Limit theorems for wobbly interval intermittent maps

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Abstract

We consider perturbations of interval maps with indifferent fixed points, which we refer to as wobbly interval intermittent maps, for which stable laws for general Hölder observables fail. We obtain limit laws for such maps and Hölder observables. These limit laws are similar to the classical semistable laws previously established for random processes. One of the considered examples is an interval map with a countable number of discontinuities, and to analyse it we need to construct a Markov/Young tower.

1 Introduction and main results

1.1 Formulating the problem and statement of the main result

In this paper we study the map

$$f_M(x) = \begin{cases} x(1 + M(x)x^\alpha), & \text{if } x \in [0, \frac{1}{2}]; \\ 2x - 1, & \text{if } x \in (1/2, 1]. \end{cases} \tag{1.1}$$

with $\alpha \in (0, 1)$, for oscillatory functions M to be described further bellow. We start by recalling that when $M(x) \equiv 2^\alpha$, f_M is the well known Liverani-Saussol-Vaienti (LSV) map [27], which preserves an absolutely continuous w.r.t. Lebesgue invariant measure μ. When $\alpha \in (0, 1)$ the measure μ can be rescaled to give an invariant probability measure, while for $\alpha \geq 1$, the measure μ is infinite, σ finite. In last two decades, several statistical properties for the LSV map have been established in both the probability setting, including [27, 36, 32, 16, 15] and the infinite measure setting, including [35, 29, 19]. This list of references is by far not exhaustive and several important references can be found in the above mentioned works.

As relevant for the present work, we recall that in the probability case of (1.1) with $M(x) \equiv 2^\alpha$, $\alpha \in (0, 1)$, the following limit theorems have been established:

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a) If $\alpha \in (0, 1/2)$, then the classical central limit theorem holds for Hölder observables on $[0,1]$ (see [27] and Young [36]). If $\alpha = 1/2$ and v is Hölder then: i) if $v(0) = 0$, the classical central limit theorem holds; ii) if $v(0) \neq 0$, then $v \notin L^2$ and we have convergence to the Gaussian $\mathcal{N}(0,1)$ but with non standard normalisation. We refer to Gouëzel [15, Section 1.1.2] for details.

b) If $\alpha \in (1/2, 1)$, the following two possibilities occur for Hölder observables $v : [0,1] \to \mathbb{R}$ with $\int v \, d\mu = 0$ (see [15]):

   i) If v is ν-Hölder with $\nu > \alpha - 1/2$ and $v(0) = 0$, then the classical central limit theorem holds for such v.

   ii) If $v(0) \neq 0$, then the Birkhoff sum $S_n^{f_M} v(x) = \sum_{k=0}^{n-1} v(f_M^j(x))$ converges in distribution to a stable law of index $\beta = 1/\alpha$. For background on stable laws we refer to [13] (see also [15, Section 1.1.2] for a brief summary).

In this work we study the type of limit laws that occur when the function $M : [0,1] \to \mathbb{R}$ appearing in (1.1) with $\alpha \in (0,1)$ is an oscillatory function and refer to these maps as *wobbly intermittent maps*. As clarified in Subsection 1.2, when $\alpha \in (1/2, 1)$, for the choices of $M = M_1, M = M_2$ introduced below (in (1.2) and (1.3)), general Hölder observables do not give rise to stable laws. The main purpose of this paper is to understand what type of limit theorems can be enjoyed by Hölder observables in the setup of wobbly intermittent maps f_{M_1}, f_{M_2} introduced below.

We focus on the wobbly map $f_{M_1} : [0,1] \to [0,1]$ defined as in (1.1) with $M = M_1 : [0,1] \to (0, \infty)$ given by

$$M_1(x) = \frac{C_0}{2^{\{c_1^{-1} \log x\}}}, \text{ where } c_1 > 0 \text{ and } C_0 \text{ so that } f_{M_1}\left((1/2)^-\right) = 1.$$
(1.2)

Here and throughout $\{\cdot\}$ denotes the fractional part. When $c_1^{-1} \log 2 < 1$, $C_0 = 2^{\alpha - \frac{1}{c_1} \log 2}$. This choice corresponds to M_1 satisfying a *log-periodic* condition with period e^{c_1}, namely $M_1(e^{c_1} x) = M_1(x)$ for all $x > 0$. Here and throughout we define $\{x\} = \{-x\}$ for $x < 0$.

As shown in Section 4, for M_1 as in (1.2) and $\alpha \in (0,1)$, the tail of the first return time of f_{M_1} to $[1/2, 1]$ is somewhat similar to the generalised St. Petersburg distribution with index $\beta = 1/\alpha \in (1, 2)$. We recall that *semistable* laws for i.i.d random variables $X, X_1, X_2 \ldots$ with a common generalised St. Petersburg distribution $\mathbb{P}(X > y) = 2^{\{\beta \log_2 y\}} y^{-\beta}$ for $y \geq 2^{1/\beta}$ with $\beta \in (0,2)$ have been obtained by Csörgő [9] (see also [8, 24, 23]). For background on semistable laws we refer to Megyesi [28], Csörgő and Megyesi [8] as well as [9, 24, 23] and references therein (see also Section 2 below for a brief introduction). It is known the induced map of the Gaspard-Wang map [14], a linearised version of the map (1.1), is isomorphic to an i.i.d. process. Limit laws of *infinite* measure preserving *Gaspard Wang* maps with a distribution of return times behaving similar to a *St. Petersburg distribution* (of index $\beta \in (0,1)$) have been obtained by Kevei and Terhesiu [25] (in particular, see [25, Section 5]). The present Theorem 1.1 for finite measure preserving wobbly maps is partially motivated by the results in [25]. We believe that a *modified Theorem 1.1* for *infinite* measure preserving wobbly maps (so, f_{M_1} as in (1.1) with M_1 as in (1.2) and $\alpha \geq 1$) can be obtained by a straightforward combination of the results in present paper with the ones in [25], but here we *focus on the finite* measure case.

A simplified wobbly map, which as explained below induces with respect to first return to a Gibbs Markov map, is $f_{M_2} : [0,1] \to [0,1]$ defined as in

(1.1) with a differentiable $M = M_2$ given by

$$M_2(x) = a\left(1 + b\sin\left(\frac{2\pi}{c_2}\log(x)\right)\right) \qquad (1.3)$$

where a, c_2 are positive real constants with $b < 1/2$ so that M_2 is bounded and bounded away from zero and M_2 is log periodic with period e^{c_2}, that is $M_2(e^{c_2}x) = M_2(x)$ for all $x > 0$.

Throughout we let $\to^d$ stand for convergence in distribution with respect to the any measure that is absolutely continuous with respect to the f_{M_1}, f_{M_2}-invariant measure μ. With these specified we provide the statement of the main result. For the statement below we let $V_{1/\alpha}(c)$ be a random variable in the domain of a semistable law of index $1/\alpha \in (1, 2)$ and parameter $c > 1$; for a precise definition of such random variable we refer to Section 2. Also, throughout $\lfloor \ \rfloor$ stands for the integer part.

Theorem 1.1 *Let M_1 and M_2 as in (1.2) and (1.3). Assume the setup of either f_{M_1} or f_{M_2} with $\alpha \in (0, 1)$.*

Let $v : [0, 1] \to \mathbb{R}$ be a Hölder observable. If either $\alpha \in (0, 1/2)$ or $\alpha \in [1/2, 1)$ and $v(0) = 0$, then the classical central limit theorem holds for f_{M_1} and f_{M_2}. Also, the following non-Gaussian limit theorem holds:

i) *Consider the setup of f_{M_1} with $\alpha \in (1/2, 1)$. Suppose that v is Hölder of exponent $\nu > \alpha - 1/2$ so that $v(0) \neq 0$. Set $c = e^{\alpha c_1}$ and $k_n = \lfloor c^n \rfloor$. Then there exists a subsequence n_r of k_n so that as $r \to \infty$*

$$\frac{\sum_{j=0}^{n_r - 1} v \circ f_{M_1}^j - B_{n_r}\int v\,d\mu}{A_{n_r}} \to^d W$$

for some random variable W with a non trivial distribution, where $A_{n_r} = (n_r)^\alpha$ and B_{n_r} is so that $\lim_{r\to\infty} B_{n_r}/n_r$ exists.

ii) *Consider the setup of f_{M_2} and the same assumptions on α and v as in item i). Set $c = e^{\alpha c_2}$, $k_n = \lfloor c^n \rfloor$ and $A_{k_n} = (k_n)^\alpha$. In this case we have the improved result*

$$\frac{\sum_{j=0}^{k_n - 1} v \circ f_{M_2}^j - B_{k_n}\int v\,d\mu}{A_{k_n}} \to^d V_{1/\alpha}(c), \quad \text{as } n \to \infty,$$

where B_{k_n} is so that $\lim_{n\to\infty} B_{k_n}/k_n$ exists.

Given Theorem 1.5 and Remark 1.6 below, we note that the ranges of α and assumptions on v which yield the central limit theorem for these wobbly maps, are essentially the same as in standard case (1.1), (i.e. with $M(x) \equiv 2^\alpha$) and the corresponding proof goes through in exactly the same way (as in the works acknowledged in a)–b) above). The proof of Theorem 1.1 for ranges of α and conditions on v that do *not* give rise to the central limit theorem is deferred to Section 6, where we also clarify that this holds for larger classes of subsequences.

Remark 1.2 It seems likely that Theorem 1.1 i) which gives the limit behaviour of the Birkoff sums for f_{M_1} can be improved to the form of Theorem 1.1 ii). But this requires a refinement of Theorem 1.5 below to $\mu(\varphi > n) = C\mu(\tau > n)(1 + o(1))$ for some $C > 0$; this type of result would require a serious amount of new work, which we cannot pursue here.

Remark 1.3 We believe that the proof of Theorem 1.1 can be extended to treat the case $\alpha = 1/2$, when v is Hölder with $v(0) \neq 0$; we expect convergence along subsequences to the Gaussian $\mathcal{N}(0, 1)$ with non standard normalisation. This seems plausible given the recent work of Berkes [3], but we do not treat this case here.

Remark 1.4 A further extension of Theorem 1.1 is to treat the case $\alpha = 1$, but with a suitably modified f, such as

$$f_M^1(x) = \begin{cases} x(1 + M(x)x(\log x)^2), & \text{if } x \in [0, \frac{1}{2}]; \\ 2x - 1, & \text{if } x \in (1/2, 1]. \end{cases} \tag{1.4}$$

with $M \equiv M_1$ as in (1.2) or $M \equiv M_2$ as in (1.3). Special versions of these maps f_M^1 with $M \equiv 1$ have been studied in see Holland [22]. The extra factor $(\log x)^2$ ensures that $f_{M_1}^1$ preserves a finite measure $\mu \ll Leb$. In particular, for $M \equiv 1$, [22] shows that the tail of the first return time of f_{M_1} to $[1/2, 1]$ behaves asymptotically as $n^{-1}(\log n)^{-2}$. We expect a semi-stable law to hold for this example, but the proof requires a more elaborate asymptotic analysis of the tail $\mu(\tau > y)$, which we do not present here.

1.2 Method of proof and main dynamical ingredient

A standard way of obtaining limit theorems, in particular stable laws, for maps with indifferent fixed points is via inducing on sets away from the indifferent fixed points along with understanding the return time distribution. To outline the difficulties in the setup of the wobbly maps f_{M_1}, and f_{M_2} (with M_1, M_2 as in (1.2) and (1.3)), we briefly recall the main steps of the involved analysis in the probability preserving LSV map $([0, 1], \mathcal{B}([0, 1]), f, \mu)$ as in (1.1) with $M \equiv 2^\alpha$. Let $Y \subset (0, 1]$ be a reference set and let $\tau : Y \to \mathbb{N}$, $\tau(x) = \inf\{n \geq 1 : f^n(x) \in Y\}$ be the first return time of f to Y. The main steps are:

a) Recall that the induced map $f^\tau : Y \to Y$, which preserves the measure $\mu_Y(E) := \mu(Y \cap E)$, has good functional analytic properties in a Banach space $\mathcal{B} \subset L^\infty(Y)$ (for instance, the space of piecewise Hölder continuous functions);

b) Recall that $\mu_Y\{\tau > n\} = Cn^{-1/\alpha}(1 + o(1))$ for some $C > 0$ depending only on f. See [27, 36] and also [22, 33, 34] for improved asymptotics.

c) Obtain a limit theorem (stable law) via the spectral method, the Nagaev-Guivarc'h (also referred to as Aaronson-Denker) method, for f^τ; we refer to the comprehensive survey [20] and to the original papers [2, 21].

d) Pull back the limit theorem from the induced system via the method of Melbourne and Török [30] (see also Sarig [31], Zweimüller [37] for details of the method for the stable law case) or the method in [15] based on techniques related to operator renewal sequences. While the 'pull back' method is more elementary and somewhat less technical, the second has the advantage of eventually providing error rates in the involved convergence (such as Berry-Essen) as well as local limit theorems as in Gouëzel [17].

In the sequel we adapt the steps a)–d) to the proof of Theorem 1.1. A first, more or less obvious, difference is that in the setup of the wobbly maps f_{M_1} and f_{M_2} *item b) does not hold*; for the asymptotics of the return time in the setup of f_{M_1}, f_{M_2} and f_M^1 defined in (1.4) we refer to equation (4.1) in

Section 4. We recall that stable laws of index $\beta \in (0, 2)$ for Gibbs Markov maps hold if and only if $\mu_Y\{\tau > n\}$ is regularly varying with index $-\beta$, that is if and only if $\mu_Y\{\tau > n\} = Cn^{-\beta}\ell(n)$ for $\beta \in (0, 2)$ and ℓ some slowly varying[1] function; we refer to [18, Theorem 1.5]. This together with equation (4.1) in Section 4 explains why stable laws cannot hold for f_{M_1} and f_{M_2}.

Items c)-d) are dealt with in an abstract setup where regular variation (in particular, item b)) fails. We refer to assumption (H3) in Section 3 and to the abstract Theorem 3.3. In Section 6 (summarising the various technical results in Sections 4 and 5) we verify that the abstract assumptions of Theorem 3.3 hold in the setup of f_{M_1}, and f_{M_2}.

It is known that the induced map of the LSV map (1.1) is Gibbs Markov, which facilitates an easy verification of item a) above. Roughly, a Gibbs Markov map is a uniformly expanding map with big images and good distortion properties; see [1, Chapter 4] (and Section 3 below) for a complete definition. While the first return map of the wobbly map f_{M_2} is Gibbs Markov, the situation is very different in the setup of the wobbly map f_{M_1}. The first difficulty in analysing the map f_{M_1} (1.2) is that there is a singularity set $\mathcal{C} = \{s_\ell = e^{-\ell c_1}, \ell \geq 1\}$, namely the set of points $x \in (0, 1/2)$ for which $-c_1^{-1} \log x \in \mathbb{N}$. We'll assume that c is chosen so that $s_1 = e^{-c_1} \ll 1$.

We write $\mathcal{D} = e^{-c_1}$, and $I_\mathcal{D} = [0, \mathcal{D}]$. Hence $\mathcal{C} \subset I_\mathcal{D}$. Thus we have to apply re-inducing schemes, and show the existence of a return time function $\varphi : Y \to Y$ for which $(f_{M_1})^\varphi$ is Gibbs Markov. Furthermore we have to show that the asymptotics of the tails of φ are comparable to those of the first return time τ.

We build upon the combinatorial arguments of Bruin, Luzzatto and van Strien [5] to establish the following result, of which the proof is deferred to Section 5.

Theorem 1.5 *Let $Y = [1/2, 1]$. Then there exists a countable partition $\mathcal{Q}$ (mod μ) of Y into subintervals $\{Y_i\}$ and a return time function $\varphi : \mathcal{Q} \to \mathbb{N}$ such that $F : Y \to Y$ where $F = f_{M_1}^\varphi$ or $F = (f_{M_1}^1)^\varphi$ is a Gibbs-Markov map with respect to $\mathcal{Q}$, preserving an absolutely continuous (w.r.t. Lebesgue) probability measure μ so that for some $C > 0$,*

$$\mu(\varphi > n) \leq C\, n^{-\frac{1}{\alpha}}. \tag{1.5}$$

Theorem 1.5 can be be regarded as a substantial refinement of the work of Diaz-Ordaz, Holland and Luzzatto [12] which adapts arguments in [5] to reinduce maps with a *finite number of singularities/discontinuities* to Gibbs Markov maps.

Remark 1.6 A similar Theorem 1.5 holds for the tails of the *first* return time τ of f_{M_2} to $Y = [1/2, 1]$; this follows from (4.1) in Section 4. The sitution is mush easier since in this case no reinducing is required as the first return map $(f_{M_2})^\tau$ is Gibbs Markov: see Remark 5.6.

After having summarized the main difficulties that occur for the considered wobbly maps, especially in the case of f_{M_1}, we turn to more details on the method of proof. In the particular setup of f_{M_2} the pull back method of Melbourne and Török mentioned in item d) above (see also [20] for a summary of the method adapted to stable laws) works with moderate adjustments to the arguments. However, in the case of f_{M_1}, which requires

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[1]A measurable function $\ell : (0, \infty) \to (0, \infty)$ is slowly varying if $\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} \to 1$ for each $\lambda > 0$.

re-inducing, it becomes more difficult to prove that the sequences (n_r) along which the scaled ergodic sums of the original map converge are subsequences of the exponential sequence $(k_n) = (\lfloor c^n \rfloor)$. Given the result in [17] along with several estimates in [7], we found it more convenient to adapt the arguments [17] to the setup of (tails of) of both f_{M_1} and f_{M_2}. We refer to the abstract Theorem 3.3 in Section 3.

An immediate notable consequence of Theorem 1.5 are upper and lower bounds on the correlation decay. For a precise statement on these bounds we need to introduce additional notation. We recall the one in [7], which is adequate to the present setup but note that this can be further refined as in the work of Bruin, Melbourne and Terhesiu [6] (which treats much more complicated classes of dynamical systems) w.r.t. the condition on the seminorm.

As in [7] we consider the following class of observables. Let $X = [0,1]$ and $\tau^*(x) := 1 + \min\{i \geq 0 : f^i(x) \in Y\}$, (here f is either f_{M_1} or f_{M_2}). Let $s(x,x')$ be the separation time of points $x,x' \in Y$ and let $\theta \in (0,1)$ be such that (3.1) below holds; equation (3.1) in Section 3.1 is used to define a Gibbs Markov map. Let $v_X : X \to \mathbb{R}$. For $\varepsilon > 0$ we define the weighted norm $\| \, \|_\theta^*$ as follows:

$$
\begin{cases}
\|v_X\|_\infty^* := \sup_{x \in X} |v_X(x)|\tau^*(x)^{1+\epsilon}, \\
|v_X|_\theta^* = \sup_{a \in \alpha} \sup_{0 \leq i < \varphi(a)} \sup_{x,x' \in a} \frac{(\tau^* \circ f^i(a))^{1+\varepsilon}}{\theta^{s(x,x')}} |v_X \circ f^i(x) - v_X \circ f^i(x')|,
\end{cases}
\tag{1.6}
$$

and $\|v_X\|_\theta^* = \|v_X\|_\infty^* + |v_X|_\theta^*$. As clarified in Subsection 5.4 (for $f \equiv f_{M_1}$), τ^* is constant on $f^i(a)$, $0 \leq i < \varphi(a)$, so the factor $\tau^* \circ f^i(a)$ in (1.6) is well-defined. We note that if v_X is supported on Y, then the weighted norms $\| \, \|_\infty^*$ and $\| \, \|_\theta^*$ coincide with $\| \, \|_{L^\infty(\mu_Y)}$ and $\| \, \|_\theta$ with $\|v\|_\theta = \|v\|_{L^\infty(\mu_Y)} + |v|_\theta$, where $v|_\theta$ is the Hölder constant of v w.r.t. the distance $d_\theta(x,x') := \theta^{s(x,x')}$.

Corollary 1.7 [7, Theorem 4.2] Assume the setup of f_{M_1} with M_1 as in (1.2) and $\alpha \in (0,1)$. Suppose that $v, w : [0,1] \to \mathbb{R}$ are such that $\|v\|_\theta^* < \infty$ and $\|w\|_\infty^* < \infty$. Let $d\mu = \frac{1}{\bar{\varphi}} d\mu$. Then

$$
\int v \, w \circ f_{M_1}^n \, d\mu - \int v \, d\mu \int w \, d\mu = \frac{1}{\bar{\varphi}} \sum_{j=n+1}^\infty \mu_0(\varphi > j) \int v \, d\mu \int w \, d\mu + E_n,
$$

where $|E_n| \leq C\|v\|_\theta^* \|w\|_\infty^* d_n$, for some $C > 0$ and

$$
d_n := \begin{cases}
n^{-1/\alpha} & \text{if } \alpha < 2; \\
n^{-2}\log n & \text{if } \alpha = 1/2; \\
n^{-(2/\alpha - 2)} & \text{if } \alpha > 2.
\end{cases}
$$

Notation For $a_n, b_n > 0$ we write $a_n = O(b_n)$ or $a_n \ll b_n$ if there is a constant $C > 0$ such that $a_n/b_n \leq C$ for all $n \geq 1$. We write $a_n \sim b_n$ if $\lim_n a_n/b_n = 1$, $a_n = o(b_n)$ if $\lim_n a_n/b_n = 0$ and $a_n \approx b_n$ if $C^{-1} \leq a_n/b_n \leq C$ for some $C > 0$ and all $n \geq 1$.

2 General background on semistable laws

Semistable laws are limits of centred and normed sums of i.i.d. random variables along subsequences k_n for which

$$
k_n < k_{n+1}, n \geq 1 \text{ and } \lim_{n \to \infty} \frac{k_{n+1}}{k_n} = c > 1.
\tag{2.1}
$$

Since $c = 1$ corresponds to the stable case ([28, Theorem 2]), we recall the background in the case $c > 1$. The simplest such a sequence is $k_n = \lfloor c^n \rfloor$.

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with distribution function $F(x) = \mathbb{P}(X \leq x)$. Given a semistable random variable V with distribution function $G(x) = \mathbb{P}(V \leq x)$, as in [28], we say that the random variable X belongs to the domain of geometric partial attraction of the semistable law G with index $\beta \in (0, 2)$ if there is a subsequence k_n for which (2.1) holds, and a norming and a centring sequence A_n, B_n, such that

$$\frac{\sum_{i=1}^{k_n} X_i}{A_{k_n}} - B_{k_n} \to^d V. \tag{2.2}$$

Without loss of generality we may assume that $A_n = n^{1/\beta} \ell_1(n)$, $\beta \in (0, 2)$, with some slowly varying function ℓ_1 (see [28, Theorem 3]). Also, B_{k_n} is so that $\lim_{r \to \infty} B_{k_n}/k_n$ exists. For the form and properties of the characteristic function of the random variable V we refer to [26, 10].

In order to characterise the domain of geometric partial attraction we need some further definitions. As $k_{n+1}/k_n \to c > 1$, for any x large enough there is a unique k_n such that $A_{k_n} \leq x < A_{k_{n+1}}$.

Let $\delta(x) = \frac{x}{A_{k_n}}$ and note that the definition of δ does depend on the norming sequence. Let $x^{-\beta}\ell(x) := \sup\{t : t^{-1/\beta}\ell_1(1/t) > x\}$ and note that $x^{1/\beta}\ell_1(x)$ and $y^\beta/\ell(y)$ are asymptotic inverses of each other. For properties of asymptotic inverse of regularly varying functions we refer to [4, Section 1.7].

By Corollary 3 in [28], (2.2) holds on the subsequence k_n with norming sequence A_{k_n} if and only if

$$\overline{F}(x) := 1 - F(x) = \frac{\ell(x)}{x^\beta}[M(\delta(x)) + h(x)], \tag{2.3}$$

where M is a log periodic function with period $c^{1/\beta}$ with c given in (2.1) (that is, $M(c^{1/\beta}x) = M(x)$) and h is right-continuous error function such that $\lim_{n \to \infty} h(A_{k_n}x) = 0$, whenever x is a continuity point of M. Moreover, if M is continuous, then $\lim_{x \to \infty} h(x) = 0$. With the parameters β and c specified, throughout the rest of the paper, V in (2.2) will be referred to as $V_\beta(c)$ and we mean a random variable distributed according to a semistable distribution of index β and parameter c. As recalled below, on different subsequences there are different limit distributions belonging to the class of semistable law.

We say that u_n *converges circularly* to $u \in (c^{-1}, 1]$, $u_n \overset{cir}{\to} u$, if $u \in (c^{-1}, 1)$ and $u_n \to u$ in the usual sense, or $u = 1$ and u_n has limit points 1, or c^{-1}, or both.

For $x > 0$ (large) define

$$\gamma_x = \frac{x}{k_n}, \quad \text{where } k_{n-1} < x \leq k_n. \tag{2.4}$$

and note that by (2.1)

$$c^{-1} = \liminf_{x \to \infty} \gamma_x < \limsup_{x \to \infty} \gamma_x = 1.$$

The definitions of the parameter γ_n and the circular convergence follow the definitions in [24, p. 774 and 776], and are slightly different from those in [28].

It follows from Theorem 1 [8] that (2.2) holds along a subsequence $(n_r)_{r=1}^{\infty}$ (instead of k_n) if and only if $\gamma_{n_r} \overset{cir}{\to} \lambda \in (c^{-1}, 1]$ as $r \to \infty$. More precisely, whenever $\gamma_{n_r} \overset{cir}{\to} \lambda$,

$$\frac{\sum_{i=1}^{n_r} X_i - B_{n_r}}{n_r^{1/\beta} \ell_1(n_r)} \to^d V_\lambda \quad \text{as } r \to \infty, \tag{2.5}$$

where B_{n_r} is so that $\lim_{r \to \infty} B_{n_r}/n_r$ exists and V_λ is a semistable random variable with distribution and characteristic function depending on λ. For the precise form of the characteristic function of V_λ we refer to [8, 23].

3 Abstract setup

3.1 Main assumptions

Let $f : X \to X$, for X a Polish space. We require that there exists $Y \subset X$ and a *general* (not necessarily first) return time $\varphi : Y \to \mathbb{N}$ such that the return map $F := f^\varphi : Y \to Y$ preserving the measure μ_Y is Gibbs Markov, as recalled below (see [1, Chapter 4] for further details). For convenience we rescale such that $\mu_Y(Y) = 1$ and assume that F has a Markov partition $\alpha = \{a\}$ such that $\varphi|_a$ is constant on each partition element so that $F : a \to Y$ is a bijection $\text{mod } \mu_Y$. Throughout, we assume that that the greatest common divisor $\gcd(\varphi(a), a \in \alpha) = 1$.

Let α_n, $n \geq 1$ be the n-th refined partition and let $p_n = \log \frac{d\mu}{d\mu \circ F^n}$ be the corresponding potential. We assume that there is $\theta \in (0, 1)$ and $C_1 > 0$ such that for all $n \geq 1$,

$$e^{p_n(y)} \leq C_1 \mu_Y(a), \qquad |e^{p_n(y)} - e^{p_n(y')}| \leq C_1 \mu_Y(a) \theta^{s(F^n y, F^n y')}, \tag{3.1}$$

for all $y, y' \in a$, $a \in \alpha_n$, where $s(y_1, y_2) = \min\{m \geq 0 : F^m(y_1) \text{ and } F^m(y_2)$ belong to different elements of $\alpha\}$ is the *separation time*. We also assume that $\inf_{a \in \alpha} \mu_Y(Fa) > 0$ (big image property).

Throughout we let $\tau : Y \to \mathbb{N}$ be the first return time to Y and write $\{\tau > n\} := \{y \in Y : \tau(y) > n\}$ and the same for $\{\varphi > n\}$. We assume that

(H0) $\mu_Y(\varphi > n) = O(n^{-\beta} \ell(n))$ for $\beta > 1$ and some slowly varying function ℓ.

We define φ in terms of consecutive first returns to Y. Since τ_k is the k-th return time to Y, *i.e.*, $\tau_0 = 0$, $\tau_{k+1}(y) = \tau_k(y) + \tau(f^{\tau_k(y)}(y))$ and we let ρ be the *reinduced time* for the general return, *i.e.*, $\varphi(y) = \tau_{\rho(y)}(y)$. As in [7], lower bounds on the correlation decay require that

(H1) $\int_{\{\varphi > n\}} \rho \, d\mu_Y = O(\mu_Y(\varphi > n))$.

Assumption (H1) will be used for simpler arguments in Subsection 3.5.

Remark 3.1 The first return time τ may be defined on a larger set than where the general return time φ is defined, but in our main example the difference in domains has measure zero, so we will ignore it.

For the purpose of simpler arguments in Subsection 3.5, we also require the following mild condition on the inducing scheme.

(H2) Either $f^i(a) \subset Y$ or $f^i(a) \cap Y = \emptyset$ for all $a \in \alpha, 0 \leq i < \varphi(a)$,

In order to obtain a non- Gaussian limit law for f_Δ (and thus for f), we assume that τ behaves according to (2.3).

(H3) Suppose that (H0) holds with $\beta \in (1, 2)$. Assume that

    i) there exists a sequence $k_n \to \infty$ so that $\lim_{n \to \infty} \frac{k_{n+1}}{k_n} = c > 1$;

    ii) there exists a right continuous, logarithmically periodic function $M : (0, \infty) \to (0, \infty)$ with period $c^{1/\beta}$, i.e. $M(c^{1/\beta}x) = M(x)$ for all $x > 0$.

    iii) Under i)-ii), suppose that for some slowly varying function ℓ,

$$\mu_Y(\tau > x) = x^{-\beta} \ell(x)(M(\delta(x)) + h(x)),$$

    where:

        •) $\delta(x) = \frac{x}{A_{k_n}}$, for $x \in [A_{k_n}, A_{k_{n+1}})$ with A_{k_n} defined by $A_n := n^{1/\beta} \ell_1(n)$, where ℓ_1 is slowly varying so that $n^{1/\beta} \ell_1(n)$ is the asymptotic inverse of $n^{\beta} \ell(n)^{-1}$.

        ••) h is some right-continuous function such that $\lim_{n \to \infty} h(A_{k_n} x) = 0$, whenever x is a continuity point of M.

Remark 3.2 The case $c = 1$ in (H3) corresponds to β-stable laws for f; it is well understood via the method in [30] (see, for instance, [31]) and omitted here.

(H4) Suppose that (H3) holds and assume that $\lim_{n \to \infty} \frac{\mu_Y(\varphi > n)}{\mu_Y(\tau > n)} = C$, for some $C > 0$.

3.2 Observables on X and Δ

The tower Δ is the disjoint union of sets $(\{\varphi = j\}, i)$, $j \geq 1$, $0 \leq i < j$ with tower map

$$f_\Delta(y, i) = \begin{cases} (y, i+1) & \text{if } 0 \leq i < \varphi(y) - 1, \\ (F(y), 0) & \text{if } i = \varphi(y) - 1. \end{cases}$$

This map preserves the measure μ_Δ defined as $\mu_\Delta(A, i) = \mu_Y(A)$ for every measurable set A, with $A \subset \{\varphi = j\}$ and $0 \leq i < j$.

Let $Y_i = \{(y, i) : \varphi(y) > i\}$ be the i-th level of the tower, so $Y = Y_0$ is the base. The restriction $\mu_\Delta|_Y = \mu_Y$ is invariant under f_Δ^φ, which is the first return map to the base. The function φ extends to the tower via $\varphi_\Delta(y, i) := \varphi(y) - i$.

Define $\pi : \Delta \mapsto X$ by $\pi(y, i) := f^i(y)$. The measure $\mu_X = \mu_\Delta \circ \pi^{-1}$ is f-invariant, and μ_X is related to the F-invariant measure μ_Y via $\mu_X(A) = \sum_{j=0}^{\infty} \mu_Y(f^{-j}A \cap \{\varphi > j\})$. Regardless of whether $\bar{\varphi} := \int_Y \varphi \, d\mu_Y$ is finite (in which case we can normalise μ_X) or not, μ_Y is absolutely continuous w.r.t. μ_X.

Let g_X be an observable supported on the original space X; it lifts to an observable on the tower $g := g_X \circ \pi$. In what follows we use the method in [15, 17] to study limit theorems for f and observables g_X supported on X via limit theorems via f_Δ and g.

Given $g = g_X \circ \pi$, let $g_Y = \sum_{j=0}^{\varphi-1} g \circ f_\Delta^j$ be its induced version (to the base of the tower). We assume

(H5) $\mu_Y(|g_Y| > t) = C\mu_Y(\varphi > t)(1 + o(1))$ as $t \to \infty$ for some $C > 0$.

3.3 Results in the abstract setup

Our main result in the abstract setup is on limit laws for f and observables supported on X.

Theorem 3.3 *Assume (H0)–(H3) with $\beta \in (1,2)$. Let $g_X : X \to \mathbb{R}$ be a bounded observable on X and assume that its induced version g_Y (to the base of the tower) satifies (H5).*

Let k_n be a subsequence satisfying (H3) i) and let $A_n := n^{1/\beta}\ell_1(n)$ be defined in (H3) iii). Then there exists a subsequence n_r of k_n so that

$$\frac{\sum_{j=0}^{n_r-1} g_X \circ f^j - B_{n_r} \int_X g_X \, d\mu}{A_{n_r}} \to^d W \text{ as } r \to \infty,$$

for some random variable W with a non trivial distribution and some centering sequence B_{n_r} is so that $\lim_{r\to\infty} B_{n_r}/n_r$ exists.

Moreover, if (H4) holds, the following holds for some centering sequence B_k is so that $\lim_{n\to\infty} B_k/k$ exists:

  i) *There is a sequence k_n such that $\frac{\sum_{j=0}^{k_n-1} g_X \circ f^j - B_{k_n} \int_X g_X \, d\mu}{A_{k_n}} \to^d V_\beta(c)$, as $n \to \infty$.*

  ii) *Given γ_x as in (2.4), whenever $\gamma_{n_r} \overset{cir}{\to} \lambda \in (c^{-1}, 1]$, there is a sequence n_r such that $\frac{\sum_{i=1}^{n_r} g_X \circ f^i - B_{n_r} \int_X g_X \, d\mu}{A_{n_r}} \to^d V_\lambda$, as $r \to \infty$, where V_λ is as in (2.5).*

3.4 Transfer operators for F, f and f_Δ

Let P be the transfer operator associated with the tower map f_Δ and potential

$$p_\Delta(y, i) := \begin{cases} 0 & \text{if } i < \varphi(y) - 1, \\ p(y) & \text{if } i = \varphi(y) - 1. \end{cases}$$

Throughout, we let $R : L^1(\mu_Y) \to L^1(\mu_Y)$ be the transfer operator associated with the base map $(F = f^\varphi, Y, \mathcal{A}, \mu_Y)$. Since F is Gibbs Markov, R has good spectral properties in the space $\mathcal{B}$ of bounded piecewise Hölder functions compactly embedded in $L^\infty(\mu_Y)$. The norm on $\mathcal{B}$ is defined by $\|v\| = |v|_\theta + |v|_\infty$, where $|v|_\theta = \sup_{a \in \mathcal{A}} \sup_{x \neq y \in a} |v(x) - v(y)|/d_\theta(x,y)$, where $d_\theta(x,y) = \theta^{s(x,y)}$ for some $\theta \in (0,1)$, and

$$s(x,y) = \min\{n : F^n(x) \text{ and } F^n(y) \text{ are in different elements of } \mathcal{A}\}$$

is the separation time. We recall that 1 is an isolated eigenvalue in the spectrum of R and that $\|R(1_{\{\varphi=n\}})\| \ll \mu_Y(\varphi = n)$ (see [2] and [32]). Throughout we let Π be the eigenprojection associated with the eigenvalue 1; in particular, $\Pi v = \int_Y v \, d\mu_Y$.

Let $g : \Delta \to \mathbb{R}$ and let g_Y be its induced version on Y. Let $s \geq 0$ and define the following transfer operators that describe the general resp. the first return to the base Y:

$$T_{n,s}v := 1_Y P^n(1_Y e^{-sg_n} v), \quad n \geq 0;$$
$$R_{n,s}v := 1_Y P^n(1_{\{\varphi=n\}} e^{-sg_n} v) = R(1_{\{\varphi=n\}} e^{-sg_Y} v), \quad n \geq 1.$$

As in [32, 16, 17], $T_{n,s} = \sum_{j=1}^n R_{j,s} T_{n-j,s}$. Note that $\sum_n R_{n,0} = R$. Also given $R(z,s)v := R(z^\varphi e^{-sg_Y} v)$ we have $R(z,s)v = \sum_{n\geq 1} R_{n,s} z^n$. As shown in [32], $\|R_{n,0}\| \ll \mu_Y(\varphi = n)$.

Since $\|R_{n,s} - R_n\| \to 0$ as $s \to 0$ and $\|R_{n,s}\| \ll \mu_Y(\varphi = n)$, [16, Theorem 2.1] ensures that: i) $R(z,s)$ is a continuous perturbation of $R(z)$ and ii) since $R(z)$ is close to $R(1)$, we have that $R(z,s)$ is close to $R(1,0)$ for z sufficiently close to 1 and s small enough. Thus, the operator $R(z,s)$ has an eigenvalue $\lambda(1,0) = 1$, which is isolated in the spectrum of $R(1,0) = R$

and the family of eigenvalue $\lambda(z, s)$ is continuous in a neighborhood of $(1, 0)$. As a consequence, there exist $\varepsilon_0 > 0$ so that $(I - R(z, s))^{-1}$ is well defined for all $z \in \bar{\mathbb{D}} \setminus \{1\}$ and $s \leq \varepsilon_0$ and $(I - R(z, s))^{-1} = \sum_{n \geq 0} T_{n,s} z^n$. As recalled in Lemma 3.4 below, the asymptotic behavior of $T_{n,s}$ as $n \to \infty$ for $s \leq \varepsilon_0$ can be understood as the n-th Fourier coefficient of $(I - R(z, s))^{-1}$.

Following [17], to understand the behaviour of $P_s^n v = P^n(e^{-sg_n} v)$ (for s small enough) when acting on functions supported on the whole of Δ via the behaviour of $T_{n,s}$ (acting on functions supported on Y), we need to define several operators that describe the action of P_s^n outside Y. Define the transfer operators associated with the end resp. beginning of an orbit on the tower as

$$A_{n,s} v := P^n(1_{\{\varphi > n\}} e^{-sg_n} v), \quad n \geq 0,$$

$$B_{n,s} v := \begin{cases} 1_Y P^n(1_{\{\varphi_\Delta = n\} \setminus Y} e^{-sg_n} v), & n \geq 1, \\ 1_Y v, & n = 0. \end{cases}$$

Also, define the transfer operator associated with orbits that do not see the base of the tower by

$$C_{n,s} v := P^n(1_{\{\varphi_\Delta > n\} \setminus Y} e^{-sg_n} v), \quad n \geq 0.$$

As noticed in [17], the following equation describes the relationship between $T_{n,s}$ and P_s^n:

$$P_s^n = \sum_{n_1 + n_2 + n_3 = n} A_{n_1,s} T_{n_2,s} B_{n_3,s} + C_{n,s}. \tag{3.2}$$

3.5 Relating limit laws for F to limit laws for f_Δ

Lemma 3.4 *[15, Theorem 2.1] Suppose that $\sum_{n \geq 1} \mu_Y(\varphi > n) < \infty$ and recall that $\bar{\varphi} = \int_Y \varphi \, d\mu_Y$. Assume that $1 - \lambda(1, s) = G(s)(1 + o(1))$ as $s \to 0$, where $G(s) : [0, \infty) \to [0, \infty)$ with $G(s) = 0$, for $s = 0$ and $G(s) > 0$, for $s > 0$. Then there exist $\varepsilon_0 > 0$ and two functions $q(s) \to 0$ as $s \to 0$, $r(n) \to 0$ as $n \to \infty$ so that for all $s \leq \varepsilon_0$ and all $n \geq 1$,*

$$\left\| T_{n,s} - \frac{1}{\bar{\varphi}} \left(1 - \frac{G(s)}{\bar{\varphi}} \right)^n \Pi \right\| \leq q(t) + r(n).$$

Proof This follows from [15, Theorem 2.1] in the setup of Young towers with summable returns to the base. As already recalled in subsection 3.4, the hypotheses of [15, Theorem 2.1] are satisfied. In particular, there exist $\varepsilon_0 > 0$ so that $(I - R(z, 0))$ is ivertible for all $z \in \bar{\mathbb{D}} \setminus \{1\}$ and for all $s \leq \varepsilon_0$.

Although the statement of [15, Theorem 2.1] is in terms of complex perturbations e^{itg} as opposed to e^{-sg} as used here and it is slightly different, the invertibility of $1 - \lambda(1, s)$ is all that is required in the proof of [15, Theorem 2.1]. Our assumption on $1 - \lambda(1, s)$ ensures that $1 - \lambda(1, s)$ is invertible for all $s \neq 0$. Hence, the argument of [15, Theorem 2.1] applies as summarized below, yielding the conclusion of the present lemma. We note that it is essential in the proof of [15, Theorem 2.1] that the first perturbation z^φ is controlled on $\bar{\mathbb{D}}$; the perturbation e^{-sg} can be controlled for $s \in \mathbb{C}$ or $s \in \mathbb{R}_+$ depending on the desired statement.

As in [16] (see also [15, Lemma 2.5]), $T_{n,0} = \frac{1}{\bar{\varphi}} \Pi + E_n$, where $\|E_n\| \to 0$; in particular, one can fix N and $\varepsilon > 0$ so that $\|E_n\| \leq \varepsilon$ for all $n > N$. It follows from [15, Proof of Proposition 2.10] that the function $r(n)$ in the statement of the present lemma is so that $r(n) \leq \varepsilon + C_0 N G(s) \left(1 - \right.$

$\frac{G(s)}{\bar\varphi}\Big)^{n-N} \le C_1\varepsilon$, for some $C_0, C_1 > 0$ as $n \to \infty$. Also, it follows from [15, Proof of Proposition 2.10] that the function $q(t)$ in the statement of the present lemma is so that $q(s) \le \sum_{j=0}^n \|F_j(s)\|$, where $F_n(s)$ is the n-th coefficient of $\left(\frac{I-R(z,s)}{1-\bar\varphi^{-1}(1-\bar\varphi^{-1}G(s)z)}\right)^{-1} - \left(\frac{I-R(z,0)}{1-z}\right)^{-1}$. By [15, Lemma 2.5], $\sum_n \|F_n(s)\| \le CF(s)$, for some $C > 0$ and $F(s) \to 0$, as $s \to 0$. Hence, $q(s) \to 0$ as $s \to 0$. ∎

As in [16, 15]) we use Lemma 3.4 and (3.2) to understand the asymptotic behaviour of $\int_\Delta P_s^n 1 \, d\mu_\Delta = \mathbb{E}_{\mu_\Delta}(e^{-sg_n})$, which is the real Laplace transform of the observable g. Provided that there exists some sequence $a_n \to \infty$ so that $\frac{1}{\bar\varphi}\left(1 - \frac{G(s/a_n)}{\bar\varphi}\right)^n$ can be related to a real Laplace transform $\hat{W}(s)$ of a non trivial distribution W, we will infer that $\mu_Y(\frac{g_n}{a_n} < a)$ can be related to $\mathbb{P}(X < a)$, where X is distributed according to W.

We recall the estimates below to be used in obtaining the asymptotic of $\int_\Delta P_s^n 1 \, d\mu_\Delta$ (as in Proposition 3.7 below).

Lemma 3.5 *Let $v : \Delta \to \mathbb{R}$ so that $v \in L^\infty(\mu_\Delta)$. There exists $C > 0$ so that for all $n \ge 0$ and for all $s \ge 0$, $|\int_\Delta C_{n,s} v \, d\mu_\Delta| \le C\mu_Y(\varphi > n)\|v\|_{L^\infty(\mu_\Delta)}$.*

Assume (H1) and suppose that v is Hölder on each level of Δ with $v \in L^\infty(\mu_\Delta)$. There exists $C > 0$ so that for all $n \ge 0$ and for all $s \ge 0$, $\sum_{j>n} \|B_{j,0}(e^{-sg_j} - 1)v\| \le C\mu_Y(\varphi > n)\|v\|_{L^\infty(\mu_\Delta)}$. Moreover, $\int_\Delta B(1,0)v \, d\mu_\Delta = \int_\Delta v \, d\mu_\Delta$.

Proof The statement on $C_{n,s}$ is immediate. For the statement on $B_{n,s}$ under (H1) see, for instance, [7, Lemma 6.5]. For a comparable estimate, exploiting $s \to 0$, without assuming (H1) we refer to [17, Lemmas 4.2 and 4.3].For the statement on the integral, see [17, Lemma 4.3]. ∎

Lemma 3.6 *Let $v, w : \Delta \to \mathbb{R}$ so that $\|(1_Y v)\|_{L^\infty(\mu_Y)} < \infty$, $w \in L^\infty(\mu_\Delta)$. Assume (H1) and (H2). Then there exists $C > 0$ so that for all $n \ge 0$ and for all $s \ge 0$,*

$$\Big|\int_\Delta \sum_{j \ge n} A_{j,0}(e^{-sg_j}-1)(1_Y v)w \, d\mu_\Delta\Big| \le C\mu_Y(\varphi > n)\|(1_Y v)\|_{L^\infty(\mu_Y)}\|w\|_{L^\infty(\mu_\Delta)}.$$

Proof This is a simplified version of [7, Lemma 6.2]. For a comparable estimate, exploiting $s \to 0$, without assuming (H1), we refer to [17, Lemmas 4.2]. ∎

We can now state

Proposition 3.7 *Assume (H1) and (H2). Assume that $1 - \lambda(1, s) = G(s)(1 + o(1))$ as $s \to 0$, where $G(s) : (0, \infty) \to (0, \infty)$ with $G(s) = 0$, for $s = 0$ and $G(s) > 0$, for $s > 0$. Then there exist $\varepsilon_0 > 0$ and two functions $q_0(s) \to 0$ as $s \to 0$, $r_0(n) \to 0$ as $n \to \infty$ so that for all $s \le \varepsilon_0$ and all $n \ge 1$,*

$$\Big|\int_\Delta P_s^n 1 \, d\mu_\Delta - \frac{1}{\bar\varphi}\Big(1 - \frac{G(s)}{\bar\varphi}\Big)^n\Big| \le q_0(s) + r_0(n).$$

Proof For $z \in \bar{\mathbb{D}}$, define the operator power series $P(z, s) = \sum_{n \ge 0} P_s^n z^n$ and similarly for $A(s, z), T(s, z), B(s, z), C(s, z)$. By (3.2), we have $P(z, s) = A(z, s)T(z, s)B(z, s) + C(z, s)$, and so,

$$\int_\Delta P(z, s) 1 \, d\mu_\Delta = \int_\Delta A(z, s)T(z, s)B(z, s) 1 \, d\mu_\Delta + \int_\Delta C(z, s) 1 \, d\mu_\Delta.$$

By Lemmas 3.6 and 3.5, both terms of the RHS are well defined for $z \in \bar{\mathbb{D}} \setminus \{1\}$ and $s \geq 0$.

Let

$$I_A(z, s) = \int_\Delta \frac{A(z, s) - A(1, s)}{z - 1}(z - 1)T(z, s)B(z, s)1 \, d\mu_\Delta,$$

$$I_B(z, s) = \int_\Delta A(1, s)(z - 1)T(z, s)\frac{B(z, s) - B(1, s)}{z - 1}1 \, d\mu_\Delta,$$

and

$$I_T(z, s) = \int_\Delta A(1, s)(z - 1)T(z, s)B(1, s)1 \, d\mu_\Delta.$$

Thus, we can write

$$\int_\Delta P(z, s)1 \, d\mu_\Delta = I_T(z, s) + I_A(z, s) + I_B(z, s) + \int_\Delta C(z, s)1 \, d\mu_\Delta.$$

By Lemma 3.5, the n-th coefficient of $\int_\Delta C(z, s)1 \, d\mu_\Delta$ is $O(\mu_Y(\varphi > n))$. By Lemma 3.4, $(z - 1)T(z, s)v = \frac{1}{\bar{\varphi}}\sum_n \left(1 - \frac{G(s)}{\bar{\varphi}}\right)^n z^n \Pi v + \sum_n (q(s) + r(n))z^n$, where $\|q(s)\| \to 0$ as $s \to 0$ and $\|r(n)\| \to 0$ as $n \to \infty$. Using this, we estimate the n-th coefficients of $I_T(z, s)$, $I_A(z, s)$ and $I_B(z, s)$.

Note that $\int_\Delta A(1, s)\Pi B(1, s)1 \, d\mu_\Delta = \int_\Delta A(1, s)1_Y \, d\mu_\Delta \int_\Delta B(1, s)1 \, d\mu_\Delta$. By Lemma 3.5 and lemma 3.6, as $s \to 0$, $\int_\Delta (A(1, s) - A(1, 0))1 \, d\mu_\Delta \to 0$ and $\int_\Delta (B(1, s) - B(1, 1))1 \, d\mu_\Delta \to 0$. Thus, $K(s) := \int_\Delta (A(1, s) - A(1, 0))1_Y \, d\mu_\Delta \int_\Delta (B(1, s) - B(1, 0))1_Y \, d\mu_\Delta \to 0$ as $s \to 0$. Putting these together,

$$I_T(z, s) - \frac{1}{\bar{\varphi}}\sum_n \left(1 - \frac{G(s)}{\bar{\varphi}}\right)^n z^n = \frac{1}{\bar{\varphi}}\sum_n \left(1 - \frac{G(s)}{\bar{\varphi}}\right)^n K(s)z^n$$

$$+ \sum_n \left(\int_\Delta A(1, s)(q(s) + r(n))B(1, s)1 \, d\mu_\Delta\right)z^n.$$

Since $K(s) \to 0$ as $s \to 0$, the n-th coefficient of the first term of the RHS of the previous displayed equality goes to 0, as $s \to 0$. The coefficient of the second term is bounded by $q(s) + r(n)$, as desired.

Next, compute that

$$I_A(z, s) = \int_\Delta \frac{A(z, s) - A(1, s)}{z - 1}\Pi(B(z, s)1) \, d\mu_\Delta$$

$$+ \int_\Delta \frac{A(z, s) - A(1, s)}{z - 1}\sum_n (q(s) + r(n))z^n \, B(z, s)1 \, d\mu_\Delta := I_A^1(z, s) + I_A^2(z, s).$$

Recall that $B(z, s)1$ is a function supported on Y and note that

$$I_A^1(z, s) = \int_\Delta \frac{A(z, s) - A(1, s)}{z - 1}1_Y \, d\mu_\Delta \int_\Delta \frac{B(z, s) - B(1, s)}{z - 1}1 \, d\mu_\Delta.$$

By Lemma 3.6, the n-th coefficient of the first factor is $O(\mu_Y(\varphi > n))$. By Lemma 3.5, the n-th coefficient of the second factor is $O(\mu_Y(\varphi > n))$. Hence, the n-th coefficient of the $I_A^1(z, s)$ is $O(\mu_Y(\varphi > n))$.

The n-th coefficient of $I_A^2(z, s)$ is bounded by the convolution of $\int_\Delta \frac{A(z,s)-A(1,s)}{z-1}1_Y \, d\mu_\Delta$ and $\sum_n (q(s) + r(n))z^n \, B(z, s)1$. Since $\|B_{n,s}1\| = O(\mu_Y(\varphi > n))$, the n-th coefficient, in norm, of $\sum_n (q(s) + r(n))z^n \, B(z, s)1$ is $O(n\mu_Y(\varphi > n)) = O(n^{-(\beta-1)}\ell(n))$. The n-th coefficient of $I_A^2(z, s)$ is $O(n^{-(\beta-1)}\ell(n))$.

Finally,

$$I_B(z,s) = \int_\Delta A(1,s)\Pi(B(z,s)1)\,d\mu_\Delta + \int_\Delta A(1,s)\sum_n (q(s)+r(n))z^n\,\frac{B(z,s)-B(1,s)}{z-1}1\,d\mu_\Delta.$$

The n-th coefficient of first term in the previous displayed equation is $O(\|B_{n,s}1\|) = O(\mu_Y(\varphi > n))$. By Lemma 3.5, the n-th coefficient of the second term in the previous displayed equation is $O(n\sum_{j>n}\|B_{j,0}(e^{-sg_j}-1)1\|) = O(n\mu_Y(\varphi > n)) = O(n^{-(\beta-1)}\ell(n))$.

The conclusion follows taking $q_0(s) = K(s) + r(s)$ and $r_0(n) = r(n) + O(n^{-(\beta-1)}\ell(n))$. ∎

3.6   Proof of Theorem 3.3

Recall that $\lambda(1,s), s \geq 0$ is the family of eigenvalues for $R(1,s)v = R(e^{-sg_Y}v)$. In order to apply Proposition 3.7 in the proof of Theorem 3.3, we need to understand the asymptotics of $1 - \lambda(1,s)$ as $s \to 0$.

Throughout this section we assume that g_Y satisfies (H5). By adding a positive constant to the bounded function g_X, we can assume without loss of generality that g_X is a positive and restrict the proofs below to the case $g_Y : Y \to \mathbb{R}_+$. With this convention, we state

Lemma 3.8 *Assume (H0) and (H3). Let $g_X : X \to \mathbb{R}_+$ be a bounded obervable and suppose that $\mu_Y(g_Y > t) = C\mu_Y(\varphi > t)(1 + o(1))$ as $t \to \infty$.*

Let $(Z_j)_{j\geq 1}$ be positive i.i.d. random variables so that $\mathbb{P}(Z_1 > t) = \mu_Y(g_Y > t)(1 + o(1))$ as $t \to \infty$. Let $(X_j)_{j\geq 1}$ be positive i.i.d. random variables so that $\mathbb{P}(X_1 > t) = (\bar\varphi)^{-1}\mu_Y(\tau > t)(1 + o(1))$, as $t \to \infty$.

For $s \geq 0$, set $G(s) = 1 - \mathbb{E}_\mathbb{P}(e^{-sZ_1})$ and $\tilde{G}(s) = 1 - \mathbb{E}_\mathbb{P}(e^{-sX_1})$. Then

  i) *for all $s > 0$, $\tilde{G}(s) > 0$ and $\tilde{G}(s) \leq (\bar\varphi)^{-1}G(s) \leq C\tilde{G}(s)$, for some $C \geq 1$;*

  ii) *$1 - \lambda(1,s) = G(s)(1 + o(1))$ as $s \to 0$.*

Moreover, if (H4) holds then $G(s) = C\bar\varphi\tilde{G}(s)(1 + o(1))$, for $C > 0$ as in (H4).

Proof Let $v(1,s)$ be the family of eigenvectors associated with $\lambda(1,s)$ and let $v(1,0)$ be the normalised eigenvector so that $\int_Y v(1,0)\,d\mu_Y = 1$. As in [2, 15], write

$$1-\lambda(1,s) = \int_Y (1-e^{-sg_Y})v(1,0)\,d\mu_Y + \int_Y (R(1,s)-R(1,0))(v(1,s)-v(1,0))\,d\mu_Y.$$

Since g_X is bounded, so is $g = g_X \circ \pi$ and thus, $g_Y \ll n$ on $\{\varphi = n\}$. Since we also know that $\mu_Y(g_Y > t) = C\mu_Y(\varphi > t)(1 + o(1)) \ll t^{-\beta}$, a direct computation (see, for instance, [17, Lemma 4,1] with $z = 1$ there) shows that

$$\|R(1,s) - R(1,0)\| \ll \sum_n \|R(1_{\{\varphi=n\}}(e^{-sg_Y}-1))\| \ll s^\beta\ell(1/s).$$

Hence, $V(s) := |\int_Y (R(1,s) - R(1,0))(v(1,s) - v(1,0))\,d\mu_Y| \ll s^{2\beta}\ell(1/s)^2$.

Given that $(Z_j)_{j\geq 1}$ are i.i.d. random variables as in the statement of the lemma,

$$\int_Y (1 - e^{-sg_Y})v(1,0)\,d\mu_Y = 1 - \mathbb{E}_\mathbb{P}(e^{-sZ_1}) = G(s).$$

It remains to estimate $G(s)$. By (H0) and (H3), $1 \leq \frac{\mu_Y(\varphi > n)}{\mu_Y(\tau > n)} \leq C$, for some $C > 1$. Recall that $\mathbb{P}(X_1 > n) = (\bar{\varphi})^{-1}\mu_Y(\tau > t)(1 + o(1))$ and $\tilde{G}(s) = 1 - \mathbb{E}_{\mathbb{P}}(e^{-sX_1})$. Clearly, $\tilde{G}(s) \leq \bar{\varphi}^{-1}G(s) \leq C\tilde{G}(s)$ and $G(s) = C\bar{\varphi}\tilde{G}(s)(1 + o(1))$ if (H4) holds.

Since X_1 satisfies (H3), [26, Theorem 1] (see also [10, Lemma 1]) ensures that there exist $C_2 > C_1 > 0$ so that $C_1 s^{\beta}\ell(1/s) \leq \tilde{G}(s) + s\mathbb{E}_{\mathbb{P}}(X_1) \leq C_2 s^{\beta}\ell(1/s)$; the analysis in [26, 10] is in terms of Fourier transforms and carries over with simplified arguments to real Laplace transforms, as required for $\tilde{G}(s)$. In particular, given $C_0 := \inf_{x>0} M(x) > 0$, we have

$$\tilde{G}(s) + s\mathbb{E}_{\mathbb{P}}(X_1) = \int_0^{\infty} (1 - e^{-sx} + sx)\,d\mathbb{P}(X_1 > x) > \int_{1/s}^{\infty} (1 - e^{-sx})\mathbb{P}(X_1 > x)\,dx$$

$$> Cs^{\beta}\int_1^{\infty} e^{-\sigma}\sigma^{-\beta}M(\sigma/s)\ell(\sigma/s)\,d\sigma > CC_0 s^{\beta}\ell(1/s)\int_1^{\infty} e^{-\sigma}\sigma^{-(\beta-\varepsilon)}\,d\sigma,$$

where in the inequality we have used Potter's bounds [4]. Since the integral is well defined, the claim for the inf follows and the reverse inequality follows similarly with sup instead of inf. It follows that $G(s) > 0$, for all $s > 0$ and $1 - \lambda(1, s) = G(s) + V(s) = G(s)(1 + O(s^{\beta}\ell(1/s)))$. ∎

Proof of Theorem 3.3 Given k_n as in (H3), g_X as in the statement of Theorem 3.3 and $g = g_X \circ \pi$, set $S_{k_n}(g_X) = \sum_{j=0}^{k_n-1} g_X \circ f^j$ and $S_{k_n}(g) = \sum_{j=0}^{k_n-1} g \circ f_{\Delta}^j$.

Let A_{k_n} as in (H3) and note that $\mathbb{E}_{\mu_X}(e^{-sA_{k_n}^{-1}S_{k_n}(g_X)}) = \mathbb{E}_{\mu_{\Delta}}(e^{-sA_{k_n}^{-1}S_{k_n}(g)})$. By Lemma 3.8, the assumption on $1 - \lambda(1, s)$ in statements of Proposition 3.7 is satisfied. It follows from Proposition 3.7 that as $n \to \infty$ and $s \to 0$,

$$\mathbb{E}_{\mu_{\Delta}}(e^{-sA_{k_n}^{-1}S_{k_n}(g)}) - \frac{1}{\bar{\varphi}}\left(1 - \frac{G(sA_{k_n}^{-1})}{\bar{\varphi}}\right)^{k_n} \to 0, \tag{3.3}$$

for $G(s) = 1 - \mathbb{E}_{\mathbb{P}}(e^{-sZ_1})$, where Z_1 is distributed like φ. By Lemma 3.8,

$$\left(1 - \tilde{G}(A_{k_n}^{-1}s)\right)^{k_n} \leq \left(1 - \frac{G(sA_{k_n}^{-1})}{\bar{\varphi}}\right)^{k_n} \leq C\left(1 - \tilde{G}(A_{k_n}^{-1}s)\right)^{k_n}, \tag{3.4}$$

for $\tilde{G}(s) = 1 - \mathbb{E}_{\mathbb{P}}(e^{-sX_1})$, where X_1 is distributed like τ and for some $C > 1$.

Given $(X_j)_{j \geq 1}$ i.i.d. random variables as in the statement of Lemma 3.8, [28, Corollary 3] ensures that for a suitable centering sequence B_{k_n},

$$A_{k_n}^{-1}\left(\sum_{j=0}^{k_n-1} X_i - B_{k_n}\right) \to^d V_{\beta}(c),$$

where $V_{\beta}(c)$ is a semistable random variable as recalled in Section 2 (in particular, see (2.3)). Multiplying equation (3.4) by $e^{-sB_{k_n}A_{k_n}^{-1}}$, we obtain

$$e^{-s\frac{B_{k_n}}{A_{k_n}}}\mathbb{E}_{\mathbb{P}}\left(\exp\left(-s\frac{\sum_{i=1}^{k_n} X_i - B_{k_n}}{A_{k_n}}\right)\right) \leq e^{-s\frac{B_{k_n}}{A_{k_n}}}\left(1 - \frac{G(sA_{k_n}^{-1})}{\bar{\varphi}}\right)^{k_n}$$

$$\leq Ce^{-s\frac{B_{k_n}}{A_{k_n}}}\mathbb{E}_{\mathbb{P}}\left(\exp\left(-s\frac{\sum_{i=1}^{k_n} X_i - B_{k_n}}{A_{k_n}}\right)\right) \tag{3.5}$$

Starting from (3.5) and extracting a subsequence n_r, we have that,

$$\frac{\mathbb{E}_{\mathbb{P}}(\exp(-sA_{n_r}^{-1}(\sum_{i=1}^{n_r} Z_i - B_{n_r})))}{\mathbb{E}_{\mathbb{P}}(\exp(-sA_{n_r}^{-1}(\sum_{i=1}^{n_r} X_i - B_{n_r})))} \to b \in (1, C) \text{ as } r \to \infty,$$

for some $b \in (1, C)$. This together with (3.3) implies that

$$\frac{\mathbb{E}_{\mathbb{P}}(\exp(-sA_{n_r}^{-1}(S_{n_r}(g) - B_{n_r}^*)))}{\mathbb{E}_{\mathbb{P}}(\exp(-s\log b\, A_{n_r}^{-1}(\sum_{i=1}^{n_r} X_i - B_{n_r}))) } \to 1 \text{ as } r \to \infty,$$

where $B_{n_r}^* = B_{n_r}^* - n_r \log b$. The previous displayed equation together with the continuity theorem for Laplace transforms (as in [13, Chapter XIII, Theorem 2 and Theorem 2(a)] yields

$$\lim_{r\to\infty} \sup_{y>0} \left| \mathbb{P}(A_{n_r}^{-1}(S_{n_r}(g) - B_{n_r}^*) > y) - \mathbb{P}(A_{n_r}^{-1}(\sum_{i=1}^{n_r} X_i - B_{n_r}) > y) \right| \to 0.$$

Since X_1 is distributed like τ and $\mu(\tau > n)$ satisfies (H3), the tail probability $\mathbb{P}(X_1 > n)$ satisfies (H3) along k_n. The merging result [8, Theorem 2] ensures that

$$\lim_{r\to\infty} \sup_{y>0} \left| \mathbb{P}(A_{n_r}^{-1}(\sum_{i=1}^{n_r} X_i - B_n) > y) - \mathbb{P}(V_{\gamma_{n_r}}^{\beta}(c) > y) \right| \to 0,$$

where the parameter γ_n is defined as in (2.4). Putting together the previous two displayed equations, we obtain

$$\lim_{r\to\infty} \sup_{y>0} \left| \mathbb{P}(A_{n_r}^{-1}(S_{n_r}(g) - B_{n_r}^*) > y) - \mathbb{P}(V_{\gamma_{n_r}}^{\beta}(c) > y) \right| \to 0.$$

As in [8, Section 1], the distribution $F_n(y) = \mathbb{P}(V_{\gamma_n}^{\beta}(c) < y)$ is stochastically compact; that is, every sequence contains a further subsequence r_n so that $F_{r_n}(y) \to G_W(y)$ as $n \to \infty$, for some random variable W with non trivial distribution G_W. Hence, $A_{n_r}^{-1}(S_{n_r}(g) - B_{n_r}) \to^d W$, where W has a non trivial distribution (although we cannot say that W is in the domain of a semistable law).

For the statement under (H4), we just need to recall that from Lemma 3.8 that $G(s) = C\bar{\varphi}\tilde{G}(s)(1 + o(1))$, for $C > 0$ as in (H4). In this case, [28, Corollary 3] applies directly to $(Z_j)_{j\geq 1}$ as in the statement of Lemma 3.8 ensuring that

$$A_{k_n}^{-1}(\sum_{i=1}^{k_n} Z_i - B_{k_n})$$

converges in distribution with respect to the measure μ_Y. This together with (3.3) gives the conclusion for the convergence in distribution with respect to the measure μ_Y of $A_{k_n}^{-1}(S_{k_n}(g) - B_{k_n}^1)$. The strong distributional convergence $A_{k_n}^{-1}(S_{k_n}(g) - B_{k_n}^1) \to^d V_\beta(c)$ follows from this together with [35, Proposition 4.1]. ∎

4 On the asymptotics of the tail of first return times τ for f_{M_1} and f_{M_2}, and verification of (H3)

Recall the wobbly maps f_{M_1}, f_{M_2} with M_1, M_2 introduced in in Subsection 1.1. In this section we prove the following tail estimates in the setup of f_{M_1} for the first return time $\tau(x) = \inf\{n \geq 1 : f_{M_1}^n(x) \in Y\}$, $x \in Y$, and $Y = [1/2, 1]$. The verification in the setup of f_{M_2} is similar. We show that there are functions $\tilde{M}, h, \delta$ and ℓ satisfying the hypothesis of Corollary 2 in [28] with

$$m_Y(\tau > y) = y^{-\beta}\ell(y)(\tilde{M}(\delta(y)) + h(y)), \text{ where } \beta = 1/\alpha. \quad (4.1)$$

In turn this allows us to verify condition (H3) for f_{M_1} along $k_n = \lfloor e^{n\alpha c_1} \rfloor$. A similar analysis holds for f_{M_2} (with c_2 instead of c_1) with a similar (but simpler) proof and we omit this.

To ease the notation, throughout this section we write f instead of f_{M_1}

Define below a monotone sequence $x_n \to 0$, such that $f^k(x_n) \in (0, 1/2)$ for all $k \leq n$, and $f^n(x_n) = 1/2$. This sequence (x_n) will allow us to study the set $\{x \in Y : \tau(x) = n\}$ for $Y = (\frac{1}{2}, 1]$. First consider the local behaviour of f near the critical set $\mathcal{C} = \{s_\ell = e^{-\ell c_1}, \ell \geq 1\}$. If $n \leq x < n+1$, then $\{x\} = x - n$. For ϵ small, consider the values $M(u_\epsilon^\pm)$, with $u_\epsilon^\pm = e^{-(\ell \pm \epsilon)c_1}$. We have:

$$M(u_\epsilon^+) = C_0 2^{-\{\ell + \epsilon\}} = C_0 2^{-\epsilon}, \quad M(u_\epsilon^-) = C_0 2^{-\{\ell - \epsilon\}} = C_0 2^{-1+\epsilon}.$$

At $\mathcal{C}$, let $s_\ell^\pm$ denote the usual upper $(+)$ and lower $(-)$ approaching limits to $s_\ell = e^{-\ell c_1}$. Then:

$$f(s_\ell^+) = s_\ell + C_0 s_\ell^{1+\alpha}, \quad f(s_\ell^-) = s_\ell + \frac{C_0}{2} s_\ell^{1+\alpha}.$$

Hence $f(s_\ell^+) > f(s_\ell^-)$, with jump size equal to $\frac{C_0}{2} s_\ell^{1+\alpha}$.

Let $x_0 = 1/2$, and define the sequence $x_n \to 0$, with the property that $f^k(x_n) \in (0, 1/2)$ for all $k \leq n$, and $f^n(x_n) = 1/2$. Due to the jumps in f at $\mathcal{C}$, the inverse map $f^{-1}(x)$ is not always defined, and so we cannot immediately set $x_n = f^{-n}(1/2)$. We begin (inductively) by setting $x_{n+1} = f^{-1}(x_n)$ in the case for which x_{n+1} is defined. This will certainly apply for small values of n. An issue arises in the case $x_n \in [f(s_\ell^-), f(s_\ell^+)]$ for some $s_\ell \in \mathcal{C}$. In this case we set $x_{n+1} = s_\ell$, and then continue iterating backwards. If $x_n \notin [f(s_\ell^-), f(s_\ell^+)]$ then again no issue arises and we set $x_{n+1} = f^{-1}(x_n)$. For this sequence we define $J_n = [x_{n+1}, x_n]$. By construction we have $\{\tau > n+1\} = f^{-1}([0, x_n]) \cap Y$.

We have the following proposition concerning the asymptotics of the sequence (x_n).

Proposition 4.1 *Consider the map f defined by* (1.2). *The sequence x_n satisfies*

$$x_n = \frac{1}{(\alpha M_0(n)n)^\beta} + O\left(\frac{\log n}{n^{1+\beta}}\right), \tag{4.2}$$

with $\beta = 1/\alpha$, and

$$M_0(n) = \frac{1}{n} \sum_{j=1}^{n} M(x_j). \tag{4.3}$$

Remark 4.2 Notice that $M_0(n)$ is the average of $M(x)$ along the sequence (x_j) up to iterate n. It is clear that $M_0(n) \in [C_0/2, C_0]$.

Proof In the case where $x_n \notin [f(s_\ell^-), f(s_\ell^+))$ then x_{n+1} is derived from $x_n = x_{n+1} + M(x_{n+1})x_{n+1}^{1+\alpha}$. If $x_n \in (f(s_\ell^-), f(s_\ell^+))$, then we set $s_\ell = x_{n+1}$ and this gives rise to an (absolute) error ξ_n given by:

$$\xi_n = \left| \frac{x_n - x_{n+1}}{x_{n+1}^{1+\alpha}} - M(x_{n+1}) \right|,$$

and this is bounded by

$$|M(s_\ell^+) - M(s_\ell^-)| s_\ell^{1+\alpha} \leq \frac{C_0}{2} s_\ell^{1+\alpha}.$$

Define

$$\widehat{M}_0(n) = M_0(n) + \frac{1}{n}\sum_{j=1}^n \xi_j,$$

with $\xi_j = 0$ in the case $x_j \notin (f(s_\ell^-), f(s_\ell^+))$. Then we have

$$x_n = x_{n+1} + [M(x_{n+1}) + \xi_{n+1}]x_{n+1}^{1+\alpha}.$$

To find the asymptotics, let $u_n = x_n^{-\alpha}$, and set $\widehat{M}(x_n) = M(x_n) + \xi_n$. Then

$$u_n = u_{n+1}\left(1 + \frac{\widehat{M}(x_n)}{u_{n+1}}\right)^{-\alpha}$$

$$= u_{n+1} - \alpha\widehat{M}(x_n) + \alpha(\alpha+1)\frac{\widehat{M}(x_n)^2}{2u_{n+1}}(1 + o(1)).$$

$$(4.4)$$

This leads to the asymptotic expansion:

$$u_n = u_0 + \sum_{j=0}^n \widehat{M}(x_j) + \alpha\sum_{j=0}^n \frac{\alpha(\alpha+1)\widehat{M}(x_j)^2}{2u_0 + 2\sum_{l=0}^j \widehat{M}(x_l)}(1 + o(1))$$

$$= u_0 + \alpha n\widehat{M}_0(n) + \sum_{j=0}^n \frac{\widehat{M}^2(u_j)}{u_0 + j\widehat{M}_0(j)}(1 + o(1)).$$

$$(4.5)$$

Since $\widehat{M}(x_n)$ is bounded, it follows that $u_n \in [\frac{n}{C_1}, C_1 n]$, for some $C_1 > 0$ depending only on α and c_1. Hence $x_n \in [\left(\frac{1}{C_1 n}\right)^\beta, \left(\frac{C_1}{n}\right)^\beta]$. These are rough bounds which we can now improve on. We claim that

$$\widehat{M}_0(n) = M_0(n) + O\left(\frac{\log n}{n}\right).$$

To show this we consider integers $j \le n$ for which $\xi_j \neq 0$. For such j, there exists $\ell \in \mathbb{N}$ with $x_j < e^{-\ell c_1} < x_{j-1}$. We claim that there can be no more than $2c_1^{-1}\log n$ such integers $j \le n$. Indeed if there are k such values of j, namely $j_1, \ldots, j_k$, then we must have $x_{j_k} < e^{-kc_1}$. This follows from the fact that the sequence (x_n) is strictly monotone decreasing, and if j_i is such that $x_{j_i} < e^{-\ell_i c_1} < x_{j_i-1}$, then we must have $x_{j_{i+1}} < e^{-(\ell_i+1)c_1}$, i.e. $\ell_{i+1} > \ell_i + 1$. However $x_n \in [\left(\frac{1}{C_1 n}\right)^{\frac{1}{\alpha}}, \left(\frac{C_1}{n}\right)^{\frac{1}{\alpha}}]$, and hence $k \le \beta c_1^{-1}\log n + C_2$, where $C_2 > 0$ is a uniform constant. The claim follows.

Returning to the asymptotic expansion for u_n in equation (4.5), we now have the refinement:

$$u_n = u_0 + \alpha n M_0(n) + O(\log n), \qquad (4.6)$$

where the third term on the right hand side of (4.5) is also $O(\log n)$ (via a harmonic series bound). Inverting for x_n gives:

$$x_n = (u_0 + \alpha n M_0(n) + O(\log n))^{-\frac{1}{\alpha}} = \frac{1}{(\alpha n M_0(n))^{\frac{1}{\alpha}}} + O\left(\frac{\log n}{n^{\frac{1}{\alpha}+1}}\right). \quad (4.7)$$

This completes the proof.

∎

Here and throughout let us extend M_0 to the positive reals so that $M_0(x) := M_0(\lfloor x \rfloor)$. We shall write $x_n = (\alpha n M_0(n))^{-1/\alpha} + n^{-1/\alpha}E_0(n)$, with $E_0(n) = o(1)$. We extend $E(x)$ to $x \in [0, \infty)$ via $E_0(x) := E_0(\lfloor x \rfloor)$.

We now examine further properties of the function M_0. To account for different versions of $M(x)$, we keep the argument general. To this end, let $p : \mathbb{R} \to \mathbb{R}$ be a piecewise Lipschitz periodic function of period c_1. Define $M(x) = a + bp(\log(x))$ where $a > 0$ and b is so small that $M(x)$ is bounded and bounded away from zero. Note that M is log-periodic with period e^{c_1}. Recall that $M_0(n) = \frac{1}{n} \sum_{j=1}^{n} M(x_j)$ is bounded and bounded away from zero. We have the following result:

Proposition 4.3 *Set $c = e^{\alpha c_1}$ and $w_k = \lfloor c^k \rfloor$. Then there exists $\zeta \in \mathbb{R}$ such that*

$$\lim_{k \to \infty} \frac{w_{k+1}}{w_k} = c \qquad \text{and} \qquad \lim_{k \to \infty} M_0(w_k) = \zeta. \qquad (4.8)$$

Proof The proof is based on the premise that M_0 is asymptotically log-periodic, where the "asymptotic" is faster than any power of the logarithm:

$$\left| \frac{M_0(cn)}{M_0(n)} - 1 \right| \le \varepsilon_n, \qquad \varepsilon_n = o((\log n)^{-r}), \qquad (4.9)$$

for any fixed $r > 0$. This implies that (4.8) holds for some sequence m_k so that $m_k \ge Cc^k$ for some $C > 0$ so that $\frac{m_{k+1}}{m_k} \to c$. In fact we'll show that (4.8) holds for $w_k = \lfloor c^k \rfloor$. Note that

$$|M_0(w_{k+1}) - M_0(w_k)| \le M_0(w_k) \left| \frac{M_0(cn)}{M_0(n)} - 1 \right| = O(\varepsilon_{w_k}) = O(\varepsilon_{\lfloor c^k \rfloor}).$$

Because $\varepsilon_n \to 0$ faster than any power of $1/\log n$, the terms $O(\varepsilon_{\lfloor c^k \rfloor})$ are summable in k, and therefore $(M_0(w_k))_{k \in \mathbb{N}}$ is a Cauchy sequence, and hence convergent.

The proof therefore relies on proving (4.9). We do this by replacing $M_0(n)$ by the solution of an integral equation. From equation (4.2), we have

$$x_n = \frac{1}{(\alpha M_0(n)n)^\beta} + O\left(\frac{\log n}{n^{1+\beta}} \right) = \frac{1 + \varepsilon_n}{(\alpha M_0(n)n)^\beta},$$

where $e_n = O(n^{-1} \log n)$. Hence, using the explicit form for $M(x)$, we have

$$M_0(n) = \frac{1}{n} \sum_{j=1}^{n} M(x_j) = \frac{1}{n} \sum_{j=1}^{n} a + bp\Big(\log((\alpha j M_0(j))^{-\beta}) + \log(1 + e_j)) \Big).$$

Since p is Lipschitz, $p(\log((\alpha j M_0(j))^{-\beta}) + \log(1+e_j))) - p(\log((\alpha j M_0(j))^{-\beta}) = O(e_j)$, which gives:

$$M_0(n) - M((\alpha j M_0(j))^{-\beta}) = \frac{1}{n} \sum_{j=1}^{n} O(e_j) = O\left(\frac{1}{n} \sum_{j=1}^{n} \frac{\log j}{j} \right) = O\left(\frac{(\log n)^2}{n} \right).$$
$$(4.10)$$

This estimate suggests to replace $M_0(n)$ by a continuous version $\overline{M}(x)$, defined implicitly by

$$\overline{M}(x) = \frac{1}{x} \int_{1}^{x} M((\alpha u \overline{M}(u))^{-\beta}) \, du. \qquad (4.11)$$

We show in the Lemma 4.4 that $\overline{M}$ is asymptotically log-periodic. To make the step back to $M_0(n)$, note that $H(x) := M((\alpha x M_0(x))^{-\beta})$ is not entirely log-periodic with period c, but still has $O(\log x)$ intervals of monotonicity

on $[0, x]$. On each such interval L, $|\int_L H(u)\,du - \sum_{j \in L \cap \mathbb{Z}} H(j)| \le \sup_L H - \inf_L H$. Therefore

$$\int_1^x H(u)\,du = \sum_{j=0}^x H(j) + O(\log x).$$

Since in our case, $H(x)$ is bounded away from zero, the sum dominates the $O(\log x)$ term.

It follows that if the integral is asymptotically log-periodic, so is the sum:

$$
\begin{aligned}
\frac{\frac{1}{cx} \sum_{j=1}^{cx} H(j)}{\frac{1}{x} \sum_{j=1}^{x} H(j)}
&= \frac{\frac{1}{cx}\left(\int_1^{cx} H(j) + O(\log cx)\right)}{\frac{1}{x}\left(\int_1^x H(j) + O(\log x)\right)} \\
&= \frac{\overline{H}(c)(1 + \frac{O(\log cx)}{cx\overline{H}(e^c x)})}{\overline{H}(x)(1 + \frac{O(\log x)}{x\overline{H}(x)})} \\
&= \frac{\overline{H}(c)}{\overline{H}(x)}\left(1 + \frac{O(\log cx)}{cx\overline{H}(cx)} + \frac{O(\log x)}{x\overline{H}(x)}\right).
\end{aligned}
$$

This error term is small enough for the purpose of (4.9). ■

Lemma 4.4 *If $M(x) = a(1 + bp(\log x))$, for $a > 0$ and $b > 0$ sufficiently small, then $\overline{M}$ satisfies (4.9) with $\varepsilon_n = O(1/n)$.*

Proof We seek an asymptotic solution of this integral equation (4.11): we will transform it to a differential equation. First let $V(x) = x\overline{M}(x)$. Then

$$V(x) = \int_1^x M((\alpha V(u))^{-\beta})\,du.$$

Differentiating with respect to x gives:

$$\frac{dV}{dx} = M((\alpha V(x))^{-\beta}) = a(1 + bp(-\beta \log V(x) - \beta \log \alpha)).$$

Now set $U(x) := \log V(x) \sim \log x$. Then

$$\frac{dU}{dx} e^U = a(1 + bp(-\beta U(x) + \log \alpha)).$$

Separating variables (as far as possible), and integrating gives:

$$g(U) := \int_0^U \frac{e^z}{a(1 + bp(-\beta z + \alpha_1))}\,dz = x + C^*,$$

with $\alpha_1 = -\beta \log \alpha$ and C^* an integration constant.

Now the left hand integral does not admit a closed form, but we can study $g(U)$ as follows: given U, choose $\ell = \lfloor \beta U / c_1 \rfloor$ and set $\hat{c} = c_1/\beta = c_1\alpha$ (so given c as in the statement, $c = e^{\hat{c}}$). Then

$$g(U) = \sum_{j=0}^{\ell-1} \int_{\hat{c}j}^{\hat{c}(j+1)} \frac{e^z}{a(1 + bp(-\beta z + \alpha_1))}\,dz + \int_{\hat{c}\ell}^U \frac{e^z}{a(1 + bp(-\beta z + \alpha_1))}\,dz.$$

For each of the integrals within the sum, let $z' = z - \hat{c}j$ be a new substitution variable. Then the periodicity of the denominator gives:

$$g(U) = \sum_{j=0}^{\ell-1} e^{j\hat{c}} g(\hat{c}) + e^{\ell\hat{c}} g(U \bmod \hat{c}) = \frac{e^{\ell\hat{c}} - 1}{e^{\hat{c}} - 1} g(\hat{c}) + e^{\ell\hat{c}} g(U \bmod \hat{c}). \quad (4.12)$$

Notice that $g(U \bmod \hat{c})$ is continuous on $[0, \hat{c})$ and bounded by $g(\hat{c})$. Thus the solution is $U(x) = g^{-1}(x + C^*)$ for some integration constant C^*. Now $\overline{M}(x) = e^{U(x)}x^{-1}$, and we want to show that

$$\lim_{x \to \infty} \frac{\overline{M}(e^{\hat{c}}x)}{\overline{M}(x)} = 1.$$

Thus asymptotic periodicity of $\overline{M}(x)$ corresponds to showing $U(e^{\hat{c}}x) - U(x) = \hat{c} + o(1)$ as $x \to \infty$.

Consider the levels $U(x) = \hat{c}\ell + z$, and $U(x') = \hat{c}(\ell + 1) + z$, with ℓ large and $z \in [0, \hat{c}]$. By geometric series, see (4.12), we have:

$$x = g(\hat{c})\frac{e^{\ell \hat{c}} - 1}{e^{\hat{c}} - 1} + e^{\ell \hat{c}}g(z), \quad x' = g(\hat{c})\frac{e^{(\ell+1)\hat{c}} - 1}{e^{\hat{c}} - 1} + e^{(\ell+1)\hat{c}}g(z).$$

We compute

$$\frac{x'}{x} = e^{\hat{c}} + \frac{g(\hat{c})}{x} = e^{\hat{c}} + O(\frac{1}{x}).$$

Thus we have found a sequence $x(\ell)$ for which $x(\ell + 1)/x(\ell) \to e^{\hat{c}}$, and

$$\frac{\overline{M}(x(\ell+1))}{\overline{M}(x(\ell))} = e^{U(x(\ell+1)) - U(x(\ell))} \cdot \frac{x(\ell)}{x(\ell+1)} = e^{\hat{c}}\frac{x(\ell)}{x(\ell+1)} = 1 + O(\frac{1}{x(\ell)}) = 1 + O(\hat{c}^{-\ell}).$$

Now given an arbitrary x (but large), let $\ell \in \mathbb{N}$ be such that $x = e^{\ell \hat{c} + z}$, for $z \in [0, \hat{c}]$. Take x' such that $U(x') = U(x) + \hat{c}$. From the above we know that $x'/x = e^{\hat{c}} + O(1/x)$, so $x' = e^{\hat{c}}x + O(1)$. Therefore, using the Mean Value Theorem,

$$
\begin{aligned}
\frac{\overline{M}(e^{\hat{c}}x)}{\overline{M}(x)} &= e^{U(e^{\hat{c}}x) - U(x)}\frac{x}{e^{\hat{c}}x} = e^{U(e^{\hat{c}}x) - U(x')} \\
&= e^{O(1)U'(\xi)} = e^{O(e^{-U(\xi)})} = e^{O(1/\xi)} = 1 + O(1/\xi),
\end{aligned}
$$

for some ξ between $e^{\hat{c}}x$ and x'. Thus $\overline{M}$ satisfies (4.9) as required. ∎

4.1 Verification of equation (4.1) and (H3) for $\mu_Y(\tau > n)$

We now explain how Proposition 4.3 and its proof can be used to verify (4.1), and in particular assumption (H3). By Proposition 4.3 equation (4.8) holds for $k_n = \lfloor c^n \rfloor$ and we can define $\tilde{M}(x) = \lim_{n \to \infty} M_0(k_n x)$, for $x \in (0, \infty)$.

As in (H3), let $A_n = n^{1/\alpha}$, and for $x \in [A_{k_n}, A_{k_{n+1}})$ define $\delta(x) = x/A_{k_n}$. To verify (4.1) and thus (H3), that is

$$\mu_Y(\tau > x) = x^{-\beta}(\tilde{M}(\delta(x)) + h(x)),$$

we need to explain why $\tilde{M}$ satisfies (H3)(ii) and define h so that (H3)(iii) holds.

From the proof of Proposition 4.3, we recall that $M_0(x) = \overline{M}(x) + O((\log x)^2/x$ as $x \to \infty$, where $\overline{M}$ is specified in equation (4.11) (and is a continuous function). We claim $M_0(k_n\delta(x))$ is a sequence of uniformly continuous functions, up to error $O\left(\log(A_{k_n})^2/A_{k_n}\right)$. This latter error can be absorbed into a right continuous function $h(x)$, with $h(x) = O((\log x)/x^2))$, as desired.

It remains to prove the claim. Consider $x, y \in [A_{k_n}, A_{k_{n+1}})$, so that $\delta(x) = x/A_{k_n}$ and $\delta(y) = y/A_{k_n}$. Recalling that $\overline{M}(x) = e^{U(x)}/x$,

$$
\begin{aligned}
|M_0(\delta(x)k_n) - M_0(\delta(y)k_n)| &= |\overline{M}(\delta(x)k_n) - \overline{M}(\delta(y)k_n)| + O\left(\log(k_n)^2/k_n\right) \\
&= \left| \frac{e^{U(xk_n)}}{xk_n} - \frac{e^{U(yk_n)}}{yk_n} \right| + O\left(\log(k_n)^2/k_n\right) \\
&\leq \frac{1}{xk_n}|e^{U(xk_n)} - e^{U(yk_n)}| + e^{U(yk_n)}\frac{|x-y|}{k_n xy} + O\left(\frac{\log(k_n)^2}{k_n}\right), \\
&\leq C_1 \frac{A_{k_n}}{x}|\delta(x) - \delta(y)| + C_2|\delta(x) - \delta(y)| + O\left(\frac{\log(k_n)^2}{k_n}\right).
\end{aligned}
$$
$$(4.13)$$

The last line follows from the fact that for $x, y \approx A_{k_n}$, $e^{U(xk_n)} \approx k_n A_{k_n}$, and $U(x)$ depends continuously on x (and hence $\delta(x)$), with C_1, C_2 independent of k_n, as required.

5 Proof of Theorem 1.5 and verification of (H0)-(H2) for f_{M_1}

In this section we prove Theorem 1.5 and for the map f_{M_1} given by equation (1.2) we verify hypotheses (H0)-(H3) (as described in Section 3.1). Hypothesis (H3) is already verified from Section 4.1. Throughout this section we work with f_{M_1}, and hence drop the subscript M_1.

To describe the organisation of proof, we explicitly construct a partition $\mathcal{Q}$ of Y, and a stopping time $\varphi : \mathcal{Q} \to \mathbb{N}$, such that for all $\omega \in \mathcal{Q}$, $f^{\varphi}(\omega) = Y$ bijectively, and with bounded distortion. Due to the presence of the singularity set $\mathcal{C}$, the stopping time φ will not (in general) coincide with the first return time τ. In the first part of the construction we define an auxiliary partition $\mathcal{P}$ and a stopping time T, such that for all $\omega \in \mathcal{P}$, we have $T(\omega) < \infty$, and $|f^{T(\omega)}(\omega)| \geq \delta$. The constant δ will be chosen to reflect that an interval $\omega \subset Y$ reaches a certain large scale under f^T, and with bounded distortion. We then estimate $\mathrm{Leb}(T > n)$. This is the key technical step. Once we have constructed $\mathcal{P}$, it is then a fairly standard argument to link the asymptotics of $\mathrm{Leb}(T > n)$ to that of $\mu(\varphi > n)$.

5.1 A combinatorial construction and tail estimates

We now define a partition $\mathcal{P}$ of Y, and a stopping time $T : \mathcal{P} \to \mathbb{N}$ such that for each $\omega \in \mathcal{P}$ we have $|f^{T(\omega)}(\omega)| \geq \delta$. The constant δ will be fixed, but for the moment we can assume that $\delta > \sqrt{\mathcal{D}}$ (and so δ is chosen so that $|f^T(\omega)|$ reaches a definite large scale). Moreover we construct $\mathcal{P}$ in such a way that $f^k \mid \omega$ is a diffeomorphism, and satisfies bounded distortion estimates for all $k \leq T(\omega)$. For the following combinatorial construction we take a finite time partition $\mathcal{P}_n$ of Y defined inductively. All $\omega \in \mathcal{P}_n$ will have $T(\omega) > n$. We set $\mathcal{P}_0 = Y$. In the following we suppose that $\omega \in \mathcal{P}_n$, and so $|f^n(\omega)| < \delta$ for $n \geq 1$.

First, we define a fixed partition $\mathcal{A}$ consisting of intervals associated to the sequence x_n described in Section 4, and for which the restriction of f to these intervals in $\mathcal{A}$ is continuous. We set $J_n = [x_{n+1}, x_n]$. If $s_\ell \notin (x_{n+1}, x_n)$ then we put $J_n \in \mathcal{A}$. If $s_\ell \in (x_{n+1}, x_n)$, then we write $J_n = J_n^+ \cup J_n^-$, with $J_n^- = [x_{n+1}, s_\ell]$ and $J_n^+ = [s_\ell, x_n]$. We put $J_n^{\pm} \in \mathcal{A}$. Thus if for all $k \leq n$, $f^k(x)$ lies in the interior of intervals in $\mathcal{A}$, then f^n is (locally) continuous and differentiable at x. We consider the following cases.

1. Suppose that $f^{n+1}(\omega) \cap (I_{\mathcal{D}} \cup \{1/2\}) = \emptyset$, and $|f^{n+1}(\omega)| < \delta$. Then we put ω into $\mathcal{P}_{n+1}$. If $|f^{n+1}(\omega)| > \delta$ then ω has reached large scale, and we put ω into $\mathcal{P}$ and set $T(\omega) = n + 1$. This component is then taken out of circulation.

2. Suppose that $f^{n+1}(\omega) \cap (I_{\mathcal{D}} \cup \{1/2\}) \neq \emptyset$. Then we have various cases. Again if $|f^{n+1}(\omega)| > \delta$ then we put ω into $\mathcal{P}$ and set $T(\omega) = n + 1$. Otherwise, we subdivide ω by intersecting the image $f^{n+1}(\omega)$ with $\mathcal{A}$ and then pull back. This is done as follows:

   • Suppose that $f^{n+1}(\omega)$ straddles at least three elements of $\mathcal{A}$. Then we subdivide ω into pieces $\omega' \subset \omega$ in such a way that $f^{n+1}(\omega') = J_r$, for some $J_r \in \mathcal{A}$. If $f^{n+1}(\omega) \cap \mathcal{C} \neq \emptyset$, then we ensure each ω' is contained in the relevant $J_r^{\pm}$ accordingly. We assign a *return depth* value r to each ω' at time $n + 1$. Now by construction of $\mathcal{A}$, the time for points $x \in f^{n+1}(\omega')$ to escape from $[0, 1/2]$ is of the order r. (The same is true for the asymptotic time to escape $I_{\mathcal{D}}$, i.e. in the case $r \to \infty$). For a component ω' such that $f^{n+1}(\omega') \subset J_r$, it may intersect $\mathcal{C}$ as it evolves under further iteration under f up to time r. The number N_r of intersections of $f^{k+n+1}(\omega')$ with $\mathcal{C}$ up to time $k \leq r$ is given by the relation:
   $$r^{-1/\alpha} \approx e^{-N_r c_1},$$
   and so $N_r \approx \frac{1}{c_1 \alpha} \log r$. For each intersection of $f^{k+n+1}(\omega')$ with a given $s_\ell \in \mathcal{C}$ the component ω' is chopped into further pieces (at most two). Hence up to time $n + 1 + r$, at most $2^{N_r} \approx r^{\log 2/\alpha c_1}$ pieces are generated per component $\omega' \subset \omega$ that satisfies $f^{n+1}(\omega') = J_r$. Each of these $\leq 2^{N_r}$ pieces $\omega'' \subset \omega'$ are then put into $\mathcal{P}_{n+1}$. They are all declared to have return depth r, and for each piece ω'', $f^{n+r} \mid \omega''$ is a diffeomorphism. The time a component spends in $I_{\mathcal{D}}$ before escaping is comparable to $r - \mathcal{D}^{-\alpha}$, and hence asymptotically equal to r. A component cannot reach large scale during this time, since we assume $\mathcal{D} \ll \delta$.

   • Suppose that $J_r \subset f^{n+1}(\omega) \subset J_{r-1} \cup J_r \cup J_{r+1}$. Then we assign a return depth value r to ω, and subdivide into pieces $\omega'' \subset \omega$ as in the previous item, and so that $f^{n+r} \mid \omega''$ is a diffeomorphism. Each ω'' is put into $\mathcal{P}_{n+1}$.

   • If $f^{n+1}(\omega) \subset J_{r-1} \cup J_r$, the we assign a return depth value r, and chop ω accordingly based upon intersections with $\mathcal{C}$ as we iterate up to time $n + r$.

   • If $f^{n+1}(\omega) \cap \{1/2\} \neq \emptyset$, then we chop ω into two pieces ω^L and ω^R with $f^{n+1}(\omega^L) \subset [0, 1/2]$ and $f^{n+1}(\omega^R) \subset [1/2, 1]$. We put ω^L and ω^R into $\mathcal{P}_{n+1}$, but assign no return depth value. This we call an *inessential* chop.

Thus for each $\omega \in \mathcal{P}_n$, there is an itinerary (t_1, r_1), $(t_2, r_2), \ldots, (t_s, r_s)$, where t_i are the sequence of times corresponding to successive returns to $I_{\mathcal{D}}$, and r_i are the associated return depths. Due to chopping procedure described above, we in fact generate many components with the same itinerary. In fact the cardinality of such components is equal to:

$$\prod_{i=1}^{s} 2^{N_{r_i}} \leq C^s \prod_{i=1}^{s} r_i^{\left(\frac{\log 2}{c_1 \alpha}\right)}, \tag{5.1}$$

for some uniform constant $C > 0$. We should also account for inessential chops. However in the case where $f^{n+1}(\omega) \cap \{1/2\} \neq \emptyset$, the left piece ω^L is mapped to an interval that contains the hyperbolic fixed point $x = 1$. Thus

in a finite number of iterations, the image of ω^L eventually covers $[1/2, 1]$ (diffeomorphically). The component ω^R is mapped into $I_\mathcal{D}$ (at time $n+2$) and it's combinatorial structure is then treated via the above algorithm. Thus intersections of components with $x = 1/2$ do not add a significant contribution to the cardinality $\mathcal{P}_n$ relative to intersections with $I_\mathcal{D}$. We can absorb the counting of such components into the constant C given in equation (5.1).

5.2 Tail estimates on the stopping time T

In this section we prove the following result:

Proposition 5.1 *The following tail estimate for T holds:*

$$\text{Leb}(\{\omega \in \mathcal{P} : T(\omega) > n\}) = O\left(n^{-\frac{1}{\alpha}}\right).$$

The idea of proof follows that of [5, Section 3.3]. In our case we have good control of the derivative along the orbit for a given itinerary. However, the complication is that we have a countable set $\mathcal{C}$ of discontinuities. This means that we generate a large number of components with a given itinerary. We show that expansion wins over chopping. By choosing c_1 sufficiently large, we ensure (on average) that components grow to large scale relative to being chopped up.

Proof To prove Proposition 5.1, we define two partitions $\mathcal{P}_n^{(1)}$ and $\mathcal{P}_n^{(2)}$ as follows. Let $\eta \in (0,1)$ be a small constant to be fixed. For $\omega \in \mathcal{P}_n$ consider its itinerary of return depths $(r_1, \ldots, r_s)$ defined up to time n. Then we set

$$\mathcal{P}_n^{(1)} = \{\omega \in \mathcal{P}_n : T(\omega) > n, \sum_{i=1}^{s} r_i \le \eta n\}, \tag{5.2}$$

and

$$\mathcal{P}_n^{(2)} = \{\omega \in \mathcal{P}_n : T(\omega) > n, \sum_{i=1}^{s} r_i > \eta n\}, \tag{5.3}$$

Note that $r_1 + \ldots + r_{s-1} \le n$, but it is possible that $r_s \ge n$. We have the following lemma (for reference compare to [5, Lemma 3.5]).

Lemma 5.2 *There exists $\theta > 0$ such that*

$$\sum_{\omega \in \mathcal{P}_n^{(1)}} |\omega| = O\left(e^{-\theta n}\right). \tag{5.4}$$

Proof Consider $\omega \in \mathcal{P}_n$ with itinerary of return depths $(r_1, \ldots, r_s)$ defined up to time n. Since a return depth value is only associated to ω on a return to $I_\mathcal{D}$, it follows that $r_i \ge \mathcal{D}^{-\alpha} > 1$ for each $i \le s$. Moreover this puts a bound on s, namely that $s \le \mathcal{D}^\alpha n$. It is also possible for this sequence to be empty if $f^k(\omega) \cap I_\mathcal{D} = \emptyset$ for all $k \le n$. For iterates evolving outside of I_Δ, the map is uniformly expanding, and $Df^k(x) \ge C\lambda^k$ if the orbit of x resides outside of $I_\mathcal{D}$ up to time k. However, the rate constant λ depends on $\mathcal{D}$. A blunt lower bound on the λ is given by

$$\lambda \ge Df_\alpha(\mathcal{D}) = 1 + C_1 \mathcal{D}^\alpha,$$

where C_1 depends only on α and c. More generally, if $x \in I_\mathcal{D}$ is such that Df^n is defined at x, $f^k(x) \in (0, 1/2)$ for all $k \le n$, and $f^n(x) \notin I_\mathcal{D}$, then we have

$$Df^n(x) \approx n^{1 + \frac{1}{\alpha}}.$$

A more refined analysis of the derivative can be obtained (but we do not need it here). Thus we have the following estimate:

$$\sum_{\omega \in \mathcal{P}_n^{(1)}} |\omega| \leq \sum_{s=1}^{n\mathcal{D}^\alpha} \sum_{k=0}^{\eta n} \sum_{\sum r_i = k} |\omega(r_1, \ldots, r_s)|$$

$$\leq \sum_{s=1}^{n\mathcal{D}^\alpha} \sum_{k=0}^{\eta n} N_{k,s} \lambda^{n-\eta n} \left(\prod_{i=1}^{s} C r_i^{-1-\frac{1}{\alpha}} \cdot 2^{\frac{1}{c_1\alpha} \log r_i} \right) \qquad (5.5)$$

$$\leq \sum_{s=1}^{n\mathcal{D}^\alpha} \sum_{k=0}^{\eta n} C^s N_{k,s} \lambda^{n-\eta n},$$

where $C > 0$ is a uniform constant. In the first line if equation (5.5), we sum over all relevant $\omega(r_1, \ldots, r_s)$ with $\sum_{i=1}^{s} r_i = k$. In the second line, we let $N_{k,s}$ denote the number of (positive) integer sequences $r_1, \ldots r_s$ with $\sum r_i = k$. The constant λ is the expansion during the iteration outside $I_\mathcal{D}$. The expression inside the term $\prod(\cdot)$ is formed by counting the number of components with itinerary r_i, and calculating the derivative along the orbit in a J_{r_i} (up to time r_i). The net contribution of each term inside the product is equal to:

$$r_i^{-1-\frac{1}{\alpha}(1-c_1^{-1}\log 2)} := r_i^{-\beta_1}, \qquad (5.6)$$

and we assume c_1 is large enough that $\beta_1 > 2$. Hence the product is insignificant in the case of $\mathcal{P}_n^{(1)}$. We now estimate $N_{k,s}$. A standard counting argument implies that

$$N_{k,s} \leq \binom{k}{s}.$$

Here $0 \leq k \leq \eta n$ and $s \leq \mathcal{D}^\alpha n$. However, this forces $s \leq \mathcal{D}^\alpha k$, since each $r_i \geq \mathcal{D}^{-\alpha}$. In particular this implies that $N_{k,s}$ is clearly an overestimate, and further optimization is possible (but the final estimate we obtain is sufficient for our purpose). An application of Stirling's formula implies that

$$N_{k,s} \leq \left(1 + \frac{2s}{k} \right)^k \cdot \left(\frac{k}{s} \right)^s \leq \exp\{2s + s(\log k - \log s)\}.$$

Since $\exp\{2s + s(\log k - \log s)\}$ is increasing in s and $s \leq \epsilon k$ for some $\epsilon < \mathcal{D}^\alpha$, we have:

$$N_{k,s} \leq \exp\{\epsilon k (2 - \log \epsilon)\}.$$

Hence

$$\sum_{\omega \in \mathcal{P}_n^{(1)}} |\omega| \leq \sum_{s=1}^{n\mathcal{D}^\alpha \eta} \sum_{k=0}^{\eta n} C^s \exp\{\mathcal{D}^\alpha k (1 - \log \epsilon)\} \cdot \lambda^{n-\eta n}$$

$$\leq n^2 \mathcal{D}^\alpha \eta^2 \cdot C^{n\mathcal{D}^\alpha \eta} \exp\{\mathcal{D}^\alpha \eta n (1 - \log \epsilon)\} \cdot \lambda^{n-\eta n}$$

$$= O(e^{-\theta n}),$$

for some $\theta > 0$. This bound follows from the fact that λ is independent of η. However, note that the lower bound on λ does depend on $\mathcal{D}$. This completes the proof of Lemma 5.2. ∎

We now consider elements in $\mathcal{P}_n^{(2)}$. We have the following lemma (for reference compare to [5, Lemma 3.6]).

Lemma 5.3 *The following estimate holds:*

$$\sum_{\omega \in \mathcal{P}_n^{(2)}} |\omega| = O\left(n^{-\frac{1}{\alpha}}\right). \tag{5.7}$$

Proof Again, consider $\omega \in \mathcal{P}_n^{(2)}$ with itinerary of return depths $(r_1, \ldots, r_s)$ defined up to time n, and with associated sequence of return times $(t_1, \ldots, t_s)$. Since $\sum_{i=1}^s r_i \geq \eta n$, a simple argument using the pigeon hole principle implies that there exists $j \leq s$ such that $r_j \geq \eta n/(2j^2)$. Consider now a set $\tilde{\omega}$ formed by a union of components in $\mathcal{P}_n^{(2)}$ which share with ω the same itinerary of return times $(t_1, \ldots, t_j)$, and itinerary of return depths $(r_1, \ldots, r_{j-1})$, but having $r_j \geq \eta n/(2j^2)$. Thus $\tilde{\omega}$ is formed by a concatenation of components (that include ω), and f^{t_j} maps $\tilde{\omega}$ diffeomorphically into an interval $(u, v) \subset [0, x_r]$ with $x_r < x_{r_j}$. Hence $|f^{t_j}(\tilde{\omega})| \leq |[0, x_{r_j}]|$. Let $\tilde{\omega}(r_1, \ldots, r_{j-1}, r) \subset \tilde{\omega}$ be a component in $\mathcal{P}_n^{(2)}$ with $f^{t_j}(\tilde{\omega}(r_1, \ldots, r_{j-1}, r)$ having depth $r \geq \eta n/(2j^2)$. We obtain

$$\sum_{\omega \in \mathcal{P}_n^{(2)}} |\omega| \leq \sum_{j=1}^n \sum_{(r_1, \ldots, r_s)} |\tilde{\omega}(r_1, \ldots, r_j)|$$

$$\leq \sum_{j=1}^n \sum_{r \geq n/2j^2} |[0, x_r]| \sum_{(r_1, \ldots, r_{j-1})} \prod_{i=1}^{j-1} r_i^{-\beta_1} \tag{5.8}$$

$$\leq \sum_{j=1}^n \sum_{r \geq n/2j^2} C r^{-1-\frac{1}{\alpha}} 2^{-\beta_1 j} \sum_{(r_1, \ldots, r_{j-1})} \prod_{i=1}^{j-1} \left(\frac{2}{r_i}\right)^{\beta_1},$$

where we recall $\beta_1 > 2$ from equation (5.6), and $C > 0$ is a uniform constant. To manipulate the product term we use the following sublemma:

Sublemma 5.4 *Suppose U_i is a sequence with $\sum_i U_i < \infty$. Then for all $\zeta > 0$ there exists n_0 such that*

$$\sum_{s \geq 1} \sum_{\substack{(n_1, \ldots, n_s) \\ n_i \geq n_0}} \prod_{n_i} \zeta U_i \leq 1.$$

We apply Sublemma 5.4 to the sequence $U_i = i^{-\beta_1}$ with $\zeta = 2^{\beta_1}$, and choose $\mathcal{D}$ small enough that $\mathcal{D}^{-\alpha} > n_0$. Thus continuing with the estimate in equation (5.8) we obtain:

$$\sum_{\omega \in \mathcal{P}_n^{(2)}} |\omega| \leq \sum_{j=1}^n (\eta n/2j^2)^{-\frac{1}{\alpha}} 2^{-\beta_1 j} = O\left(n^{-\frac{1}{\alpha}}\right). \tag{5.9}$$

∎

The proof of Proposition 5.1 now follows from Lemmas 5.2 and 5.3. We obtain that $|\{T(\omega) > n\}| = O(n^{-1/\alpha})$. We note that the singularity set $\mathcal{C}$ does not affect this power law asymptotic. The singularity set affects the $O(\cdot)$ multiplier through the constant β_1. ∎

5.3 Completing the proof of Theorem 1.5

Given the tail estimate of Proposition 5.1, the remaining part of the proof follows a standard argument, e.g. [5, 36]. These arguments don't tend to

rely heavily on the precise form of the map. In the following lemma, the significance of reaching size δ is made transparent. In particular we show that for $\omega \in \mathcal{P}$, a definite fraction of ω diffeomorphically maps onto Y in finite time beyond $T(\omega)$. We state the following.

Lemma 5.5 *There exists $t_0 > 0$, $\xi > 0$, such that for all $\omega \in \mathcal{P}$, there exists $\tilde{\omega} \subset \omega$ with the following properties:*

1. $f^{T(\omega)+t}(\tilde{\omega}) = Y$, *for some $t \leq t_0$, and this action is a diffeomorphism with bounded distortion.*

2. $|\tilde{\omega}| \geq \xi |\omega|$;

3. *Both components of $f^{T(\omega)}(\omega \setminus \tilde{\omega})$ are of size $\delta/3$.*

Proof The proof of the lemma follows from the fact that $\mathcal{D} \ll \delta$, e.g. $\delta > \sqrt{\Delta}$ will suffice. The cases to consider include i) when $f^{T(\omega)}(\omega) \cap \{1/2\} \neq \emptyset$, ii) when $f^{T(\omega)}(\omega) \cap I_{\mathcal{D}} \neq \emptyset$, or iii) when $f^{T(\omega)}(\omega) \cap (\{1/2\} \cup I_{\mathcal{D}}) = \emptyset$. For example in the case $f^{T(\omega)}(\omega) \cap I_{\mathcal{D}} \neq \emptyset$ we consider the intersection of $f^{T(\omega)}(\omega)$ with intervals of the form $[x_n, x_{n+1}]$ with $x_n > \mathcal{D}$. For n large we have $x_n \approx n^{-1/\alpha}$. Given $\xi > 0$, we can choose n so that $n^{-1/\alpha - 1} \approx \xi$, and moreover that both components of $f^{T(\omega)}(\omega) \setminus [x_{n+1}, x_n]$ are larger than $\delta/3$. The other cases reduce to similar scenarios.

For the distortion, we note that the Schwarzian derivative $Sf = f''' f' - \frac{3}{2}(f'')^2 = -6$ for $x > \frac{1}{2}$ and $= C_0(\tilde{\alpha}^2 - 1)\tilde{\alpha} x^{\tilde{\alpha}-2} - C^2(\tilde{\alpha}+1)^2 \tilde{\alpha} \frac{2+5\tilde{\alpha}}{2} x^{2(\tilde{\alpha}-1)} < 0$ for $x < \frac{1}{2}$ inside each interval where f is continuous. Here $\tilde{\alpha} = \alpha - c_1^{-1} \log 2 \in (0,1)$. Therefore $f^{T(\omega)}(\omega)$ has negative Schwarzian derivative for each ω in the statement of this lemma. Since both components of $f^{T(\omega)}(\omega \setminus \tilde{\omega})$ are of size $\delta/3$, the Koebe Principle (see [11]) gives boundedness of distortion of $f^{T(\omega)}$ on $\tilde{\omega}$, and the remaining $\leq t_0$ will not destroy this. ■

Remark 5.6 An argument similar to the Proof of 5.5 shows that the Schwarzian derivative of f_{M_1}, $f_{M_1}^1$ is negative, yielding the required distortion properties. Since the big image property is automatically satisfies in the setup of the first induced maps F with $F = (f_{M_1})^\tau$ and $F = (f_{M_1}^1)^\tau$, in these cases the map F is Gibbs Markov.

Tail of the return time function φ

Thus, in Lemma 5.5 the left and right components ω^L, ω^R of $\omega \setminus \tilde{\omega}$ are treated as new starting intervals. We then apply the combinatorial algorithm to each of the two components separately, i.e. we put $\mathcal{P}_0 = \omega^{L,R}$, and then find $|\{x : T(x) > n | x \in \omega^{L,R}\}|$. For the central component $\tilde{\omega}$, this now becomes an element of $\mathcal{Q}$ for the Gibbs-Markov map $F := f^\varphi$, and we define $\varphi(\tilde{\omega}) = T(\tilde{\omega}) + t(\omega)$. This process is repeated, and thus each element of $\mathcal{Q}$ has an associated itinerary of large scale times $T_1(\omega), \ldots T_n(\omega), \ldots$, before return time $\varphi(\omega)$. For $x \in Y$, we write $T_1(x) := T(x)$, and define $\{T_n\}$ recursively via $T_{n+1} = T_n(x) + T(f^{T_n(x)}(x))$. Thus for all $\omega \in \mathcal{Q}$, there exists $s \geq 1$, and $t \leq t_0$ such that we have $\varphi(\omega) = T_s(\omega) + t(\omega)$. and gives rise to a further sequence of partitions $\mathcal{Q}_n$ and stopping times $T_1, \ldots, T_n, \ldots$ We have the following lemma:

Lemma 5.7 *For every i,*

$$\text{Leb}(\{x : T_{i+1}(x) \text{ exists and} T_{i+1} > T_i + k \mid T_i\}) \leq D_\delta \text{Leb}(T > k),$$

where $D_\delta > 0$ is a constant.

Proof This is a standard argument using bounded distortion, the definition of T and Lemma 5.5. See [5, Lemma 4.4.], and also [36]. ∎

Now recall each $\omega \in \mathcal{Q}$ has the sequence:

$$0 = T_0 < T_1 < \ldots < T_{s(\omega)} < \varphi(\omega).$$

Let $\mathcal{Q}^{(n)} = \{\omega \in \mathcal{Q} : \varphi(\omega) > n\}$, and let

$$\mathcal{Q}_i^{(n)} = \{\omega \in \mathcal{Q}^{(n)} : T_{i-1} < n \leq T_i\}$$

denote the set of elements in $\mathcal{Q}^{(n)}$ which have exactly $i - 1$ large scale times before n. We let

$$|\mathcal{Q}_i^{(n)}| = \sum_{\omega \in \mathcal{Q}_i^{(n)}} |\omega|, \quad |\mathcal{Q}^{(n)}| = \sum_{i=1}^{n} |\mathcal{Q}_i^{(n)}|.$$

Completing the proof of Theorem 1.5.

Let $\eta > 0$ be a small constant, and write:

$$|\{\varphi > n\}| = |\mathcal{Q}^{(n)}| = \sum_{i=1}^{\eta n} |\mathcal{Q}_i^{(n)}| + \sum_{i=\eta n}^{n} |\mathcal{Q}_i^{(n)}|.$$

By Lemma 5.5 we have:

$$|\mathcal{Q}_i^{(n)}|/|\mathcal{Q}_{i-1}^{(n)}| \leq 1 - \xi,$$

and therefore

$$\sum_{i=\eta n}^{n} |\mathcal{Q}_i^{(n)}| \leq \frac{1}{\xi}(1 - \xi)^{\eta n}.$$

To estimate $\sum_{i=1}^{\eta n} |\mathcal{Q}_i^{(n)}|$ we again use a combinatorial argument (and see [5, 36] for similar). For each element of $\mathcal{Q}_i^{(n)}$, we can assign an itinerary $(k_1, \ldots, k_i)$ with $\sum_j k_j = n$, and $k_j = T_j - T_{j-1}$ for all $j \leq i - 1$. We put $k_i = n - T_{i-1}$. A pigeon hole argument implies that for any such itinerary, there exists k_j with $k_j \geq n/i$. Let

$$\mathcal{Q}_{i,j}^{(n)} = \{\omega \in \mathcal{Q}_i^{(n)}, k_{j'} < n/i, \text{ for } j' < j \text{ and } k_j \geq n/i\}.$$

We have the following

$$\sum_{i=1}^{\eta n} |\mathcal{Q}_i^{(n)}| = \sum_{i=1}^{\eta n} \sum_{j=1}^{i} |\mathcal{Q}_{i,j}^{(n)}| \leq \sum_{i=1}^{\eta n} i(1-\xi)^{i-1}|\{T > n/i\}$$

$$\leq \sum_{i=1}^{\eta n} i(1-\xi)^{i-1} \cdot \left(\frac{i}{n}\right)^{\frac{1}{\alpha}} = O(n^{-\frac{1}{\alpha}}),$$

which gives the required bound. This completes the tail estimate for the return time function, and hence the proof of Theorem 1.5.

5.4 Verification of (H0)-(H2) for f_{M_1}.

We now verify (H0)-(H3) for the map $f_{M_1}(x)$. Firstly, (H0) immediately follows from the statement of Theorem 1.5. To verify (H1), we can adapt

[5, Lemma 3.6] via Lemma 5.3 to study the asymptotics of the re-inducing time $\rho : Y \to \mathbb{N}$, (see [7]). We obtain

$$\mu_0(\rho > L : \varphi > n) = O\left(\lambda_0^L n^{-\alpha}\right),$$

for some $\lambda_0 \in (0,1)$. This then leads to

$$\int_{\varphi>n} \rho \, d\mu_0 \le C_1 \sum_{L \ge 0} \lambda_0^L n^{-\alpha} = O(n^{-\alpha}) \le C_2 \mu_0(\varphi > n),$$

for uniform constants C_1, C_2. To verify (H2), we note that the combinatorial algorithm described in Section 5.1 requires that for $\omega \in \mathcal{P}$ we either have $f^k(\omega) \subset Y$ of $f^k(\omega) \cap Y = \emptyset$.

6 Proof of Theorem 1.1 for larger classes of sequences

Recall f_M with $M = M_1$ or $M = M_2$ defined in (1.2) and (1.3). Let $k_n = \lfloor c^n \rfloor$ with $c = e^{\alpha c_1}$ and $c = e^{\alpha c_2}$, respectively. As shown in Subsection 4.1, (H3) holds along k_n. We recall that in the setup of f_{M_2}, the stronger assumption (H4) holds (by an argument similar to the one Subsection 4.1 with an easier analysis since $\varphi \equiv \tau$).

Regarding the verification of (H5) on the tail of the induced observable we recall the following result. Recall that $g = g_X \circ \pi$ and that $g_Y = \sum_{j=0}^{\varphi-1} g \circ f_\Delta^j$.

Lemma 6.1 *Let $x_0 \in X$ be the indifferent periodic point of f_M. Let $g_X : X \to \mathbb{R}$ be a Hölder observable of exponent $\nu > 1/\beta - 1/2$ and suppose $g_X(x_0) \neq 0$. Then $\mu_Y(|g_Y| > t) = C\mu_Y(\varphi > t)(1+o(1))$ as $t \to \infty$ for some $C > 0$.*

Proof The argument goes as in [15, Proof of Theorem 1.3]. We recall the main elements for completeness since the argument there is written for first return times.

Set $\tilde{g}_Y = \sum_{j=0}^{\varphi-1} g_X \circ f^j$. Let y, k so that $\varphi(y) = k$, assume $g_X(x_0) = C > 0$ and write

$$\tilde{g}_Y(y) = Ck + \sum_{j=0}^{k-1} \left(g_X(f^j(y)) - g_X(f^j(x_0))\right).$$

Since g_X is ν-Hölder, $\sum_{j=0}^{k-1} |g_X(f^j(y)) - g_X(f^j(x_0))| \le |g_X|_\nu \sum_{j=0}^{k-1} \left|f^j(y) - f^j(x_0)\right|^\nu$. Since x_0 is the indifferent periodic point, $\sum_{j=1}^{k} \left|f^j(y) - f^j(x_0)\right|^\nu \ll \sum_{j=1}^{k}(k-j)^{-\beta\nu} \ll k^{1-\nu\beta}$ for all $1 \le j \le k$. Hence, $\tilde{g}_Y(y) = C\varphi(y) + O(\varphi(y)^{1-\nu\beta})$.

Since $\nu > 1/\beta - 1/2$, $O(\varphi(y)^{-\nu\beta}) = O(\varphi(y)^{-\beta(1/\beta-1/2)}) = O(\varphi(y)^{-(1-\beta/2)}) = o(1)$, where in the last equality we have used that and $\beta < 2$. Thus, $\mu_Y(|\tilde{g}_Y| > t) = C'\mu_Y(\varphi > t)(1 + o(1))$, as $t \to \infty$. This gives the statement on $\tilde{g}_Y$ and, thus, on g_Y in the case $C > 0$. The case $C < 0$ goes similarly. ∎

Given the already introduce terminology, here we obtain Theorem 1.1 ii) for larger classes of sequences. More precisely, similar to (2.4), for $x > 0$ (large) define

$$\gamma_x = \frac{x}{k_n}, \qquad \text{where } k_{n-1} < x \le k_n. \tag{6.1}$$

As explained below, given γ_x as in (6.1), whenever $\gamma_{q_r} \xrightarrow{cir} \lambda \in (c^{-1}, 1]$,

$$\frac{\sum_{j=0}^{q_r-1} v \circ f_{M_2}^j}{(q_r)^{1/\alpha}} \xrightarrow{d} V(\alpha, \lambda), \text{ as } r \to \infty, \tag{6.2}$$

where V_λ is as in (2.5).

Proof of Theorem 1.1 For f_{M_1}, the abstract assumptions (H0)–(H3) are verified in Subsections 5.4 and 4.1. Thus, Theorem 3.3 applies to f_{M_1}, f_{M_2} with norming sequence $(k_n)^{1/\alpha}$ giving also the additional result (6.2). We note that the statement on the non trivial distribution limit for f_{M_1} is along some sequence m_r and no identification of m_r is claimed. ∎

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