoting $f^\flat$ as in Remark 2.1.17, we can give the definition of strict morphism in the context of logarithmic geometry. Morphisms like this will play a crucial role in our investigation. Informally, they are morphisms between logarithmic schemes which do not alter the logarithmic structures and therefore are only interesting from the scheme-theoretical point of view.

Definition 2.1.18. A morphism of logarithmic schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is strict if the induced morphism of logarithmic structures $f^\flat : f_{\log}^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

Example 2.1.19. A classical example of strict morphisms is given by the following construction. Given any morphisms on logarithmic schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, it can be factorised as a morphism whose scheme-theoretical component is the identity (i.e. it is the data of a morphism of sheaves of monoids on X) followed by a strict morphism:

$$f : (X, \mathcal{M}_X) \to (X, f_{\log}^*\mathcal{M}_Y) \to (Y, \mathcal{M}_Y).$$

Logarithmic schemes, a priori, can be hard to treat because of the lack of structure of monoids and the lack of "regularity" that a sheaf may present. The first issue is addressed by requiring the local conditions presented in Subsection 2.1.1. For the second one, we ask the logarithmic schemes to be locally described by charts. We give the definition below.

Definition 2.1.20. Let X be a logarithmic scheme and P a monoid. We define a chart for X subordinated to P to be a strict morphism of logarithmic schemes $X \to \operatorname{Spec} k[P]$ where the target has logarithmic structure induced by the constant pre-logarithmic structure P. A logarithmic scheme is coherent (resp. integral, fine, fs) if locally on X it admits charts subordinated to a finitely generated (resp. integral, fine, fs) monoid.

These features define full subcategories of the category of logarithmic schemes. Most of the time, we will work in the category of fs logarithmic schemes.

Warning 2.1.21. It is well known that the fibered product in the category of fs logarithmic schemes differs from the one in the wider category of logarithmic schemes. To avoid confusion, we will denote by $\times, \ulcorner$ the fibered products in the category of logarithmic schemes, by $\times^{\ell}, \ulcorner\ell$ the fibered products in the category of logarithmic schemes and by $\times^{\not\ell}, \ulcorner\not\ell$ the latter when it coincides with the first one. This is always the case when pulling back along strict morphisms and when the fibered product is taken over a trivial logarithmic scheme (e.g. the k-point with trivial logarithmic structure).

We now adapt some properties of morphisms of schemes to logarithmic schemes giving their definitions in this context.

Definition 2.1.22. A morphism of logarithmic schemes $i : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a closed immersion of logarithmic schemes if the scheme-theoretical data $i : X \to Y$ is a closed immersion of schemes and the morphism of monoids $i^\flat : i^*_{\log}\mathcal{M}_X \to \mathcal{M}_Y$ is surjective. An exact closed immersion of logarithmic schemes is a closed immersion which is strict.

Definition 2.1.23. Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of logarithmic schemes. We say that f is formally smooth (resp. formally unramified, formally étale) if, for every commutative solid diagram

$$\begin{tikzcd}
(T, \mathcal{M}_T) \arrow[r, "g"] \arrow[d, "i"] & (X, \mathcal{M}_X) \arrow[d, "f"] \\
(T', \mathcal{M}_{T'}) \arrow[r, "h"] \arrow[ur, dashed] & (Y, \mathcal{M}_Y)
\end{tikzcd}$$

with T and T' affine schemes and i an exact closed immersion such that the underlying morphism of schemes is defined by a square-zero ideal, there exists at least (resp. at most, exactly) one dashed arrow making the two triangles commute. We say that f is smooth (resp. unramified, étale) if it is formally smooth (resp. formally unramified, formally étale) and the underlying morphism of schemes is of finite presentation.

The following statement justifies the idea of extending the scheme-theoretical notions that we mentioned in introducing the concept.

Proposition 2.1.24. A strict morphism of logarithmic schemes is logarithmically smooth (resp. logarithmically unramified, logarithmically étale) if and only if the underlying morphism of schemes is smooth (resp. unramified, étale).

Proof. For brevity, we prove the statement for the smooth case. The others are analogous. Suppose that $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is logarithmically smooth and strict. Take any commutative diagram of schemes

$$\begin{tikzcd} T \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T' \arrow[r] & Y \end{tikzcd}$$

with i a square-zero extension of T. Define logarithmic structures on T and T' by pulling back the logarithmic structure on Y. The existence of a lifting of this diagram is given by the existence of a lifting as logarithmic morphism. Vice versa, let the underlying morphism f be smooth. For any diagram as in Definition 2.1.23, any scheme-theoretical lifting $k : T' \to T$ is equipped with the structure of logarithmic morphism by setting $k^\flat := g^\flat$. $\square$

Lastly, we remind the reader of the definition of logarithmic Kähler differentials and we relate them to logarithmically smooth and logarithmically unramified morphisms.

Definition 2.1.25. Let $f : (X, \mathcal{M}_X, \alpha_X) \to (Y, \mathcal{M}_Y, \alpha_Y)$ be a morphism of logarithmic schemes. We define the sheaf of logarithmic differentials

$$\Omega^{1,\log}_{X/Y} := \Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes \mathcal{M}_X^{gp})/\sim$$

where the relation $\sim$ is generated by identifying

- $(d(\alpha_X(a)), 0) \sim (0, \alpha_X(a) \otimes a)$ for every local section a of $\mathcal{M}_X$;
- $(0, 1 \otimes a) \sim 0$ for every local section a of $\operatorname{im}(f^*_{\log}\mathcal{M}_Y \to \mathcal{M}_X)$.

The sheaf of logarithmic differentials enjoys the following universal property. More is true: the sheaf of logarithmic differentials is characterized by it. In other texts, the definition is given in terms of the universal property and then it is shown that the construction above satisfies it.

Proposition 2.1.26. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$- modules. A logarithmic derivation with values in $\mathcal{E}$ is a pair $(D : \mathcal{O}_X \to \mathcal{E}, \delta : \mathcal{M}_X \to \mathcal{E})$ where D is a derivation with values in $\mathcal{E}$ and δ is a homomorphism of monoids satisfying:

- $D(\alpha_X(m)) = \alpha_X(m)\delta(m)$ for m local section of $\mathcal{M}_X$;
- $\delta(f^\flat(n)) = 0$ for n local section of $f^*_{\log}(\mathcal{M}_Y)$.

The sheaf of logarithmic differentials represents the functor taking $\mathcal{O}_X$-modules to logarithmic derivations with values in it, with logarithmic derivation given by

$$\begin{aligned} D = d : \mathcal{O}_X &\to \Omega^1_{X/Y} \to \Omega^{1,\log}_{X/Y} \\ f &\mapsto df \end{aligned}$$

and

$$\begin{aligned}\delta = dlog : \mathcal{m}_X &\to \Omega^{1,\log}_{X/Y}\\ a &\mapsto (1 \otimes a).\end{aligned}$$

Lemma 2.1.27. For $f : X \to Y$ a strict morphism of logarithmic schemes,

$$\Omega^{1,\log}_{X/Y} \simeq \Omega^1_{X/Y}.$$

Proof. This follows recalling the definition of logarithmic Kähler differentials, for which the homomorphism $\delta = 0$ when f is strict. In this case, logarithmic derivations are derivations and therefore the logarithmic Kähler differentials are the usual ones. ☐

Remark 2.1.28. Commonly, the monoid part of the logarithmic differential δ is denoted *dlog*, by identifying $(1 \otimes a)$ with $\frac{da}{a}$. For logarithmic schemes as in Example 2.1.14, the logarithmic differentials "allow poles along the boundary". Let us suggest the precise meaning of this idea through the following example.

Example 2.1.29. Consider $(\mathbb{A}^1, \{0\})$ as a logarithmic scheme over Spec k with the trivial logarithmic structure as in Example 2.1.15. The global Kähler differentials of the affine line can be identified with $k[t]dt$. Let us consider the conditions in 2.1.25. The second type of relation does not appear in this example as the base Spec k has a trivial logarithmic structure. The first one identifies $d(\alpha(t))$ with $\alpha(t) \otimes t$. Writing the logarithmic Kähler differentials as in Proposition 2.1.26,

$$\delta(t) = 1 \otimes t = \frac{dt}{t} = dlog(t).$$

Similarly to what happens with the usual differentials, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of logarithmic schemes, there is an exact sequence of sheaves on X

$$f^*\Omega^{1,\log}_{Y/Z} \to \Omega^{1,\log}_{X/Z} \to \Omega^{1,\log}_{X/Y} \to 0. \tag{2.1}$$

Let us finally relate the logarithmic properties of morphisms to the logarithmic Kähler differentials. The first and third items constitute the statement of [Kat89, Proposition (3.12)], while the second one is part of the content of [Ogu18, Theorem IV.3.4.2].

Proposition 2.1.30. Let $f : X \to Y$ be a morphism of fine logarithmic S-schemes. The following conditions hold:

- if f is logarithmically smooth, $f^*\Omega^{1,\log}_{Y/S} \to \Omega^{1,\log}_{X/S}$ is injective and the exact sequence (2.1) is split (in particular, $\Omega^{1,\log}_{X/Y}$ is locally free);
- f is unramified if and only if $\Omega^{1,\log}_{X/Y} = 0$;
- if f is logarithmically étale, $f^*\Omega^{1,\log}_{Y/S} \to \Omega^{1,\log}_{X/S}$ is an isomorphism.

If moreover X and Y are logarithmically smooth over S, the reversed implications in the first and third items hold.

Definition 2.1.31. A k-scheme with a logarithmic structure is logarithmically smooth if it is logarithmically smooth over $\operatorname{Spec} k$ with the trivial logarithmic structure.

2.2 The stacks $\mathscr{L}$ and the logarithmic cotangent complex

What has been presented so far is the classical approach to logarithmic geometry. More modern techniques consist in encoding the logarithmic structure on a scheme (or stack) via a morphism towards a stack classifying fine logarithmic structures $\mathscr{L}og$. When working in the category of fs logarithmic scheme, like in our case, one may look at an open substack $\mathscr{L}$, classifying such logarithmic structures. The foundations on which this new method relies are due to Olsson and are presented in [Ols03, Ols05]. In this section, we resume some definitions and properties that we will use later in this thesis. In this section, we will always work with fine logarithmic schemes, which are, as anticipated, a convenient class to work with.

2.2.1 The stacks of logarithmic structures $\mathscr{L}og$ and $\mathscr{L}$

Given a logarithmic scheme X, we denote by $\underline{X}$ its underlying scheme. We will define the stack of logarithmic structures whose functor of points takes a scheme to the set of logarithmic structures on it. In general, this is constructed relatively to a base logarithmic scheme Y, and is denoted by $\mathscr{L}og_Y$. We will denote $\mathscr{L}og = \mathscr{L}og_{(\operatorname{Spec} k, k^\times)}$ its absolute version.

Definition 2.2.1. Let X be a fine logarithmic scheme. We define the category $\mathscr{L}og_X$ whose objects are morphisms of fine logarithmic schemes $h : Y \to X$ and morphisms are commutative diagrams

$$(h' : Y' \to X) \to (h : Y \to X) \quad : \quad \begin{tikzcd} Y' \arrow[rr, "f"] \arrow[dr, "h'"'] & & Y \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

where f is a strict morphism of logarithmic schemes.

Moreover, a morphism of logarithmic schemes $f : X \to Y$ naturally induces a functor between the fibered categories

$$\mathscr{L}og(f) : \mathscr{L}og_X \to \mathscr{L}og_Y.$$

We spell out a tautological prerequisite for this construction to give an algebraic stack.

Lemma 2.2.2 ([Ols03])**.** The category $\mathscr{L}og_X$ is fibered over the category of $\underline{X}$-schemes.

Proof. Let $\underline{f} : \underline{Y'} \to \underline{Y}$ be a morphism of $\underline{X}$-schemes and $\mathcal{M}_Y$ a fine logarithmic structure with a morphism of logarithmic structures $h^\flat : h^*_{\log}\mathcal{M}_X \to \mathcal{M}_Y$ (where h denotes the structure morphism $Y \to X$). Because morphisms in $\mathcal{L}og_X$ are strict, one may construct the morphism $f : (Y', \underline{f}^*_{\log}\mathcal{M}_Y) \to (Y, \mathcal{M}_Y)$ above $\underline{f}$. This morphism is strongly Cartesian since morphisms in $\mathcal{L}og_X$ are determined by the underlying morphism of schemes. □

Theorem 2.2.3 ([Ols03])**.** Let X be a fine logarithmic scheme. Then $\mathcal{L}og_X$ is a stack for the fppf topology and is of finite presentation over $\underline{X}$.

The proof of this fact is elaborated in [Ols03, Section 3]. In the paper, it is shown the existence of a flat cover of finite presentation over $\underline{X}$ of $\mathcal{L}og_X$. This allows us to define a universal logarithmic structure as in [Ols03, Definition 5.1] on $\mathcal{L}og_X$ which tautologically makes all morphisms towards it strict as morphisms of logarithmic algebraic stacks.

Remark 2.2.4. The functor of points of $\mathcal{L}og_{(X,\mathcal{M}_X)}$ as an algebraic stack takes a scheme T to

$$\mathcal{L}og_{(X,\mathcal{M}_X)}(T) = \left\{ (\mathcal{M}_T, \varphi) : \begin{array}{c} \mathcal{M}_T \text{ is a logarithmic structure on } T \\ \text{and } \varphi : (T, \mathcal{M}_T) \to (X, \mathcal{M}_X) \\ \text{is a morphism of logarithmic schemes} \end{array} \right\}.$$

Throughout this thesis, we will mostly be working with the open substack $\mathcal{L}_X$ (in the original literature denoted by $\mathcal{T}or_X$).

Definition 2.2.5. Let X be a fine and saturated logarithmic scheme. We define the fibered category over the category of $\underline{X}$-schemes $\mathcal{L}_X$ whose objects are morphisms of fine and saturated logarithmic schemes $h : Y \to X$ and morphisms are commutative diagrams

$$(h' : Y' \to X) \to (h : Y \to X) \quad : \quad \begin{tikzcd} Y' \arrow[rr, "f"] \arrow[dr, "h'"'] & & Y \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

where f is a strict morphism of logarithmic schemes.

Proposition 2.2.6 ([Ols03])**.** Let X be a fine and saturated scheme, then $\mathcal{L}_X$ is an open substack of $\mathcal{L}og_X$.

The algebraic stack $\mathcal{L}_X$ can be equipped with the restriction of the universal logarithmic structure on $\mathcal{L}og_X$, and thus be considered as a logarithmic stack.

It is clear from the construction of $\mathcal{L}og_X$ (resp. $\mathcal{L}_X$) that any morphism of fine (resp. fs) logarithmic schemes $Y \to X$ induces a morphism $Y \to \mathcal{L}og_X$ (resp. $Y \to \mathcal{L}_X$). The morphism induced by the identity is subject to the following useful property.

Lemma 2.2.7 ([Ols03])**.** Let X be a fine logarithmic scheme. The morphism $\underline{X} \to \underline{\mathcal{L}og}_X$ induced by the identity is an open immersion.

As anticipated, the stacks $\mathscr{L}og$ and $\mathscr{L}$ are useful to make properties of morphisms more handy. The following equivalence is proved in [Ols03, Section 4].

Proposition 2.2.8. Let $f : X \to Y$ be a morphism of fine logarithmic schemes whose underlying morphism of schemes is of finite presentation. Then f is logarithmically smooth (resp. logarithmically étale) if and only if $\underline{\mathscr{L}og_X} \to \underline{\mathscr{L}og_Y}$ is formally smooth (resp. formally étale), if and only if $\underline{X} \to \underline{\mathscr{L}og_Y}$ is formally smooth (resp. formally étale).

An analogous result may be proved with the same technique for logarithmically unramified morphisms.

Motivated by this fact, we can more or less interestingly extend properties in algebraic geometry to logarithmic algebraic geometry, following the definition below.

Definition 2.2.9. Let $\mathscr{P}$ be a property. A morphism of fine logarithmic schemes $X \to Y$ satisfies logarithmic $\mathscr{P}$ if $X \to \mathscr{L}og_Y$ satisfies $\mathscr{P}$.

Since $\mathscr{L}_X \subset \mathscr{L}og_X$ is an open substack, one can replace $\mathscr{L}og_X$ with $\mathscr{L}_X$ when working in the category of fs logarithmic schemes most of the time having the same properties. In particular, an analogous result to Proposition 2.2.8 holds and by replacing the stack classifying logarithmic structures in Definition 2.2.9 one obtains an equivalent definition when $\mathscr{P}$ is a local property. In this thesis, the property $\mathscr{P}$ will be taken to be smooth, étale, flat, separated, and having reduced geometric fibers. Therefore, we will mostly work with the stack classifying fs logarithmic structures.

Remark 2.2.10. In [Ols03, chapter 5], it is given the definition of logarithmic algebraic stacks, where the logarithmic structures via lisse-étale descent from logarithmic schemes covering the stack. All the content of this subsection may be revisited for logarithmic algebraic stacks. Olsson's paper goes through the details of the adaptation of the constructions and results. We skip this treatment for brevity and because in this thesis the only logarithmic algebraic stacks that will be treated are orbifolds (see Chapter 5), for which many technicalities are reduced.

2.2.2 The logarithmic cotangent complex

The stack classifying fine (resp. fs) logarithmic structures is the base structure used to define the logarithmic cotangent complex, as in the work by Olsson. In this subsection, we review some content from [Ols05] to recall the definition and basic properties of the logarithmic cotangent complex. This construction will occupy a position in the landscape of logarithmic algebraic geometry which is analogous to the one of the cotangent complex introduced and developed in the early age of modern algebraic geometry by Cartier, Grothendieck, Berthelot and Illusie ([Car56, Gro68, BGI71, Ill71]). We warn the reader that a different version of the logarithmic cotangent complex exists alongside the work by Olsson. The latter is due to Gabber ([Ols05, Section 8], [SSV16]), and Olsson presents it detailing the differences with his version. With the expression "logarithmic cotangent complex" we will always refer to Olsson's version, but the fact that another version exists might be recalled for context.

The logarithmic cotangent complex is an object associated with logarithmic morphisms. It behaves nicely with respect to compositions and pullbacks under a rather technical condition (called condition (T) in [Ols05]). We introduce it immediately to set the framework for its thinking.

Condition 2.2.11. For $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphisms of fine logarithmic schemes, there exists a family of commutative diagrams

$$\begin{tikzcd}
X \arrow[d, "f"] & X \times_Z Z_i \arrow[l] \arrow[d] & X \times_Y Y_i \arrow[l] \arrow[d] & X_i \arrow[l, "\pi_{X,i}"'] \arrow[dl, "f_i"] \\
Y \arrow[d, "g"] & Y \times_Z Z_i \arrow[l] \arrow[d] & Y_i \arrow[l, "\pi_{Y,i}"'] \arrow[dl, "g_i"] & \\
Z & Z_i \arrow[l, "\pi_{Z,i}"'] & &
\end{tikzcd}$$

satisfying

1. the schemes $\underline{X}_i$, $\underline{Y}_i$ and $\underline{Z}_i$ are affine;
2. the morphisms $\pi_{X,i}$, $\pi_{Y,i}$ and $\pi_{Z,i}$ are strict, flat and locally of finite presentation;
3. the induced morphism $\bigsqcup_i \underline{X}_i \to \underline{X}$ is surjective;
4. there exist charts

   $$Q_{X_i} \to \mathcal{M}_{X_i} \qquad Q_{Y_i} \to \mathcal{M}_{Y_i} \qquad Q_{Z_i} \to \mathcal{M}_{Z_i}$$

   and injective maps

   $$Q_{Z_i} \to Q_{Y_i} \qquad Q_{Y_i} \to Q_{X_i}$$

   such that the diagrams

   $$\begin{tikzcd}
   Q_{Y_i} \arrow[r] \arrow[d] & Q_{X_i} \arrow[d] \\
   f^*_{i,\log}\mathcal{M}_{Y_i} \arrow[r] & \mathcal{M}_{X_i}
   \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
   Q_{Z_i} \arrow[r] \arrow[d] & Q_{Y_i} \arrow[d] \\
   g^*_{i,\log}\mathcal{M}_{Z_i} \arrow[r] & \mathcal{M}_{X_i}
   \end{tikzcd}$$

   commute and

   $$\operatorname{Tor}_j^{\mathcal{O}_{Z_i} \otimes_{\mathbb{Z}[Q_{Z_i}]} \mathbb{Z}[Q_{Y_i}]}(\mathcal{O}_{Z_i} \otimes_{\mathbb{Z}[Q_{Z_i}]} \mathbb{Z}[Q_{X_i}], \mathcal{O}_{Y_i}) = 0 \quad \text{for } j > 0.$$

This condition restricts the set of composites and the Cartesian diagrams for which the logarithmic cotangent complex is well-behaved. Typically, it is ensured by logarithmic flatness-type assumptions. Let us now introduce the logarithmic cotangent complex.

Definition 2.2.12. Let $f : X \to Y$ be a morphism of fine logarithmic schemes. The logarithmic cotangent complex of f is

$$\mathbb{L}_f^{\log} := \mathbb{L}_{\underline{X}\to\underline{\mathcal{L}og}_Y} \in \mathcal{D}^{\leq 0}(\underline{X}).$$

where we are considering the underlying morphism of algebraic stacks to the strict morphism $X \to \mathcal{L}og_Y$ induced by f.

Proposition 2.2.13. The logarithmic cotangent complex satisfies the following properties:

1. if f is a strict morphism of fine logarithmic scheme, $\mathbb{L}_f^{\log} = \mathbb{L}_f$;
2. if f is logarithmically smooth, $\mathbb{L}_f \simeq \Omega_f^{1,\log}[1]$;
3. a) given a commutative diagram of fine logarithmic schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

   there is an induced morphism

$$g'^*\mathbb{L}_f^{\log} \to \mathbb{L}_{f'};$$

   b) if moreover f is logarithmically flat (as in Definition 2.2.9) and the diagram is Cartesian, the morphism above is an isomorphism;

   c) if additionally $X' \xrightarrow{f'} X \xrightarrow{g} Y$ satisfies condition 2.2.11, there is an isomorphism

$$f'^*\mathbb{L}_g^{\log} \oplus g'^*\mathbb{L}_f^{\log} \to \mathbb{L}_{g\circ f'}^{\log};$$

4. if $X \xrightarrow{f} Y \xrightarrow{g} Z$ satisfies condition 2.2.11, there exists a morphism $\mathbb{L}_f^{\log} \to f^*\mathbb{L}_g^{\log}[1]$ fitting in the distinguished triangle

$$f^*\mathbb{L}_g^{\log} \to \mathbb{L}_{g\circ f}^{\log} \to \mathbb{L}_f^{\log} \to f^*\mathbb{L}_g^{\log}[1].$$

Remark 2.2.14. Let $f : X \to Y$ be a morphism of fine and saturated logarithmic schemes. It follows from Proposition 2.2.6 and the fact that open immersions are étale that equivalently $\mathbb{L}_f^{\log} = \mathbb{L}_{\underline{X}\to\mathcal{L}_X}$.

Remark 2.2.15. As in Remark 2.2.10, this construction may be performed also for logarithmic algebraic stacks, without giving rise to any substantial difference.

2.3 Artin fans

The stack classifying logarithmic structures above a logarithmic scheme is a crucially important foundational tool, but it tends to be hard to work with concretely because of its lack of separateness (see [Ols03, Remark 3.17]). For this reason, we introduce the Artin fans, and we prove a technical result ([HHL23, Theorem A]) that will be crucial in the definition and investigation of the notions of logarithmic Hochschild homology and cohomology.

In this subsection, we will assume all logarithmic schemes and logarithmic algebraic stacks to be fs. Thus it will be more convenient to work with the stack $\mathcal{L}_X$.

We begin by recalling a fact that will be useful in this and the next sections.

Fact 2.3.1 ([ACG$^+$13])**.** Any fs logarithmically smooth logarithmic scheme admits charts to toric varieties.

We first define Artin fans abstractly, and then the Artin fan associated with a logarithmic scheme in steps. Similarly to the stacks $\mathcal{L}og_X$ and $\mathcal{L}_X$, the Artin fan encodes the logarithmic structure, but looks like a quotient stack étale locally. For a more detailed treatment of the topic, the reader may want to consult [ACMW14], [ACM$^+$15] and [AW18] (the latter for the construction in the logarithmically smooth case).

Definition 2.3.2. An Artin fan is a logarithmic algebraic stack logarithmically étale over $\operatorname{Spec} k$ (with the trivial logarithmic structure). A morphism of Artin fans is a morphism of logarithmical algebraic stacks between Artin fans. We denote the 2-category of Artin fans by $\mathbf{AF}_k$.

For a toric variety, the Artin fan has a direct description. Given fact 2.3.1, it is reasonable to expect this construction to provide the "local" picture of Artin fans in wider generality.

Definition 2.3.3. Let V be a toric variety. Denote its dense torus by $T \simeq \mathbb{G}_m^k \subset V$. We set the Artin fan of V to be

$$\Theta_V = [V/T].$$

Remark 2.3.4. Topologically, the Artin fan of a toric variety has an open point for every cone in the fan and specializations in the places of the inclusions of the cones (which are the reversed inclusions of the corresponding subvarieties). Each point has stacky structure $B\mathbb{G}_m^j$ for $j = 0, \dots, k$.

Example 2.3.5. Consider the toric variety $\mathbb{A}^1$. Its dense torus $\mathbb{G}_m$ acts on it by multiplication fixing the origin (its complement, i.e. the divisor giving the logarithmic structure). Its Artin fan is $\Theta_{\mathbb{A}^1} = [\mathbb{A}^1/\mathbb{G}_m]$, which has a point corresponding to its dense open torus with stacky structure $[\mathbb{G}_m/\mathbb{G}_m] \simeq B\{*\} \simeq \mathbb{Z}$ and an open point corresponding to the origin with stack structure $[*/\mathbb{G}_m] \simeq B\mathbb{G}_m$. See Figure 2.1. Similarly, consider the toric variety $\mathbb{A}^n$. Its dense torus $\mathbb{G}_m^n$ complement the union of the n hyperplanes intersecting transversely. It is easy to compute that

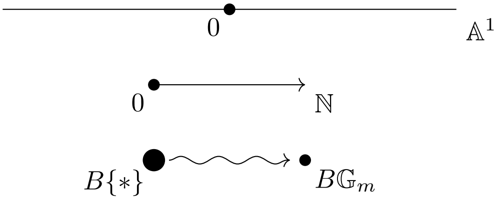

Figure 2.1: The affine line as a toric variety, its fan and its Artin fan

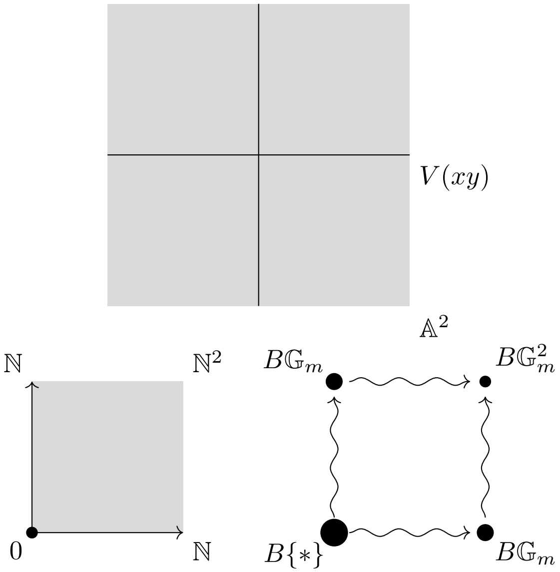

Figure 2.2: The affine plane as a toric variety, its fan and its Artin fan

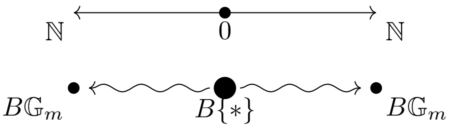

Figure 2.3: The fan and the Artin fan of $\mathbb{P}^1$.

for $j = 0, \ldots, n$ the Artin fan has $\binom{n}{j}$ components with stacky structure $B\mathbb{G}_m^j$. Figure 2.2 describes the situation for $n = 2$.

Example 2.3.6. Consider the toric variety $\mathbb{P}^1$. Its dense torus is $\mathbb{G}_m$ and has complement $\{0, \infty\}$. Its Artin fan is made of three points. The point corresponding to the dense torus has stacky structure $B\{*\}$, whilst the points corresponding to 0 and ∞ have stacky structure $B\mathbb{G}_m$. Figure 2.3 illustrates the inclusions.

Following the presentation of this topic in [ACM$^+$15], we implicitly define the Artin and for small (elsewhere called atomic) logarithmic schemes, that we introduce here.

Definition 2.3.7. A logarithmic scheme X is said to be small with respect to one of its points $x \in X$ if $\overline{\mathcal{M}}_X(X) \to \overline{\mathcal{M}}_{X,x}$ is an isomorphism and the logarithmic stratum $\{y \in X | \overline{\mathcal{M}}_{X,x} \simeq \overline{\mathcal{M}}_{X,y}\}$ is connected. A logarithmic scheme is said to be small if it is small with respect to some of its points.

Lemma 2.3.8. Affine toric varieties are small as logarithmic schemes.

Proof. Let X be an affine toric variety. The fan of X is given by a cone. Let x be the point corresponding to the whole cone in the toric variety. Since affine toric varieties are defined by sharp monoids, the restriction morphism in Definition 2.3.10 is $\mathbb{N}^k \to \mathbb{N}^k$. Because the fan has one maximal face, the deepest logarithmic stratum is made by one point, and therefore it is connected. ☐

For toric varieties, the following property holds and defines Artin fans.

Proposition 2.3.9 ([ACMW14])**.** Let X be a small logarithmic scheme. Then the category of morphisms from X to Artin fans has an initial object $X \to \Theta_X$.

Definition 2.3.10. Let X be a small logarithmic scheme. We call Θ_X as in Proposition 2.3.9 the Artin fan of X.

Remark 2.3.11. It follows from the construction in [ACMW14] that Artin fans are functorial with respect to strict morphisms.

Lemma 2.3.12. For X an affine toric variety, Definition 2.3.3 and Definition 2.3.10 agree.

Proof. This follows from the construction in the proof of [ACMW14, Proposition 3.2.1]. ☐

We are ready to give the most general definition.

Definition 2.3.13. Let X be a logarithmic algebraic stack and denote by $\mathcal{C}$ the category of groupoid presentations $V \rightrightarrows U \to X$ where all the morphisms are strict and V and U are the disjointed union of small logarithmic schemes. By Remark 2.3.11 that such presentations induce morphisms of the Artin fans $\Theta_V \to \Theta_U$. We define the Artin fan of X as

$$\Theta_X := \operatorname{colim}_{\mathcal{C}}(\Theta_V \to \Theta_U).$$

The existence of the colimit is discussed in the proof of [ACMW14, Proposition 3.2.1].

This completes the discussion around the construction of the Artin fan Θ_X associated with a logarithmic algebraic stack X, which will be a scheme in most cases we will consider.

Remark 2.3.14. Working with the Artin fan will be convenient because of its rather explicit local picture. However, an unpleasant fact differentiates it from the stacks classifying logarithmic structures $\mathcal{L}og_X$ and $\mathcal{L}_X$. It is argued in [ACM$^+$15] that the construction of the Artin fan is not functorial. The rest of this chapter is devoted to the study of the behaviour of the Artin fan with respect to the fiber product.

To state the main result of this chapter in its widest generality, we introduce the relative Artin fan.

Definition 2.3.15 (due to Herr, Molcho, Pandharipande, Wise – in progress)**.** Let $X \to Y$ be a morphism of logarithmic schemes or logarithmic algebraic stacks. If it exists, the relative Artin fan of $X \to Y$ is the target of the initial factorization $X \to \Theta_{X/Y} \to \mathcal{L}_Y$, where the second morphism is a strict, étale and representable morphism of logarithmic algebraic stacks.

For the practical use that we will make of this tool, we will integrate the following assumption in the definition.

Assumption 2.3.16. The logarithmic algebraic stack $\Theta_{X/Y}$ is a family of cone stacks over Y.

Conjecturally, relative Artin fans have this feature as soon as their existence is guaranteed. This is being proved by Herr, Molcho, Pandharipande, and Wise. In this thesis, we will be interested to the absolute Artin fans of logarithmically smooth schemes, for which this property holds by construction.

The relative Artin fan exists for logarithmically flat morphisms of finite presentation with logarithmically reduced geometric fibers. These conditions are needed for it to be endowed with a logarithmic structure (see [Ols03, Section 5] and [Sta24, 0786]). Having logarithmically reduced geometric fibers is a technical condition whose basic properties will be investigated later in this chapter (see Subsection 2.3.1).

Remark 2.3.17. In [AW18], the definition of relative Artin fan is given for logarithmically smooth and quasicompact morphisms of logarithmic schemes $X \to Y$. In this case, the Artin fan is given by $\pi_0(X/\mathcal{L}_Y)$ to the sense of [LMB00, Construction 6.9]. This is a special case of the notion in Definition 2.3.15. This follows by the universal property that it inherits by its quotient nature and [LMB00, 6.9.i] that $\pi_0(X/\mathcal{L}_Y)$ satisfies the more general definition.

Remark 2.3.18. For X a logarithmically smooth quasicompact logarithmic algebraic stack over k,

$$\Theta_X = \pi_0(X/\mathcal{L}).$$

2.3.1 Geometric fibers in logarithmic geometry

As technical preliminary to the statement and proof of the main theorem in this section, we need to discuss a rather technical assumption, which is reducedness of geometric fibers of the strict morphisms induced by logarithmic morphisms.

In particular, we repeatedly use a lemma that lets us compare geometric fibers. We consider when a map $X \to Y$ of algebraic stacks has reduced or connected geometric fibers.

The property "is reduced" is smooth-local, so a map $X \to S$ of algebraic stacks has reduced geometric fibers if there is a square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

with $V \to S$, $U \to X \times_S V$ smooth surjections from schemes such that $U \to V$ has reduced fibers.

Connectedness of an algebraic stack X means that of its underlying topological space $|X|$ of equivalence classes of points [Sta24, 04XG].

Lemma 2.3.19. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of algebraic stacks. Suppose $g : Y \to Z$ is étale and let $v \to Z$ be a geometric point. Ranging over lifts $v \dashrightarrow Y$ of the point to Y, to geometric fibers of f form an étale cover of those of $g \circ f$

$$\bigsqcup_{v \dashrightarrow Y} X \times_Y v \to X \times_Z v. \tag{2.2}$$

If f has reduced geometric fibers, so does $g \circ f$. If g is surjective and representable by algebraic spaces, we have a converse. The map (2.2) is the identity. If $g \circ f$ has reduced or connected geometric fibers, so does f.

Proof. Assume $Z = v$ is a geometric point. An étale map $Y \to Z$ is automatically of DM type (as it has finite automorphisms group). Choose an étale cover $Y' \to Y$ by a scheme Y'. The map $Y' \to Z$ is a disjoint union of copies of the points $\bigsqcup_I Z \to Z$ by [Sta24, 02GL].

Any geometric point $v \to Y$ lifts to the cover Y'. Write X_i for the preimage of the i-th inclusion, with i ranging in I, of Z into Y':

$$\begin{tikzcd} X_i \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner", very near start] & & X \arrow[d] \\ Z \arrow[r] & Y' = \bigsqcup_I Z \arrow[r] & Y. \end{tikzcd}$$

The family $\{X_i \to X\}_I$ is an étale cover.

If g is representable by algebraic spaces and we assume $Z = v$, then g is representable by schemes [Sta24, 03KX]. Take $Y' = Y$ above.

□

We define when a map $X \to S$ has logarithmically reduced geometric fibers following the philosophy of Definition 2.2.9.

Definition 2.3.20. We say that a morphism of logarithmic algebraic stacks $X \to Y$ has logarithmically reduced geometric fibers if the induced morphism $X \to \mathcal{L}_Y$ has reduced geometric fibers.

The property of having logarithmically reduced geometric fibers enjoys the following property, analogous to the content of Proposition 2.2.8.

Lemma 2.3.21. For a morphism $X \to S$ of logarithmic algebraic stacks, the map $X \to \mathscr{L}_S$ has reduced geometric fibers if and only if $\mathscr{L}_X \to \mathscr{L}_S$ does. If a pair of maps $X, Y \to \mathscr{L}_S$ both have reduced geometric fibers, so does the fiber product $X \times_S^{\ell} Y \to \mathscr{L}_S$.

Proof. By definition of reduced geometric fibers (in particular, their smooth-local nature), we can replace X by a smooth cover to assume it is a scheme. The second statement results from the first because of the induced strict pullback square

$$\begin{tikzcd}
\mathscr{L}_{X \times_S^{\ell} Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & \mathscr{L}_X \arrow[d] \\
\mathscr{L}_Y \arrow[r] & \mathscr{L}_S
\end{tikzcd}.$$

Consider the pullback square

$$\begin{tikzcd}
\mathscr{L}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & \mathscr{L}_{\mathscr{L}_S} \arrow[r, "d_1"] \arrow[d, "d_2"] & \mathscr{L}_S \\
X \arrow[r] & \mathscr{L}_S &
\end{tikzcd}$$

where the stack $\mathscr{L}_{\mathscr{L}_S}$ [Ols03] parameterizes chains of logarithmic structures

$$\mathscr{M}_S \to \mathscr{M}_1 \to \mathscr{M}_2.$$

The map d_1 forgets M_1 and d_2 forgets M_2.

If $X \to \mathscr{L}_S$ has reduced geometric fibers, so does the pullback $\mathscr{L}_X \to \mathscr{L}_{\mathscr{L}_S}$. The composite $\mathscr{L}_X \to \mathscr{L}_{\mathscr{L}_S} \to \mathscr{L}_S$ also does, by Lemma 2.3.19. The converse is immediate by Lemma 2.2.7.

□

Lemma 2.3.22. Composites of morphisms with logarithmically reduced geometric fibers also have them. The pullback of a morphism with logarithmically reduced geometric fibers in the category of fs logarithmic algebraic stacks also has logarithmically reduced geometric fibers.

Proof. Take $X \xrightarrow{f} Y \xrightarrow{g} Y$, where both f and g have logarithmically reduced geometric fibers. They induce $X \to \mathscr{L}_Y \to \mathscr{L}_Z$. It follows from Lemma 2.3.21 that the induced morphism has reduced geometric fibers. Let $X \to S$ be a morphism with logarithmically reduced geometric fibers, and consider the induced Cartesian diagram

$$\begin{tikzcd}
\mathscr{L}_{X \times_S^{\ell} Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & \mathscr{L}_X \arrow[d] \\
\mathscr{L}_Y \arrow[r] & \mathscr{L}_S
\end{tikzcd}.$$

The conclusion follows from Lemma 2.3.21 and [Sta24, 0576] after taking smooth covers by schemes. $\square$

The following example proved the need for the assumption of logarithmically reduced geometric fibers for the existence of the relative Artin fan.

Example 2.3.23 ([FG22, Proposition 1.4.1])**.** Suppose $X \to S$ is strict; then the relative Artin fan is the initial factorization of $X \to S$ through an étale S-algebraic space. If $X = \operatorname{Spec} \mathbb{Z}[i]$ and $S = \operatorname{Spec} \mathbb{Z}$, there is no such factorization. If $p \in \mathbb{Z}$ is an odd prime and $T_p = \operatorname{Spec} \mathbb{Z} \cup \operatorname{Spec} \mathbb{Z}$ is the bug-eyed line at p, there is a factorization

$$X \to T_p \to S.$$

But this is true for all odd primes p, so there can be no initial such factorization.

2.3.2 Weakly logarithmically separated

We introduce a technical condition that we will use later, that of being "weakly logarithmically separated".

Definition 2.3.24. Let $f : X \to S$ be a morphism of logarithmic algebraic stacks which admits an Artin fan $\Theta_{X/S}$. Say f is weakly logarithmically separated over S if the map $X \to \Theta_{X/S}$ is separated, i.e., if the diagonal map

$$X \to X \times_{\Theta_{X/S}} X$$

is proper. A logarithmic algebraic stack X is weakly logarithmically separated if the map $X \to \mathrm{pt}$ is.

Remark 2.3.25. The condition that $X \to \mathcal{L}S$ is separated (i.e. $X \to S$ is logarithmically separated, according to Definition 2.2.9) is stronger. There is a pullback diagram

$$\begin{tikzcd}
X \arrow[r] & X \times_{\Theta_{X/S}} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & X \times_{\mathcal{L}S} X \arrow[d] \\
 & \Theta_{X/S} \arrow[r] & \Theta_{X/S} \times_{\mathcal{L}S} \Theta_{X/S}.
\end{tikzcd}$$

Because $\Theta_{X/S} \to \mathcal{L}S$ is étale and representable by algebraic spaces, its diagonal is an open embedding. If the composite $X \to X \times_{\mathcal{L}S} X$ is proper, so is the map $X \to X \times_{\Theta_{X/S}} X$, which was claimed.
Vice versa, all Artin fans $\mathscr{B}$ are weakly logarithmically separated, but the map $\mathscr{B} \to \mathcal{L}$ need not be separated. Take the union $X = \mathcal{L} \cup \mathcal{L}$ along the open point $\mathrm{pt} \in \mathcal{L}$. Then X is its own Artin fan and weakly logarithmically separated even though the map $X \to \mathcal{L}$ is not separated.

Example 2.3.26. Toric varieties V are weakly logarithmically separated, as the diagonal map

$$V \to V \times_{\Theta_V} V$$

is the inclusion

$$V \to V \times T, \qquad v \mapsto (v, 0).$$

If X is a logarithmic scheme, it is weakly logarithmically separated if and only if $X \to X \times_{\Theta_X} X$ is a closed immersion.

Lemma 2.3.27. Let $X \to Y$ be a strict, separated map of logarithmic algebraic stacks admitting Artin fans over S. If Y is weakly logarithmically separated over S, so is X.

Proof. Construct the diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, swap, "\Delta_{X/Y}"] & & X \times_{\Theta_{X/S}} X \arrow[d, "t"] \\
& X \times_Y X \arrow[r, "j'"] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell", very near start] & X \times_{\Theta_{Y/S}} X \arrow[d] \\
& Y \arrow[r, "j"] & Y \times_{\Theta_{Y/S}} Y
\end{tikzcd}$$

with Cartesian square. By assumption, j, j', and $\Delta_{X/Y}$ are proper. The map $\Theta_{X/S} \to \Theta_{Y/S}$ is étale and representable, so its diagonal is open and the same is true for its pullback t. Thus $X \to \Theta_{X/S}$ is separated.

□

Being weakly logarithmically separated and separated are independent, as we now give examples of each condition without the other.

Example 2.3.28. Let $X = \mathbb{A}^1 \cup \mathbb{A}^1$ be the bug-eyed line at the origin, the union of two copies of $\mathbb{A}^1$ along the complements of their origins. Endow it with the divisorial logarithmic structure at both origins. The Artin fan $\Theta_X = \Theta \cup \Theta$ is the union of two copies of $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ along their open point. The map $X \to \Theta_X$ is the quotient by $\mathbb{G}_m$, so it is separated. Thus, this toric bug-eyed line is weakly logarithmically separated.

The stack $\mathscr{L}$ is weakly logarithmically separated but not separated.

2.3.3 Factorization of Artin fans

With the notions elaborated above, we are able to state the main result in this section ([HHL23, Theorem 3.1]).

Theorem 2.3.29. Consider a pair of logarithmic schemes or logarithmic algebraic stacks X, Y over a base S that admit relative Artin fans over S. Suppose the maps $X, Y \to S$ are quasicompact, logarithmically flat, and have logarithmically reduced geometric fibers. Then the Artin fan of the fiber product $X \times_S^\ell Y$ is the fiber product of the Artin fans

$$\Theta_{X \times_S^\ell Y/S} \xrightarrow{\sim} \Theta_{X/S} \times_S^\ell \Theta_{Y/S}.$$

This is an isomorphism of factorizations of $X \times_S^\ell Y \to \mathcal{L}S$ to the sense of Definition 2.3.15.

For the proof, it is enough that the fiber product $X \times_S^\ell Y \to \Theta_{X\times_S^\ell Y/S}$ and one of $X \to \Theta_{X/S}, Y \to \Theta_{Y/S}$, have connected, nonempty geometric fibers. This is the content of Proposition 2.3.37, which is the main result on which this theorem relies.

For the use that we will make of this result in the thesis, we state Corollary 2.3.30. This statement applies for the logarithmically smooth absolute case (for which the Artin fan may be expressed as in Remark 2.3.18).

Corollary 2.3.30. If X, Y are logartihmically smooth and quasicompact, the Artin fan of their product is the product of the Artin fans

$$\Theta_{X\times Y} \xrightarrow{\sim} \Theta_X \times \Theta_Y.$$

The theorem concerns a fiber product. The product of two maps $X \to S, Y \to T$ is simpler.

Proposition 2.3.31. Let $f : X \to S$, $g : Y \to T$ be a maps of logarithmic algebraic stacks admitting relative Artin fans. Suppose f, g are quasicompact, logarithmically flat, and have logarithmically reduced geometric fibers. The product of the Artin fans is the Artin fan of the product:

$$\Theta_{X\times Y/S\times T} \xrightarrow{\sim} \Theta_{X/S} \times \Theta_{Y/T}.$$

This is an isomorphism of factorizations of

$$X \times Y \to \mathcal{L}S \times \mathcal{L}T.$$

We establish a few lemmas to help show Artin fans are isomorphic.

Lemma 2.3.32. An étale morphism $g : Y \to Z$ representable by schemes between locally noetherian algebraic stacks locally of finite type which is injective on geometric points is an open immersion. If it is bijective on geometric points, it is an isomorphism.

Proof. Localize to assume Y, Z are schemes. Morphisms locally of finite type which are surjective on geometric points are surjective by [Sta24, 0487]. It remains to prove the first statement.

The map $Y \to Z$ is injective on geometric points if and only if $Y \to Y \times_Z Y$ is surjective on geometric points, hence surjective by [Sta24, 0487] again. Then $Y \to Z$ is universally injective [Sta24, 01S4]. A universally injective étale map is an open immersion [Sta24, 02LC].

□

Lemma 2.3.33. Let $f : \mathscr{B} \to \mathscr{C}$ be a strict étale map (locally of finite type) representable by algebraic spaces that is a bijection on geometric points. Suppose $\mathscr{B}$ is a family of cone stacks over S. Then f is an isomorphism.

Proof. Localize to assume $\mathscr{C} = \Theta_Q \times S$ for some monoid Q. We argue the map f is proper by lifting maps from $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$

$$\begin{tikzcd} & \mathscr{B} \arrow[d] \\ \Theta \arrow[r] \arrow[ur, dashed] & \Theta_Q \times S \end{tikzcd} \tag{2.3}$$

by [AW18, Theorem 2.4.1]. Write $\mathscr{B}' \coloneqq \mathscr{B} \times_{\Theta_Q \times S} \Theta$ for the pullback.

Let v be a geometric point with logarithmic structure $M_v = \mathcal{O}_v^* \oplus \mathbb{N}$ and strict map $v \to \Theta$. Lifts of (2.3) are equivalent to strict lifts of the logarithmic geometric point v and in turn to lifts of the underlying geometric point v°

$$\begin{tikzcd} & \mathscr{B}' \arrow[d] & & & \mathscr{B}' \arrow[d] \\ v \arrow[r] \arrow[ur, dashed] & \Theta & \equiv & v^\circ \arrow[r] \arrow[ur, dashed] & \Theta. \end{tikzcd}$$

But lifts of geometric points are unique by assumption.

The map f is proper étale, hence finite and representable by schemes. We conclude by Lemma 2.3.32.

□

Example 2.3.34. The assumption that $\mathscr{B} \to \mathscr{C}$ is representable by algebraic spaces is necessary. Let $\mathscr{B}$ be any Artin fan and $\mathscr{C}$ the Artin fan of $\mathscr{B}$. The two need not coincide, as argued in [ACM⁺15, example 5.4.1].

Proposition 2.3.35. Let $\mathscr{B} \to \mathscr{C}$ be a strict étale map representable by algebraic spaces, with $\mathscr{B}$ a family of cone stacks over S. Suppose some map $X \to \mathscr{B}$ and the composite $X \to \mathscr{B} \to \mathscr{C}$ are surjective with geometrically connected fibers. Then $\mathscr{B} \xrightarrow{\sim} \mathscr{C}$ is an isomorphism.

Proof. As in the proof of Lemma 2.3.19, the fibers $\mathscr{B} \times_{\mathscr{C}} v$ over a geometric point v are disjoint unions $\bigsqcup v$. If $\mathscr{B} \times_{\mathscr{C}} v \neq v$, the fibers of $X \to \mathscr{C}$ are not connected. The map $\mathscr{B} \to \mathscr{C}$ is bijective on geometric points, so Lemma 2.3.33 asserts it is an isomorphism.

□

Example 2.3.36. Let $D \subseteq X$ be an effective Cartier divisor, with both X, D smooth and geometrically connected. We claim that the map $X \to \Theta$ is the Artin fan for the D-divisorial logarithmic structure on X.

There is a sequence

$$X \to \Theta_X \to \Theta.$$

The fibers of $X \to \Theta$ are geometrically connected. Indeed, D is by assumption, and $X \setminus D$ is geometrically irreducible. So Proposition 2.3.35 equates $\Theta_X \xrightarrow{\sim} \Theta$.

Now let $D \subseteq X$ be an s.n.c. divisor and write D_i for the n components. All the strata $D_I = \bigcap_{i \in I} D_i$ are smooth, and suppose they are geometrically connected;

this includes $D_\varnothing = X$. Suppose all the components meet in a single nonempty deepest stratum D_I. There is a map

$$X \to \Theta^n,$$

where each coordinate parameterizes a component D_i. Proposition 2.3.35 identifies this map with the Artin fan of X

$$\Theta_X \simeq \Theta^n.$$

If the strata are not geometrically connected, this is false. Take a smooth, disconnected divisor $D = D_1 \sqcup D_2$ such as a pair of points

$$\{0, \infty\} \subseteq \mathbb{P}^1.$$

The Artin fan of $\mathbb{P}^1$ with the divisorial log structure is the union of two copies of Θ along their open points (see Example 2.3.6).

The map $\mathscr{L}S \to S$ is quasiseparated in the sense of the stacks project but not in the sense of Olsson's article or Laumon-Moret-Bailly [CHL23, Remark 1.1]. So if $X \to S$ is quasicompact, so is $X \to \mathscr{L}S$ by [Sta24, 050Y].

Proposition 2.3.37. Let $X \to S$ be quasicompact, logarithmically flat, and have logarithmically reduced geometric fibers. Then the map $X \to \Theta_{X/S}$ is surjective and has connected geometric fibers. It coincides with the space of relative connected components $\pi_0(X/\mathscr{L}S)$ [Rom09].

If $X \to S$ is quasicompact and logarithmically smooth, the hypotheses are satisfied. Compare with [FG22, Lemme 1.2.1].

Proof. We assume all our stacks are locally noetherian and locally of finite type, so $X \to \mathscr{L}S$ is flat and of finite presentation. All logarithmically flat maps $X \to S$ of finite presentation yield surjective maps $X \to \Theta_{X/S}$, because the map to the Artin fan is open.

There is a quasicompact, étale factorization

$$X \to \pi_0(X/\mathscr{L}S) \to \mathscr{L}S$$

under our hypotheses by [Rom09, Theorem 2.5.2]. There is a unique factorization

$$X \to \Theta_{X/S} \dashrightarrow \pi_0(X/\mathscr{L}S) \to \mathscr{L}S$$

by universal property. The dashed arrow is an étale cover representable by algebraic spaces.

Apply Lemma 2.3.19 to $X \to \Theta_{X/S} \to \pi_0(X/\mathscr{L}S)$, to see the geometric fibers of $X \to \Theta_{X/S}$ are connected. Proposition 2.3.35 shows the dashed arrow is an isomorphism

$$\Theta_{X/S} \xrightarrow{\sim} \pi_0(X/\mathscr{L}S).$$

□

Corollary 2.3.38. Let $X \to S$ be quasicompact, logarithmically flat, and have logarithmically reduced geometric fibers. Consider a strict étale map $\mathscr{B} \to \Theta_{X/S}$ from a family of cone stacks to the Artin fan of X that is representable by algebraic spaces. Take the pullback

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell", very near start] & X \arrow[d] \\
\mathscr{B} \arrow[r] & \Theta_{X/S}.
\end{tikzcd}$$

Then $X' \to \mathscr{B}$ is the relative Artin fan of X'.

If S is atomic and $\mathscr{B} = \Theta_P \times S$ is an S-Artin cone, then X' is also atomic.

Proof. Because $X \to \Theta_{X/S}$ is quasicompact, flat, and has reduced fibers, the morphism $X' \to \mathscr{B}$ has the same properties. There is a unique, strict étale surjection $\Theta_{X'/S} \to \mathscr{B}$ representable by algebraic spaces. Lemma 2.3.19 applied to $X' \to \Theta_{X'/S} \to \mathscr{B}$ shows X' has logarithmically reduced geometric fibers.

The map $X' \to \Theta_{X'/S}$ is surjective and has connected geometric fibers by Proposition 2.3.37. Then Proposition 2.3.35 equates $\Theta_{X'/S} \xrightarrow{\sim} \mathscr{B}$. If $\mathscr{B} = \Theta_P \times S$ and S is atomic, so is X'.

□

We are ready to prove the main theorem of this section.

Proof of Theorem 2.3.29. Form the fs pullback diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell", very near start] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell", very near start] & Y \arrow[d] \\
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell", very near start] & \Theta_{X/S} \times^{\ell}_{S} \Theta_{Y/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell"] & \Theta_{Y/S} \arrow[d] \\
X \arrow[r] & \Theta_{X/S} \arrow[r] & S,
\end{tikzcd}$$

naming $W = X \times^{\ell}_{S} Y$.

We assumed $X \to \Theta_{X/S}$ was flat and quasicompact. It is of finite presentation by our assumptions, so [Sta24, 06R7] asserts it is universally open.

Consider the composite

$$W \xrightarrow{f} Y' \xrightarrow{g} \Theta_{X/S} \times^{\ell}_{S} \Theta_{Y/S}.$$

We want to show $g \circ f$ has geometrically connected fibers. We know g does, and f is open and has geometrically connected fibers. It follows that $g \circ f$ has geometrically connected fibers by [Sta24, 0387].

The map $\Theta_{X/S} \times^{\ell}_{S} \Theta_{Y/S} \to S$ is étale and representable by algebraic spaces, engendering a map from the Artin fan

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & \Theta_{X/S} \times^{\ell}_{S} \Theta_{Y/S} \arrow[d] \\
\Theta_W \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathscr{L}S.
\end{tikzcd}$$

By Proposition 2.3.37, the map $W \to \Theta_W$ is surjective with geometrically connected fibers. Proposition 2.3.35 equates the Artin fans

$$\Theta_W \xrightarrow{\sim} \Theta_{X/S} \times^{\ell}_{S} \Theta_{Y/S}.$$

$\square$

Proof of Proposition 2.3.31. Obtain maps

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] & \Theta_{X/S} \times \Theta_{Y/T} \arrow[d] \\
\Theta_{X \times Y/S \times T} \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathscr{L}S \times \mathscr{L}T.
\end{tikzcd}$$

Argue the dashed arrow is an isomorphism using Proposition 2.3.37 and Proposition 2.3.35 as in the proof of Theorem 2.3.29.

$\square$

CHAPTER 3

Logarithmic Hochschild homology and cohomology

In this chapter, we offer a construction of logarithmic Hochschild homology and cohomology and equate them with Olsson's version [Ols24].

Using the machinery of formality of derived self-intersections [AC12], recalled in Section 1.4, we derive versions of the celebrated Hochschild-Konstant-Rosenberg (HKR) isomorphism recalled in Corollary 1.3.9 and conjecture over the Duflo isomorphism for logarithmic schemes [Kon03, CdB10].

We show our notion of logarithmic Hochschild homology and cohomology is invariant under logarithmic alterations (in particular, under compactifications), generalising the technology of Hochschild homology and cohomology to non-proper spaces. We also define and compute logarithmic versions of cyclic homology.

The content of this chapter is presented in [HHL23, section 4].

3.1 Definition of logarithmic Hochschild homology and cohomology

Fix a finite type, quasicompact, quasiseparated logarithmic algebraic stack X. By Theorem 2.3.29, we obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times X \arrow[d] \\
\Theta_X \arrow[r] & \Theta_{X \times X}
\end{tikzcd} .$$

This diagram is almost never Cartesian. If it were, the diagonal $X \to X \times X$ would be logarithmically étale.

We consider the fiber product $B = B_X$

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell", very near start] & X \times X \arrow[d] \\
& \Theta_X \arrow[r] & \Theta_{X \times X}
\end{tikzcd} . \tag{3.1}$$

This is both an fs and ordinary fiber product because the map $X \times X \to \Theta_{X\times X}$ is strict. The map $i : X \to B$ is the logarithmic diagonal.

Proposition 3.1.1. The map $f : B \to X \times X$ is logarithmically étale and representable by algebraic spaces. It factors through an open embedding $B \subseteq X \times_{\mathscr{L}} X$.

The logarithmic diagonal $i : X \to B$ is strict, quasicompact, quasiseparated, and representable by algebraic spaces.

Proof. The map $B \to X \times X$ is logarithmically étale as it is the pullback of a map between Artin fans, which is logarithmically étale by [AMW14, Lemma A.7]. To see that i is strict, recall that the logarithmic structure of a fiber product is the coproduct of the logarithmic structures and to use the local explicit construction of the Artin fan of [ACM$^+$15]. The diagonal $X \to X \times X$ is representable, so i is once we show $B \to X \times X$ is.
The diagonal of the algebraic stack Θ_X is étale and representable by algebraic spaces; the same is true for its pullback $B \to X \times X$.
There is a variant of (3.1)

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell", very near start] & X \times_{\mathscr{L}} X \arrow[d] \\
& \Theta_X \arrow[r] & \Theta_X \times_{\mathscr{L}} \Theta_X
\end{tikzcd} .$$

Because $\Theta_X \to \mathscr{L}$ is strict, étale, and representable by algebraic spaces, its diagonal is open.

$\square$

Remark 3.1.2. Equivalently, $B = X \times_{\Theta_X} X$ where the map $X \to \Theta_X$ is the Artin fan map.

Remark 3.1.3. The logarithmic diagonal $X \to B$ is proper if and only if X is weakly logarithmically separated. If X is a weakly logarithmically separated logarithmic scheme, $X \to B$ is an exact closed immersion.

Example 3.1.4. If X is a toric variety with dense torus T, the description $B = X \times_{\Theta_X} X$ with $\Theta_X = [X/T]$ gives the identification

$$B \simeq X \times_{[X/T]} X = X \times T.$$

The map $B \to X \times X$ is the product of the projection and action maps $X \times T \rightrightarrows X$. The factorization of $X \to X \times X$ through B is the inclusion

$$X \subseteq X \times T; \qquad x \mapsto (x, 0).$$

Remark 3.1.5. The map $X \to B$ is essentially the "logarithmic diagonal" [KS04, Definition 4.2.9]. If X is small (recall Definition 2.3.7), then $(X \times_S X)^{\sim}$ from [KS04, §5.2] coincides with B_X when it is defined. Before the advent of Artin fans, one worked locally with charts $[P] \coloneqq \Theta_P$. Their definition also involves choices of charts and "framing", which can mean that their logarithmic diagonal

$$X \to \text{“}X \times^{\log}_{S\times[P]} X\text{”}$$

is only exact [KS04, Corollary 4.2.8].

Lemma 3.1.6. If X is a logarithmically smooth logarithmic scheme, then B is also logarithmically smooth and $i : X \to B$ is a local complete intersection morphism. If X is also weakly logarithmically separated, $X \to B$ is a regular embedding.

Proof. The morphism $\theta : B \to \Theta_X$ is strict and smooth, with a logarithmically étale target. Its source is logarithmically smooth. The associated distinguished triangle is

$$\theta^*\mathbb{L}^{\log}_{\Theta_X} \to \mathbb{L}^{\log}_{B} \to \mathbb{L}^{\log}_{B/\Theta_X} \xrightarrow{[+1]} .$$

Using Definition 2.2.9 and Definition 2.2.12, we have that $\mathbb{L}^{\log}_{\Theta_X} = 0$, so $\mathbb{L}^{\log}_{B} \simeq \mathbb{L}^{\log}_{B/\Theta_X} \simeq \mathbb{L}_{B/\Theta_X}$ which is zero in non-zero degrees.

Since $X \to B$ is representable by algebraic spaces, being a local complete intersection is smooth-local in the source and target. Replace X, B by schemes.

Consider the distinguished triangle between the cotangent complexes coming from the sequence of maps $X \to B \to \Theta_X$:

$$i^*\mathbb{L}_{B/\Theta_X} \to \mathbb{L}^{\log}_{X} \to \mathbb{L}_{X/B} \xrightarrow{[+1]} .$$

Since $X \to \Theta_X$ is flat, so is $X \times X \to \Theta_X \times \Theta_X$, and thus B is equivalent to the derived fiber product $(X \times X) \times^h_{\Theta_{X\times X}} \Theta_X$, meaning that $i^*\mathbb{L}_{B/\Theta_X} = \mathbb{L}^{\log}_X \oplus \mathbb{L}^{\log}_X$.

By Proposition 2.2.13 logarithmically smooth schemes have logarithmic cotangent complex isomorphic to the shifted module of logarithmic Kähler differential forms. From the diagram above, we have that $\mathbb{L}_{X/B}$ is concentrated in degree -1, meaning that $X \to B$ is a local complete intersection morphism.

If X is further weakly logarithmically separated, $i : X \to B$ is a closed and hence regular embedding.

□

Now, we are ready to define logarithmic Hochschild homology and cohomology via endofunctors analogous to those in Definition 1.2.6.

Definition 3.1.7. Let X be a quasicompact, quasiseparated, weakly logarithmically separated, finite type logarithmic algebraic stack. Let $\mathscr{D}(X)$ be a dg enhancement of the unbounded derived category of coherent sheaves on X. Define the log Hochschild homology endofunctor to be

$$i^*i_* : \mathscr{D}(X) \to \mathscr{D}(X)$$

and the logarithmic Hochschild homology of X to be the graded space

$$\mathrm{HH}^{\ell}_{\bullet}(X) = R^{\bullet}\Gamma(X, i^*i_*\mathcal{O}_X).$$

If X is a logarithmic scheme, we define the logarithmic Hochschild cohomology endofunctor to be

$$i^!i_* : \mathcal{D}(X) \to \mathcal{D}(X)$$

and the logarithmic Hochschild cohomology of X to be the graded space

$$\mathrm{HH}^{\bullet}_{\ell}(X) = R^{\bullet}\Gamma(X, i^!i_*\mathcal{O}_X)$$

Here $i^!$ denotes the right adjoint of i_*, which exists in our case by [Nee96].

Remark 3.1.8. In the definition, we do not consider the logarithmic structure on X, since $X \to B$ is a strict morphism.

Remark 3.1.9. For logarithmic schemes with trivial logarithmic structure, the Artin fan is trivial and the diagonal between the Artin fans is clearly an isomorphism. Thereby Definition 3.1.7 coincides with the one of Hochschild homology and cohomology as in Definition 1.2.6.

Along the lines of the remark above, logarithmic Hochschild homology and cohomology extend the usual notions respecting the boundaries, in the sense that the next statement makes precise. Let us recall the framework for it as motivation for the investigation of logarithmic algebraic geometry. By [Nag62], any separated scheme U of finite type over the ground field admits a (non-unique a priori) morphism j to a proper scheme X with dense image $j(U) \subset X$ and with complement $D = X \setminus j(U)$ being an s.n.c. divisor D. When equipping U with the trivial logarithmic structure and X with the divisorial logarithmic structure defined by D, one obtains that

$$j : (U, \varnothing) \to (X, D)$$

is a strict morphism. The logarithmic Hochschild homology and cohomology endofunctors are functorial with respect to the pullback along the compactification morphisms.

Proposition 3.1.10. *Let U be a separated scheme of finite type over the ground field k. For any compactification X via an s.n.c. divisor D, and any $\mathcal{F} \in \mathcal{D}(X)$, we have that*

$$\Delta^*\Delta_*j^*\mathcal{F} \simeq j^*i^*i_*\mathcal{F}, \qquad \Delta^!\Delta_*j^*\mathcal{F} \simeq j^*i^!i_*\mathcal{F}$$

where $\Delta : U \to U \times U$ is the diagonal and so $\Delta^*\Delta_*$ and $\Delta^!\Delta_*$ are the Hochschild homology and cohomology endofunctors respectively, and $i : X \to X \times_{\Theta_X} X$ is the logarithmic diagonal for X. In particular

$$\Delta^*\Delta_*\mathcal{O}_U \simeq j^*i^*i_*\mathcal{O}_X, \qquad \Delta^!\Delta_*\mathcal{O}_U \simeq j^*i^!i_*\mathcal{O}_X.$$

Proof. First, consider the commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d] \\
U \times_{\Theta_X} U \arrow[r] \arrow[d] & X \times_{\Theta_X} X \arrow[d] \\
U \times U \arrow[r] & X \times X
\end{tikzcd} .$$

The bottom and outer squares are clearly Cartesian. By diagram chasing, the top square is Cartesian too. Moreover $U \times U \simeq U \times_{\Theta_X} U$ since U is the locus where the logarithmic structure is trivial. Denote the morphisms in the top square

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\ulcorner \ell", very near start] & X \arrow[d, "i"] \\
U \times U \arrow[r, "j'"'] & X \times_{\Theta_X} X
\end{tikzcd} .$$

By flat base change along the open immersion j',

$$\Delta^* \Delta_* j^* \simeq \Delta^* j'^* i_* \simeq j^* i^* i_* .$$

For the second part of the statement, recall that being j and j' open immersions, $j^* \simeq j^!$ and $j'^* \simeq j'^!$ [Sta24, 0AU0]. Using this fact and flat base change again

$$\Delta^! \Delta_* j^* \simeq \Delta^! j'^* i_* \simeq \Delta^! j'^! i_* \simeq j^! i^! i_* \simeq j^* i^! i_* .$$

□

Remark 3.1.11. We do not claim that the above result shows that the logarithmic Hochschild homology and cohomology are invariant under compactification, and they a priori depend on it. In Example 3.3.5 we will see a divisorial logarithmic scheme with logarithmic Hochschild homology in finitely many degrees, whose logarithmically trivial open subscheme has Hochschild homology concentrated in infinitely many degrees.

Remark 3.1.12. We defined Hochschild homology and cohomology as an endofunctor instead of using the categorical approach sketched in the Section 1.2 because of the coexistence of multiple different notions of coherent sheaves on logarithmic schemes in the literature, for instance, using parabolic bundles [MS80, Yok95], considering sheaves on root stacks [TV18], or considering toroidal compactifications [Vai17]. A subtlety is that $\mathcal{O}_X$ and other vector bundles on X need not be "logarithmically coherent sheaves" by these definitions, as $\mathcal{O}_X$ does not satisfy descent for logarithmic alterations.

Remark 3.1.13. If X is a logarithmically smooth scheme in Definition 3.1.7, both $i^* i_*$ and $i^! i_*$ restrict to endofunctors of the derived category of perfect complexes on X. This is a consequence of Proposition 3.2.3.

In this thesis, we will mostly have to do with logarithmic schemes. The only algebraic stacks which will be explicitly treated are orbifolds, in Chapter 5. For those, the existence of the exceptional inverse image functor $i^!$ will be argued.

Assumption 3.1.14. Assume X is a finite type, weakly logarithmically separated, logarithmically smooth logarithmic scheme.

Note X is automatically Noetherian, finite presentation, quasicompact, and quasiseparated.

To conclude the presentation of these tools, we show that our definition of logarithmic Hochschild homology and cohomology recovers Olsson's construction.

Proposition 3.1.15. Let X satisfy assumption 3.1.14. The value of the logarithmic Hochschild homology endofunctor on the structure sheaf recovers Olsson's construction of logarithmic Hochschild homology

$$\mathrm{HH}^{\ell}_{\bullet}(X) \simeq HH(X/\mathscr{L})$$

as defined in [Ols24].

Proof. Olsson's explicit chain complex computes the derived self-intersection of the diagonal $X \to X \times_{\mathscr{L}} X$ [Ols24]. The map $u : B \to X \times_{\mathscr{L}} X$ is open by Proposition 3.1.1, hence using [Sta24, 08ED] we see that the natural transformation $u^* u_* \Rightarrow \mathrm{Id}$ is an equivalence. This makes the natural transformation $i^* u^* u_* i_* \Rightarrow i^* i_*$ an equivalence, that identifies Olsson's definition with the one given in this thesis. ☐

3.2 The logarithmic HKR theorem

In this subsection, we prove Theorem 3.2.7.

Given a closed embedding $i : X \to B$ of schemes, one can consider the derived self-intersection $W := X \times^h_B X$. The notion of formality concerns the simplicity of this derived self-intersection [AC12, Gri14, Yu16].

Recall Definition 1.4.1. Formality for the derived self-intersection W means formality for $p_* \mathcal{O}_W$ where $p : W \to X$ is the pullback map of i. From a geometric point of view, formality means that W is the total space of a (shifted) vector bundle. From an algebraic point of view, formality assures that the structure sheaf of the derived self-intersection is as simple as possible.

Unlike in the non-derived case, the structure sheaf of W (over X) is given by the derived tensor product (see, for instance, [Toë14])

$$\mathcal{O}_W = \mathcal{O}_X \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{O}_X.$$

For a local complete intersection $i : X \to B$, the cohomology sheaves of the derived tensor product are given by exterior powers of the corresponding conormal bundle

$$\mathscr{H}^{-i}(\mathcal{O}_W) = \wedge^i N^{\vee}_{X/B}. \tag{3.2}$$

Consider the formal dg symmetric algebra of the conormal sheaf

$$\mathrm{Sym}(N^{\vee}_{X/B}[1]) := \bigoplus_i \wedge^i N^{\vee}_{X/B}[i],$$

equipped with the standard wedge product (and 0 differentials). The map $i : X \to B$ is formal if formula (3.2) can be lifted to the derived level. This can be interpreted as:

- An isomorphism of dg algebras

$$h_*\mathcal{O}_W \xrightarrow{\sim} \Delta_*\mathrm{Sym}(N^{\vee}_{X/B}[1]) \qquad \text{in } \mathcal{D}(X \times X), \tag{3.3}$$

  with the natural maps $h : W \to X \times X$, $\Delta : X \to X \times X$, or

- An isomorphism of dg schemes

$$W \xrightarrow{\sim} N_{X/B}[-1] \tag{3.4}$$

  over $X \times X$ via the diagonal.

The formality of the derived self-intersection is governed by the first infinitesimal neighborhood $X^{(1)}$ of X inside B. In this discussion, we use the notion of quantized cycles.

Definition 3.2.1. A quantized cycle [Gri14] is a retraction $\sigma : X^{(1)} \to X$ of the closed embedding $j : X \to X^{(1)}$ of X into its first infinitesimal neighborhood.

Notice that the results in [AC12] do not require the existence of a quantized cycle splitting the closed embedding, but the extension of the cotangent bundle is sufficient. Under the condition of the existence of a quantized cycle, the proof of Theorem 1.4.5 is easier than in the original paper [AC12] and more can be proved. We state the comprehensive result presented in [ACH19, Gri14] under this assumption and sketch the proof.

Theorem 3.2.2 ([ACH19, Theorem 1.8])**.** *Let $i : X \to B$ be a regular embedding (i.e. l.c.i. closed immersion) with a quantized cycle $\sigma : X^{(1)} \to X$. Then, we have the following statements:*

1. *the derived self-intersection is formal in the algebraic sense (3.3), meaning that there exists an isomorphism of dg-algebras*

$$h_*\mathcal{O}_W \xrightarrow{\sim} \Delta_*\mathrm{Sym}(N^{\vee}_{X/B}[1]) \qquad \text{in } \mathcal{D}(X \times X);$$

2. *the derived self-intersection is formal in the geometric sense (3.4): there exists an isomorphism $W \xrightarrow{\sim} N_{X/B}[-1]$ of dg schemes over $X \times X$;*

3. *the dg endofunctors of $\mathcal{D}(X)$ are isomorphic*

$$i^*i_*(-) \simeq (-) \otimes \mathrm{Sym}(N^{\vee}_{X/B}[1]).$$

Proof. The equivalence of (1), (2), and (3) are proven in [ACH19] in the case of a closed embedding of smooth schemes. The proof is the same in the case of a local complete intersection.
Now, we prove (3). We consider the factorization of $i : X \to B$ given

$$X \xrightarrow{j} X^{(1)} \xrightarrow{h} B.$$

We have the following sequence of functors

$$i^*i_*(-) \Rightarrow j^*j_*(-) \simeq j^*j_*j^*\sigma^*(-) \simeq j^*(\sigma^*(-) \otimes j_*\mathcal{O}_X) \simeq (-) \otimes j^*j_*\mathcal{O}_X$$

given by first the adjunction (h^*, h_*), then using that σ is a quantized cycle and so $\sigma \circ j = id$, then using the projection formula, and again using that $\sigma \circ j = id$.

In [AC12], a morphism of complexes $j^*j_*\mathcal{O}_X \to \mathrm{Sym}(N^\vee_{X/B}[1])$ was constructed via the exponential map in the smooth case that can be easily adapted to the case of a local complete intersection. By composition, we obtain a natural transformation of functors

$$i^*i_*(-) \Rightarrow (-) \otimes \mathrm{Sym}(N^\vee_{X/B}[1]). \tag{3.5}$$

To show (3.5) is an isomorphism, we only need to show that for every locally free sheaf $\mathcal{E}$, the natural transformation yields a quasi-isomorphism $i^*i_*\mathcal{E} \to \mathcal{E} \otimes \mathrm{Sym}(N^\vee_{X/B}[1])$. This is verified in theorem A of [Gri20, Theorem A]. ☐

Now, we consider X a logarithmic scheme satisfying assumption 3.1.14 and $i : X \to B$ to be the logarithmic diagonal constructed in Diagram (3.1).

Proposition 3.2.3. If X satisfies assumption 3.1.14, Theorem 3.2.2 applies to the closed immersion $i : X \to B$.

Proof. The map $X \to B$ is a local complete intersection morphism by Lemma 3.1.6. Thus, in order to apply Theorem 3.2.2, we need to show that there exists a quantized cycle σ for the closed embedding $X \to B$. Such a quantized cycle can be obtained by considering the sequence of maps

$$X^{(1)} \to B \to X \times X \to X$$

where the first map is the closed immersion of the first infinitesimal neighborhood in the ambient space, the second map comes from the Cartesian Diagram (3.1) as pullback of the diagonal of the Artin fan $\Theta_X \to \Theta_X \times \Theta_X$, and the last map is any of the projections $\pi_1, \pi_2 : X \times X \to X$.

☐

Remark 3.2.4. In the case of the logarithmic diagonal map $i : X \to B$, a global splitting $p : B \to X$ exists. The quantized cycle in the proof of Proposition 3.2.3 is the restriction of such a splitting to the first infinitesimal neighborhood.

Remark 3.2.5. If X is not weakly logarithmically separated, $X \to B$ is not a closed immersion. But Lemma 3.1.6 still shows the cotangent complex of $X \to B$ is concentrated in degree -1. It is an unramified local complete intersection, which is étale locally a regular immersion by [Sta24, 04HH].

As a corollary of Proposition 3.2.3, we get an explicit expression for the logarithmic Hochschild homology and cohomology of X.

Corollary 3.2.6. For X satisfying assumption 3.1.14, we have natural isomorphisms of functors

$$i^* i_*(-) \xrightarrow{\sim} - \otimes \operatorname{Sym}(\Omega_X^{1,\log}[1])$$

and

$$i^! i_*(-) \xrightarrow{\sim} - \otimes \operatorname{Sym}(T_X^{\log}[-1]).$$

Proof. Using Proposition 3.2.3 and Theorem 3.2.2, we have an isomorphism of functors

$$i^* i_*(-) \simeq - \otimes \operatorname{Sym}(N_{X/B}^{\vee}[1]).$$

The cotangent bundle of the logarithmic diagonal may be identified with

$$\begin{aligned} N_{X/B}^{\vee} \simeq N_{X/B}^{\log\vee} &= \ker(i^*\Omega_B^{1,\log} \to \Omega_X^{1,\log}) \simeq \ker(\Delta^*\Omega_{X\times X}^{1,\log} \to \Omega_X^{1,\log}) \\ &\simeq \ker(\Omega_X^{1,\log} \oplus \Omega_X^{1,\log} \to \Omega_X^{1,\log}) \simeq \Omega_X^{1,\log}, \end{aligned}$$

where the first isomorphism is given by strictness of the morphism i, the second by definition of logarithmic cotangent complex, the third by the morphism $B \to X\times X$ being logarithmically étale, the fourth by the behaviour of the cotangent sheaf with respect to the fiber product and the last by definition of the diagonal morphism.

Similarly for the tangent bundle

$$T_X^{\log} \simeq N_{X/X\times X}^{\log} \simeq N_{X/B}.$$

For a regular embedding $X \to B$ of codimension d, we have that the functor $i^!$ is isomorphic to the functor $i^*(-) \otimes \wedge^d N_{X/B}[-d]$ implying that

$$i^! i_*(-) \xrightarrow{\sim} - \otimes \operatorname{Sym}(T_X^{\log}[-1]).$$

□

Applying Corollary 3.2.6 to the structure sheaf we obtain the logarithmic version of the classical HKR isomorphism [HKR62].

Theorem 3.2.7 (Logarithmic HKR)**.** If X satisfies assumption 3.1.14, the logarithmic Hochschild homology (resp. cohomology) of $\mathcal{O}_X$ is computed in terms of the logarithmic cotangent (resp. tangent) bundle, namely, there exists an isomorphism of graded k-vector spaces

$$\mathrm{HH}_\bullet^{\ell}(X) = R^\bullet\Gamma(X, i^* i_* \mathcal{O}_X) \simeq \bigoplus_{q-p=\bullet} H^p(X, \Omega_X^{q,\log})$$

$$(\text{resp. } \mathrm{HH}_{\ell}^\bullet(X) = R^\bullet\Gamma(X, i^! i_* \mathcal{O}_X) \simeq \bigoplus_{p+q=\bullet} H^p(X, \wedge^q T_X^{\log})).$$

Formality also shows logarithmic Hochschild homology and cohomology are invariant under logarithmic alterations. In particular, this fact recovers Proposition 3.1.10.

Theorem 3.2.8. Let $\pi : X \to Y$ be a logarithmically étale map between logarithmic schemes satisfying assumption 3.1.14. The logarithmic Hochschild homology and cohomology complexes of an object $\mathcal{F} \in \mathcal{D}(Y)$ pull back to that on X:

$$\pi^* i_Y^* i_{Y,*}\mathcal{F} \simeq i_X^* i_{X,*}\pi^*\mathcal{F}, \qquad \pi^* i_Y^! i_{Y,*}\mathcal{F} \simeq i_X^! i_{X,*}\pi^*\mathcal{F}.$$

Proof. The logarithmic tangent and cotangent bundles pull back along logarithmically étale morphisms as

$$\pi^*\Omega_Y^{1,\log} \simeq \Omega_X^{1,\log}, \qquad \pi^* T_Y^{\log} \simeq T_X^{\log}.$$

By Corollary 3.2.6 and the fact that the inverse image commutes distributes with respect to the tensor product and commutes with the symmetric algebra functor

$$\begin{aligned}\pi^* i_Y^* i_{Y,*}\mathcal{F} &\simeq \pi^*(\mathcal{F} \otimes \operatorname{Sym}(\Omega_Y^{1,\log}[1])) \simeq \pi^*\mathcal{F} \otimes \operatorname{Sym}(\pi^*\Omega_Y^{1,\log}[1]) \\ &\simeq \pi^*\mathcal{F} \otimes \operatorname{Sym}(\Omega_X^{1,\log}[1]) \simeq i_X^* i_{X,*}\pi^*\mathcal{F}\end{aligned}$$

and

$$\begin{aligned}\pi^* i_Y^! i_{Y,*}\mathcal{F} &\simeq \pi^*(\mathcal{F} \otimes \operatorname{Sym}(T_Y^{\log}[1])) \simeq \pi^*\mathcal{F} \otimes \operatorname{Sym}(\pi^* T_Y^{\log}[1]) \\ &\simeq \pi^*\mathcal{F} \otimes \operatorname{Sym}(T_X^{\log}[1]) \simeq i_X^! i_{X,*}\pi^*\mathcal{F}.\end{aligned}$$

□

As a consequence, we deduce the invariance of logarithmic Hochschild homology as vector spaces for logarithmic blow-ups.

Definition 3.2.9. Let $f : X \to Y$ be a morphism of logarithmic schemes. We say that f is a logarithmic alteration if, strict étale locally on Y, f is the pullback of a morphism of Artin fans which is of Deligne-Mumford type, proper and birational. We say that f is a logarithmic blow-up if it is a logarithmic alteration which is also representable and proper.

Example 3.2.10. Let $X = (\mathbb{P}^1, 0)$ as logarithmic scheme. The blow-up $\pi : Bl_{(0,0)}X^2 \to X^2$ is a logarithmic blow-up when $Bl_{(0,0)}X^2$ is equipped with the logarithmic structure induced by the divisor $\mathbb{P}^1 \times \{0\} \cup \{0\} \cup \mathbb{P}^1 \cup E$ where E denotes the exceptional divisor. In fact, it is the morphism of Artin fans

$$\Sigma \to \Theta_X \times \Theta_X \simeq \Theta^2$$

where Σ is the Artin fan of $Bl_{(0,0)}X^2$.

Corollary 3.2.11. Let $f : X \to Y$ be a logarithmic blow-up, with X and Y satisfying assumption 3.1.14. Then there are isomorphisms of vector spaces

$$\mathrm{HH}_\bullet^\ell(X) \simeq \mathrm{HH}_\bullet^\ell(Y) \quad \text{and} \quad \mathrm{HH}_\ell^\bullet(X) \simeq \mathrm{HH}_\ell^\bullet(Y).$$

Proof. We prove the statement for the homology spaces. Being f logarithmically étale, we have that $f^*\Omega_Y^{1,\log} \simeq \Omega_X^{1,\log}$. By the projection formula and the fact that f is a blow-up

$$\begin{aligned} f_*\Omega_X^{1,\log} \simeq f_*f^*\Omega_Y^{1,\log} &\simeq f_*(f^*\Omega_Y^{1,\log} \otimes \mathcal{O}_X) \\ &\simeq \Omega_Y^{1,\log} \otimes f_*\mathcal{O}_X \simeq \Omega_Y^{1,\log} \otimes \mathcal{O}_Y \simeq \Omega_Y^{1,\log}. \end{aligned} \tag{3.6}$$

The Leray spectral sequence [Sta24, 01F2] associated with f is

$$E_2^{p,q} = R^p\Gamma(Y, R^q f_*f^*j^*j_*\mathcal{O}_X) \Rightarrow R^{p+q}\Gamma(X, f^*j^*j_*\mathcal{O}_Y)).$$

The right-hand side term is

$$R^{p+q}\Gamma(X, f^*j^*j_*\mathcal{O}_Y)) \simeq R^{p+q}\Gamma(X, i^*i_*\mathcal{O}_X) \simeq \bigoplus_{q'-p'=p+q} H^{p'}(X, \Omega_X^{q',\log})$$

by Theorem 3.2.7. The left-hand side may be rewritten as

$$R^p\Gamma(Y, R^q f_*f^*j^*j_*\mathcal{O}_X) \simeq R^p\Gamma(Y, R^q f_*i^*i_*\mathcal{O}_X) \simeq R^p\Gamma(Y, R^q f_*\mathrm{Sym}(\Omega_X^{1,\log}[1]))$$

which by the equation 3.6 and the Leray spectral sequence associated with the identity converges to

$$\bigoplus_{q'-p'=p+q} H^{p'}(Y, \Omega_Y^{q',\log}).$$

The proof of the isomorphism for the cohomology spaces follows the same strategy. ☐

Remark 3.2.12. Corollary 3.2.11 suggests a relation between the logarithmic derived categories of logarithmic blow-ups. This relation may be comparing this evidence with the result of Bondal and Orlov [BO95], which provide semiorthogonal decompositions of the derived categories of blow-ups in terms of the derived categories of a twisted line bundle of the exceptional divisor and of the space before blowing up.

We conclude this section by displaying another classical property that the logarithmic Hochschild homology satisfies.

Proposition 3.2.13. Let X and Y be proper logarithmic schemes satisfying assumption 3.1.14 and $\mathcal{F} \in \mathcal{D}^b(X)$ and $\mathcal{G} \in \mathcal{D}^b(Y)$ bounded derived complexes. We have that

$$\mathrm{HH}_\bullet^\ell(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \simeq \mathrm{HH}_\bullet^\ell(X, \mathcal{F}) \otimes \mathrm{HH}_\bullet^\ell(Y, \mathcal{G}) \tag{3.7}$$

as graded vector spaces.

Proof. We repeatedly use the Künneth formula [Sta24, 0FLQ] and Theorem 3.2.7. For a fixed degree $n \in \mathbb{Z}$, we compare the two sides of the isomorphism in the statement. Denote $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ the natural projections. On the left-hand side, we have

$$\begin{aligned}
\mathrm{HH}^{\ell}_{n}(X \times Y, \mathscr{F} \boxtimes \mathscr{G}) &\simeq \bigoplus_{q-p=n} H^p\left(X \times Y, \pi_1^*\mathscr{F} \otimes \pi_2^*\mathscr{G} \otimes \Omega^{q,\log}_{X\times Y}\right) \\
&\simeq \bigoplus_{q-p=n} H^p\left(X \times Y, \bigoplus_{r+s=q} \left(\pi_1^*\left(\mathscr{F} \otimes \Omega^{r,\log}_X\right) \otimes \pi_2^*\left(\mathscr{G} \otimes \Omega^{s,\log}_Y\right)\right)\right) \\
&\simeq \bigoplus_{q-p=n}\bigoplus_{r+s=q} H^p\left(X \times Y, \pi_1^*\left(\mathscr{F} \otimes \Omega^{r,\log}_X\right) \otimes \pi_2^*\left(\mathscr{G} \otimes \Omega^{s,\log}_Y\right)\right) \\
&\simeq \bigoplus_{q-p=n}\bigoplus_{r+s=q}\bigoplus_{t+u=p} \left(H^t\left(X, \mathscr{F} \otimes \Omega^{r,\log}_X\right) \otimes H^u\left(X, \mathscr{G} \otimes \Omega^{s,\log}_Y\right)\right) \\
&\simeq \bigoplus_{p}\bigoplus_{r}\bigoplus_{t} \left(H^t\left(X, \mathscr{F} \otimes \Omega^{r,\log}_X\right) \otimes H^{p-t}\left(X, \mathscr{G} \otimes \Omega^{n+p-r,\log}_Y\right)\right).
\end{aligned}$$

The right-hand side is

$$\begin{aligned}
(\mathrm{HH}^{\ell}_{\bullet}(X,\mathscr{F}) \otimes \mathrm{HH}^{\ell}_{\bullet}(Y,\mathscr{G}))_n &= \bigoplus_{i+j=n} \left(\mathrm{HH}^{\ell}_{i}(X,\mathscr{F}) \otimes \mathrm{HH}^{\ell}_{j}(Y,\mathscr{G})\right) \\
&\simeq \bigoplus_{i+j=n} \left(\left(\bigoplus_{k-h=i} H^h(X, \mathscr{F} \otimes \Omega^{k,\log}_X)\right) \otimes \left(\bigoplus_{m-\ell=j} H^\ell(Y, \mathscr{G} \otimes \Omega^{m,\log}_Y)\right)\right) \\
&\simeq \bigoplus_{i+j=n}\bigoplus_{k-h=i}\bigoplus_{m-\ell=j} \left(H^h(X, \mathscr{F} \otimes \Omega^{k,\log}_X) \otimes H^\ell(Y, \mathscr{G} \otimes \Omega^{m,\log}_Y)\right) \\
&\simeq \bigoplus_{i}\bigoplus_{h}\bigoplus_{\ell} \left(H^h(X, \mathscr{F} \otimes \Omega^{h+i,\log}_X) \otimes H^\ell(Y, \mathscr{G} \otimes \Omega^{n-i+\ell,\log}_Y)\right).
\end{aligned}$$

By shifting the indices in the last expression by setting $s = h, p - s = \ell$ and $r - s = i$ we obtain the same expression as for the left-hand side.

□

Remark 3.2.14. Applying Proposition 3.2.13 to the structure sheaf that one can redundantly rewrite as

$$\mathscr{O}_{X\times Y} \simeq \mathscr{O}_{X\times Y} \otimes \mathscr{O}_{X\times Y} \simeq \pi_1^*\mathscr{O}_X \otimes \pi_2^*\mathscr{O}_Y \simeq \mathscr{O}_X \boxtimes \mathscr{O}_Y$$

we obtain the isomorphism of graded vector spaces

$$\mathrm{HH}^{\ell}_{\bullet}(X \times Y) \simeq \mathrm{HH}^{\ell}_{\bullet}(X) \otimes \mathrm{HH}^{\ell}_{\bullet}(Y),$$

generalizing [Lod98, Theorem 4.2.5].

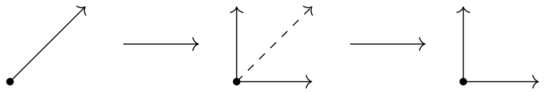

Figure 3.1: The cone complexes corresponding to the Artin fans $[\mathbb{A}^1/\mathbb{G}_m] \to \Sigma \to [\mathbb{A}^1/\mathbb{G}_m]^2$. These are the Artin fans of $X \to Bl_{\{0\}}\mathbb{A}^2 \to \mathbb{A}^2$. The blow-up is produced by subdividing $\Theta_{\mathbb{A}^2}$ along the image of the diagonal $\Theta_{\mathbb{A}^1}$.

3.3 Smooth pairs

As previously highlighted, schemes with logarithmic structures induced by a (simple) normal crossing divisor are a geometrically interesting class of logarithmic schemes. For the most basic examples of normal crossings pairs (X, D), we say when X is weakly logarithmically separated.

Proposition 3.3.1. Let (X, D) be a normal crossings pair with X quasicompact. Assume all the strata $D_I = \bigcap_I D_i$ are smooth, including $D_\varnothing = X$. Endowing X with the divisorial logarithmic structure, X is weakly logarithmically separated if and only if $\underline{X}$ is separated.

This applies, for example, to smooth pairs (X, D).

Proof. The map $X \to \Theta_X$ is smooth and quasicompact. Localize in Θ_X to assume $\Theta_X = \Theta^n$ and X is small by Corollary 2.3.38.
The map $X \to \Theta^n$ is weakly logarithmically separated if and only if its base change $L \to \mathbb{A}^n$ along the quotient $\mathbb{A}^n \to \Theta^n$ is separated. This L is the direct sum of the line bundles $\mathcal{O}(D_i)$ for the components of D. Argue L is separated if and only if X is. Use the fact that sections of a vector bundle are closed immersions.

□

Our construction of the logarithmic diagonal can be described explicitly when the logarithmic structure on a smooth scheme X is given by a smooth effective Cartier divisor D. Then B is an open subset of the blow-up $Bl_{D\times D}(X \times X)$ of $D \times D$ inside $X \times X$. First, notice that $\Theta_{(X,D)} = \Theta$. The product of the logarithmic scheme associated with the pair (X, D) with itself is given by the pair $(X \times X, X \times D \cup D \times X)$. The bottom arrow in Diagram (3.1) is the diagonal $\Theta \to \Theta^2$ and factors through the subdivision giving the Artin fan of the the blow-up of X^2 at the codimension 2 stratum $D \times D$ as in Figure 3.1. The construction in Diagram (3.1) may be preformed in two steps. The first pullback gives the blow-up $Bl_{D\times D}(X \times X)$ by [Sta24, 01OF] because the pair (X, D) is logarithmically flat. The second pullback gives an open into the blow-up, obtained by removing the strict transforms of $X \times D$ and $D \times X$ in the blow-up.

The diagonal map $\Delta : X \to X \times X$ factors through $Bl_{D\times D}(X \times X)$, providing Cartesian squares

$$\begin{tikzcd}
D \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "j"] \\
E \arrow[r] \arrow[d] \arrow[r, phantom, very near start, "\ulcorner"'] & Bl_{D\times D}(X \times X) \arrow[d] \\
D \times D \arrow[r] & X \times X.
\end{tikzcd}$$

We denote the map $X \to Bl_{D\times D}(X \times X)$ by j. Since B is an open subset of $Bl_{D\times D}(X \times X)$, we have a natural isomorphism of functors i^*i_* and j^*j_*. For the same reason, $N^\vee_{X/Bl_{D\times D}(X\times X)} \simeq N^\vee_{X/B} \simeq \Omega^{1,\log}_X$.

Recall that (X, D) is weakly logarithmically separated if and only if $\underline{X}$ is separated by Proposition 3.3.1. For pairs, one could define logarithmic Hochschild homology and cohomology and repeat our proofs with the classical blow-up $Bl_{D\times D}X \times X$ instead of the log diagonal B. To summarize the above discussion, we have the following statement which is an explicit version of Theorem 3.2.7.

Proposition 3.3.2. *For a pair (X, D) of a smooth, separated scheme X and D a simple normal crossing divisor on X, we obtain*

$$R^n\Gamma(X, \mathrm{HH}^{\ell}_X(\mathcal{O}_X)) = \oplus_{q-p=n} H^p(X, \Omega^q_X(\log D))$$

and

$$R^n\Gamma(X, \mathrm{HH}_{\ell}^X(\mathcal{O}_X)) = \oplus_{p+q=n} H^p(X, \wedge^q T_X(-\log D)).$$

Example 3.3.3. Consider the pair $(X, D) = (\mathbb{A}^1, \{0\})$. The blow-up $Bl_{D\times D}(X \times X) = Bl_{\{0\}}\mathbb{A}^2$ is the blow-up of the affine plane at the origin. The Artin fans of $\mathbb{A}^1, \mathbb{A}^2$ are their quotients by dense tori $[\mathbb{A}^1/\mathbb{G}_m], [\mathbb{A}^1/\mathbb{G}_m]^2$ as in Figure 3.1. We have a Cartesian diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner\ell"] & Bl_{\{0\}}\mathbb{A}^2 \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner\ell"] & \mathbb{A}^2 \arrow[d] \\
& {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[r] & \Sigma \arrow[r] & {[\mathbb{A}^2/\mathbb{G}_m^2].}
\end{tikzcd}$$

Here Σ is the stack quotient of $Bl_{\{0\}}\mathbb{A}^2$ by its dense torus, the Artin fan corresponding to the first quadrant subdivided along the diagonal.

Write $E' \simeq \mathbb{P}^1$ for the exceptional divisor in $Bl_{\{0\}}\mathbb{A}^2$ and $E \simeq \mathbb{G}_m \subseteq E'$ for the complement of its two points of intersection with the strict transforms of the $x-$ and $y-$ axes. The space B in this case is the complement in $Bl_{\{0\}}\mathbb{A}^2$ of the strict transforms of the axes

$$B = (\mathbb{A}^2 \setminus V(xy)) \cup (E \setminus \{0, \infty\}).$$

This is obtained by decomposing the blow-up into its $\mathbb{G}_m^2$-invariant components

$$Bl_{\{0\}}\mathbb{A}^2 = (\mathbb{A}^2 \setminus Z(xy)) \cup (Z(x) \setminus \{0\}) \cup (Z(y) \setminus \{0\}) \cup (E \setminus \{0, \infty\}) \cup \{0\} \cup \{\infty\}$$

and taking the components to which the diagonal maps.

The computation of the logarithmic Hochschild homology and cohomology complexes are particularly easy for such pairs given the simplicity of their cotangent and tangent bundles.

Example 3.3.4. Let us apply Theorem 3.2.7 to compute the logarithmic Hochschild homology and cohomology of $\mathbb{A}^1$ with the logarithmic structure induced by 0. Recall that $\Omega^{1,log}_{\mathbb{A}^1} \simeq k[t]\dfrac{dt}{t}$ and dually $T^{\log}_{\mathbb{A}^1} \simeq t \cdot k[t]\dfrac{\partial}{\partial t}$ are locally free of rank 1. The complexes obtained by taking wedge products have no higher terms. Thus the log Hochschild homology (respectively cohomology) is concentrated in degrees 0 and 1 (respectively 0 and -1).

In the above example, the Hochschild homology and cohomology spaces coincide, making it not more interesting than being the first example that one can make following the construction step by step. Let us now consider an example for which the logarithmic Hochschild homology differs from the Hochschild homology of the underlying space.

Example 3.3.5. Consider $X = \mathbb{P}^1$ with logarithmic structure given by an effective Cartier divisor of n distinct points. In this case, we have that $\Omega^{1,\log}_X = \mathcal{O}_{\mathbb{P}^1}(n-2)$, and thus if $n \geq 2$, we have that the 1st logarithmic Hochschild homology space is of positive dimension,

$$\dim R^1\Gamma(X, i^*i_*(\mathcal{O}_X)) > 0.$$

In fact the naïve logarithmic Hodge diamond given by the collection of the integers $\dim H^p(\mathbb{P}^1, \Omega^{q,\log}_{\mathbb{P}^1})$ for $p = 0, 1$, $q = 0, 1$ (compare this with [GHH⁺24]) for $n \geq 2$ is given by

$$\begin{array}{ccc} & 0 & \\ 0 & & n-1 \\ & 1 & \end{array}$$

Recalling that the Hodge diamond of $\mathbb{P}^1$ is

$$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$$

we realize that this result is in contrast to the case of ordinary Hochschild homology of $\mathbb{P}^1$, which is concentrated in degree 0:

$$\mathrm{HH}_i(\mathbb{P}^1) = 0 \qquad i \neq 0.$$

To conclude our investigation about smooth pairs, we display how logarithmic Hochschild homology spaces of pairs are related to the Hochschild homology spaces of the underlying scheme and those of the divisor.

Lemma 3.3.6. *For X a smooth variety and $D \subset X$ a smooth divisor, there is a long exact sequence*

$$\cdots \to \mathrm{HH}_\bullet(D) \to \mathrm{HH}_\bullet(X) \to \mathrm{HH}^\ell_\bullet(X, D) \to \mathrm{HH}_{\bullet-1}(D) \to \dots .$$

Proof. Denote the inclusion by $j : D \hookrightarrow X$. By [Sta24, 0FMU] this gives rise to a short exact sequence of de Rham complexes

$$0 \to \Omega^\bullet_X \to \Omega^\bullet_X(\log D) \to j_*\Omega^\bullet_D[-1] \to 0.$$

Observe that

$$\Delta^*_X \Delta_{X,*} j_* \simeq \Delta^*_X (j \times j)_* \Delta_{D,*} \simeq j_* \Delta^*_D \Delta_{D,*}$$

by flat base change of the diagram

$$\begin{tikzcd} D \arrow[r, "j"] \arrow[d, "\Delta_D"] & X \arrow[d, "\Delta_X"] \\ D \times D \arrow[r, "j \times j"] & X \times X \end{tikzcd} .$$

For this reason, $\mathrm{Sym}^\bullet(j_*\Omega^\bullet_D[1]) \simeq j_*\mathrm{Sym}^\bullet(\Omega^\bullet_D[1])$. Because the Hochschild complex and its logarithmic counterpart have zero differentials, the short exact sequence above is a sequence of mixed complexes, and [Sta24, 010H] implies that the short exact sequence upgrades to a distinguished triangle

$$\mathrm{Sym}^\bullet(\Omega^1_X[1]) \to \mathrm{Sym}^\bullet(\Omega^1_X(\log D)[1]) \to j_*\mathrm{Sym}^\bullet(\Omega^1_D[1])[-1] \xrightarrow{[+1]} .$$

By applying the derived global sections functor, one obtains the claimed long exact sequence. ☐

Remark 3.3.7. Lemma 3.3.6 gives an additional property supporting the interpretation of logarithmic Hochschild homology as Hochschild homology "with boundary". Indeed, the long exact sequence can be interpreted as the exactness Eilenberg–Steenrod axiom.

Corollary 3.3.8 (of HKR)**.** If X and D are both smooth and proper, then $\mathrm{HH}_{-\dim X}(X) \simeq \mathrm{HH}^\ell_{-\dim X}(X, D)$.

Proof. By the ordinary HKR theorem for smooth and proper varieties,

$$\dim \mathrm{HH}_n(X) = \sum_{q-p=n} h^{p,q}_X \quad \text{and} \quad \dim \mathrm{HH}_n(D) = \sum_{q-p=n} h^{p,q}_D.$$

This implies that $\mathrm{HH}_n(X) = 0$ for $n < -\dim X$ or $n > \dim X$ and $\mathrm{HH}_n(D) = 0$ for $n < -\dim D$ or $n > \dim D$. In particular, $\mathrm{HH}_{-\dim X}(D) = 0$. This proves that the morphism

$$\mathrm{HH}_{-\dim X}(X) \to \mathrm{HH}^\ell_{-\dim X}(X, D)$$

in the long exact sequence above is an isomorphism. ☐

Example 3.3.9. Let E be an elliptic surface and consider its trivial deformation, whose geometric realization is the trivial elliptic fibration $S = E \times \mathbb{P}^1$. We want to compute the logarithmic Hochschild homology of the logarithmic scheme given by S with the logarithmic structure induced by E. As usual, we denote it by the pair (S, E). Recall the Hodge diamond of any elliptic curve is

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array},$$

and the Hodge diamond of S is

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}.$$

In particular, these provide the dimensions of the Hochschild homology:

$$\begin{aligned} \dim_k \mathrm{HH}_0(E) &= 2 \\ \dim_k \mathrm{HH}_{\pm 1}(E) &= 1 \\ \dim_k \mathrm{HH}_{\pm 2}(E) &= 0 \\ \dim_k \mathrm{HH}_0(S) &= 4 \\ \dim_k \mathrm{HH}_{\pm 1}(S) &= 2 \\ \dim_k \mathrm{HH}_{\pm 2}(S) &= 0. \end{aligned}$$

Plugging the vector spaces obtained in the long exact sequence of Lemma 3.3.6, one obtains

$$\begin{aligned} 0 &\to \mathrm{HH}^\ell_2(S, E) \to k \to k^2 \to \mathrm{HH}^\ell_1(S, E) \to k^2 \to k^4 \to \\ &\to \mathrm{HH}^\ell_0(S, E) \to k \to k^2 \to \mathrm{HH}^\ell_{-1}(S, E) \to 0 \to 0 \to \mathrm{HH}^\ell_{-2}(S, E) \to 0. \end{aligned}$$

In particular, this implies that $\mathrm{HH}^\ell_{\pm 2}(S, E) = 0$, $\mathrm{HH}^\ell_{\pm 1}(S, E) \simeq k$ and $\mathrm{HH}^\ell_0(S, E) \simeq k^2$, agreeing with the Hochschild homology of the underlying scheme.

Example 3.3.10. Let us now consider a non-trivial deformation of an elliptic curve, with singular fiber. Following the discussion about the Hodge diamond of elliptic surfaces contained in [SS10], for such surfaces they are of the form

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ \chi - 1 & & 10\chi & & \chi - 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

where $\chi = \chi(S)$. In particular, $\dim_k \mathrm{HH}_{\pm 1}(S) = 0$. The long exact sequence in Lemma 3.3.6 for this class of examples is

$$\begin{aligned} 0 &\to k^{\chi - 1} \to \mathrm{HH}^\ell_2(S, E) \to k \to 0 \to \mathrm{HH}^\ell_1(S, E) \to k^2 \to k^{10\chi + 2} \to \\ &\to \mathrm{HH}^\ell_0(S, E) \to k \to 0 \to \mathrm{HH}^\ell_{-1}(S, E) \to 0 \to k^{\chi - 1} \to \mathrm{HH}^\ell_{-2}(S, E) \to 0. \end{aligned}$$

This sequence allows to compute

$$\begin{aligned}
&\mathrm{HH}_0^{\ell}(S,E) = k^{10\chi+1}\\
&\mathrm{HH}_{\pm 1}^{\ell}(S,E) = 0\\
&\mathrm{HH}_{-2}^{\ell}(S,E) = k^{\chi-1}\\
&\mathrm{HH}_2^{\ell}(S,E) = k^{\chi}.
\end{aligned}$$

3.4 The nodal cubic

In this section, we consider the degeneration locus of a family of smooth curves consisting of a nodal curve. We contrast the logarithmic Hochschild homology of the latter with its ordinary one.

Let $X \subseteq \mathbb{P}^2$ be the nodal cubic cut out by the equation

$$y^2 z = x^3 + x^2 z.$$

The nodal cubic X admits a unique logarithmic structure making it an integral, saturated, vertical*, logarithmically smooth curve over a point v with rank one logarithmic structure $\overline{M}_v = \mathbb{N}$. If $p \in X$ is the node $[0:0:1]$, the stalk $\overline{M}_{X,p} = \mathbb{N}^2$ has rank two, but all the other points $x \in X$ have rank one $\overline{M}_{X,x} = \mathbb{N}$†.

The map on characteristic monoids from $X \to v$ is

$$\overline{M}_v \xrightarrow{\sim} \overline{M}_x, \qquad \overline{M}_v \to \overline{M}_p; \quad t \mapsto (t,t)$$

at the node p and general point $x \in X$. Write $q \in \Gamma(v, M_v^{gp})$ for an element mapping to $1 \in \mathbb{N} = \Gamma(v, \overline{M}_v^{gp})$.

Étale locally around p, X is isomorphic to the union of the axes $V(xy) \subseteq \mathbb{A}^2$. Recalling that the Artin fan construction is functorial for strict morphisms, the Artin fan has an étale cover by $\Theta_{\mathbb{A}^2} = \Theta^2$ around its singular point p. As you traverse the loop in X, the $x-$ and $y-$axes turn out to be a single divisor, corresponding to a single ray of Θ_X. So the Artin fan Θ_X of X is the quotient of Θ^2 by identifying the $x-$ and $y-$axes. See Figure 3.2.

The logarithmic Hochschild homology of $X \to v$ is computed by Theorem 3.2.7, once we compute the logarithmic Kähler differentials $\Omega^{1,\log}_{X/v}$.

Lemma 3.4.1. The logarithmic cotangent bundle of X is trivial

$$\Omega^{1,\log}_{X/v} \simeq \mathcal{O}_X.$$

---

[*] A morphism of monoids $\theta : P \to Q$ is said to be vertical if $\mathrm{im}(Q \to \mathrm{coker}(\theta^{gp}))$ is a group. A morphism of coherent logarithmic schemes is said to be vertical if étale locally it admits charts to vertical morphisms of monoids.

[†] This is locally the same as equipping $\mathbb{P}^2$ with divisorial logarithmic structure along X and then restricting the logarithmic structure to X. But that logarithmic structure would encode the nontrivial normal bundle $N_{X/\mathbb{P}^2}$ of the nodal cubic, as opposed to the trivial bundle pulled back from $t \in \Gamma(\overline{M}_v^{gp})$.

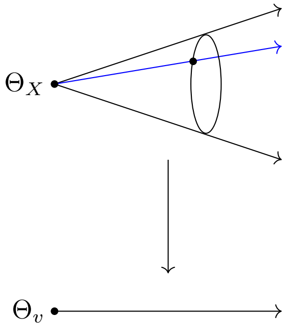

Figure 3.2: The Artin fan of the nodal cubic Θ_X is the cone over a circle with a single vertex. It is obtained from Θ^2 by glueing together the two axes. It has one ray, depicted in blue.

Proof. The logarithmic cotangent bundle for nodal curves is the dualizing sheaf [Kat99, Proposition 1.13]

$$\omega_X = \Omega_X^{1,\log}.$$

Under the normalization map $\nu : \mathbb{P}^1 \to X$, the dualizing sheaf pulls back to

$$\nu^*\omega_X = \Omega_{\mathbb{P}^1}^{1,\log}(2) = \mathcal{O}_{\mathbb{P}^2}.$$

It is of degree zero, giving a section of Pic^0_X.

Consider a flat family $\widetilde{X} \to \mathbb{A}^1$ of smooth genus-one curves degenerating to $\widetilde{X} \times_{\mathbb{A}^1} \{0\} = X$. For example, take the Legendre family

$$V\left(y^2z = x(x+z)(x+\lambda z)\right) \qquad \subseteq \mathbb{P}^2 \times \mathbb{A}^1_t$$

and restrict to $\mathbb{A}^1_t \setminus \{1\}$. The two line bundles $\Omega^{1,\log}_{\widetilde{X}/\mathbb{A}^1}$ and $\mathcal{O}_{\widetilde{X}}$ on $\widetilde{X}$ give sections of $\mathrm{Pic}^0_{\widetilde{X}/\mathbb{A}^1}$ over $\mathbb{A}^1$ that agree away from the origin $\{0\} \in \mathbb{A}^1$. The sheaf $\mathrm{Pic}^0_{\widetilde{X}/\mathbb{A}^1}$ is representable by a separated scheme [BLR12, Theorem 3, §8.4], so the two sections coincide.

□

Explicitly, we have

$$\dim R^i\Gamma(X, i^*i_*(\mathcal{O}_X)) = \begin{cases} 2 & i = 0 \\ 1 & i = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

agreeing with the ordinary Hochschild homology of a smooth cubic curve (in particular, with those appearing in the proof of Lemma 3.4.1).

The ordinary Hochschild homology of the map $\underline{X} \to \underline{v} = \mathrm{pt}$ is quite complicated with infinitely many non-zero terms (see [He23]).

3.5 Cyclic homology of a logarithmic scheme

We develop the logarithmic version of cyclic homology parallel to the non-logarithmic approach [TV07, BZN12, BM93]. We begin with the definition of the logarithmic loop space.

Definition 3.5.1. Let X be a logarithmic scheme with Artin fan $X \to \Theta_X$. We define the logarithmic derived loop space of X as the derived stack

$$LX^{\log} := \mathbf{Map}_{\mathbf{dSt}/\Theta_X}(S^1, X).$$

Here, S^1 denotes the stack associated with the constant simplicial presheaf

$$\mathbf{alg}/\Theta_X \to \mathbf{SSets}.$$

Since the homotopy pushout of

$$\begin{tikzcd}\star \sqcup \star \arrow[r] \arrow[d] & \star \\ \star & \end{tikzcd},$$

is S^1, we obtain that the logarithmic derived loop space can be realized as a derived self-intersection

$$LX^{\log} = X \times^h_{X \times^h_{\Theta_X} X} X.$$

If X is logarithmically flat, the derived and underived logarithmic fibre products coincide

$$X \times^h_{\Theta_X} X \simeq B.$$

The logarithmic derived loop space of a logarithmically smooth scheme X is the derived self-intersection

$$LX^{\log} \simeq X \times^h_B X.$$

Using either of the projection maps $p : LX^{\log} \to X$ and [Toë12, Proposition 1.4], we have

$$p_* \mathcal{O}_{LX^{\log}} = i^* i_* \mathcal{O}_X.$$

For logarithmically smooth schemes X, we have identified the functions on the logarithmic derived loop space with the logarithmic Hochschild homology of X evaluated at the structure sheaf.

The logarithmic derived loop space is equipped with an S^1-action induced by the action on the first argument $\mathbf{Map}_{\mathbf{dSt}/\Theta_X}(S^1, X)$ called *rotating the loops*. Therefore $R\Gamma(X, i^* i_*(\mathcal{O}_X))$ can be seen as an S^1-equivariant complex, in other words, as a mixed complex [Kas87, Lod98].

We are ready to define the versions of the cyclic homology of a logarithmic scheme.

Definition 3.5.2. For a logarithmically smooth scheme X, we define

- the k-th cyclic homology as the complex of homotopy orbits of the S^1-action
$$HC_k^{\ell}(X) := \pi_k(R\Gamma(X, i^* i_*(\mathcal{O}_X))_{hS^1}),$$
- the k-th negative cyclic homology as the complex of homotopy fixed points of the S^1-action
$$HC_k^{-,\ell}(X) := \pi_k(R\Gamma(X, i^* i_*(\mathcal{O}_X))^{hS^1}),$$
- and finally, the k-th periodic cyclic homology as the complex of Tate fixed points of the S^1-action (the reader may want to consult [Hoy15] for the definition)
$$HC_k^{per,\ell}(X) := \pi_k(R\Gamma(X, i^* i_*(\mathcal{O}_X))^{tS^1}).$$

Remark 3.5.3. M. Olsson defines an equivalent version of logarithmic cyclic homology using explicit chain complexes [Ols24].

From the universal property of the pullback, we obtain a natural S^1-equivariant map $X \to LX^{\log}$ called the constant loop map. The structure morphism $\Theta_X \to k$ induces a forgetful map $\mathcal{F}_{LX} : LX^{\log} \to LX$ from the logarithmic derived loop space to the loop space of X forgetting the logarithmic loops.

$$\begin{tikzcd}
LX \arrow[drr, red] \arrow[ddr, red] & & & \\
& LX^{\log} \arrow[ul, dotted] \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, red] & \\
& X \arrow[r] \arrow[drr, red] & B \arrow[dr] & \\
& & & X \times X
\end{tikzcd}$$

This forgetful map is S^1-equivariant, providing an S^1-equivariant map of complexes (thus, a map of mixed complexes)

$$R\Gamma(X, \Delta^* \Delta_*(\mathcal{O}_X)) \to R\Gamma(X, i^* i_*(\mathcal{O}_X)). \tag{3.8}$$

This is the same as the derived global sections of the morphism of sheaves given by the adjunction along the logarithmically étale map $B \simeq X \times_{\Theta_X} X \to X \times X$.

Example 3.5.4. In the case of a logarithmic scheme given by a pair (X, D) with X a smooth scheme and D a simple normal crossing divisor on X, the map of mixed complexes

$$R\Gamma(X, \Delta^* \Delta_*(\mathcal{O}_X)) \to R\Gamma(X, i^* i_*(\mathcal{O}_X))$$

induced by the forgetful map $\mathcal{F}_{LX}$ is given by the inclusion $\Omega_X^1 \to \Omega_X^1(\log D)$.

The degree -1 differential on the mixed complex $R\Gamma(X, i^*i_*(\mathcal{O}_X))$ is given by the logarithmic de Rham differential as in the case of $R\Gamma(X, \Delta^*\Delta_*(\mathcal{O}_X))$. As a consequence, the variants of the cyclic homology can be computed in terms of the logarithmic de Rham complex.

For instance, the periodic cyclic homology computes the even (or odd) part of the logarithmic de Rham cohomology,

$$HC_0^{per,\ell}(X) = \prod_{i\geq 0} H^{2i}_{DR,\log}(X),$$

and

$$HC_1^{per,\ell}(X) = \prod_{i\geq 0} H^{2i+1}_{DR,\log}(X).$$

Using the logarithmic Hodge-to-de Rham degeneration [Kat89, EV85, Hab20], we can compute the periodic cyclic homology explicitly:

$$HC_0^{per,\ell}(X) = \bigoplus_{p+q \text{ is even}} H^q(X, \Omega^p_X(\log D)),$$

and

$$HC_1^{per,\ell}(X) = \bigoplus_{p+q \text{ is odd}} H^q(X, \Omega^p_X(\log D)).$$

3.6 Around the Duflo isomorphism

In this section, we describe the state of work around extra structure on the logarithmic Hochschild homology and cohomology spaces, which we have referred to as k-vector spaces until now.

Both sides of both the isomorphisms appearing in Theorem 3.2.7 have natural algebra structures.

We start with the homological logarithmic HKR isomorphism

$$\mathrm{HH}^{\ell}_{\bullet}(X) = R^{\bullet}\Gamma(X, i^*i_*\mathcal{O}_X) \simeq \bigoplus_{q-p=\bullet} H^p(X, \Omega^{q,\log}_X).$$

The algebra structure on $R\Gamma(X, i^*i_*(\mathcal{O}_X))$ is given by the natural algebra structure on $i^*i_*\mathcal{O}_X$ and the algebra structure on $\oplus_{q-p=\star} H^p(X, \Omega^{q,\log}_X)$ is given by the usual wedge product on $\mathrm{Sym}(\Omega^{1,\log}_X[1])$. It was proven in [AC12] that these two algebra structures coincide in the smooth case, even on the level of the dg algebras $i^*i_*\mathcal{O}_X$ and $\mathrm{Sym}(\Omega^{1,\log}_X[1])$. The proof can be adapted to the case of local complete intersections, showing that the logarithmic Hochschild homology as a dg algebra is isomorphic to the dg algebra of logarithmic 1-forms, $\oplus_{q-p=\star} H^p(X, \Omega^{q,\log}_X)$.

Now, let us look at the isomorphism for the logarithmic Hochschild cohomology

$$\mathrm{HH}^{\bullet}_{\ell}(X) = R^{\bullet}\Gamma(X, i^!i_*\mathcal{O}_X) \simeq \bigoplus_{p+q=\bullet} H^p(X, \wedge^q T^{\log}_X).$$

The algebra structure on logarithmic Hochschild cohomology $R^\star\Gamma(X, i^! i_*(\mathcal{O}_X))$ comes from the Yoneda product on the derived Hom space $\mathrm{RHom}_B(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$ while the algebra structure on the dg algebra of logarithmic polyvector fields $\oplus_{p+q=\star} H^p(X, T_X^{q,\log})$ is given by the usual wedge product. Contrary to the case of logarithmic Hochschild homology, these algebra structures may differ. For ordinary Hochschild cohomology [Kon03, CdB10], the algebra structures become isomorphic after contracting with the square root of the Todd class.

For logarithmic Hochschild cohomology, we likewise conjecture the algebra structures are isomorphic after contraction with the square root of the "logarithmic Todd class" that we define in the rest of this chapter. We follow [Gri14, Yu19].

3.6.0.1 Logarithmic Todd class

Since the map $i : X \to B$ is a local complete intersection morphism of codimension d, Verdier duality gives

$$\wedge^d T_X^{\log}[-d] \simeq \mathcal{RH}om_X(\mathcal{O}_X, i^! \mathcal{O}_B).$$

Using the natural map $\mathcal{O}_B \to i_*\mathcal{O}_X$, we obtain a map

$$\mathcal{RH}om_X(\mathcal{O}_X, i^! \mathcal{O}_B) \to \mathcal{RH}om_X(\mathcal{O}_X, i^! i_* \mathcal{O}_X)$$

where the latter (using Verdier duality again) can be identified with

$$\mathcal{RH}om_X(\mathcal{O}_X, i^! i_*\mathcal{O}_X) \simeq \mathcal{RH}om_X(\mathcal{O}_X, i^* i_*\mathcal{O}_X \otimes \wedge^d T_X^{\log}[-d]) \simeq \mathrm{Sym}(T_X^{\log}[-1])$$

where the last isomorphism comes from the logarithmic HKR map of Corollary 3.2.6. The composite of these maps is a map

$$\wedge^d T_X^{\log}[-d] \to \mathrm{Sym}(T_X^{\log}[-1])$$

in the (bounded) derived category $\mathcal{D}^b(X)$. We call the corresponding element in

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(\wedge^d T_X^{\log}[-d], \mathrm{Sym}(T_X^{\log}[-1])) = R^0\Gamma(X, \mathrm{Sym}(\Omega_X^{1,\log}[1])) =$$

$$= \bigoplus_n H^n(X, \Omega_X^{n,\log})$$

the *logarithmic Todd class* of X. We denote this class by $Td_X^{\log}$.

Remark 3.6.1. Via the logarithmic Hodge-to-de Rham degeneration [Kat89, EV85, Hab20], we see that the Todd class can be interpreted as a class in the even part of the logarithmic Hodge cohomology, $\bigoplus_n H_{dR}^{2n,\log}(X)$ as one would expect [SST20].

Remark 3.6.2. The degree 0 part of the Todd class $Td_X^{\log}$ is always 1. This follows from the identification of the degree d part of $\mathrm{Sym}(T_X^{\log}[-1])$ with $\wedge^d T_X^{\log}$, see [Gri14]. As a consequence, it makes sense to consider the square root of the Todd class.

With the notation established above, we can state the conjecture around the Duflo isomorphism fo logarithmica Hochschild cohomology.

Conjecture 3.6.3. The composite morphism $I : HKR^{-1} \circ \iota_{\sqrt{Td_X^{\log}}}$ provides an isomorphism of dg-algebras $\oplus_{p+q=\star} H^p(X, \wedge^q T_X^{\log}) \to R^\star\Gamma(X, i^! i_*(\mathcal{O}_X))$ where HKR^{-1} is the inverse of the composite

$$HKR : \operatorname{Ext}^\star_B(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \xrightarrow{\sim} \operatorname{Ext}^\star_X(i^* i_*\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \oplus_k \operatorname{Ext}^{\star-k}(\Omega_X^{k,\log}, \mathcal{O}_X) \xrightarrow{\sim} \oplus_{p+q=\star} H^p(X, \wedge^q T_X^{\log})$$

induced by adjunction and the formality isomorphism of Corollary 3.2.6, and $\iota_{\sqrt{Td_X^{\log}}}$ is contraction with the square root of the logarithmic Todd class.

CHAPTER 4

Invariance of logarithmic Hochschild homology and cohomology

One of the features of Hochschild homology and cohomology recalled in Chapter 1 is the invariance under derived equivalence, proved in [Cal03]. One issue that one naturally encounters trying to adapt the proof to the logarithmic framework is that the notion of derived equivalence has no natural logarithmic counterpart because of the lack of a notion derived category of coherent sheaves in any logarithmic sense, as in Remark 3.1.12. To introduce the main technique used in the original proof by Căldăraru, recall Definition 1.2.7 for Hochschild homology and cohomology. The proof of the invariance result relies on the relation of the two graded vector spaces via the Serre functor as in [Kuz09], that we recall in the first section of this chapter. The main theorem in [Cal03] is stated for smooth and proper Deligne-Mumford stacks for which Serre duality holds. Sufficient conditions for the assumption to be true are given in [Lev22] and the same will be adopted as assumptions in this thesis. In this chapter, we will deal with logarithmic schemes and Deligne-Mumford stacks with additional properties of smoothness and properness on the underlying spaces. The standard example that the reader may want to keep in mind is of a smooth and proper variety with a s.n.c. divisor inducing the logarithmic structure as in Example 2.1.14.

Our discussion begins by defining the logarithmic Serre functor for schemes, which will play the analogous role to the one of the Serre functor in [Kuz09]. Then, we adapt the technicalities involved in the proof to Deligne-Mumford stacks as in

[Lev22] to extend the result for orbifolds, foreseeing the discussion in Chapter 5. Finally, we define *strict logarithmic derived equivalence*, which is a condition under which the invariance of logarithmic Hochschild homology and cohomology holds, and we prove this fact.

4.1 The logarithmic Serre functor for logarithmic schemes

Let X be a smooth and proper algebraic variety (or a scheme, algebraic space, Deligne-Mumford stack) over a field k and $\mathscr{E} \in \mathscr{D}^b(X)$ a bounded complex. Following Definition 1.2.7 ([Kuz09]), Hochschild cohomology and homology spaces of X with coefficients in $\mathscr{E}$ are given respectively by

$$\mathrm{HH}^\bullet(X) := \mathrm{Hom}^\bullet_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{E})$$

and

$$\mathrm{HH}_\bullet(X) := \mathbb{H}^\bullet(X \times X, \Delta_*\mathscr{O}_X \otimes \Delta_*\mathscr{E})$$

where $\Delta : X \to X \times X$ is the diagonal morphism.

The next lemma provides an equivalent definition of Hochschild homology space of an algebraic variety is a generalization [BFK23] of the one proved by Kuznetsov in [Kuz09] with coefficients in the structure sheaf.

Lemma 4.1.1. There is a canonical isomorphism

$$\mathbb{H}^\bullet(X \times X, \Delta_*\mathscr{O}_X \otimes \Delta_*\mathscr{E}) \simeq \mathrm{Hom}^\bullet_{X\times X}(\Delta_*\mathscr{O}_X, S_X(\mathscr{E})).$$

In the statement above, we denote by S_X the Serre functor of X, explicitly given by $S_X = - \otimes \Delta_*\omega_X[\dim X]$, where $\omega_X = \bigwedge^{\dim X} \Omega^1_X$ is the canonical bundle of X. In other words, the Hochschild homology space of X is the Hochschild cohomology space with coefficients twisted by the Serre functor. We introduce a logarithmic Serre functor fitting in a similar equivalence for logarithmic Hochschild homology.

Let us recall that for a logarithmic scheme X, we construct the diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell" very near start] & X \times X \arrow[d] \\
& \Theta_X \arrow[r] & \Theta_{X\times X}
\end{tikzcd} .$$

Recall that the morphism $i : X \to B = X \times_{\Theta_X} X$ is the strict diagonal. We use it to define the logarithmic Hochschild homology and cohomology complexes as

$$i^* i_* \mathscr{O}_X \quad \text{and} \quad i^! i_* \mathscr{O}_X$$

respectively. The logarithmic Hochschild homology and cohomology spaces are given by applying the global section functor to the complexes above. Similarly to the classical case, logarithmic Hochschild homology and cohomology spaces may be interpreted as

$$\mathrm{HH}^\ell_\bullet(X) = \mathbb{H}^\bullet(B, i_*\mathscr{O}_X \otimes i_*\mathscr{O}_X)$$

and

$$\mathrm{HH}^\bullet_\ell(X) = \mathrm{Hom}^\bullet_B(i_*\mathscr{O}_X, i_*\mathscr{O}_X).$$

Preliminarily, we introduce the logarithmic Serre functor. We will see that this functor plays the expected role, relating logarithmic Hochschild homology and cohomology. However, we do not state that this has any more general interpretation as an intrinsic Serre functor of a reasonable logarithmic derived category, that one could expect knowing that the ordinary Serre functor is the intrinsic Serre functor of $\mathscr{D}^b(X)$.

Definition 4.1.2. Let X be a smooth, weakly logarithmically separated and logarithmically flat logarithmic scheme. The logarithmic Serre functor of X is

$$S_X^{\log} := - \otimes \omega_X^{\log}[\dim X] : \mathscr{D}^b(X) \to \mathscr{D}^b(X)$$

where $\omega_X^{\log} = \bigwedge^{\dim X} \Omega_X^{1,\log}$.

Remark 4.1.3. This notation is classical and follows the discussion in [Cal03], which we will refer to often in this work. Nonetheless, it is important to realise that it differs from other literature (see, for instance, [Kuz09] and [BFK23]): consistently with this other choice, we should have denoted $-\otimes i_*\omega_X^{\log}[\dim X]$ by $S_X^{\log}$. However, the projection formula shows that

$$- \otimes i_*\omega_X^{\log}[\dim X] \simeq i_*(i^*(-) \otimes \omega_X^{\log}[\dim X]),$$

allowing a more general discussion for our purposes via the functor in Definition 4.1.2.

Lemma 4.1.4. The logarithmic Serre functor is a Fourier-Mukai functor of kernel $\Delta_*\omega_X^{\log}[\dim X] \in \mathscr{D}^b(X \times X)$ (resp. $i_*\omega_X^{\log}[\dim X] \in \mathscr{D}^b(X \times_{\Theta_X} X)$).

Proof. This statement follows from the fact that the diagonal Δ (resp. the strict diagonal i) satisfies the universal property of the Cartesian product and the projection formula. Explicitly, for the statement on $\Delta_*\omega_X^{\log}$, given the diagram

$$\begin{tikzcd}
& X \arrow[ddl, bend right, "\mathrm{Id}"'] \arrow[d, dashed, "\Delta"] \arrow[ddr, bend left, "\mathrm{Id}"] & \\
& X \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X & & X
\end{tikzcd}$$

and a complex $\mathscr{E} \in \mathscr{D}^b(X)$ we have that

$$\Phi_{X\to X}^{\Delta_*\omega_X^{\log}[\dim X]}(\mathscr{E}) := p_{2,*}(p_1^*\mathscr{E} \otimes \Delta_*\omega_X^{\log}[\dim X]) \simeq \underbrace{p_{2,*}\Delta_*}_{\mathrm{Id}_*}(\underbrace{\Delta^*p_1^*}_{\mathrm{Id}^*}\mathscr{E} \otimes \omega_X^{\log}[\dim X])$$

$$\simeq \mathscr{E} \otimes \omega_X^{\log}[\dim X] = S_X^{\log}(\mathscr{E}).$$

The same computation may be performed over the base Θ_X to obtain the remaining part of the statement. ☐

Lemma 4.1.5. Let X be a smooth, proper, logarithmically smooth and weakly logarithmically separated logarithmic scheme. Then $\omega_X^{\log}[\dim X] \simeq (i^!\mathcal{O}_B)^{-1}$.

Proof. Let $u : B \hookrightarrow \overline{B}$ be a compactification over X, with the induced logarithmic structure. This compactification exists by [CLO12] by recovering B as in the diagram

$$\begin{tikzcd}
X \times_{\Theta_X} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\not\ell", very near start] & X \arrow[d] \\
X \arrow[r] & \Theta_X
\end{tikzcd}$$

and observing that since $X \to \Theta_X$ is strict, separated and smooth, so are the projections $X \times_{\Theta_X} X \rightrightarrows X$. The compactification morphism factors as

$$B \xrightarrow{u} \overline{B} \to X$$

and the logarithmic structure on $\overline{B}$ is obtained by the pullback of the one on X. The morphism u is a strict open immersion, therefore logarithmically étale. Because u is open,

$$i^!\mathcal{O}_B \simeq i^!u^! \simeq i^!u^*\mathcal{O}_{\overline{B}} \simeq \mathcal{O}_{\overline{B}}$$

by [Sta24, 0AU0]. Moreover, $\Omega_B^{1,\log} \simeq u^*\Omega_{\overline{B}}^{1,\log}$ as u is logarithmically étale. By the general expression of the exceptional inverse image of a proper morphism between proper schemes, we have that

$$i^!\mathcal{O}_B \simeq i^!u^!\mathcal{O}_B = i^*u^*\omega_{\overline{B}}^{-1}[-\dim \overline{B}] \otimes \omega_X[\dim X] \simeq \omega_{X/\overline{B}}[\dim u \circ i].$$

Being both u and i strict, their composite $u \circ i$ is also strict, and $\omega_{X/\overline{B}} \simeq \omega_{X/\overline{B}}^{\log} \simeq (\omega_X^{\log})^{-1}$. Moreover $\dim u \circ i = \dim X - \dim \overline{B}$ and $\dim \overline{B} = \dim B = 2\dim X$ as u is open and dense and X is logarithmically flat. ☐

Analogously to Lemma 4.1.1, the following statement holds for the logarithmic Hochschild homology space.

Lemma 4.1.6. Let X be a smooth, proper, logarithmically smooth and weakly logarithmically separated logarithmic scheme. There is a canonical isomorphism

$$\mathbb{H}^\bullet(B, i_*\mathcal{O}_X \otimes i_*\mathcal{E}) \simeq \operatorname{Hom}_B^\bullet(i_*\mathcal{O}_X, i_*S_X^{\log}(i_*\mathcal{E})).$$

Proof. The proof follows the lines of the one of 4.1.1. By definition of logarithmic Hochschild homology

$$\mathrm{HH}_\bullet^\ell(X, \mathcal{E}) = \operatorname{Hom}_B^\bullet(\mathcal{O}_B, i_*\mathcal{O}_X \otimes i_*\mathcal{E}) \simeq \operatorname{Hom}_B^\bullet((i_*\mathcal{O}_X)^\vee, i_*\mathcal{E}).$$

By Grothendieck duality and Lemma 4.1.5

$$\begin{aligned}
(i_*\mathcal{O}_X)^\vee &\simeq \mathcal{H}om_B(i_*\mathcal{O}_X, \mathcal{O}_B) \simeq \\
&\simeq i_*\mathcal{H}om_X(\mathcal{O}_X, i^!\mathcal{O}_B) \simeq i_*(\omega_X^{\log})^{-1}[-\dim X].
\end{aligned}$$

Plugging this in the computation above

$$\mathrm{HH}^{\ell}_{\bullet}(X,\mathcal{E}) \simeq \mathrm{Hom}^{\bullet}_{B}(i_*(\omega_X^{\log})^{-1}[-\dim X], i_*\mathcal{E}).$$

Taking the tensor product with $i_*\omega_X^{\log}[\dim X]$ on both sides we obtain that

$$\mathrm{HH}^{\ell}_{\bullet}(X,\mathcal{E}) \simeq \mathrm{Hom}^{\bullet}_{B}(i_*\mathcal{O}_X, i_*S_X^{\log}(\mathcal{E})).$$

□

4.2 The logarithmic Serre functor for logarithmic Deligne-Mumford stacks

In Section 4.1, we developed some technical tools, namely the logarithmic Serre functor (Definition 4.1.2) and understood its pivotal role in the conjugation of logarithmic Hochschild homology and cohomology (Lemma 4.1.6). We carried on the detailed discussion for schemes. However, the proofs may be generalized to orbifolds (and, more in general, to Deligne-Mumford stacks).

Grothendieck duality in terms of a dualizing complex has been proved in [Nir09] and [Lev22] for tame separated Deligne-Mumford stacks. Because our work is limited to ground fields of characteristic 0, the assumption that we will make in the rest of this chapter for Deligne-Mumford stacks is just of being separated. If the stack is additionally proper, the dualizing sheaf is given by the canonical bundle. If Y is a smooth and proper scheme with a G-action, then $X = [Y/G]$ satisfies the assumption and the canonical bundle can be expressed as

$$\omega_{[Y/G]} = \bigwedge^{\dim Y} \Omega^1_{[Y/G]} \simeq \bigwedge^{\dim Y} \left(\Omega^1_Y\right)^G .$$

Equip now X with a logarithmic structure. Logarithmic quotient stacks require an additional property on the action to be well-defined.

Definition 4.2.1. We say that the action of G on the logarithmic scheme X is logarithmically compatible if the action morphism

$$G \times X \to X$$

is a morphism of logarithmic schemes when G is equipped with the trivial logarithmic structure.

Example 4.2.2. Let X be a toric variety, regarded as a logarithmic scheme, and G its dense torus. The action extended from the multiplication of G

$$G \times X \to X$$

is logarithmically compatible. Notice that this action gives a quotient stack which is not Deligne-Mumford.

Example 4.2.3. The reflextion action of $\mathbb{Z}/2\mathbb{Z}$ on the logarithmic affine line $(\mathbb{A}^1, 0)$ given by

$$\begin{aligned}\mathbb{Z}/2\mathbb{Z} \times \mathbb{A}^1 &\to \mathbb{A}^1\\ (x,t) &\to -t\end{aligned}$$

where $x \in \mathbb{Z}/2\mathbb{Z}$ is the non-unit element is logarithmically compatible.

Example 4.2.4. The shuffle action by the symmetric group Σ_n on X^n where X is a logarithmic scheme and $n > 0$ an integer is logarithmically compatible.

Example 4.2.5. The translation action of $\mathbb{G}_a \simeq \mathbb{A}^1$ on $(\mathbb{A}^1, 0)$ is not logarithmically compatible.

Remark 4.2.6. A logarithmically compatible action on the logarithmic scheme induces an action on the Artin fan.

If X is a smooth and proper logarithmic scheme with a logarithmically compatible G-action, the logarithmic canonical bundle is given by

$$\omega^{\log}_{[Y/G]} = \bigwedge^{\dim Y} \Omega^{1,\log}_{[Y/G]} \simeq \bigwedge^{\dim Y} \left(\Omega^{1,\log}_{Y}\right)^{G}.$$

The only step where one has to be careful is the existence of compactifications, that we used in Lemma 4.1.5. Indeed, compactifications are known to exist for schemes but are not as straightforward for stacks. We prove the following lemma.

Lemma 4.2.7. *Let X be a proper separated logarithmic Deligne-Mumford stack. Then B_X admits a compatification.*

Proof. The criterion proved in [Ryd09] reduces the problem to show that B_X is separated. Since X is separated, $X \times X$ is as well. We have to prove that the diagonal morphism

$$\begin{aligned}X \times_{\Theta_X} X \to &(X \times_{\Theta_X} X) \times (X \times_{\Theta_X} X)\\ &\simeq (X \times X) \times_{\Theta_{X\times X}} (X \times X)\end{aligned}$$

is closed. This morphism fits in the Cartesian diagram

$$\begin{tikzcd}
X \times_{\Theta_X} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (X \times X) \times_{\Theta_{X\times X}} (X \times X) \arrow[d] \\
X \times X \arrow[r] & X \times X \times X \times X
\end{tikzcd}$$

and is therefore closed being the base change of a closed morphism.

□

This holds in particular for $X = [Y/G]$ as above. This lemma allows us to generalize Lemma 4.1.6 to the case of proper separated Deligne-Mumford stacks.

4.3 The proof of invariance

To conclude this chapter, we prove its main result, that is the invariance of logarithmic Hochschild homology and cohomology, according to the definition given in [Ols24] and [HHL23] under a condition that we call *strict logarithmic derived equivalence*. Contrarily to what the name may suggest, we do not claim that the mentioned condition should be a satisfactory definition of logarithmic derived equivalence for reasons that will be clarified later. Remark 4.3.3 provides evidence that a good notion of logarithmic derived equivalence should include more cases than those satisfying the definition introduced below. Nevertheless, we believe that strict logarithmic derived equivalence should imply logarithmic derived equivalence by analogy.

Definition 4.3.1. Let X and Y be smooth and proper logarithmic schemes. Assume that there is an Artin fan $\mathscr{A}$ and strict morphisms $\Theta_X \to \mathscr{A}$ and $\Theta_Y \to \mathscr{A}$, and a smooth and proper logarithmic scheme Z and a commutative diagram

$$\begin{tikzcd}
& Z \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& \mathscr{A} &
\end{tikzcd}$$

where p_X and p_Y are strict and proper. We say that X and Y are *strictly logarithmically derived equivalent* if there is an equivalence of type

$$p_{Y,*}(p_X^*(-) \otimes \mathscr{P}) : \mathscr{D}^b(X) \to \mathscr{D}^b(Y)$$

for some $\mathscr{P} \in \mathscr{D}^b(Z)$. We call such a functor a *strict logarithmic derived equivalence.*

Lemma 4.3.2. We have natural morphisms $Z \to X \times_{\mathscr{A}} Y \to X \times Y$, with the first one being strict.

Proof. The morphisms above are given by the universal property of the Cartesian product for the diagrams

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& X \times_{\mathscr{A}} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \not\ell" very near start] & Y \arrow[d] \\
& X \arrow[r] & \mathscr{A}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \times_{\mathscr{A}} Y \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & Y \arrow[d] \\
& X \arrow[r] & *
\end{tikzcd} .$$

Moreover, the morphisms $X \times_{\mathscr{A}} Y \to X$ and $X \times_{\mathscr{A}} Y \to Y$ obtained in the left-hand side diagram are strict being the base-change of strict morphisms. The morphisms $X \to X$ and $Z \to Y$ are strict by assumption. Thus also the universal morphism is. □

Remark 4.3.3. The setting described in 4.3.1 implies derived equivalence, with explicit Fourier-Mukai kernel $h_*\mathscr{P}$ where $h : Z \to X \times Y$ the universal morphism. This is a straightforward application of the projection formula [Sta24, 01E6]. In particular, any strict logarithmic derived equivalence is a derived equivalence. This consequence is a reason why 4.3.1 does not aim to be a definition of logarithmic derived equivalence. With this in mind, there will be no confusion in referring to $\mathscr{P}$ as the kernel of a Fourier-Mukai equivalence, omitting to specifying that it is after pushforward.

Remark 4.3.4. Morally, one would like to admit all derived coherent sheaves in $\mathscr{D}^b(X \times_{\mathscr{A}} Y)$ as Fourier-Mukai kernel in order to satisfy Definition 4.3.1. Unfortunately, the failure of properness for $X \times_{\mathscr{A}} Y$ raises problems in Fourier-Mukai theory. Therefore we restrict the choice of kernels to the compactly supported ones, i.e. those that can be realized as pushforward of Fourier-Mukai kernels in $\mathscr{D}^b(Z)$ for some smooth and proper Z.

Remark 4.3.5. A broad class of examples of strict logarithmic derived equivalences can be realized by replacing the Artin fan with a logarithmic scheme S. Consider X and Y smooth and proper logarithmic schemes and S a logarithmic scheme with strict and proper morphisms $X \to S$ and $Y \to S$. A Fourier-Mukai equivalence $q_{X,*}(q_Y^*(-) \otimes \mathscr{P})$ with $X \xleftarrow{q_X} X \times_S Y \xrightarrow{q_Y} Y$ and $\mathscr{P} \in \mathscr{D}^b(X \times_S Y)$ is a strict logarithmic derived equivalence. Here it suffices to take $\mathscr{A} = \Theta_S$ and $Z = X \times_S Y$.
Moreover, if X and Y are smooth and proper derived equivalent schemes over S (i.e. the Fourier-Mukai equivalence has a kernel in $\mathscr{D}^b(X \times_S Y)$), and $\mathcal{M}_S$ is a logarithmically smooth logarithmic structure on S. Pulling it back via $f : X \to S$ and $g : Y \to S$, we obtain the strictly logarithmically derived equivalent pair $(X, f_{\log}^*\mathcal{M}_S)$ and $(Y, g_{\log}^*\mathcal{M}_S)$.

We are now able to state the main result of this chapter.

Theorem 4.3.6. If X and Y are strictly logarithmically derived equivalent separated, weakly logarithmically separated and logarithmically flat logarithmic schemes or separated logarithmic Deligne-Mumford stacks, then their logarithmic Hochschild homology and cohomology spaces are isomorphic.

We prove this theorem following the strategy used in [Cal03] to show that ordinary Hochschild homology and cohomology are invariant under derived equivalence.

Let X and Y be as in Definition 4.3.1. Denote by $\Phi : \mathscr{D}^b(X) \to \mathscr{D}^b(Y)$ a Fourier-Mukai equivalence providing a strict logarithmic derived equivalence, with Fourier-Mukai kernel $\mathscr{P} \in \mathscr{D}^b(Z)$, which we assumed to come with the equivalence. A categorical interpretation for this encounters the same issues highlighted in [Cal03, 1.10] and should be addressed as in [Cal03, Appendix A, Appendix B]. Denote by Φ^* and $\Phi^!$ its left and right adjoints respectively. Consider the sequence of natural transformations

$$1_{\mathscr{D}^b(X)} \to \Phi^! \circ \Phi \to \Phi^! \circ S_Y^{\log} \circ \Phi \to S_X^{\log} \circ \Phi^* \circ \Phi \to S_X^{\log} \tag{4.1}$$

constructed as in [Cal03].

Lemma 4.3.7. The functors in (4.1) are Fourier-Mukai functors and the natural transformations

$$1_{\mathscr{D}^b(X)} \to \Phi^! \circ \Phi \quad \text{and} \quad \Phi^* \circ \Phi \to 1_{\mathscr{D}^b(X)}$$

correspond to natural morphisms of their Fourier-Mukai kernels.

Proof. To prove that all the functors are of Fourier-Mukai type, a classical construction of the kernels of the adjoints gives explicit kernels for Φ^* and $\Phi^!$ in terms of the kernel of Φ (see [Huy07]). The logarithmic Serre functors are of Fourier-Mukai type by Lemma 4.1.4. Compositions of Fourier-Mukai functors are Fourier-Mukai kernels explicitly given by the convolutions of the kernels.

A proof of the correspondence of the natural transformations to natural morphisms of the Fourier-Mukai kernels is given by [Cal03, Proposition 5.1].

□

A technical tool that we will use is given by the statement below and relates the relative Serre functor of a projection and the Serre functor of the left-out factor.

Lemma 4.3.8. Let X and Y be smooth and proper Deligne-Mumford and $\pi_X, \pi_Y : X \times Y \to X, Y$ the natural projections. Then

$$\pi_X^! \mathscr{O}_X \simeq \pi_Y^* \omega_Y [\dim Y].$$

Proof. Since X and Y are smooth and proper, their product $X \times Y$ and the projection morphisms are as well. Therefore

$$\begin{aligned} \pi_X^! \mathscr{O}_X &\simeq \omega_{X\times Y/X}[\dim \pi_X] \simeq \pi_X^* \omega_X^{-1}[-\dim X] \otimes \omega_{X\times Y}[-\dim X \times Y] \\ &\simeq \pi_X^* \omega_X^{-1}[-\dim X] \otimes \pi_X^* \omega_X [\dim X] \otimes \pi_Y^* \omega_Y [\dim Y] \simeq \pi_Y^* \omega_Y [\dim Y]. \end{aligned}$$

□

The following proposition is the crucial result on which Theorem 4.3.6 relies.

Proposition 4.3.9. Let Φ be a strict logarithmic derived equivalence between X and Y are strictly logarithmically derived equivalent separated, weakly logarithmically separated and logarithmically flat logarithmic schemes or separated logarithmic Deligne-Mumford stacks. Then $\Phi^! \circ S_Y^{\log} \to S_X^{\log} \circ \Phi^*$ is an equivalence.

Proof. Let $\mathscr{P} \in \mathscr{D}^b(Z)$ be the kernel of a strict logarithmic derived equivalence $\Phi = p_{Y,*}(p_X^*(-) \otimes \mathscr{P})$. In the notation of Remark 4.3.3, Φ may be written as Fourier-Mukai functor through $X \times Y$ as $\Phi = \Phi^{h_*\mathscr{P}}_{X\to Y} \simeq \pi_{Y,*}(\pi_X^*(-) \otimes h_*\mathscr{P})$, where

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd} .$$

Set $\mathscr{K} := h_*\mathscr{P}$ and define

$$\begin{aligned}\mathscr{K}^* &:= \mathscr{K}^\vee \otimes \pi_Y^*\omega_Y[\dim Y];\\ \mathscr{K}^! &:= \mathscr{K}^\vee \otimes \pi_X^*\omega_X[\dim X].\end{aligned}$$

The above complexes give the Fourier-Mukai kernels of the left and right adjoint of Φ respectively: set $\Phi^* := \Phi_{Y\to X}^{\mathscr{K}^*}$ and $\Phi^! := \Phi_{Y\to X}^{\mathscr{K}^!}$, then $\Phi^* \dashv \Phi \dashv \Phi^!$ (see, for instance, [Huy07]). By Grothendieck duality, the dual of a pushforward is also a pushforward. More precisely,

$$\begin{aligned}(h_*\mathscr{P})^\vee &= \mathscr{H}om_{X\times Y}(h_*\mathscr{P}, \mathcal{O}_{X\times Y}) \simeq h_*\mathscr{H}om_Z(\mathscr{P}, h^!\mathcal{O}_{X\times Y})\\ &\simeq h_*(\mathscr{P}\otimes (h^!\mathcal{O}_{X\times Y})^\vee, \mathcal{O}_Z) \simeq h_*(\mathscr{P}\otimes (h^!\mathcal{O}_{X\times Y})^\vee)^\vee\\ &\simeq h_*(\mathscr{P}^\vee \otimes h^!\mathcal{O}_{X\times Y}).\end{aligned}$$

Refer to the following notation for the diagonals and the projections

$$\begin{tikzcd}
 & Y \arrow[d, "\Delta_Y"] \arrow[ddl, equal, bend right] \arrow[ddr, equal, bend left] & & Z \arrow[d, "h"] \arrow[ddl, bend right, "p_Y"'] \arrow[ddr, bend left, "p_X"] & & X \arrow[d, "\Delta_X"] \arrow[ddl, equal, bend right] \arrow[ddr, equal, bend left] & \\
 & Y\times Y \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & & X\times Y \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_X"] & & X\times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
Y & & Y & & X & & X
\end{tikzcd} .$$

The functor $S_X^{\log}\circ\Phi^*$ may be expanded as

$$\begin{aligned}S_X^{\log}\circ\Phi^* &\simeq p_{2,*}(\Delta_{X,*}\omega_X^{\log}[\dim X]\otimes p_1^*(\pi_{X,*}(\mathscr{K}^*\otimes\pi_Y^*(-))))\\ &\simeq p_{2,*}\Delta_{X,*}(\omega_X^{\log}[\dim X]\otimes \Delta_X^* p_1^*(\pi_{X,*}(\mathscr{K}^*\otimes\pi_Y^*(-))))\\ &\simeq \omega_X^{\log}[\dim X]\otimes \pi_{X,*}(\mathscr{K}^*\otimes\pi_Y^*(-))\\ &\simeq \pi_{X,*}(\pi_X^*\omega_X^{\log}[\dim X]\otimes(\mathscr{K}^*\otimes\pi_Y^*(-)))\\ &\simeq \pi_{X,*}(\pi_X^*\omega_X^{\log}[\dim X]\otimes(h_*\mathscr{P})^\vee\otimes\pi_Y^*\omega_Y[\dim Y]\otimes\pi_Y^*(-))\\ &\simeq \pi_{X,*}(\pi_X^*\omega_X^{\log}[\dim X]\otimes h_*(\mathscr{P}^\vee\otimes h^!\mathcal{O}_{X\times Y})\otimes\pi_Y^*\omega_Y[\dim Y]\otimes\pi_Y^*(-))\\ &\simeq \pi_{X,*}h_*(h^*\pi_X^*\omega_X^{\log}[\dim X]\otimes\mathscr{P}^\vee\otimes h^!\mathcal{O}_{X\times Y}\otimes h^*\pi_Y^*\omega_Y[\dim Y]\otimes h^*\pi_Y^*(-))\\ &\simeq p_{X,*}(p_X^*\omega_X^{\log}[\dim X]\otimes\mathscr{P}^\vee\otimes h^!\mathcal{O}_{X\times Y}\otimes p_Y^*\omega_Y[\dim Y]\otimes p_Y^*(-)).\end{aligned}$$

In other words, $S_X^{\log}\circ\Phi^*$ has Fourier-Mukai kernel in

$$p_X^*\omega_X^{\log}[\dim X]\otimes\mathscr{P}^\vee\otimes h^!\mathcal{O}_{X\times Y}\otimes p_Y^*\omega_Y[\dim Y]\in\mathscr{D}^b(Z).$$

Analogous computations for $\Phi^!\circ S_Y^{\log}$ give the Fourier-Mukai kernel

$$p_Y^*\omega_Y^{\log}[\dim Y]\otimes\mathscr{P}^\vee\otimes h^!\mathcal{O}_{X\times Y}\otimes p_X^*\omega_X[\dim X]\in\mathscr{D}^b(Z).$$

We consider the first kernel, that we denote by $\mathscr{H}$. By Lemma 4.3.8 and the explicit expression of the exceptional inverse image we get that

$$\begin{aligned}\mathscr{H} &\simeq p_X^*\omega_X^{\log}[\dim X]\otimes\mathscr{P}^\vee\otimes h^!\mathcal{O}_{X\times Y}\otimes h^*\pi_Y^*\omega_Y[\dim Y]\simeq\\ &\simeq p_X^*\omega_X^{\log}[\dim X]\otimes\mathscr{P}^\vee\otimes h^!\mathcal{O}_{X\times Y}\otimes h^*\pi_X^!\mathcal{O}_X\\ &\simeq p_X^*\omega_X^{\log}[\dim X]\otimes\mathscr{P}^\vee\otimes h^!\pi_X^!\mathcal{O}_X.\end{aligned}$$

Since p_X is smooth and proper between smooth and proper separated Delioqne-Mumford stacks, and strict as a morphism of logarithmic schemes, we have that

$$h^!\pi_X^!\mathcal{O}_X\simeq p_X^!\mathcal{O}_X\simeq p_X^*(\omega_X^{\log})^{-1}[-\dim X]\otimes\omega_Z^{\log}[\dim Z].$$

Plugging this in the above expression of $\mathscr{H}$ we get that

$$\mathscr{H}\simeq\mathscr{P}^\vee\otimes\omega_Z^{\log}[\dim Z].$$

A similar computation gives the same expression for the computed kernel of the composite $\Phi^!\circ S_Y^{\log}$. □

With the aid of Proposition 4.3.9, we are able to Theorem 4.3.6 following the steps prescribed by [Cal03, Section 8].

Proof of Theorem 4.3.6. Let $\mathscr{P}\in\mathscr{D}^b(Z)$ be the kernel of a strict logarithmic derived equivalence from X to Y. Set $\mathscr{K}=h_*\mathscr{P}\in\mathscr{D}^b(X\times Y)$ and $\mathscr{K}^*=\mathscr{K}^\vee\otimes\pi_Y^*S_Y$. It is shown in [Orl02] that

$$\mathscr{H}:=\mathscr{K}^*\boxtimes\mathscr{K}=\sigma_{1,2}^*\mathscr{K}^*\otimes\sigma_{3,4}^*\mathscr{K}\in\mathscr{D}^b(X\times Y\times X\times Y)$$

is a Fourier-Mukai kernel for a derived equivalence from $X\times X$ to $Y\times Y$ (here $\sigma_{1,2},\sigma_{3,4}:X\times Y\times X\times Y\to X\times Y$ are the projections onto the enumerated factors).

It follows directly from [Cal03] that the Fourier-Mukai equivalence $\Phi^{\mathscr{H}}_{X\times X\to Y\times Y}$ morphisms $f_{X,*}(i_{X,*}\mathcal{O}_X)=\Delta_{X,*}\mathcal{O}_X$ to $f_{Y,*}(i_{Y,*}\mathcal{O}_Y)=\Delta_{Y,*}\mathcal{O}_Y$. Therefore the logarithmic Hochschild cohomology spaces of X and Y are isomorphic.

To prove that the logarithmic Hochschild homology spaces of X and Y are isomorphic as well, we have to prove that $\Phi^{\mathscr{H}}_{X\times X\to Y\times Y}$ morphisms $f_{X,*}(i_{X,*}\omega_X^{\log})$ to $f_{Y,*}(i_{Y,*}\omega_Y^{\log})$. In other words, we have to prove that $\Phi^{\mathscr{H}}_{X\times X\to Y\times Y}$ preserves the logarithmic Serre functor. In the rest of this proof, we show that

$$\Phi^{\mathscr{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\log})\simeq\mathscr{K}\circ\Delta_{X,*}\omega_X^{\log}\circ\mathscr{K}^*.$$

Once this expression is proved, one can use Proposition 4.3.9 to equate it to $\Delta_{Y,*}\omega_Y^{\log}$.

By definition,

$$\Phi^{\mathscr{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\log})=\sigma_{2,4,*}(\sigma_{1,3}^*(\Delta_{X,*}\omega_X^{\log})\otimes\mathscr{H}).$$

Applying flat base change [Sta24, 02KH] to the Cartesian diagram

$$\begin{tikzcd}
Y \times X \times Y \arrow[r, "\Delta_{1,3}"] \arrow[d, "\rho_2"] \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \times X \times Y \arrow[d, "\sigma_{1,3}"] \\
X \arrow[r, "\Delta_X"] & X \times X
\end{tikzcd} \tag{4.2}$$

we obtain that

$$\Phi^{\mathcal{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\mathrm{log}}) \simeq \sigma_{2,4,*}(\Delta_{1,3,*}\rho_2^*\omega_X^{\mathrm{log}} \otimes \mathcal{H}).$$

By the projection formula [Sta24, 01E6], the latter is isomorphic to

$$\begin{aligned}
\Phi^{\mathcal{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\mathrm{log}}) &\simeq \sigma_{2,4,*}\Delta_{1,3,*}(\rho_2^*\omega_X^{\mathrm{log}} \otimes \Delta_{1,3}^*\mathcal{H}) \\
&\simeq \rho_{1,3,*}(\rho_2^*\omega_X^{\mathrm{log}} \otimes \Delta_{1,3}^*\mathcal{H}).
\end{aligned}$$

where $\rho_{1,3} : Y \times X \times Y \to Y \times Y$ is the natural projection onto the product of the indexed components (and later $\rho_{1,2}, \rho_{2,3} : Y \times X \times Y \to X \times Y$). Consider the complex $\Delta_{1,3}^*\mathcal{H}$. By definition of $\mathcal{H}$ it is equal to

$$\Delta_{1,3}^*(\sigma_{1,2}^*\mathcal{K}^* \otimes \sigma_{3,4}^*\mathcal{K}) \simeq \rho_{1,2}^*\mathcal{K} \otimes \rho_{2,3}^*\mathcal{K}^*.$$

In order to express

$$\Phi^{\mathcal{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\mathrm{log}}) \simeq \rho_{1,3,*}(\rho_2^*\omega_X^{\mathrm{log}} \otimes \rho_{1,2}^*\mathcal{K} \otimes \rho_{2,3}^*\mathcal{K}^*)$$

in a more suitable way we consider the projection morphism $\tau_{2,3,4} : X \times Y \times X \times Y \to Y \times X \times Y$. As $\tau_{2,3,4} \circ \Delta_{1,3} = 1_{Y\times X\times Y}$,

$$\Phi^{\mathcal{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\mathrm{log}}) \simeq \rho_{1,3,*}(\tau_{2,3,4,*}\Delta_{1,3,*}\rho_2^*\omega_X^{\mathrm{log}} \otimes \rho_{1,2}^*\mathcal{K} \otimes \rho_{2,3}^*\mathcal{K}^*).$$

By flat base change of Diagram (4.2),

$$\Phi^{\mathcal{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\mathrm{log}}) \simeq \rho_{1,3,*}(\tau_{2,3,4,*}\sigma_{1,3}^*\Delta_{X,*}\omega_X^{\mathrm{log}} \otimes \rho_{1,2}^*\mathcal{K} \otimes \rho_{2,3}^*\mathcal{K}^*).$$

By projection formula,

$$\Phi^{\mathcal{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\mathrm{log}}) \simeq \rho_{1,3,*}\tau_{2,3,4,*}(\sigma_{1,3}^*\Delta_{X,*}\omega_X^{\mathrm{log}} \otimes \tau_{2,3,4}^*\rho_{1,2}^*\mathcal{K} \otimes \tau_{2,3,4}^*\rho_{2,3}^*\mathcal{K}^*).$$

Since

$$\begin{aligned}
\rho_{1,2} \circ \tau_{2,3,4} &= \sigma_{2,3} \\
\rho_{1,3} \circ \tau_{2,3,4} &= \sigma_{2,4} \\
\rho_{2,3} \circ \tau_{2,3,4} &= \sigma_{3,4}
\end{aligned}$$

one obtains the desired convolution

$$\begin{aligned}
\Phi^{\mathcal{H}}_{X\times X\to Y\times Y}(\Delta_{X,*}\omega_X^{\mathrm{log}}) &\simeq \sigma_{2,4,*}(\sigma_{1,3}^*\Delta_{X,*}\omega_X^{\mathrm{log}} \otimes \sigma_{2,3}^*\mathcal{K} \otimes \sigma_{3,4}^*\mathcal{K}^*) \\
&\simeq \mathcal{K} \circ \Delta_X\omega_X^{\mathrm{log}} \circ \mathcal{K}^*
\end{aligned}$$

□

CHAPTER 5

Logarithmic Hochschild homology and cohomology for orbifolds

In this chapter, we prove a logarithmic version of the orbifold HKR [ACH19, Corollary 1.17] recalled in Subsection 1.4.2 of this thesis. The result inspiring us offers a further decomposition of the Hochschild homology and cohomology spaces of orbifolds (see Definition 1.4.9) obtained by the quotient of a smooth, quasi-projective scheme modulo a finite group. In this case, the Hochschild homology and cohomology endofunctors decompose as the direct sum of the intersection endofunctors for the action of each of the elements of the group. This decomposition allows them to compute the global sections as the fixed locus of a direct sum of Hodge homology spaces.

The problem in the logarithmic case requires some additional assumptions. First of all, in order to induce a logarithmic structure on the quotient, we need the group to act on X logarithmically compatibly to the sense of Definition 4.2.1. This makes the quotient morphism

$$X \to [X/G]$$

tautologically strict. However, even how the action is logarithmically compatible, there are additional difficulties making the very general case hard to treat. The main obstacle consists of the delay between the quotient of the Artin fan and the Artin fan of the quotient (see Remark 5.1.2) and that, for this reason, the action may not lift to the strict diagonal. To avoid this issue, we restrict to a class of actions that we call *firm*. Whilst the first assumption is rather reasonable in the context of logarithmic geometry, the second one constitutes a relevant restriction to the scenarios that we address. In Section 5.1, we prove the logarithmic HKR for such orbifolds, following the strategy of [ACH19]. The content of this section is part of [HHL23].

In Section 5.2, we give an example of non-firm orbifold and we compute its logarithmic Hochschild homology. The example consists of the action of the symmetric group Σ_n on the n-fold product X^n of a logarithmic scheme X. This computation is made possible by the explicitness of the logarithmic intersection loci.

In this chapter, we will conventionally make the following assumption for all logarithmic schemes that we will consider.

Assumption 5.0.1. All logarithmic schemes are logarithmically smooth and quasicompact.

In particular, Corollary 2.3.30 will always apply.

5.1 The logarithmic HKR theorem for firm orbifolds

In this section, we describe the logarithmic Hochschild homology and cohomology of logarithmic orbifolds $[X/G]$ in the case of a *firm* action (Definition 5.1.3). Using formality results, we understand the logarithmic derived loop space associated with a logarithmic orbifold and we derive formulae for the logarithmic Hochschild homology and cohomology. We closely follow [ACH19]. We will expand the details of the proofs when they differ from the reference and omit those already contained in the paper.

5.1.1 Logarithmic orbifolds and firm logarithmic orbifolds

Let G be a finite group acting on a logarithmic scheme (X, M_X) logarithmically compatibly (see Definition 4.2.1).

Remark 5.1.1. Let X be a smooth scheme equipped with divisorial logarithmic structure from a normal crossings divisor D as in Example 2.1.14. The action is logarithmically compatible if and only if it restricts to D. Assume that there exists an element $g \in G$ and a point $x \in D$ such that $g.x \notin D$. Then the induced morphism $(g.^{-1} j_* \mathcal{O}_U)_x \to (j_* \mathcal{O}_U)_x$ (here $U = X \setminus D$)) has a non-zero source and a zero target, thus it is not bijective.

The quotient stacks are naturally logarithmic stacks by descent. This makes quotient morphism $X \to [X/G]$ a strict morphism. In particular, there is an induced morphism of Artin fans

$$\Theta_X \to \Theta_{[X/G]}.$$

Remark 5.1.2. The main technical difficulty here is that the Artin fan of an orbifold is not isomorphic to the corresponding orbifold of Artin fan. More precisely, consider the stack quotient $Y = [X/G]$ with its natural logarithmic structure. The morphisms

$$G \times X \rightrightarrows X \to Y$$

are strict, as is each morphism

$$g : X \to X$$

by which g acts on X. These morphisms induce morphisms on Artin fans

$$\Theta_{G\times X} = G \times \Theta_X \rightrightarrows \Theta_X \to \Theta_Y.$$

This diagram yields a morphism

$$[\Theta_X/G] \to \Theta_Y$$

which one might expect to be an isomorphism. Unfortunately, it never is unless G is trivial. The vertices of Θ_X are always fixed points for the G-action. The Artin fan Θ_Y is always representable over $\mathscr{L}$ by construction, whereas $[\Theta_X/G]$ need not be. Nevertheless the morphism is a partial coarse moduli space, in particular finite and flat.

To circumvent this difficulty, we consider logarithmic orbifolds obtained from actions which not only respect the logarithmic structures, but are rigid with respect to it. We make this precise in the definition below.

Definition 5.1.3. Let X be a logarithmic scheme and G a finite group acting logarithmically compatibly on X. We say that the action is firm and $[X/G]$ is a firm orbifold if the projection and action morphisms to the Artin fan

$$G \times X \rightrightarrows X \to \Theta_X$$

coincide. Equivalently, the induced action of G on the Artin fan is trivial (the morphism $X \to \Theta_X$ is G-invariant).

Assumption 5.1.4. For the rest of this section, we assume that G is a finite group acting firmly on a logarithmic scheme X.

Lemma 5.1.5. For firm actions, we have $[\Theta_X/G] = \Theta_X \times BG$.

Proof. It suffices to write

$$[\Theta_X/G] \simeq [\Theta_X \times \{*\}/e \times G] \simeq [\Theta_X/e] \times [\{*\}/G] \simeq \Theta_X \times BG$$

where e denotes the trivial group and the trivial action of G on Θ_X is considered as the product of the action of the trivial group on Θ_X and the action of G on a point.

□

Remark 5.1.6. If the logarithmic structure is induced by a normal crossing divisor D, a firm action means that G acts on each stratum of the logarithmic structure separately.

Example 5.1.7. Consider $\mathbb{P}^1$ equipped with the logarithmic structure induced by the divisor that is the union of $0 = [0 : 1]$ and $\infty = [1 : 0]$, and let the nontrivial element x in $G := \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{P}^1$ by $[x : y] \mapsto [y : x]$. This is the inversion action $z \mapsto \frac{1}{z}$. It swaps $0 = [0 : 1]$ and $\infty = [1 : 0]$. The action is not firm.

The lack of firmness can be seen on the Artin fan, computed in 2.3.6. The action on $\mathbb{P}^1$ induces an action of G on the $\Theta_{\mathbb{P}^1}$. The action swaps the point of highest rank in $\Theta_{\mathbb{P}^1}$, and thus the Artin fan is not invariant.

Lemma 5.1.8. If an action $G \circlearrowleft X$ is firm, the Artin fan of the quotient stack $[X/G]$ is the same as that of X:

$$\Theta_X \xrightarrow{\sim} \Theta_{[X/G]}.$$

Proof. Because $X \to \Theta_X$ is G-invariant, it factors through $[X/G]$. We claim the factorization $[X/G] \to \Theta_X$ is the Artin fan of the source, checking its universal property. For any factorization

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & {[X/G]} \arrow[r] \arrow[d] & \mathscr{B} \arrow[d] \\
 & \Theta_X \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathscr{L}
\end{tikzcd}$$

through an étale representable morphism $\mathscr{B} \to \mathscr{L}$, there is a unique dashed arrow. This results from the analogous universal property for $X \to \Theta_X$ and G-invariance. ☐

In the case of a firm action, the logarithmic diagonal $[X/G] \to B_{[X/G]}$ records not only the image of the diagonal $X \to X \times X$, but also the images of the "twisted diagonals" $\Delta_g : X \to X \times X, x \mapsto (x, g.x)$.

We now present the explicit expression for the target of the logarithmic diagonal, in order to compute the logarithmic Hochschild homology and cohomology of firm orbifolds. The firmness assumption allows us to lift the action of G to the space B (Lemma 5.1.9), and additional comparison results (Lemma 5.1.10 and Lemma 5.1.11) show that $B_{[X/G]}$ is isomorphic to an orbifold itself.

Lemma 5.1.9. Let $[X/G]$ be a firm logarithmic orbifold. There is an induced action of $G \times G$ on the scheme $B = B_X$ and the natural morphism $B \to X \times X$ is $G \times G$-equivariant.

Proof. The action of $G \times G$ on B is induced by the pullbacks

$$\begin{tikzcd}
G \times G \times B \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, very near start, "\ulcorner \not\ell"] & G \times G \times X \times X \arrow[d, shift left] \arrow[d, shift right] & \\
B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner \not\ell"] & X \times X \arrow[d] & . \\
\Theta_X \arrow[r] & \Theta_X \times \Theta_X &
\end{tikzcd}$$

where the two right vertical arrows $G \times G \times X \times X \to X \times X$ are given by the action and the projection morphisms. Since $[X/G]$ is firm, the composition $G \times G \times X \times X \to X \times X \to \Theta_X \times \Theta_X$ does not depend on whether we use the action morphism or the projection morphism. This means that both the Cartesian products are

$$G \times G \times X \times X \times_{\Theta_X \times \Theta_X} \Theta_X \simeq G \times G \times B$$

fitting in the diagram drawn above. Thus we get a morphism $G \times G \times B \to B$ compatible with the action morphism $G \times G \times X \times X \to X \times X$, and it is straightforward to show that the morphism $G \times G \times B \to B$ induces a group action of $G \times G$ on B. $\square$

As a consequence of the lemma above, we obtain a morphism of logarithmic orbifolds $[B/G \times G] \to [X \times X/G \times G]$ by descending the morphism $B \to X \times X$.

Lemma 5.1.10. The morphism $[B/G \times G] \to [X \times X/G \times G]$ is logarithmically étale.

Proof. In the Cartesian diagram of Lemma 5.1.9, the morphisms $B \to X \times X$ and $G \times G \times B \to G \times G \times X \times X$ are logarithmically étale, being the pullbacks of a logarithmically étale morphism. The statement follows from this. $\square$

The commutative diagram of the comparison of the group actions of G on X and of $G \times G$ on $X \times X$

$$\begin{tikzcd}
G \times X \arrow[r, "\Delta_G \times \Delta_X"] \arrow[d, shift left] \arrow[d, shift right] & G \times G \times X \times X \arrow[d, shift left] \arrow[d, shift right] \\
X \arrow[r, "\Delta_X"] \arrow[d] & X \times X \arrow[d] \\
\Theta_X \arrow[r] & \Theta_X \times \Theta_X
\end{tikzcd}.$$

provides a morphism $G \times X \to G \times G \times B$, and subsequently a natural morphism of orbifolds

$$h : [X/G] \to [B/G \times G]. \tag{5.1}$$

Lemma 5.1.11 will show that there is no confusion in calling it (again) logarithmic diagonal.

Lemma 5.1.11. There is an isomorphism of logarithmic algebraic stacks

$$[B/G \times G] \xrightarrow{\sim} [X/G] \times_{\Theta_X} [X/G].$$

Proof. By assumption 5.1.4, there is an isomorphism

$$[B/G \times G] = [X \times_{\Theta_X} X/G \times G] \simeq [X/G] \times_{\Theta_X} [X/G].$$

By Lemma 5.1.8 we have

$$[X/G] \times_{\Theta_X} [X/G] \simeq [X/G] \times_{\Theta_{[X/G]}} [X/G].$$

$\square$

This ensures that the logarithmic Hochschild homology of a firm orbifold is equivalent to the endofunctor h^*h_* where

$$h : [X/G] \to [B/G \times G] \simeq [X/G] \times_{\Theta_{[X/G]}} [X/G].$$

Since by [Kaw04] the functor i_* is of Fourier-Mukai type, h_* has both left and right adjoints, the logarithmic Hochschild cohomology for a firm orbifold as above can be defined similarly as in 3.1.7 by $h^!h_*$.

From now on, we follow [ACH19] to describe the functors h^*h_*, $h^!h_*$ explicitly using twisted diagonals.

First, we change the presentation of $[X/G]$ to the presentation $[X \times G/G \times G]$ where the action is given as $(g_1, g_2).(x, h) \mapsto (g_1.x, g_2hg_1^{-1})$. With this presentation, the diagonal morphism $[X/G] \to [X \times X/G \times G]$ becomes the quotient by $G \times G$ of the twisted diagonal morphisms

$$\Delta_g : X \to X \times X$$

defined as $x \mapsto (x, g.x)$. Since the diagonal morphism $[X/G] \to [X \times X/G \times G]$ factors through $[B/G \times G]$, we see that the twisted diagonal morphisms factors through B, obtaining morphisms $i_g : X \to B$ that we call the twisted logarithmic diagonal morphisms. From the discussion above, we see that the twisted logarithmic diagonal morphisms can be explicitly described, namely $i_g(x) = (1, g).i(x)$.

Thus, we have

$$h^*h_*(-) \simeq \bigoplus_{g \in G} i_g^* i_*(-). \tag{5.2}$$

and

$$h^!h_*(-) \simeq \bigoplus_{g \in G} i_g^! i_*(-) \tag{5.3}$$

where the functors on the right-hand side are naturally equipped with an action of G.

5.1.2 Formality of orbifold Hochschild co/homology

Similar results to those recalled in Subsection 1.4.2 hold for firm orbifolds. The reason why we restrict the investigation to such rigid actions is that in order to have the decomposition of the Hochschild homology and cohomology spaces we want to use the decompositions of the endofunctors defining them as in equations (5.2) and (5.3). To obtain those decompositions we relied on the explicit expression for the strict diagonal obtained in Lemma 5.1.11 and the lifting of the action to it as in Lemma 5.1.9. By replacing $[X \times X/G \times G]$ by $[B/X \times X]$ and adapting the proofs in [ACH19], we will obtain the desired formality result for orbifolds.

Preliminarily, let us describe intersection diagrams and their associated vector bundles, that we will use to study intersections of the strict diagonal with the twisted strict diagonals to compute their logarithmic intersection loci.

Let C be a variety with closed subvarieties A and B. We assume that both the embeddings $i : A \to C$ and $j : B \to C$ are local complete intersections.

$$\begin{tikzcd}
A \cap B \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "i"] \\
B \arrow[r, "j"] & C
\end{tikzcd} \tag{5.4}$$

Similar to the case of self-intersections in Section 3.2, one can ask when the derived intersection $W := A \times^h_C B$ is as simple as possible (i.e. formal).

Definition 5.1.12. The excess intersection bundle E is the cokernel of the natural morphism of normal bundles fitting into a short exact sequence

$$0 \to N_{A\cap B/B} \to N_{A/C}|_{A\cap B} \to E \to 0.$$

Analogously to Section 3.1, when E is locally free we have notions of formality for intersection diagrams, that is formality for their derived intersection locus, i.e. formality of its structure sheaf. We recall it here.

- Algebraic formality is the condition of the dg-algebra of the structure sheaf of W being isomorphic to the dg-algebra of the symmetric algebra of the shifted dual excess bundle (3.3).
- Geometric formality is the condition of the derived intersection W being isomorphic to the total space of the shifted excess bundle (3.4).

We state an alternative version of [ACH19, Proposition 3.3] to local complete intersections between schemes which are not smooth a priori (later we will apply this to l.c.i. morphisms between logarithmically smooth schemes). The proof is straightforward.

Theorem 5.1.13. Consider the situation of Diagram (5.4) and assume that the associated excess bundle E is locally free. Assume that the local complete intersection $A \to C$ admits a quantized cycle and the short exact sequence of locally free sheaves

$$0 \to N_{A\cap B/B} \to N_{A/C}|_{A\cap B} \to E \to 0$$

is split.

1. The derived intersection is formal in the geometric sense, i.e, there exists an isomorphism of dg-schemes $W \simeq E[-1]$ over $A \times B$.
2. The derived intersection is formal in the algebraic sense, i.e, there exists an isomorphism of dg-algebras $\mathcal{O}_W \simeq \mathrm{Sym}(E^\vee[1])$ (once it is pushed forward to $A \times B$).
3. There exists an isomorphism of dg endofunctors $\mathcal{D}(A) \to \mathcal{D}(B)$

$$j^* i_*(-) \simeq q_* \left(p^*(-) \otimes \mathrm{Sym}(E^\vee[1])\right).$$

Proof. This follows as in the original paper [ACH19], that only relies on the smoothness assumption to imply that the excess bundle is locally free as argued in [ACH19, 1.3] using [CKS03, Proposition A.3]. Because we assumed E to satisfy this property, one can see the theorem to still hold by performing the same proof. □

We use the theorem above in the case of the logarithmic twisted diagonals

$$i_g : X \to X \times X; \qquad x \mapsto (x, g.x)$$

of a firm action $G \circlearrowright X$. These factor through B.

Definition 5.1.14. The logarithmic fixed point locus $X^g_{\log}$ is the underived intersection of $i : X \to B$ and $i_g : X \to B$

$$\begin{tikzcd}
X^g_{\log} \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner \ell\mkern-9mu-"] & X \arrow[d, "i"] \\
X \arrow[r, "i_g"'] & B.
\end{tikzcd} \tag{5.5}$$

Denote the excess bundle of this intersection by E. For such diagrams, E may be computed explicitly.

Lemma 5.1.15. The schemes $X^g_{\log}$ are logarithmically smooth, and we have an isomorphism of bundles

$$E^\vee \simeq \Omega^{1,\log}_{X^g_{\log}} = \left(\Omega^{1,\log}_X\right)_g$$

where $\left(\Omega^{1,\log}_X\right)_g = \Omega^{1,\log}_X / \langle \omega - g\omega \rangle$ denotes the g-coinvariants of $\Omega^{1,\log}_X$.

Proof. Consider the short exact sequence

$$0 \to E^\vee \to N^\vee_{X/B}|_{X^g_{\log}} \to N^\vee_{X^g_{\log}/X} \to 0$$

arising from Diagram (5.5). Identify the middle term with $\Omega^{1,\log}_X$. Since i_g is obtained from i via the action with a group element g of finite order, we see that the last term is given by the g-invariants of $\Omega^{1,\log}_X$ by Cartan's lemma.

Thus, the excess bundle can be identified with $\left(\Omega^{1,\log}_X\right)_g$ which is also a locally free sheaf. Using the embedding $q : X^g_{\log} \to X$, we see that $E^\vee \simeq \Omega^{1,\log}_{X^g_{\log}} = \left(\Omega^{1,\log}_X\right)_g$. Writing the distinguished triangle associated to the morphism q one sees that the higher terms vanish, thus the schemes $X^g_{\log}$ are log smooth. □

Since the averaging morphism

$$\left(\Omega^{1,\log}_X\right)_g \to \Omega^{1,\log}_X$$

defined as

$$\omega \mapsto \sum_{i=1}^{o(g)} \frac{1}{o(g)} g^i \omega$$

splits the excess bundle short exact sequence, the conditions of Theorem 5.1.13 are satisfied.

Corollary 5.1.16. The intersection depicted in Diagram (5.5) is formal in the sense of Theorem 5.1.13. In particular, there are natural isomorphisms of endofunctors

$$i_g^* i_*(-) \simeq q_* \left(p^*(-) \otimes \operatorname{Sym}\left(\left(\Omega_X^{1,\log} \right)_g [1] \right) \right)$$

and

$$i_g^! i_*(-) \simeq q_* \left(p^!(-) \otimes \operatorname{Sym}\left(\left(T_X^{\log} \right)^g [-1] \right) \right).$$

□

In the realm of derived algebraic geometry, the corollary above says that the logarithmic derived loop space of the orbifold $[X/G]$ defined as the derived fiber product $L^{\log}[X/G] := [X/G] \times^h_{[B/G\times G]} [X/G]$ is isomorphic to the shifted tangent bundle on the logarithmic inertia stack defined as the underived fiber product $I^{\log}[X/G] := [X/G] \times_{[B/G\times G]} [X/G]$ as dg-schemes:

$$\mathbb{T}_{I^{\log}[X/G]}[-1] \simeq L^{\log}[X/G].$$

As a corollary, we obtain a logarithmic version of the decomposition of orbifold Hochschild homology (see for instance, [Bar03, Gan11] for the classical non-logarithmic decomposition).

Corollary 5.1.17. We have isomorphisms of k-vector spaces

$$\mathrm{HH}^{\ell}_{\bullet}([X/G]) = \left(\bigoplus_{g\in G} \mathrm{HH}^{\ell}_n \left(X^g_{\log} \right) \right)^G = \left(\bigoplus_{g\in G} \bigoplus_{q-p=n} H^p \left(X^g_{\log}, \Omega^{q,\log}_{X^g_{\log}} \right) \right)^G.$$

Proof. The proof is parallel to the proof of [ACH19, Corollary 1.17]. □

We conclude this section with a simple example in which there is no contribution to the Hochschild homology from the twisted sectors corresponding to the orbifold.

Example 5.1.18. Consider $\mathbb{A}^1$ with the negation action of $\mu_2 = \mathbb{Z}/2\mathbb{Z}$. We equip $\mathbb{A}^1$ with the toric logarithmic structure given by the Cartier divisor of the origin (0) as in Example 2.1.15. Then $B \subseteq Bl_{\vec{0}}\mathbb{A}^2$ is the complement of the strict transforms of the $x-$ and $y-$axes. The inclusions $i, i_{-1} : \mathbb{A}^1 \to B$ are the diagonal and anti-diagonal, which do not meet in the blow-up.
Because the twisted diagonal i_{-1} does not meet i, there is no contribution from $-1 \in \mu_2$.

$$h^* h_*(-) \simeq i^* i_*(-) \simeq (-) \otimes_{k[x]} k[x,t]$$

where t has degree -1, and $\mathbb{Z}/2\mathbb{Z}$ acts as $x \mapsto -x$ and $t \mapsto t$.

5.2 Computations for a non-firm orbifold

We conclude by computing explicitly the logarithmic Hochschild homology and cohomology spaces of the orbifold given by the symmetric group acting by shuffling on the n-fold product of a logarithmic scheme. This computation in the ordinary setup is presented in [BFK23].

Let $n > 0$ be a positive integer and X a logarithmically smooth quasicompact logarithmic scheme. Let us denote by Σ_n the symmetric group, that is the group of permutations of a set of n elements. By X^n we mean the Cartesian product in the category of logarithmic schemes, which is the same as the Cartesian product in the category of fs schemes

$$X^n = \underbrace{X \times \cdots \times X}_{n \text{ times}} \simeq \underbrace{X \times^{\ell} \cdots \times^{\ell} X}_{n \text{ times}}$$

being the fibered product being on the point with the trivial logarithmic structure.

We consider the action

$$\begin{aligned} \Sigma_n \times X^n &\to X^n \\ (\sigma, (x_1, \dots, x_n)) &\mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}). \end{aligned}$$

Remark 5.2.1. The main difficulty of this case is that for $n > 1$, unless X only has one logarithmic stratum, the action is not firm. In fact, if the Artin fan Θ_X has at least two points, by Corollary 2.3.30 $\Theta_{X^n} \simeq \Theta_X^n$ and Σ_n acts non-trivially on it.

Because of Remark 5.2.1, we cannot apply the techniques developed in Section 5.1. Below, we circumvent this problem by explicitly computing the intersection loci and using them to describe the logarithmic Hochschild homology and cohomology endofunctors for $[X^n/\Sigma_n]$.

By Definition 3.1.7, we have to compute the self-intersection of the logarithmic diagonal

$$\iota : [X^n/\Sigma_n] \to [X^n/\Sigma_n] \times_{\Theta_{[X^n/\Sigma_n]}} [X^n/\Sigma_n] \simeq [X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n/\Sigma_n \times \Sigma_n],$$

where the identification is given by the lift of the action of Σ_n on X^n to the fiber product over $\Theta_{[X^n/\Sigma_n]}$, which is Σ_n-invariant.

The endofunctors $\iota^*\iota_*$ and $\iota^!\iota_*$ may be decomposed as

$$\iota^*\iota_* \simeq \bigoplus_{\sigma \in \Sigma_n} i_\sigma^* i_* \quad \text{and} \quad \iota^!\iota_* \simeq \bigoplus_{\sigma \in \Sigma_n} i_\sigma^! i_*$$

where

$$i, i_\sigma : X^n \to X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n$$

are the factorizations of the diagonal and the twisted diagonal

$$\Delta, \Delta_\sigma : X^n \to X^n \times X^n$$

given by $x \mapsto (x, x)$ and $x \mapsto (x, \sigma.x)$ respectively.

The factorization of the diagonal and the twisted diagonal $\Delta, \Delta_\sigma : X^n \to X^n \times X^n$ though $X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n$ instead of $X^n \times_{\Theta_X^n} X^n$ maintains the good features that allow to compute logarithmic Hochschild homology and cohomology along them. We recall them and prove them below.

Lemma 5.2.2. The logarithmic algebraic stack $X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n$ is representable by logarithmic algebraic spaces.

Proof. The logarithmic algebraic stack is defined by the pullback diagram

$$\begin{tikzcd}
X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \not{\ell}" near start] & X^n \times X^n \arrow[d] \\
\Theta_{[X^n/\Sigma_n]} \arrow[r] & \Theta_{[X^n/\Sigma_n]} \times \Theta_{[X^n/\Sigma_n]}
\end{tikzcd}.$$

The bottom morphism is representable by logarithmic algebraic spaces as it is morphisms of Artin fans, having

$$\Theta_{[X^n/\Sigma_n]} \times \Theta_{[X^n/\Sigma_n]} \simeq \Theta_{[X^n/\Sigma_n]\times[X^n/\Sigma_n]}$$

from Corollary 2.3.30. Thus, its pullback is representable by logarithmic algebraic spaces, and its target is a logarithmic scheme, therefore its source is representable by algebraic spaces. ☐

Lemma 5.2.3. The morphism $X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n \to X^n \times X^n$ is logarithmically étale.

Proof. The morphism $X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n \to X^n \times X^n$ is obtained by the Cartesian diagram

$$\begin{tikzcd}
X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \not{\ell}" near start] & X^n \times X^n \arrow[d] \\
\Theta_{[X^n/\Sigma_n]} \arrow[r] & \Theta_{[X^n/\Sigma_n]} \times \Theta_{[X^n/\Sigma_n]}
\end{tikzcd}$$

which is strict as the morphism $X^n \to [X^n/\Sigma_n] \to \Theta_{[X^n/\Sigma_n]}$ is the composite of strict morphisms, and thus so is the right-most vertical morphism. Since

$$\Theta_{[X^n/\Sigma_n]} \times \Theta_{[X^n/\Sigma_n]} \simeq \Theta_{[X^n/\Sigma_n]\times[X^n/\Sigma_n]} \simeq \Theta_{[X^n\times X^n/\Sigma_n\times\Sigma_n]}$$

the bottom morphism is logarithmically étale, being a morphism of Artin fans. Therefore the top morphism is logarithmically étale as well. ☐

Lemma 5.2.4. The morphisms $i, i_\sigma : X^n \to X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n$ are strict.

Proof. The morphisms i, i_σ are defined by the diagram

$$\begin{tikzcd}
X^n \arrow[r, "{i,i_\sigma}"'] \arrow[rr, bend left, "{\Delta,\Delta_\sigma}"] \arrow[dr] & X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \not{\ell}" near start] & X^n \times X^n \arrow[d] \\
& \Theta_{[X^n/\Sigma_n]} \arrow[r] & \Theta_{[X^n/\Sigma_n]} \times \Theta_{[X^n/\Sigma_n]}
\end{tikzcd}.$$

The right-most vertical morphism is strict, because $X^n \to [X^n/\Sigma_n] \to \Theta_{[X^n/\Sigma_n]}$ is. Hence, its pullback $X^n \times_{[\Theta_{X^n/\Sigma_n}]} X^n \to \Theta_{[X^n/\Sigma_n]}$ is also strict. We already recalled that the oblique morphism $X^n \to \Theta_{[X^n/\Sigma_n]}$ is strict. Therefore i, i_σ are strict too. ☐

Lemma 5.2.5. The logarithmic algebraic space $X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n$ is also logarithmically smooth.

Proof. Since X is logarithmically smooth, so is any iteration of its products with itself. In particular $X^n \times X^n$ is logarithmically smooth. The morphism

$$f : X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n \to X^n \times X^n$$

induces the distinguished triangle of logarithmic cotangent complexes as guaranteed by Proposition 2.2.13

$$f^*\mathbb{L}^{\log}_{X^n} \to \mathbb{L}^{\log}_{X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n} \to \mathbb{L}^{\log}_f \xrightarrow{+1} .$$

Since the morphism f is logarithmically étale, $\mathbb{L}^{\log}_f = 0$ by Proposition 2.2.8. Therefore

$$\mathbb{L}^{\log}_{X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n} \simeq f^*\mathbb{L}^{\log}_{X^n \times X^n},$$

that is concentrated at most in degree 0 as $X^n \times X^n$ is logarithmically smooth. Using Proposition 2.2.8 again, we obtain that $X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n$ is logarithmically smooth. ☐

Lemma 5.2.6. If X is logarithmically separated, then the strict diagonal and the twisted strict diagonals $i, i_\sigma : X^n \to X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n$ are l.c.i. closed immersions.

Proof. The morphism i induces the distinguished triangle

$$i^*\mathbb{L}^{\log}_{X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n} \to \mathbb{L}^{\log}_{X^n} \to \mathbb{L}^{\log}_i \xrightarrow{+1}$$

as in Proposition 2.2.13. As i is strict, $\mathbb{L}^{\log}_i \simeq \mathbb{L}_i$. Since the source and target of i are logarithmically smooth, their logarithmic cotangent complexes are concentrated at most in degree 0, and then $\mathbb{L}_i$ is concentrated at most in cohomological degrees 0 and -1. This implies that i is a l.c.i. morphism. The same argument can be performed once one replaces i with i_σ.

To show that i and i_σ are closed immersions when X is logarithmically separated we alter the diagram defining the strict diagonal and we look at

$$\begin{tikzcd}
X^n \arrow[r, "{i,i_\sigma}"] & X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell"] & X^n \times_{\mathcal{L}} X^n \arrow[d] \\
 & \Theta_{[X^n/\Sigma_n]} \arrow[r] & \Theta_{[X^n/\Sigma_n]} \times_{\mathcal{L}} \Theta_{[X^n/\Sigma_n]}
\end{tikzcd} .$$

The top composite can be written as $X^n \to (X \times_{\mathcal{L}} X)^n$ and is proper if X is logarithmically separated. Since $\Theta_{[X^n/\Sigma_n]} \to \mathcal{L}$ is strict, étale and representable by algebraic spaces, its diagonal is an open immersion. Therefore the morphisms i and i_σ are proper as well. ☐

We can compute the endofunctors $i_\sigma^* i_*$ and $i_\sigma^! i_*$ using [ACH19] through the self-intersection diagram

$$\begin{tikzcd}
Z^\sigma_{\log} \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner \not\ell", very near start] & X^n \arrow[d, "i"] \\
X^n \arrow[r, "i_\sigma"] & X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n
\end{tikzcd} .$$

We obtain

$$i_\sigma^* i_*(-) \simeq q_*(p^*(-) \otimes \mathrm{Sym}(E^\vee[1])) \qquad i_\sigma^! i_*(-) \simeq q_*(p^!(-) \otimes \mathrm{Sym}(E[-1]))$$

where E denotes the excess bundle associated with the diagram above.

The main result for the orbifold $[X^n/\Sigma_n]$ is given by the following proposition, that, together with the decompositions in (5.2), give the sheafy HKR theorem in this case.

Proposition 5.2.7. Let $\sigma \in \Sigma_n$ be a permutation and N_σ the number of orbits of σ. In the previous notation, we have that

$$i_\sigma^* i_*(-) \simeq q_*\left(p^*(-) \otimes \mathrm{Sym}\left(\oplus^{N_\sigma} \Omega_X^{1,\log}[1]\right)\right);$$

$$i_\sigma^! i_*(-) \simeq q_*\left(p^!(-) \otimes \mathrm{Sym}\left(\oplus^{N_\sigma} T_X^{\log}[-1]\right)\right).$$

The feature that makes the case of the symmetric powers still computable despite the lack of firmness is that the intersection locus is much nicer than in other cases. Informally, this is a consequence of the rigidity of the permutations, which do not "stretch" the space along the logarithmic structure. The computation of the intersection locus is a local problem and therefore can be reduced to its treatment for the toric case. What one finds is somehow predictable, which is that the intersection locus and the logarithmic intersection locus coincide. Although this is not true in general, as Example 5.1.18 displays.

Lemma 5.2.8. Let X be a toric variety and let

$$\begin{tikzcd}
Y^\sigma_{\log} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \not\ell", very near start] & X^n \arrow[d, "j"] \\
X^n \arrow[r, "j_\sigma"] & X^n \times_{\Theta_{X^n}} X^n
\end{tikzcd} . \tag{5.6}$$

Then $Y^\sigma_{\log} \simeq X^{N_\sigma}$ where N_σ is the number of orbits of the action of σ on $\{1, \dots, n\}$. In particular, the intersection locus $Y^\sigma_{\log}$ is logarithmically smooth.

Proof. First recall that by Corollary 2.3.30 $\Theta_{X^n} \simeq \Theta_X^n$. This implies that

$$X^n \times_{\Theta_{X^n}} X^n \simeq (X \times_{\Theta_X} X)^n,$$

and, by Example 3.1.4, the strict diagonal is known to be

$$X \times_{\Theta_X} X \simeq X \times T$$

for toric varieties, where $\mathbb{G}_m^k \simeq T \subset X$ denotes the dense torus. The diagonal and the twisted diagonals factor through the components as

$$\begin{aligned} X^n &\to X^n \times T^n \\ j : (x_1, \ldots, x_n) &\mapsto ((x_1, \ldots, x_n), (1, \ldots, 1)) \\ j_\sigma : (x_1, \ldots, x_n) &\mapsto ((x_{\sigma(1)}, \ldots, x_{\sigma(n)}), (1, \ldots, 1)) \end{aligned}$$

as σ acts by permutations on the torus. This implies that $Y^\sigma_{\log}$ is determined by the intersection in the underlying scheme X^n, so $Y^\sigma_{\log} \simeq X^{N_\sigma}$. $\square$

Proof of Proposition 5.2.7. Recall that, being the morphism $\Theta_{X^n} \to \Theta_{[X^n/\Sigma_n]}$ a morphism of Artin fans induced by a strict morphism of logarithmic algebraic stacks, it is strict, étale and representable. Therefore its diagonal

$$\Theta_{X^n} \to \Theta_{X^n} \times_{\Theta_{[X^n/\Sigma_n]}} \Theta_{X^n}$$

is an open immersion. Its pullback

$$u : X^n \times_{\Theta_{X^n}} X^n \to X^n \times_{\Theta_{[X^n/\Sigma_n]}} X^n$$

is also an open immersiona and by [Sta24, 08ED] it induces equivalences $u^*u_* \simeq id$ and $u^!u_* \simeq id$. In the notation of Lemma 5.2.8, there are induced equivalences $i_\sigma^* i_* \simeq j_\sigma^* j_*$ and $j_\sigma^! j_* \simeq i_\sigma^! i_*$. Applying Theorem 5.1.13 to Diagram (5.6) one deduces

$$j_\sigma^* j_*(-) \simeq q_*\left(p^*(-) \otimes \mathrm{Sym}\left(E^\vee[1]\right)\right) \qquad j_\sigma^! j_*(-) \simeq q_*\left(p^!(-) \otimes \mathrm{Sym}\left(E[-1]\right)\right)$$

where the excess bundle is computed by the exact sequence

$$0 \to E^\vee \to N^\vee_{X^n/X^n \times_{\Theta_{X^n}} X^n}|_{Y^\sigma_{\log}} \to N^\vee_{Y^\sigma_{\log}/X^n} \to 0. \tag{5.7}$$

It remains to compute the excess bundle E. Because the morphisms in Diagram 5.6 are strict,

$$N^{\log}_{X^n/X^n \times_{\Theta_{X^n}} X^n} \simeq N_{X^n/X^n \times_{\Theta_{X^n}} X^n} \quad \text{and} \quad N^{\log}_{Y^\sigma_{\log}/X^n} \simeq N_{Y^\sigma_{\log}/X^n}.$$

By identifying $Y^\sigma_{\log} \simeq X^{N_\sigma}$ we have the isomorphism

$$\Omega^{1,\log}_{Y^\sigma_{\log}} \simeq \oplus^{N_\sigma} \Omega^{1,\log}_X|_{Y^\sigma_{\log}}.$$

The morphism $X^n \times_{\Theta_{X^n}} X^n \to X^n \times X^n$ is logarithmically étale, hence

$$\Omega^{1,\log}_{X^n \times_{\Theta_{X^n}} X^n} \simeq \Omega^{1,\log}_{X^n \times X^n}|_{X^n \times_{\Theta_{X^n}} X^n} \simeq \Omega^{1,\log}_{X^n}|_{X^n \times_{\Theta_{X^n}} X^n} \oplus \Omega^{1,\log}_{X^n}|_{X^n \times_{\Theta_{X^n}} X^n}.$$

Using the facts recalled above, we can compute the conormal bundles by

$$\begin{aligned} N^{\log,\vee}_{X^n/X^n\times_{\Theta_{X^n}}X^n} &= \ker\left(\Omega^{1,\log}_{X^n\times_{\Theta_{X^n}}X^n}|_{X^n} \to \Omega^{1,\log}_{X^n}\right) \\ &\simeq \ker\left(\Omega^{1,\log}_{X^n}\oplus\Omega^{1,\log}_{X^n} \to \Omega^{1,\log}_{X^n}\right) \simeq \Omega^{1,\log}_{X^n} \simeq \oplus^n\Omega^{1,\log}_{X}|_{X^n} \end{aligned}$$

and

$$\begin{aligned} N^{\log,\vee}_{Y^\sigma_{\log}/X^n} &= \ker\left(\Omega^{1,\log}_{X^n}|_{Y^\sigma_{\log}} \to \Omega^{1,\log}_{Y^\sigma_{\log}}\right) \\ &\simeq \ker\left(\oplus^n\Omega^{1,\log}_{X}|_{Y^\sigma_{\log}} \to \oplus^{N_\sigma}\Omega^{1,\log}_{X}|_{Y^\sigma_{\log}}\right) \simeq \oplus^{n-N_\sigma}\Omega^{1,\log}_{X}|_{Y^\sigma_{\log}}. \end{aligned}$$

One can compute the kernels in the exact sequence (5.7) to compute

$$E^\vee \simeq \bigoplus^{N_\sigma}\Omega^{1,\log}_X|_{Y^\sigma_{\log}} \quad\text{and}\quad E \simeq \bigoplus^{N_\sigma} T^{\log}_X|_{Y^\sigma_{\log}}.$$

□

The HKR decomposition for the action of the symmetric group by shuffling the product is given by the following formula.

Corollary 5.2.9. In the previous notation,

$$\begin{aligned} \mathrm{HH}^\ell_\bullet([X^n/\Sigma_n]) &\simeq \left(\bigoplus_{\sigma\in\Sigma_n}\bigoplus_{q-p=\bullet} H^p\left(X^{N_\sigma},\Omega^{q,\log}_{X^{N_\sigma}}\right)\right)^{\Sigma_n} \\ &\simeq \bigoplus_{\nu\dashv n}\left(\bigotimes_{i\geq 0}\mathrm{Sym}^{\lambda_i}\mathrm{HH}^\ell_\bullet(X)\right) \simeq \bigoplus_{\nu\dashv n}\left(\bigotimes_{i\geq 0}\mathrm{Sym}^{\lambda_i}\left(\bigoplus_{q-p=\bullet} H^p\left(X,\Omega^{q,\log}_X\right)\right)\right). \end{aligned}$$

where ν runs across all partitions of n and λ_i counts the number of i's in the partition ν.

Proof. The first isomorphism is obtained using Proposition 5.2.7 and taking the global sections. The second one follows by [BFK23, Theorem 3.7] applied to the logarithmic Kähler differentials. It is the logarithmic analogous of [BFK23, Theorem 3.9] for the structure sheaf.

□

Along the lines of [BFK23], we state a conjecture on the vector spaces computed above.

Conjecture 5.2.10. Let X be a quasicompact, weakly logarithmically separated, logarithmically smooth logarithmic scheme of dimension at most 2 and $n\geq 1$ an integer. The logarithmic Hochschild homology of the logarithmic scheme of points $X^{[n],\log}$ [Ken23] coincides with the logarithmic Hochschild homology of $[X^n/\Sigma_n]$.

Example 5.2.11. Let $X = (\mathbb{P}^1, \{0, \infty\})$ be the toric projective line and $n = 2$. The symmetric group Σ_n acts on

$$X^2 = (\mathbb{P}^1 \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1 \cup \{\infty\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{0\} \cup \mathbb{P}^1 \times \{\infty\})$$

by swapping the components. The intersection locus for the action by the trivial element is X^2 itself, whilst the intersection locus for the action by the non-trivial element is isomorphic to X.

Preliminarily, let us recall that the sheaf of logarithmic Kähler differentials for the toric projective line $(\mathbb{P}^1, \{0, \infty\})$ is the trivial line bundle (see Example 3.3.5). It follows that

$$\Omega^{1,\log}_{X^2} \simeq \pi_1^* \Omega^{1,\log}_{\mathbb{P}^1} \oplus \pi_2^* \Omega^{1,\log}_{\mathbb{P}^1} \simeq \mathcal{O}^{\oplus 2}_{\mathbb{P}^2}$$

where $\pi_1, \pi_2 : X^2 \to X$ denote the natural projections.

Let us compute the logarithmic Hochschild homology of the trivial intersection locus, whose derived complex is

$$i^*_{\mathrm{Id}} i_* \mathcal{O}_{X^2} = i^* i_* \mathcal{O}_{X^2} \simeq \mathrm{Sym}(\Omega^{1,\log}_{X^2}[1]) \simeq [0 \to \mathcal{O}_{\mathbb{P}^2} \xrightarrow{0} \mathcal{O}^{\oplus 2}_{\mathbb{P}^2} \xrightarrow{0} \mathcal{O}_{\mathbb{P}^2} \to 0].$$

We compute the direct summands giving the logarithmic Hochschild homology via [Sta24, 0BED], obtaining

$$H^0(X^2, \mathcal{O}_{X^2}) \simeq k, \quad H^p(X^2, \mathcal{O}_{X^2}) = 0 \text{ for } p > 0,$$

$$H^0(X^2, \Omega^{1,\log}_{X^2}) \simeq k^2, \quad H^p(X^2, \Omega^{1,\log}_{X^2}) = 0 \text{ for } p > 0,$$

$$H^0(X, \Omega^{2,\log}_{X^2}) \simeq k, \quad H^p(X^2, \Omega^{2,\log}_{X^2}) = 0 \text{ for } p > 0.$$

The derived complex associated with the non-trivial element σ of Σ_n is

$$i^*_\sigma i_* \mathcal{O}_X \simeq q_* \mathrm{Sym}(\Omega^{1,\log}_X[1]) \simeq [0 \to \mathcal{O}_{\mathbb{P}^1} \xrightarrow{0} \mathcal{O}_{\mathbb{P}^1} \to 0].$$

Proceeding as above, we obtain

$$H^0(X, \mathcal{O}_X) \simeq k, \quad H^p(X, \mathcal{O}_X) = 0 \text{ for } p > 0,$$

$$H^0(X, \Omega^{1,\log}_X) \simeq k, \quad H^p(X, \Omega^{1,\log}_X) = 0 \text{ for } p > 0.$$

Applying Corollary 5.2.9, it results that

$$\mathrm{HH}^\ell_0 \left([X^2/\Sigma_2]\right) \simeq \left(H^0(X^2, \mathcal{O}_{X^2}) \oplus H^0(X, \mathcal{O}_X)\right)^{\Sigma_2} \simeq \left(k^2\right)^{\Sigma_2} \simeq k^2.$$

$$\mathrm{HH}^\ell_1 \left([X^2/\Sigma_2]\right) \simeq \left(H^0(X^2, \Omega^{1,\log}_{X^2}) \oplus H^0(X, \Omega^{1,\log}_X)\right)^{\Sigma_2} \simeq \left(k^3\right)^{\Sigma_2} \simeq k^2.$$

$$\mathrm{HH}^\ell_2 \left([X^2/\Sigma_2]\right) \simeq \left(H^0(X, \Omega^{2,\log}_{X^2})\right)^{\Sigma_2} \simeq (k)^{\Sigma_2} \simeq k.$$

Bibliography

[AC12] Dima Arinkin and Andrei Căldăraru. When is the self-intersection of a subvariety a fibration? *Advances in Mathematics*, 231(2):816–842, 2012.

[ACG$^+$13] Dan Abramovich, Qile Chen, Danny Gillam, Yuhao Huang, Martin Olsson, Matthew Satriano, and Shenghao Sun. Logarithmic geometry and moduli. In Gavril Farkas and Ian Morrison, editors, *Handbook of Moduli, Volume III*. International Press, 2013.

[ACH19] Dima Arinkin, Andrei Caldararu, and Márton Hablicsek. Formality of derived intersections and the orbifold HKR isomorphism. *Journal of Algebra*, 540:100–120, 2019.

[ACM$^+$15] Dan Abramovich, Qile Chen, Steffen Marcus, Martin Ulirsch, and Jonathan Wise. Skeletons and fans of logarithmic structures. *arXiv e-prints*, page arXiv:1503.04343, March 2015.

[ACMW14] Dan Abramovich, Qile Chen, Steffen Marcus, and Jonathan Wise. Boundedness of the space of stable logarithmic maps. *Journal of the European Mathematical Society*, 19, 08 2014.

[AMW14] Dan Abramovich, Steffen Marcus, and Jonathan Wise. Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenerations. *Annales de l'Institut Fourier*, 64(4):1611–1667, 2014.

[AW18] Dan Abramovich and Jonathan Wise. Birational invariance in logarithmic Gromov-Witten theory. *Compositio Mathematica*, 154(3):595–620, 2018.

[Bar03] Vladimir Baranovsky. Orbifold cohomology as periodic cyclic homology. *International Journal of Mathematics*, 14(08):791–812, 2003.

[BFK23] Pieter Belmans, Lie Fu, and Andreas Krug. Hochschild cohomology of Hilbert schemes of points on surfaces. *arXiv preprint arXiv:2309.06244*, 2023.

[BGI71] Pierre Berthelot, Alexandre Grothendieck, and Luc Illusie. *Séminaire de Géométrie Algébrique du Bois Marie - 1966-67 - Théorie des intersections et théorème de Riemann-Roch (SGA 6)*, volume 225 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin; New York, 1971.

[BLPØ23] Federico Binda, Tommy Lundemo, Doosung Park, and Paul Arne Østvær. A Hochschild-Kostant-Rosenberg theorem and residue sequences for logarithmic Hochschild homology. *Advances in Mathematics*, 435:109354, 2023.

[BLR12] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron Models*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2012.

[BM93] Marcel Bökstedt and Ib Madsen. *Topological cyclic homology of the integers*. Aarhus Universitet. Matematisk Institut, 1993.

[BO95] Alexei Bondal and Dmitri Orlov. Semiorthogonal decomposition for algebraic varieties. *arxiv preprint arXiv:math.AG/9506012*, 1995.

[BZN12] David Ben-Zvi and David Nadler. Loop spaces and connections. *Journal of Topology*, 5(2):377–430, 2012.

[Cal03] Andrei Caldararu. The Mukai pairing, I: the Hochschild structure. *arXiv preprint math/0308079*, 2003.

[Cal05] Andrei Caldararu. The Mukai pairing, II: the Hochschild–Kostant–Rosenberg isomorphism. *Advances in Mathematics*, 194(1):34–66, June 2005.

[Car56] Pierre Cartier. Dérivations dans les corps. In *Séminaire Cartan*, volume 8. 1956.

[CdB10] Damien Calaque and Michel Van den Bergh. Hochschild cohomology and Atiyah classes. *Advances in Mathematics*, 224(5):1839–1889, 2010.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological Algebra*. Princeton University Press, Princeton, N. J., 1956.

[CHL23] You–Cheng Chou, Leo Herr, and Yuan–Pin Lee. The log product formula in quantum K-theory. *Mathematical Proceedings of the Cambridge Philosophical Society*, page 1–28, 2023.

[CKS03] Andrei Căldăraru, Sheldon Katz, and Eric Sharpe. D-branes, B fields, and Ext groups. *Advances in Theoretical and Mathematical Physics*, 7(3):381–404, 2003.

[CLO12] Brian Conrad, Max Lieblich, and Martin Olsson. Nagata compactification for algebraic spaces. *Journal of the Institute of Mathematics of Jussieu*, 11(4):747–814, 2012.

[EV85] Hélene Esnault and Eckart Viehweg. Logarithmic de Rham complexes and vanishing theorems. 1985.

[FG22] Daniel Ferrand and Ofer Gabber. Enveloppe étale de morphismes plats. *Algebra & Number Theory*, 16(3), July 2022.

[Gan11] Nora Ganter. Inner products of 2-representations. *arXiv preprint arXiv:1110.1711*, 2011.

[Ger64] Murray Gerstenhaber. On the deformation of rings and algebras. *Annals of Mathematics*, 79(1):59–103, 1964.

[GHH$^+$24] Andreas Gross, Leo Herr, David Holmes, Pim Spelier, and Jesse Vogel. The log Grothendieck ring of varieties. *arxiv preprint arXiv:2412.07715*, 2024.

[Gri14] Julien Grivaux. The Hochschild–Kostant–Rosenberg isomorphism for quantized analytic cycles. *International Mathematics Research Notices*, 2014(4):865–913, 2014.

[Gri20] Julien Grivaux. Derived geometry of the first formal neighbourhood of a smooth analytic cycle. *Advances in Mathematics*, 361:106924, 2020.

[Gro68] A. Grothendieck. *Categories Cofibrees Additives Et Complexe Cotangent Relatif.* Lecture notes in mathematics - vol. 79. Springer, 1968.

[Hab20] Márton Hablicsek. Hodge theorem for the logarithmic de Rham complex via derived intersections. *Research in the Mathematical Sciences*, 7:1–21, 2020.

[Har10] Robin Hartshorne. *Deformation Theory*, volume 257 of *Graduate Texts in Mathematics.* Springer Science & Business Media, New York, 2010.

[He23] Yunfan He. Hodge to de rham degeneration of nodal cubic curve. *arXiv preprint*, 2311.08477, 2023. Available at `https://arxiv.org/abs/2311.08477`.

[HHL23] Márton Hablicsek, Leo Herr, and Francesca Leonardi. Logarithmic Hochschild co/homology via formality of derived intersections. *arXiv preprint arXiv:2308.09447*, 2023.

[HKR62] Gerard Hochschild, Bertram Kostant, and Alex Rosenberg. Differential forms on regular affine algebras. *Transactions of the American Mathematical Society*, 102(3):383–408, 1962.

[Hoy15] Marc Hoyois. The homotopy fixed points of the circle action on Hochschild homology. *arXiv preprint arXiv:1506.07123*, 2015.

[Hua20] Shengyuan Huang. Functoriality of HKR isomorphisms. *arXiv preprint arXiv:2002.00017*, 2020. Available at: https://arxiv.org/abs/2002.00017.

[Huy07] Daniel Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry*. Oxford University Press, 2007.

[Ill71] Luc Illusie. *Complexe Cotangent et Déformations I*, volume 239 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin; New York, 1971.

[Kas87] Christian Kassel. Cyclic homology, comodules, and mixed complexes. *Journal of Algebra*, 107(1):195–216, 1987.

[Kat89] Kazuya Kato. Logarithmic structures of Fontaine-Illusie, in Algebraic Analysis, Geometry, and Number Theory, Proceedings of the JAMI Inaugural Conference. *Supplement to Aner. J. Math.*, pages 191–224, 1989.

[Kat99] Fumiharu Kato. Log smooth deformation and moduli of log smooth curves. 11, 11 1999.

[Kaw04] Yujiro Kawamata. Equivalences of Derived Categories of Sheaves on Smooth Stacks. *American Journal of Mathematics*, 126(5):1057–1083, 2004.

[Kel06] Bernhard Keller. On differential graded categories. *arXiv preprint math/0601185*, 2006.

[Kel21] Bernhard Keller. Hochschild (co)homology and derived categories. *Bulletin of the Iranian Mathematical Society*, page online, Apr 2021. Received: 5 March 2020 / Revised: 19 February 2021 / Accepted: 16 March 2021 / Published online: 24 April 2021.

[Ken23] Patrick Kennedy-Hunt. The Logarithmic Quot space: foundations and tropicalisation. *arXiv preprint arXiv:2308.14470*, 2023.

[Kon03] Maxim Kontsevich. Deformation Quantization of Poisson Manifolds. *Letters in Mathematical Physics*, 66:157–216, 2003.

[KS04] Kazuya Kato and Takeshi Saito. On the conductor formula of Bloch. *Publications Mathématiques de l'IHÉS*, 100:5–151, 2004.

[Kuz09] Alexander Kuznetsov. Hochschild homology and semiorthogonal decompositions. *arXiv e-prints*, page arXiv::0904.4330, April 2009.

[Lev22] Denis Levchenko. Serre duality for tame Deligne–Mumford stacks. *Res Math Sci*, 9:67, 2022.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, Heidelberg, 2000.

[Lod98] Jean-Louis Loday. Cyclic homology. Appendix E by Marıa O. Ronco. chapter 13 by the author in collaboration with Teimuraz Pirashvili. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 301, 1998.

[LVdB05] Wendy Lowen and Michel Van den Bergh. Hochschild cohomology of abelian categories and ringed spaces. *Advances in Mathematics*, 198(1):172–221, Dec 2005. Dedicated to Michael Artin on the occasion of his 70th birthday.

[McC94] Randy McCarthy. The cyclic homology of an exact category. *Journal of Pure and Applied Algebra*, 93(3):251–296, May 1994.

[Mit72] Barry Mitchell. Rings with several objects. *Advances in Mathematics*, 8(1):1–161, Feb 1972.

[MS80] Vikram Bhagvandas Mehta and Conjeevaram Srirangachari Seshadri. Moduli of vector bundles on curves with parabolic structures. *Mathematische annalen*, 248(3):205–239, 1980.

[Nag56] Masayoshi Nagata. On the imbedding problem of abstract varieties in projective varieties. *Mem. College Sci. Univ. Kyoto Ser. A Math.*, 30(1):71–82, 1956.

[Nag62] Masayoshi Nagata. Imbeddings of an abstract variety in a complete variety. *Journal of Mathematics of Kyoto University*, 2:1–10, 1962.

[Nee96] Amnon Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *Journal of the American Mathematical Society*, 9(1):205–236, 1996.

[Nir09] Fabio Nironi. Grothendieck Duality for Deligne-Mumford Stacks. *arXiv preprint arXiv:0811.1955*, 2009.

[Ogu18] Arthur Ogus. *Lectures on Logarithmic Algebraic Geometry.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2018.

[Ols03] Martin C. Olsson. Logarithmic geometry and algebraic stacks. *Annales scientifiques de l'École Normale Supérieure*, Ser. 4, 36(5):747–791, 2003.

[Ols05] Martin Olsson. The logarithmic cotangent complex. *Mathematische Annalen*, 333, 2005.

[Ols24] Martin Olsson. Hochschild homology for log schemes. *arXiv preprint arXiv:2406.08629*, 2024.

[Orl02] Dmitri O Orlov. Derived categories of coherent sheaves on Abelian varieties and equivalences between them. *Izvestiya: Mathematics*, 66(3):569–594, 2002.

[Rom09] Matthieu Romagny. Composantes connexes et irréductibles en familles. *Manuscripta Mathematica*, 136, 12 2009.

[Ryd09] David Rydh. Compactifications of tame Deligne–Mumford stack. 2009. Preprint.

[SS10] Matthias Schütt and Tetsuji Shioda. Elliptic Surfaces. *Advanced Studies in Pure Mathematics*, 60:51–160, 2010.

[SST20] Sarah Scherotzke, Nicolò Sibilla, and Mattia Talpo. Parabolic semi-orthogonal decompositions and Kummer flat invariants of log schemes. *Documenta Mathematica*, 25:955–1009, 2020.

[SSV16] Steffen Sagave, Timo Schürg, and Gabriele Vezzosi. Derived logarithmic geometry I. *Journal of the Institute of Mathematics of Jussieu*, 15(2):367–405, 2016.

[Sta24] The Stacks Project Authors. *Stacks Project.* https://stacks.math.columbia.edu, 2024.

[Swa96] Richard G. Swan. Hochschild cohomology of quasiprojective schemes. *Journal of Pure and Applied Algebra*, 110:57–80, 1996.

[Tab05] Gonçalo Tabuada. Invariants additifs de dg-catégories. *International Mathematics Research Notices*, 2005(53):3309–3339, 2005.

[Tem23] Michael Temkin. Introduction to logarithmic geometry. In *New Techniques in Resolution of Singularities*, pages 87–122. Springer, 2023.

[Tod09] Yukinobu Toda. Deformations and Fourier-Mukai transforms. *Journal of Differential Geometry*, 81(1):197–224, January 2009.

[Toë12] Bertrand Toën. Proper local complete intersection morphisms preserve perfect complexes. *arXiv preprint arXiv:1210.2827*, 2012.

[Toë14] Bertrand Toën. Derived algebraic geometry. *EMS Surveys in Mathematical Sciences*, 1(2):153–240, 2014.

[TV07] Bertrand Toën and Gabriele Vezzosi. A note on Chern character, loop spaces and derived algebraic geometry. available at. *arXiv preprint arXiv:0804.1274*, 2007.

[TV18] Mattia Talpo and Angelo Vistoli. Infinite root stacks and quasi-coherent sheaves on logarithmic schemes. *Proceedings of the London Mathematical Society*, 116(5):1187–1243, 2018.

[Vai17] Dmitry Vaintrob. Categorical Logarithmic Hodge Theory, I. *arXiv preprint arXiv:1712.00045*, 2017.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Yek01] Amnon Yekutieli. The Continuous Hochschild Cochain Complex of a Scheme. *Canadian Journal of Mathematics*, 54:1319 – 1337, 2001.

[Yok95] Kôji Yokogawa. Infinitesimal deformation of parabolic Higgs sheaves. *International Journal of Mathematics*, 6(1):125, 1995.

[Yu16] Shilin Yu. The Dolbeault dga of a formal neighborhood. *Transactions of the American Mathematical Society*, 368(11):7809–7843, 2016.

[Yu19] Shilin Yu. Todd class via homotopy perturbation theory. *Advances in Mathematics*, 352:297–325, 2019.

Summary

The concept of intersection is one of the first mathematical notions for which we develop a natural intuition. In school, we learnt that the set of multiples of 2 and the set of multiples of 3 intersect in the set of multiples of 6. We understand that intersections may also be empty (think of the intersection of odd numbers and even numbers), "very large" (it may coincide with one of the sets that we are intersecting), or of any intermediate size.

We also have some natural intuition of geometric intersections. Instead of sets, we now want to intersect algebraic varieties, like curves, surfaces, etc. For such intersections to make sense, we need to consider these varieties in an ambient space. For two abstract lines, it makes no sense to intersect, but depending on how we place them in a plane, we know that they may be parallel, be incident or coincide, corresponding to three different situations in terms of intersections. In a more sophisticated language, intersections are Cartesian products. Let X and Y be two algebraic varieties, Z the ambient variety where we consider them and $i : X \to Z$ and $j : Y \to Z$ their immersions in the larger spaces. Their intersection is such that the diagram

$$\begin{tikzcd}
X \cap Y = X \times_Z Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "i"] \\
Y \arrow[r, "j"] & Z
\end{tikzcd} \tag{5.8}$$

commutes, meaning that regarding $X \cap Y$ into X via the immersion p and then into Z via i is the same as regarding $X \cap Y$ into Y via q and then into Z via j: shortly, $i \circ p = j \circ q$. Moreover, this property is universal, meaning that whenever there is another variety W fitting in a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "p'"] \arrow[d, "q'"] & X \arrow[d, "i"] \\
Y \arrow[r, "j"] & Z
\end{tikzcd} , \tag{5.9}$$

there is a unique morphism $h : W \to X \cap Y$ such that $p' = p \circ h$ and $q' = q \circ h$. Informally, all diagrams like (5.13) "pass through" diagram (5.12), making

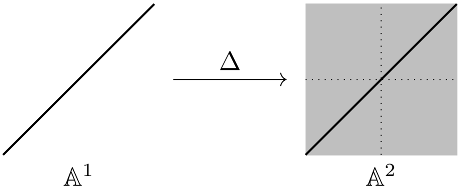

Figure 1: The immersion of the affine line into the affine plane via the diagonal.

the latter special. This is how we understand intersections and why they are interesting.

Hochschild homology and cohomology are geometrically defined via a similar procedure. We will not give their definitions here because their level of technicality clashes with the purpose of this summary of transmitting the ideas on which this thesis is based. Instead, we sketch the construction on which they rely, which is actually sufficient for their study. Consider the situation above for a variety X, included in its product with itself $X \times X$ along the diagonal, and intersect it with itself, embedded along the diagonal again. Figure 1 displays the diagonal morphism for the affine line. The reader may rightfully object that intersecting a space with itself, when embedded in an ambient space along the same morphism, is not of great interest. The intersection obtained as described above is just the same space. In spite of self-intersections being uninteresting for algebraic varieties, they tend to be more often interesting for the same spaces when regarded as derived varieties. Derived self-intersections are interesting as soon as the embedding along the diagonal is not flat, which is typically the case for diagonals as soon as the space has positive dimension. Once this construction is set, the Hochschild homology of the given space is just a piece of data attached to this construction. In this sense, the study of derived self-intersections is the core of the study of Hochschild homology.

In this thesis, we study the logarithmic Hochschild homology and cohomology spaces, they are obtained in a similar fashion as above. In broader generality, logarithmic schemes are pairs given by a scheme together with a sheaf of monoids satisfying some properties. For this summary, picture a logarithmic scheme as a variety compactified along a boundary (as obtained via Nagata's compactification). For example, as illustrated by Figure 2, the affine line $\mathbb{A}^1$ may be compactified by taking the two ends at infinity and adding a point that will serve as a common end, which is the boundary for this case. We want to highlight the boundary, pictorially marking it, and mathematically putting a logarithmic structure on it.

To record the compactification, we think of pairs (X, D) given by the compactification X and the boundary D. Unless D is empty, if X has dimension n, then D has dimension $n-1$: more precisely, it is an effective Cartier divisor and can be chosen with simple normal crossing singularities.

The definition of logarithmic Hochschild homology and cohomology is given by the self-intersection of the logarithmic scheme into an alteration of its product. Let us try to suggest a picture for this construction under suitable assumptions. Consider the pair (X, D) and let n be the dimension of X. Its product with itself

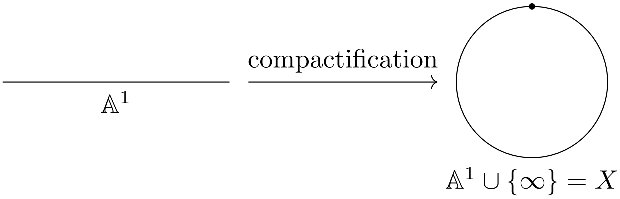

Figure 2: The compactification of the affine line $\mathbb{A}^1$ at infinity gives a compact space X. We denote the space obtained like so as a pair $(X, \{\infty\})$.

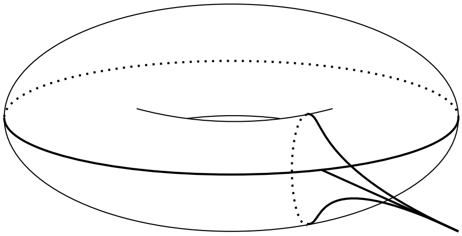

Figure 3: The real section of the alteration of the product $(X, \{\infty\}) \times (X, \{\infty\}) = (X \times X, (X \times \{\infty\}) \cup (\{\infty\} \times X))$. The boundary is highlighted and given by three components. The two circular sections are given by the two copies of $X \times \{\infty\}$. The third line is the additional component arising at their intersection $\{\infty\} \times \{\infty\}$, and is the exceptional divisor of the blow-up.

as a pair is

$$(X, D) \times (X, D) = (X \times X, (X \times D) \cup (D \times X)),$$

made by the usual product $X \times X$, of dimension $2n$, together with a boundary given by two copies of $D \times X$, of dimension $2n-1$, meeting at $D \times D$, of dimension $2n-2$. We alter it by blowing up the locus $D \times D$, obtaining an additional component of dimension $2n-1$, given by the exceptional divisor. Figure 3 attempts to illustrate the resulting space for $(X, \{\infty\})$ as compactification of $\mathbb{A}^1$.

Once again, we can embed the given space, which now appears as a pair, in the altered product sketched above. Logarithmic Hochschild homology is defined in a similar fashion as the ordinary one via derived self-intersections along the altered diagonal.

This thesis is about logarithmic Hochschild homology and cohomology and their properties via the approach illustrated in this summary.

In Chapter 1, we revisit the background of ordinary Hochschild homology and cohomology. We recall some classical results using the methods that will be adapted to the logarithmic case in the thesis.

In Chapter 2, we provide background on logarithmic geometry. This variation of algebraic geometry studies logarithmic spaces, which are more general objects than the pairs like in this summary. We give the background on logarithmic schemes and give an idea of how the same notions and results work for logarithmic stacks. We revisit the topic via historical subsequent approaches, concluding

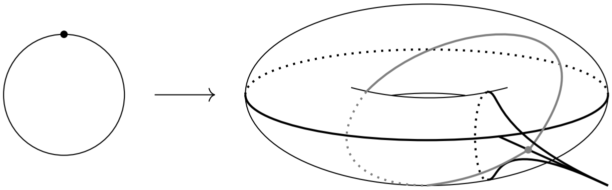

Figure 4: The altered diagonal for the pair $(X, \{\infty\})$ to the alteration of the product $(X, \{\infty\}) \times (X, \{\infty\})$. On the left side, we have the pair (X, ∞) obtained as in figure 2. On the right side, its image (in grey) is embedded in the altered product obtained as in figure 3. The marked point of $(X, \{\infty\})$ lands on the marked locus of $(X, \{\infty\}) \times (X, \{\infty\})$ and is the only point where they intersect.

with the one via the Artin fans. In the same chapter, we prove that the construction of the Artin fan respects products, which is a crucial fact for the following investigation of logarithmic Hochschild homology and cohomology.

In Chapter 3, we define logarithmic Hochschild homology and cohomology results and prove some of their properties, analogously to those of the original theory. The most important of these results is a logarithmic version of the celebrated HKR theorem and states formality for the complex defining logarithmic Hochschild homology and cohomology. The other properties are mostly consequences of this fact and include invariance under logarithmic blow-ups, the Künneth formula, a long exact sequence relating logarithmic Hochschild homology of smooth pairs and Hochschild homology of the underlying space and divisor and definition and role of the logarithmic loop space.

In Chapter 4, we show the invariance of logarithmic Hochschild homology and cohomology under a certain type of equivalence. Unlike the properties proved in the previous chapter, this one requires a different approach, the development of a technical tool (the logarithmic Serre functor) and the description of some "logarithmically rigid" derived equivalences (strict logarithmic derived equivalences).

In Chapter 5, we refine our understanding of logarithmic Hochschild homology and cohomology for quotient spaces obtained up to the action of finite groups (orbifolds). Nice orbifolds already satisfy the assumptions of the (logarithmic) HKR theorem. However, their stacky nature makes it complicated to work with them. In classical theory, the Hochschild homology and cohomology sheaves decompose as the direct sum of intersection endofunctors on the schemes on which the group is acting. The logarithmic version of this result is stated and proved for actions acting trivially on the Artin fan. Finally, we perform explicit computations for the orbifold obtained by the symmetric group acting on the n-fold product of a scheme with itself, which is a case where the action on the Artin fan is non-trivial and determines the action on the whole logarithmic scheme.

This thesis aims to be a broad, despite incomplete, treatment of a variation of Hochschild homology and cohomology theory via a geometric approach. Some possible directions are mentioned in the document and may be pursued in the future to complete this investigation.

Samenvatting

Het concept van doorsnede is een van de eerste wiskundige begrippen waarvoor we een natuurlijke intuïtie ontwikkelen. Op school leerden we dat de verzameling van veelvouden van 2 en de verzameling van veelvouden van 3 elkaar snijden in de verzameling van veelvouden van 6. We begrijpen dat doorsneden ook leeg kunnen zijn (denk aan de doorsnede van oneven en even getallen), "zeer groot" (de doorsnede kan samenvallen met een van de twee verzamelingen) of een tussenliggende grootte hebben.

We hebben ook een natuurlijke intuïtie voor meetkundige doorsneden. In plaats van verzamelingen willen we nu algebraïsche variëteiten snijden, zoals krommen, oppervlakken, enzovoorts. Om zulke doorsneden zinvol te maken, moeten we deze variëteiten beschouwen binnen een gezamelijkeruimte. Twee abstracte lijnen kunnen niet zomaar gesneden worden, maar afhankelijk van hoe we ze plaatsen in een vlak, weten we dat ze parallel, snijdend of samenvallend kunnen zijn, wat overeenkomt met drie verschillende situaties in termen van doorsneden. In een verfijnder taalgebruik beschouwt men doorsneden als Cartesische producten. Laat X en Y twee algebraïsche variëteiten zijn, Z de omgevingsvariëteit waarin we ze beschouwen en $i : X \to Z$ en $j : Y \to Z$ hun immersies in deze grotere ruimte. Hun doorsnede is dan zo dat het diagram

$$\begin{tikzcd}
X \cap Y = X \times_Z Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "i"] \\
Y \arrow[r, "j"] & Z
\end{tikzcd} \tag{5.10}$$

commuteert, wat betekent dat $X \cap Y$ via de immersie p in X en vervolgens via i in Z gaat, op dezelfde manier in Z terechtkomt als via q naar Y en vervolgens via j naar Z: kort gezegd, $i \circ p = j \circ q$. Bovendien is deze eigenschap universeel, wat betekent dat telkens wanneer er een andere variëteit W is die past in een commutatief diagram

$$\begin{tikzcd}
W \arrow[r, "p'"] \arrow[d, "q'"] & X \arrow[d, "i"] \\
Y \arrow[r, "j"] & Z
\end{tikzcd}, \tag{5.11}$$

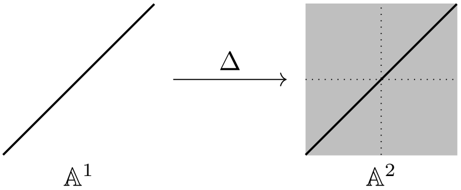

Figure 5: De immersie van de affiene lijn in het affiene vlak via de diagonaal.

er een unieke morfisme $h : W \to X \cap Y$ bestaat zodat $p' = p \circ h$ en $q' = q \circ h$. Informeel gezegd, alle diagrammen zoals (5.13) “gaan door” het diagram (5.12), wat dit laatste bijzonder maakt. Dit is hoe we doorsneden begrijpen en waarom ze interessant zijn.

Hochschild-homologie en -cohomologie worden meetkundig gedefinieerd via een gelijkaardige procedure. We geven hier hun definities niet, omdat hun technisch niveau botst met het doel van deze samenvatting: het overbrengen van de ideeën waarop dit proefschift is gebaseerd. In plaats daarvan schetsen we de constructie waarop ze gebaseerd zijn, wat eigenlijk voldoende is voor hun studie. Beschouw de situatie hierboven voor een variëteit X, opgenomen in zijn product met zichzelf $X \times X$ langs de diagonaal, en snij die met zichzelf, opnieuw ingebed langs de diagonaal. Figuur 5 toont de diagonaalmorfisme voor de affiene lijn. De lezer zou terecht kunnen opmerken dat het snijden van een ruimte met zichzelf, wanneer die in een omgevingsruimte wordt ingebed via dezelfde morfisme, niet zo interessant is. De verkregen doorsnede is dan gewoon dezelfde ruimte. Hoewel zelfdoorsneden niet zo boeiend zijn voor algebraïsche variëteiten, zijn ze dat vaak wel voor dezelfde ruimten wanneer we ze beschouwen als afgeleide variëteiten. Afgeleide zelfdoorsneden worden interessant zodra de inbedding langs de diagonaal niet plat is, wat typisch het geval is zodra de ruimte positieve dimensie heeft. Eens deze constructie is opgezet, is de Hochschild-homologie van de gegeven ruimte eenvoudigweg een stukje informatie dat aan deze constructie wordt gehecht. In die zin is de studie van afgeleide zelfdoorsneden de kern van de studie van Hochschild-homologie.

In dit proefschift bestuderen we de logaritmische versies van Hochschild-homologie en -cohomologie. Deze worden op een gelijkaardige manier verkregen als hierboven. Algemeen gesproken zijn logaritmische schema’s paren die bestaan uit een schema samen met een schoof van monoïden die aan bepaalde eigenschappen voldoet. Voor deze samenvatting kan men een logaritmisch schema zien als een variëteit die is gecompactificeerd langs een rand (zoals verkregen via Nagata’s compactificatie). Bijvoorbeeld, zoals geïllustreerd in Figuur 6, kan de affiene lijn $\mathbb{A}^1$ gecompactificeerd worden door de twee uiteinden op oneindig te nemen en een punt toe te voegen dat als gemeenschappelijk einde dient—de rand in dit geval. We willen deze rand markeren, visueel aanduiden, en er wiskundig een logaritmische structuur op leggen.

Om de compactificatie vast te leggen, denken we in termen van paren (X, D), gegeven door de compactificatie X en de rand D. Tenzij D leeg is, geldt: als X dimensie n heeft, dan heeft D dimensie $n-1$: preciezer, D is een effectieve Cartier-

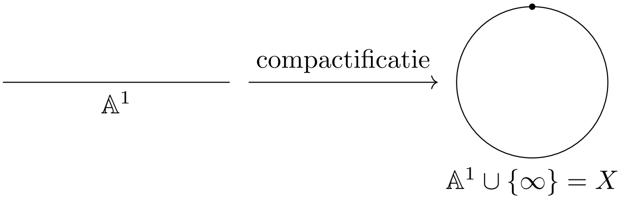

Figure 6: De compactificatie van de affiene lijn $\mathbb{A}^1$ aan oneindig geeft de compacte ruimte X. De verkregen ruimte noteren we als het paar $(X, \{\infty\})$.

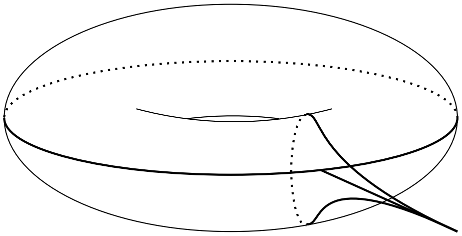

Figure 7: De reële doorsnede van de aanpassing van het product $(X, \{\infty\}) \times (X, \{\infty\}) = (X \times X, (X \times \{\infty\}) \cup (\{\infty\} \times X))$. De rand wordt benadrukt en bestaat uit drie componenten. De twee cirkelvormige doorsneden worden gegeven door de twee kopieën van $X \times \{\infty\}$. De derde lijn is de bijkomende component die ontstaat bij hun snijpunt $\{\infty\} \times \{\infty\}$ en is de exceptionele divisor van de blow-up.

divisor en kan gekozen worden met eenvoudige normale kruising-singulariteiten.

De definitie van logaritmische Hochschild-homologie en -cohomologie wordt gegeven via de zelfdoorsnede van het logaritmische schema in een aanpassing van zijn product. Laten we proberen een beeld te schetsen van deze constructie onder geschikte aannames. Beschouw het paar (X, D) en laat n de dimensie van X zijn. Het product van het paar met zichzelf is:

$$(X, D) \times (X, D) = (X \times X, (X \times D) \cup (D \times X)),$$

bestaande uit het gebruikelijke product $X \times X$ met dimensie $2n$, samen met een rand bestaande uit twee kopieën van $D \times X$, met dimensie $2n-1$, die elkaar snijden in $D \times D$, met dimensie $2n - 2$. We wijzigen dit door het centrum $D \times D$ op te blazen, en verkregen zo een bijkomende component van dimensie $2n - 1$, namelijk de exceptionele divisor. Figuur 7 poogt deze aangepasde ruimte te illustreren voor $(X, \{\infty\})$ als compactificatie van $\mathbb{A}^1$.

Opnieuw kunnen we de gegeven ruimte, nu als paar, inbedden in het aangepasde product zoals hierboven geschetst. Logaritmische Hochschild-homologie wordt gedefinieerd op een gelijkaardige manier als de gewone, via afgeleide zelfdoorsneden langs de aangepasde diagonaal.

Dit proefschift gaat over logaritmische Hochschild-homologie en -cohomologie en hun eigenschappen, volgens de benadering geïllustreerd in deze samenvatting.

In Hoofdstuk 1 herhalen we de achtergrond van gewone Hochschild-homologie

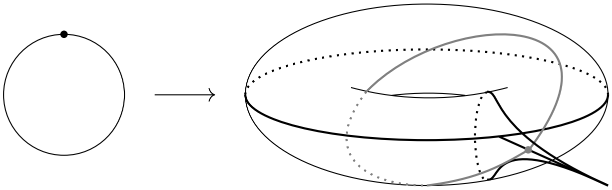

Figure 8: De aangepasde diagonaal voor het paar $(X, \{\infty\})$ naar de aanpassing van het product $(X, \{\infty\}) \times (X, \{\infty\})$. Aan de linkerkant staat het paar (X, ∞) zoals in Figuur 6. Aan de rechterkant is het beeld (in grijs) ingebed in het aangepasde product uit Figuur 7. Het gemarkeerde punt van $(X, \{\infty\})$ landt op het gemarkeerde centrum van $(X, \{\infty\}) \times (X, \{\infty\})$ en is het enige snijpunt.

en -cohomologie. We herhalen enkele klassieke resultaten met behulp van methoden die we zullen aanpassen aan de logaritmische context.

In Hoofdstuk 2 geven we achtergrondinformatie over logaritmische meetkunde. Deze variatie van de algebraïsche meetkunde bestudeert logaritmische ruimten, die algemenere objecten zijn dan de paren zoals in deze samenvatting. We introduceren logaritmische schema's en geven een idee van hoe gelijkaardige noties en resultaten werken voor logaritmische stapels. We bekijken het onderwerp via historische benaderingen en eindigen met die via de Artin-fans. In hetzelfde hoofdstuk bewijzen we dat de constructie van de Artin-fan producten respecteert—een cruciaal feit voor de verdere studie van logaritmische Hochschild-homologie en -cohomologie.

In Hoofdstuk 3 definiëren we logaritmische Hochschild-homologie en -cohomologie en bewijzen enkele van hun eigenschappen, analoog aan die van de oorspronkelijke theorie. Het belangrijkste resultaat is een logaritmische versie van het beroemde HKR-stelling, dat de formaliteit van het complex vastlegt dat logaritmische Hochschild-homologie en -cohomologie definieert. De andere eigenschappen volgen grotendeels hieruit, en omvatten invariantie onder logaritmische blow-ups, de Künneth-formule, een lange exacte rij die logaritmische Hochschild-homologie van gladde paren relateert aan die van de onderliggende ruimte en divisor, en de definitie en rol van de logaritmische lusruimte.

In Hoofdstuk 4 tonen we de invariantie van logaritmische Hochschild-homologie en -cohomologie onder een bepaald type equivalentie. In tegenstelling tot de eigenschappen uit het vorige hoofdstuk vereist dit een andere benadering, de ontwikkeling van een technisch hulpmiddel (de logaritmische Serre-functor) en de beschrijving van enkele "logaritmisch rigide" afgeleide equivalenties (strikte logaritmische afgeleide equivalenties).

In Hoofdstuk 5 verfijnen we ons begrip van logaritmische Hochschild-homologie en -cohomologie voor quotiënt-ruimten verkregen onder werking van eindige groepen (orbifolds). Orbifolds met goede eigenschappen voldoen reeds aan de voorwaarden van het (logaritmisch) HKR-stelling. Toch maakt hun stapel-structuur het lastig om ermee te werken. In de klassieke theorie splitsen de Hochschild-homologie- en -cohomologie-scheden zich als directe sommen van doorsnede-endofunctoren op de schema's waarop de groep werkt. De logaritmische versie van dit resultaat wordt geformuleerd en bewezen voor acties die triviaal werken op de Artin-fan. Tot slot voeren we expliciete berekeningen uit voor de orbifold verkregen door de symmetrische groep die werkt op het n-voudig product van een schema met zichzelf—een geval waarin de actie op de Artin-fan niet triviaal is en de hele logaritmische structuur bepaalt.

Dit proefschift beoogt een brede, hoewel onvolledige, behandeling van een variatie op de Hochschild-homologie en -cohomologie via een meetkundige benadering. Sommige mogelijke vervolgrichtingen worden genoemd en kunnen in de toekomst worden uitgewerkt om dit onderzoek te voltooien.

Sommario

Il concetto di intersezione è una delle prime nozioni matematiche per cui sviluppiamo un'intuizione naturale. A scuola, abbiamo imparato che l'insieme dei multipli di 2 e l'insieme dei multipli di 3 si intersecano nell'insieme dei multipli di 6. Comprendiamo che le intersezioni possono anche essere vuote (pensiamo all'intersezione dei numeri dispari e dei numeri pari), "molto grandi" (potrebbero coincidere con uno degli insiemi che stiamo intersecando), o di dimensioni intermedie.

Abbiamo anche un'intuizione naturale delle intersezioni geometriche. Invece di insiemi, ora vogliamo intersecare varietà algebriche, come curve, superfici, ecc. Affinché tali intersezioni abbiano senso, dobbiamo considerare queste varietà in uno spazio ambiente. Per due linee astratte, non ha senso intersecarle, ma a seconda di come le disponiamo in un piano, sappiamo che possono essere parallele, incidenti o coincidere, corrispondendo a tre situazioni diverse in termini di intersezioni. In un linguaggio più sofisticato, le intersezioni sono prodotti cartesiani. Siano X e Y due varietà algebriche, Z la varietà ambientale in cui le consideriamo e $i : X \to Z$ e $j : Y \to Z$ le loro immersioni negli spazi più grandi. La loro intersezione è tale che il diagramma

$$\begin{tikzcd} X \cap Y = X \times_Z Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "i"] \\ Y \arrow[r, "j"] & Z \end{tikzcd} \tag{5.12}$$

è commutativo, il che significa che considerare $X \cap Y$ in X tramite l'immersione p e poi in Z tramite i è lo stesso che considerare $X \cap Y$ in Y tramite q e poi in Z tramite j: in breve, $i \circ p = j \circ q$. Inoltre, questa proprietà è universale, il che significa che ogni volta che c'è un'altra varietà W con mappe p' e q' che rendono commutativo il diagramma

$$\begin{tikzcd} W \arrow[r, "p'"] \arrow[d, "q'"] & X \arrow[d, "i"] \\ Y \arrow[r, "j"] & Z \end{tikzcd}, \tag{5.13}$$

esiste un unico morfismo $h : W \to X \cap Y$ tale che $p' = p \circ h$ e $q' = q \circ h$. Informalmente, tutti i diagrammi come (5.13) "passano attraverso" il diagramma (5.12),

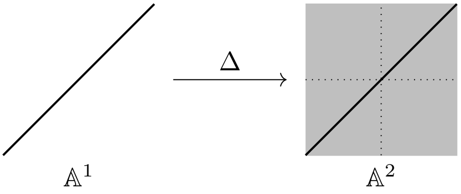

Figure 9: L'immersione della retta affine nel piano affine tramite la diagonale.

rendendo quest'ultimo speciale. Questo è il mdo in cui intendiamo le intersezioni e perché sono interessanti.

L'omologia e la coomologia di Hochschild sono definite geometricamente tramite una procedura simile. Non daremo le loro definizioni in questa sede perché il loro livello di tecnicismo collide con lo scopo di questo sommario di trasmettere le idee attorno a cui si sviluppa questa tesi. Invece, descriviamo la costruzione su cui si basano, che è effettivamente sufficiente per il loro studio. Consideriamo la situazione sopra per una varietà X, inclusa nel suo prodotto con se stessa $X \times X$ lungo la diagonale, e intersechiamola con se stessa, immessa lungo la diagonale di nuovo. La Figura 9 mostra il morfismo diagonale per la retta affine. Il lettore potrebbe giustamente obiettare che intersecare uno spazio con se stesso, quando immesso in uno spazio ambientale tramite lo stesso morfismo, non è di grande interesse. L'intersezione ottenuta come descritto sopra è solo lo stesso spazio. Nonostante le auto-intersezioni siano poco interessanti per le varietà algebriche, tendono a essere più interessanti per gli stessi spazi quando considerati come varietà derivate. Le auto-intersezioni derivate sono interessanti non appena l'immersione lungo la diagonale non è piatta, che è tipicamente il caso non appena lo spazio ha dimensione positiva. Una volta che questa costruzione è realizzata, l'omologia di Hochschild dello spazio dato è semplicemente un dato allegato a questa costruzione. In questo senso, lo studio delle auto-intersezioni derivate è il nucleo dello studio dell'omologia di Hochschild.

In questa tesi, studiamo gli spazi di omologia e coomologia di Hochschild logaritmica, che sono ottenuti in modo simile a quanto sopra. In maggiore generalità, gli schemi logaritmici sono coppie date da uno schema insieme a un fascio di monoidi che che soddisfa alcune proprietà. Per questo sommario, immaginiamo uno schema logaritmico come una varietà compatificata lungo un bordo (come ottenuto tramite la compatificazione di Nagata). Ad esempio, come illustrato dalla Figura 10, la retta affine $\mathbb{A}^1$ può essere compatificata prendendo i due estremi all'infinito e aggiungendo un punto che servirà come punto comune a cui incollare le estremità, che è il bordo per questo caso. Vogliamo evidenziare il bordo, graficamente segnalandolo, e matematicamente mettendo una struttura logaritmica su di esso.

Per registrare la compatificazione, pensiamo a coppie (X, D) date dalla compatificazione X e dal bordo D. A meno che D non sia vuoto, se X ha dimensione n, allora D ha dimensione $n-1$: più precisamente, è un divisore di Cartier effettivo e può essere scelto con singolarità di tipo "simple normal crossing".

La definizione dell'omologia e coomologia di Hochschild logaritmica è data

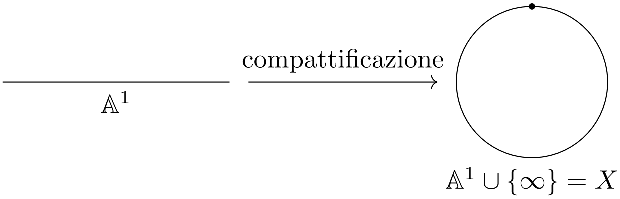

Figure 10: La compatificazione della retta affine $\mathbb{A}^1$ all'infinito dà lo spazio compatto X. Denotiamo lo spazio ottenuto in questo modo come una coppia $(X, \{\infty\})$.

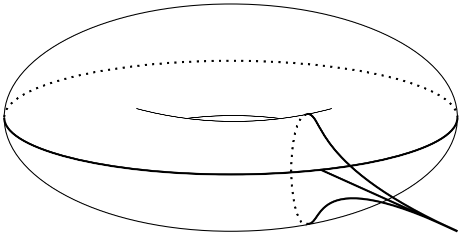

Figure 11: La sezione reale dell'alterazione del prodotto $(X, \{\infty\}) \times (X, \{\infty\}) = (X \times X, (X \times \{\infty\}) \cup (\{\infty\} \times X))$. Il bordo è evidenziato e dato da tre componenti. Le due sezioni circolari sono date dalle due copie di $X \times \{\infty\}$. La terza linea è la componente aggiuntiva che sorge dalla loro intersezione $\{\infty\} \times \{\infty\}$ ed è il divisore eccezionale del blow-up.

dall'auto-intersezione dello schema logaritmico in una alterazione del suo prodotto. Cerchiamo di suggerire un'immagine per questa costruzione sotto opportune ipotesi. Considera la coppia (X, D) e sia n la dimensione di X. Il suo prodotto con se stesso come coppia è

$$(X, D) \times (X, D) = (X \times X, (X \times D) \cup (D \times X)),$$

composto dal prodotto usuale $X \times X$, di dimensione $2n$, insieme a un bordo dato da due copie di $D \times X$, di dimensione $2n-1$, che si incontrano in $D \times D$, di dimensione $2n-2$. Lo alteriamo scoppiando il luogo $D \times D$, ottenendo una componente aggiuntiva di dimensione $2n-1$, data dal divisore eccezionale. La Figura 11 tenta di illustrare lo spazio risultante per $(X, \{\infty\})$ come compatificazione di $\mathbb{A}^1$.

Ancora una volta, possiamo immergere lo spazio dato, che ora si presenta come una coppia, nel prodotto alterato descritto sopra. L'omologia di Hochschild logaritmica è definita in modo simile a quella ordinaria tramite auto-intersezioni derivate lungo la diagonale alterata.

Questa tesi riguarda l'omologia e la coomologia di Hochschild logaritmica e le loro proprietà attraverso l'approccio illustrato in questo sommario.

Nel Capitolo 1, rivediamo il background dell'omologia e coomologia di Hochschild ordinaria. Ripercorriamo alcuni risultati classici utilizzando i metodi che saranno adattati al caso logaritmico nella tesi.

Nel Capitolo 2, forniamo il background sulla geometria logaritmica. Questa variazione della geometria algebrica studia gli spazi logaritmici, che sono oggetti

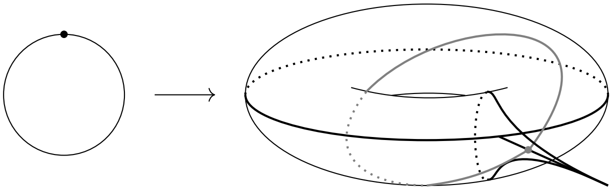

Figure 12: Ulteriore visualizzazione dell'alterazione del prodotto in alto, con il bordo evidenziato e dato dalle tre componenti, inclusa la componente aggiuntiva data dal blow-up.

più generali rispetto alle coppie come quelle illustrate in questo sommario. Forniamo il background sugli schemi logaritmici e diamo un'idea di come le stesse nozioni e i risultati funzionano per gli stack logaritmici. Rivediamo l'argomento attraverso approcci storici successivi, concludendo con quello tramite i fan di Artin. Nello stesso capitolo, dimostriamo che la costruzione del fan di Artin rispetta i prodotti, il che è un fatto cruciale per l'indagine successiva sull'omologia e la coomologia di Hochschild logaritmica.

Nel Capitolo 3, definiamo i risultati dell'omologia e della coomologia di Hochschild logaritmica e proviamo alcune delle loro proprietà, analogamente a quelle della teoria originale. Il più importante di questi risultati è una versione logaritmica del celebre teorema HKR, che afferma la formalità per il complesso che definisce l'omologia e la coomologia di Hochschild logaritmica. Le altre proprietà sono per lo più conseguenze di questo fatto e includono l'invarianza a meno di blow-up logaritmici, la formula di Künneth, una lunga sequenza esatta che collega l'omologia di Hochschild logaritmica di coppie lisce con l'omologia di Hochschild dello spazio sottostante e del divisore, e la definizione e il ruolo dello spazio dei lacci logaritmico.

Nel Capitolo 4, mostriamo l'invarianza dell'omologia e della coomologia di Hochschild logaritmica sotto un certo tipo di equivalenza. A differenza delle proprietà dimostrate nel capitolo precedente, questa richiede un approccio diverso, lo sviluppo di uno strumento tecnico (il funtore di Serre logaritmico) e la descrizione di alcune equivalenze derivate "logaritmicamente rigide" (equivalenze derivate logaritmiche strette).

Nel Capitolo 5, perfezioniamo la nostra comprensione dell'omologia e della coomologia di Hochschild logaritmica per spazi quozienti ottenuti fino all'azione di gruppi finiti (orbifold). Gli orbifold con buone proprietà soddisfano già le ipotesi del teorema HKR (logaritmico). Tuttavia, la loro natura di stack rende complicato lavorare con essi. Nella teoria classica, i fasci di omologia e coomologia di Hochschild si decompongono come la somma diretta di endofunctori di intersezione sugli schemi sui quali il gruppo agisce. La versione logaritmica di questo risultato è enunciata e provata per azioni che agiscono banalmente sul fan di Artin. Infine, eseguiamo calcoli espliciti per l'orbifold ottenuto dal gruppo simmetrico che agisce sul prodotto n-volte di uno schema con se stesso, caso in cui l'azione sul fan di Artin non è banale e determina l'azione sull'intero schema logaritmico.

Questa tesi ha l'obiettivo di essere un trattamento ampio, sebbene incompleto, di una variazione della teoria dell'omologia e coomologia di Hochschild attraverso un approccio geometrico. Alcune possibili direzioni sono menzionate e potrebbero essere perseguite in futuro per completare questa indagine.

Acknowledgments

This thesis is the outcome of four years of work and would certainly not exist without the many people who directly and indirectly contributed to it. The limited space allotted here will not allow me to thank them all as they deserve, and I apologize in advance for this.

First and foremost, I want to express my gratitude to my supervisor, Márton Hablicsek. Thank you for introducing me to the topic, for your patience in explaining things to me more than once when my understanding was limited, and for your support on difficult days. Your contribution to this work is undeniable.

To Ronald van Luijk, thank you not only for overseeing the administrative aspects of my PhD but also for your enthusiasm and comprehension, always being available with your insightful advice.

To Leo Herr, from whom I learned a lot via our collaboration. Moreover, thank you for taking part in the committee and giving your clever feedback.

To Tyler Kelly and Bertrand Toën, I am deeply grateful for the valuable discussions we had and for agreeing to participate in the doctoral committee, reading, and providing enlightening feedback on this thesis.

To the members of the Algebra, Geometry, and Number Theory group, who created a lively and stimulating work environment, allowing me to grow academically and personally. A special thanks goes to David Holmes, who gave the course in logarithmic geometry which was my first introduction to the topic.

To the PhD community, who created a space where it was a pleasure to work. I cannot avoid mentioning Paolo Bordignon, with whom I had the honour of sharing an office and discussing more number-theoretical and non-mathematical questions. I am also grateful to Margherita Pagano, who listened to several of my practice talks and many of my questions. Many more of my PhD fellows have my gratitude for contributions that extend beyond the academic world. Their presence has been invaluable.

La mia più profonda gratitudine va alla mia famiglia e al supporto incondizionato datomi da sempre. A mia madre, che non ha mai dubitato di nessuna delle mie scelte. A mio padre, che più di chiunque altro al mondo mi ha mostrato l'importanza dell'istruzione e al contempo la sua inutilità quando non accompagnata dalla gentilezza e dal rispetto dell'altro. A mio fratello Marco, con cui ho

condiviso tutto, e che mi ha permesso di guardarmi come in uno specchio e di riconoscermi [*].

I am unable to express how grateful I am to Federico, whose unconditional support has been the cornerstone of this journey.

---

[*] My deepest gratitude goes to my family and the unconditional support they have always given me. To my mother, who never doubted any of my choices. To my father, who more than anyone else in the world showed me the importance of education and, at the same time, its futility when not accompanied by kindness and respect for others. To my brother Marco, with whom I have shared everything, and who has allowed me to mirror in him and recognize myself.

Curriculum Vitae

Francesca Leonardi was born in Rome on November 21, 1997, and raised there. Between 2011 and 2016, she was enrolled in Liceo Scientifico Statale “Augusto Righi”, earning her high school diploma with a grade of 100/100. During those years, she was a member of the school’s team in the Italian Mathematics Olympiad and an active member of the student political scene of the institute.

Afterward, she started a Bachelor’s degree in Mathematics at Sapienza Univesità di Roma, partially supported by the INdAM scholarship and following the Percorso d’Eccellenza. She graduated *cum laude* in 2019 with a thesis written under the supervision of Prof.dr. Paolo Buttà with the title “Il gas di Lorentz nel limite di bassa densità”. She continued her studies at the same university with a Master’s degree in Mathematics, specializing in Algebra and Geometry. She spent the academic year 2020-2021 at the Université de Toulouse (at the time “Université Toulouse III – Paul Sabatier”) through the Erasmus+ program. There she wrote her thesis “The Homotopy Theory of Grothendieck: An Introduction to Homotopical Algebra” under the supervision of Prof.dr. Bertrand Toën and Dr. Domenico Fiorenza. She graduated in 2021 *cum laude*. During both her Bachelor’s and Master’s, she has been an active member of the student community, and she worked as a private tutor, teaching assistant, and waitress.

She began her PhD in 2021 at Leiden University under the supervision of Dr. Márton Hablicsek and Prof.dr. Ronald van Luijk in the research group of Algebra, Geometry and Number Theory. She spent four years carrying on research, teaching and assisting courses, participating in local and international events, presenting her work via talks and posters in some of them, and organizing reading groups and local and international workshops.