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Stable large deviations for deterministic dynamical systems

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We obtain large deviations for a class of non-square integrable dependent random variables in the domain of attraction of an α -stable law, $\alpha \in (0, 1) \cup (1, 2]$. This class includes ergodic sums of observables in the domain of attraction of an α -stable law driven by Gibbs–Markov maps.

Keywords: Large deviations; stable law; dependent processes.

1. Introduction

The purpose of this work is to obtain large deviations (LDs) for a class of dependent random variables satisfying α -stable limit laws, $\alpha \in (0, 1) \cup (1, 2]$. In particular, we obtain stable large deviations for a class of deterministic dynamical systems (including Gibbs–Markov maps, see Sec. 4.1 for details). As far as we are aware, this is the first stable LD result for dynamical systems.

We start by recalling the statement of LDs for real-valued i.i.d. random variables $(X_i)_{i \geq 1}$ characterized by stable laws with index $\alpha \in (0, 2]$; for the $\alpha = 2$ case, we still assume that the variance is infinite. Suppose that

$$\mathbb{P}(X_1 > x) = p\ell(x)x^{-\alpha}(1 + o(1)), \quad \mathbb{P}(X_1 < -x) = q\ell(x)x^{-\alpha}(1 + o(1)), \quad (1.1)$$

as $x \rightarrow \infty$, where ℓ is a slowly varying function and $p, q > 0$ with $p + q = 1$. Then the sequence $(X_i)_{i \geq 1}$ satisfies the following convergence result in distribution:

$$(S_n - b_n)/a_n \rightarrow_d Y_\alpha, \quad (1.2)$$

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where $S_n = \sum_{i=1}^n X_i$, the limit random variable Y_α is an α -stable law (with the parameters α , p and q),

$$a_n^\alpha(1 + o(1)) = \begin{cases} n\ell(a_n) & \text{if } \alpha \in (0, 2), \\ n\hat{\ell}(a_n) & \text{if } \alpha = 2 \quad \text{for } \hat{\ell}(x) := 1 + \int_1^{1+x} \frac{\ell(u)}{u} du \end{cases} \quad (1.3)$$

and

$$b_n = \begin{cases} 0 & \text{if } \alpha \in (0, 1), \\ n\mathbb{E}(X_1) & \text{if } \alpha \in (1, 2], \\ n\mathbb{E}(X_1 1_{\{|X_1| \leq a_n\}}) & \text{if } \alpha = 1 \end{cases} \quad (1.4)$$

is the centering sequence. It is known that in the i.i.d. setup, (1.1) and (1.2) are equivalent.

We recall that when $\alpha = 2$, up to redefining X_1 as $X_1 - \mathbb{E}(X_1)$ and rescaling a_n by a constant, the limit random variable Y_2 is the standard Gaussian. Let $\bar{\Phi}(x) = \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}(1 + o(1))$ be the distribution function of the standard Gaussian. Let $\sigma^2(x) = \mathbb{E}((X_1 - \mathbb{E}(X_1))^2 1_{\{|X_1 - \mathbb{E}(X_1)| \leq x\}})$ be the truncated variance of X_1 . One can easily verify that $n\sigma^2(a_n) = a_n^2(1 + o(1)) = n\hat{\ell}(a_n)(1 + o(1))$.

With these specified, we recall the precise statement for large deviations for i.i.d. sequences.

Theorem 1.1. *Assume (1.1) with $\alpha \in (0, 2]$. Let $N : \mathbb{N} \rightarrow \mathbb{R}_+$ be so that $N(n)/a_n \rightarrow \infty$, as $n \rightarrow \infty$. Then the following hold as $n \rightarrow \infty$:*

If $\alpha \in (0, 2)$, then

$$\begin{aligned} \mathbb{P}(S_n - b_n > N) &= n\mathbb{P}(X_1 > N)(1 + o(1)), \\ \mathbb{P}(S_n - b_n < -N) &= n\mathbb{P}(X_1 < -N)(1 + o(1)). \end{aligned} \quad (1.5)$$

If $\alpha = 2$, then

$$\mathbb{P}(S_n - b_n > N) = \left(\bar{\Phi}\left(\frac{N}{a_n}\right) + n\mathbb{P}(X_1 > N) \right) (1 + o(1)), \quad (1.6)$$

and a similar statement holds for $\mathbb{P}(S_n - b_n \leq -N)$.

Theorem 1.1 for $\alpha \in (0, 1) \cup (1, 2)$ has been known since the works of [9, 17, 18]. The more difficult (general) case $\alpha = 1$ has only very recently been dealt with in [4, Theorem 2.1]. The statement for $\alpha = 2$ is contained in [19]: see [5, Theorem 2.1], contained within [19, Theorem 6]. We remark that [5, Condition 2.4 of Theorem 2.1] in Theorem 1.1 holds as soon as $\mathbb{P}(X_1 > x) \sim c\mathbb{P}(X_1 < -x)$ for some $c \geq 0$ and $a_n^2 = n\sigma^2(a_n)$: see [5, Examples 2.2 and A.10].

Remark 1.2. To illustrate the role of $\bar{\Phi}$, we can take for example $\ell(x) \equiv 1$, so $\hat{\ell}(x) = \log x$, and $N \sim a_n \log N$. Then the term with $\bar{\Phi}(N/a_n)$ dominates the other term $n\mathbb{P}(X_1 > N)$ by approximately a factor $\log N$.

We are mainly interested in obtaining a version of Theorem 1.1 for a class of deterministic dynamical systems. We postpone the case $\alpha = 1$ to later work.

A version of Theorem 1.1 for a class of dependent random variables satisfying certain clustering and limit conditions has been obtained in [16]. To our knowledge, this is the only previous work in which a version “close” to the general form of Theorem 1.1 for dependent random variables has been considered; by “close”, we mean that (1.5) is shown to hold in a certain range of $N/a_n \rightarrow \infty$ as opposed to the whole region of n and N so that $N/a_n \rightarrow \infty$. It is not clear to us how to verify the limit conditions in [16, Theorem 3.1] for the class of dynamical systems considered here and in this work, we use a different method of proof.

The somewhat related problem of stable local large deviations (LLDs)^a for deterministic dynamical systems (including the Gibbs–Markov maps) has been treated in [15]. An optimal LLD has been very recently obtained in [13] for (much more challenging) dynamical systems (with very heavy dependencies) of physical interest known as the Lorentz gases. We note that stable LLD does not imply stable LD. Even for i.i.d. random variables, in the generality of (1.1), stable LLD does not imply^b Theorem 1.1; though under ideal smoothness of the tail probabilities (1.1), this is in fact possible (see [4, Theorem 2.4]). For deterministic dynamical systems, the transition from the stable LLD obtained in [13, 15] to a version of Theorem 1.1 seems impossible. As we shall explain in the sequel, obtaining a version of Theorem 1.1 for dynamical systems is in many respects much more delicate than stable LLD. In this paper, we restrict ourselves to the class of dynamical systems treated in [15].

Throughout the rest of the paper, we let $N : \mathbb{N} \rightarrow \mathbb{R}_+$ be so that $N(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $N(n)/a_n \rightarrow \infty$ as $n \rightarrow \infty$.

We state the main result and refer to Theorem 3.1 below for a more general version.

Theorem 1.3. *Let (T, Ω, μ) be a probability-preserving mixing Gibbs–Markov map. Let $v : \Omega \rightarrow \mathbb{R}$ be a measurable observable with $\int_{\Omega} v^2 d\mu = \infty$ and suppose that v satisfies (1.1) (with μ instead of $\mathbb{P}$). Assume further that v is locally Lipschitz and satisfies a “nice” technical condition (more precisely, (4.1) stated in Sec. 4.1).*

Write $v_n = \sum_{j=0}^{n-1} v \circ T^j$ for the ergodic sum of v . Assume that v_n satisfies (1.2) (with v_n instead of S_n) with a_n and b_n as in (1.3) and (1.4).

Let $\delta > 0$ be arbitrarily small and set

$$D(x) = \begin{cases} (\log x)\ell(x)x^{-\alpha} & \text{if } \alpha \in (0, 1), \\ x^{-(\alpha-\delta)} & \text{if } \alpha \in (1, 2]. \end{cases}$$

^aA stable LLD result is concerned with bounds on $\mathbb{P}(S_n - b_n \in (N - h, N + h))$, for $h > 0$, as $N/a_n \rightarrow \infty$.

^bAs shown in [4, 6], under (1.1) with $\alpha \in (0, 2)$ (see also [15] for a different proof and for the case $\alpha = 2$), for any $h > 0$, there exists $C > 0$ so that $\mathbb{P}(S_n - b_n \in (x - h, x + h)) \leq C \frac{n}{a_n} \frac{\ell(|x|)}{1+|x|^\alpha}$, for all $n \geq 1$ and for all $x \in \mathbb{R}$.

Take any $N = N(n)$ such that $N/a_n \rightarrow \infty$. Then, for $\alpha \in (0, 1) \cup (1, 2)$, the following holds as $n \rightarrow \infty$:

$$|\mu(v_n > N) - n\mu(v > N)| = O(D(N)) + o(n\ell(N)N^{-\alpha}),$$

and a similar statement holds for $\mu(v_n < -N)$.

Also, for $\alpha = 2$, as $n \rightarrow \infty$,

$$|\mu(v_n > N) - n\mu(v > N)| = O(D(N)) + O(n \log NN^{-2}) + O(n\hat{\ell}(N)N^{-2}),$$

and a similar statement holds for $\mu(v_n \leq -N)$.

The corollary below gives the range of $N = N(n)$ for which $\mu(v_n > N) \sim n\mu(v > N)$ holds when $\alpha \in (0, 1) \cup (1, 2)$. However, in the case $\alpha = 2$, note that if $\ell \equiv 1$ then the asymptotic equality fails (see Remark 1.2). For $\alpha \in (1, 2)$, the probability that $v_n > N$ is equivalent to the probability that there is exactly one jump larger than N , i.e. there exists exactly one $j = 0, \dots, n-1$ so that $v \circ T^j > N$.

Corollary 1.4. *Assume the setup of Theorem 3.1. Then $\mu(v_n > N) \sim n\mu(v > N)$ holds in the following cases:*

- for $\alpha \in (0, 1)$ and $N \in (a_n, e^{an})$, with a small and independent of n ;
- for $\alpha \in (1, 2)$ and $N \in (a_n, n^{1/\delta})$ for any $\delta > 0$.

While in this work we focus on observables taking the values in $\mathbb{R}$, we believe that at the expense of cumbersome notation throughout, Theorem 1.3 can be extended to $\mathbb{R}^d$ -valued observables.

Although the proof of Theorem 1.3 below is quite technical, the main steps can be summarized as follows:

1. Rephrase Theorem 1.1 for $\alpha \in (0, 1) \cup (1, 2)$ (and (1.6) for $\alpha = 2$) for i.i.d. sequences in terms of the characteristic functions: see Sec. 2.
2. Decompose the Fourier transform (analog of characteristic function) for the ergodic sum $v_n = \sum_{j=0}^{n-1} v \circ T^j$ into the characteristic function of an i.i.d. sequence and “some good” quantities: see Secs. 3.1 and 3.2.
3. Estimate the various integrals appearing (from the “good” quantities) in the analytic expression of $\mu(v_n - b_n > x)$ via the “modulus of continuity”-type arguments (in the sense of [10, Chap. 1]): see Sec. 3.3.
4. Put items 1 and 3 together to conclude.

Notation. We use “big O ” and $\ll$ notation interchangeably, writing $b_n = O(c_n)$ or $b_n \ll c_n$ if there are constants $C > 0$, $n_0 \geq 1$ such that $b_n \leq Cc_n$ for all $n \geq n_0$. As usual, $b_n = o(c_n)$ means that $\lim_{n \rightarrow \infty} b_n/c_n = 0$ and $b_n \sim c_n$ means that $\lim_{n \rightarrow \infty} b_n/c_n = 1$.

2. Rephrasing Theorem 1.1 in Terms of Characteristic Functions

We start with a general lemma, valid whenever $N(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Lemma 2.1. *Let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of random variables and Y another random variable, defined on the same probability space but independent of all the Z_n . Assume that there exist two sequences $N = N(n)$ and $h_N = h_{N(n)}$ such that $\mathbb{P}(|Y| > h_N) = o(\mathbb{P}(Z_n > N))$ and $\mathbb{P}(Z_n > N \pm h_N) = \mathbb{P}(Z_n > N)(1 + o(1))$ as $n \rightarrow \infty$.*

Then for $\tilde{Z}_n = Z_n + Y$, we have

$$\mathbb{P}(\tilde{Z}_n > N) = \mathbb{P}(Z_n > N)(1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

Proof. Since Z_n and Y are independent, we have (for $h = h_{N(n)}$)

$$\begin{aligned} \mathbb{P}(Z_n > N + h) &= \mathbb{P}(Z_n > N + h, |Y| > h) + \mathbb{P}(Z_n > N + h, |Y| \leq h) \\ &\leq \mathbb{P}(|Y| > h) + \mathbb{P}(\tilde{Z}_n > N). \end{aligned}$$

Therefore,

$$\mathbb{P}(\tilde{Z}_n > N) \geq \mathbb{P}(Z_n > N + h) - \mathbb{P}(|Y| > h) = \mathbb{P}(Z_n > N)(1 + o(1)).$$

For the other inequality, we have

$$\begin{aligned} \mathbb{P}(Z_n \leq N - h) &= \mathbb{P}(Z_n \leq N - h, |Y| > h) + \mathbb{P}(Z_n \leq N - h, |Y| \leq h) \\ &\leq \mathbb{P}(|Y| > h) + \mathbb{P}(\tilde{Z}_n \leq N). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}(\tilde{Z}_n > N) &= 1 - \mathbb{P}(\tilde{Z}_n \leq N) \leq 1 - \mathbb{P}(Z_n \leq N - h) + \mathbb{P}(|Y| > h) \\ &= \mathbb{P}(Z_n > N - h) + \mathbb{P}(|Y| > h) = \mathbb{P}(Z_n > N)(1 + o(1)). \end{aligned}$$

This finishes the proof. □

For $t \in \mathbb{R}$, let $\Psi(t) = \mathbb{E}(e^{itX_1})$ be the characteristic function of X_1 . In this section, we translate Theorem 1.1 for $\alpha \in (0, 1) \cup (1, 2]$ into a statement on the characteristic function $\Psi(t)$. Such a translation is captured in Eq. (2.1) below, which is the starting point for the dynamic setting. Since $\alpha \neq 1$, with no loss of generality we can assume that $b_n = 0$. For $\alpha \in (0, 1)$ this is given, while for $\alpha \in (1, 2]$, we simply replace X_1 by $X_1 - \mathbb{E}(X_1)$.

Proposition 2.2. *Assume the setup of Theorem 1.1 with $\alpha \neq 1$. Let Y be an L^2 random variable, independent of S_n (for each n), with real-valued, even and C^2 characteristic function ψ_Y , supported in $[-\epsilon, \epsilon]$ for some^c small $\epsilon > 0$.*

^cThe existence of such a random variable Y on the same probability space and with characteristic function ψ_Y follows from, for instance, [8, Proposition 3.8; 11, Proof of Theorem 3; 15, Footnote 1].

Take any $N = N(n)$ such that $a_n = o(N(n))$ and let $g(n)$ be so that $nN = o(g(n))$. Then:

- (i) If $\alpha \in (0, 1) \cup (1, 2)$, then the first asymptotic equivalence in (1.5) holds if and only if

$$\left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\Psi(t)^n - n\Psi(t)}{it} dt \right| = o(n\mathbb{P}(X_1 > N)), \quad (2.1)$$

as $n \rightarrow \infty$.

- (ii) If $\alpha = 2$, then (1.6) holds if and only if

$$\begin{aligned} & \left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\Psi(t)^n - n\Psi(t)}{it} dt - \bar{\Phi}\left(\frac{N}{a_n}\right) \right| \\ &= o\left(n\mathbb{P}(X_1 > N) + \bar{\Phi}\left(\frac{N}{a_n}\right)\right), \end{aligned} \quad (2.2)$$

as $n \rightarrow \infty$.

Remark 2.3. (a) The second asymptotic equivalence in (1.5) can be dealt with in the same way using the fact that $\mathbb{P}(S_n < -N) = \mathbb{P}(-S_n > N)$.

- (b) If the random variables X_i are $\mathbb{Z}$ -valued (so, S_n is $\mathbb{Z}$ -valued), then one could exploit a simpler formula, bypassing the presence of ψ_Y in the statement of Proposition 2.2. Indeed,

$$\mathbb{P}(S_n > N) = \sum_{j>N} \mathbb{P}(S_n = j) = \sum_{j>N} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itj} \Psi(t)^n dt.$$

Proof. To start with, recall the inversion formulas

$$\mathbb{P}(X_1 \in (N, N + g(n)]) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \Psi(t) dt$$

and

$$\mathbb{P}(S_n \in (N, N + g(n)]) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \Psi(t)^n dt,$$

for any choice of $g(n)$. We want to allow that $N(n), g(n) \rightarrow \infty$ and obtain the desired formula for $\mathbb{P}(S_n > N)$. Recall that $nN = o(g(n))$ and compute that

$$\begin{aligned} \mathbb{P}(X_1 > N + g(n)) &= p\ell(N + g(n)) \frac{1}{g(n)^\alpha} \frac{1 + o(1)}{\left(1 + \frac{N}{g(n)}\right)^\alpha} \\ &= o\left(\frac{n\ell(N + g(n))}{g(n)^\alpha}\right) = o(\mathbb{P}(X_1 > N)). \end{aligned}$$

Since $N = o(\frac{N+g(n)}{n})$,

$$\mathbb{P}(S_n > N + g(n)) \leq \sum_{j=0}^{n-1} \mathbb{P}\left(|X_j| > \frac{N + g(n)}{n}\right) = o(n\mathbb{P}(X_1 > N)).$$

Thus, given the choice of $g(n)$, we have $\mathbb{P}(X_1 \in (N, N+g(n)]) = \mathbb{P}(X_1 > N)(1-o(1))$ and

$$\begin{aligned} \mathbb{P}(S_n \in (N, N + g(n)]) &= \mathbb{P}(S_n > N) - \mathbb{P}(S_n > N + g(n)) \\ &= \mathbb{P}(S_n > N) - o(n\mathbb{P}(X_1 > N)). \end{aligned}$$

Therefore, as $n \rightarrow \infty$,

$$\mathbb{P}(S_n > N) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \Psi(t)^n dt + o(n\mathbb{P}(X_1 > N)) \quad (2.3)$$

and

$$\mathbb{P}(X_1 > N) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \Psi(t) dt + o(\mathbb{P}(X_1 > N)). \quad (2.4)$$

Next, we argue that one can adjust the domain of integration in (2.3). Let $\tilde{S}_n = S_n + Y$, where Y is as in the statement, i.e. an L^2 random variable, independent of S_n , with real-valued, even and C^2 Fourier transform ψ_Y , supported in $[-\epsilon, \epsilon]$ for some small $\epsilon > 0$.

The analogs of (2.4) and (2.3) are

$$\mathbb{P}(X_1 + Y > N) = \frac{1 + o(1)}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \psi_Y(t) \Psi(t) dt \quad (2.5)$$

and

$$\begin{aligned} \mathbb{P}(S_n + Y > N) &= \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \psi_Y(t) \Psi(t)^n dt \\ &\quad + o\left(n\mathbb{P}\left(X_1 + \frac{Y}{n} > N\right)\right). \end{aligned} \quad (2.6)$$

To compare the tails of X_1 and $X_1 + Y$, and of S_n and $\tilde{S}_n = S_n + Y$, we will use Lemma 2.1 several times, making different choices for Z_n appearing in that lemma.

The case $\alpha \in (0, 1) \cup (1, 2)$. Take $\delta \in (0, 2 - \alpha)$. Choose $h_N = N^{\frac{\alpha+\delta}{2}} = o(N)$, so $\mathbb{P}(X_1 > N \pm h_N) = \mathbb{P}(X_1 > N)(1 + o(1))$. Because Y is L^2 , the tail $\mathbb{P}(|Y| > h_N) = o(h_N^{-2}) = o(N^{-\alpha-\delta}) = o(\ell(N)N^{-\alpha}) = o(\mathbb{P}(X_1 > N))$. Therefore, Lemma 2.1 for $Z_n \equiv X_1$ gives

$$\begin{aligned} \mathbb{P}(X_1 > N) &= \mathbb{P}(X_1 + Y > N)(1 + o(1)) \\ &= \frac{1 + o(1)}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \psi_Y(t) \Psi(t) dt. \end{aligned} \quad (2.7)$$

Next, we want to obtain a version with S_n in (2.6).

From (1.5) to (2.1). If S_n satisfies (1.5), i.e. $\mathbb{P}(S_n > N) = n\mathbb{P}(X_1 > N)(1 + o(1))$, then the previous computation and choice of h_N give also $\mathbb{P}(S_n > N + h_N) = \mathbb{P}(S_n > N)(1 + o(1))$ and $\mathbb{P}(|Y| > h_N) = o(\mathbb{P}(X_1 > N)) = o(\mathbb{P}(S_n > N))$. Therefore, Lemma 2.1 for $Z_n = S_n$ and $\tilde{S}_n = S_n + Y$ gives

$$\mathbb{P}(S_n > N) = \mathbb{P}(\tilde{S}_n > N)(1 + o(1)).$$

From (2.1) to (1.5). Note that $\mathbb{P}(\frac{|Y|}{n} > h_N) < \mathbb{P}(|Y| > h_N) = o(\mathbb{P}(X_1 > N))$.

If (2.1) holds, then using (2.6), and also (2.7) together with (2.5), we obtain that $\mathbb{P}(\tilde{S}_n > N) = n\mathbb{P}(X_1 > N)(1 + o(1))$. Again using Lemma 2.1 with $Z_n = \tilde{S}_n$, $S_n = \tilde{S}_n - Y$ and $h_N = N^{\frac{\alpha+\delta}{2}} = o(N)$, we can conclude that

$$\mathbb{P}(\tilde{S}_n > N) = \mathbb{P}(S_n > N)(1 + o(1)). \quad (2.8)$$

Hence, after subtracting (2.5) from (2.6) and apply (2.7) and (2.8), we obtain that

$$\begin{aligned} & |\mathbb{P}(S_n > N) - n\mathbb{P}(X_1 > N)| \\ &= \frac{1 + o(1)}{2\pi} \left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\Psi(t)^n - n\Psi(t)}{it} dt \right| \\ &+ o(n\mathbb{P}(X_1 > N)) = o(n\mathbb{P}(X_1 > N)), \end{aligned}$$

as $n, N \rightarrow \infty$, so that $a_n = o(N)$.

Next we made the adjustments for the case $\alpha = 2$. Let $m(N)$ be so that $\mathbb{P}(|Y| > N) = m(N)N^{-2}$. So, $m(N) = o(\ell(N))$. The choice of $m(N)$ ensures that there is a function $\tilde{m}(N) \rightarrow \infty$ such that also $\tilde{m}(N)^2 m(N/\tilde{m}(N)) = o(\ell(N))$. Now take $h_N = N/\tilde{m}(N) = o(N)$, so $\mathbb{P}(X_1 > N + h_N) = \mathbb{P}(X_1 > N)(1 + o(1))$. Moreover,

$$\begin{aligned} \mathbb{P}(|Y|/n > h_N) &< \mathbb{P}(|Y| > h_N) \ll N^{-2} \tilde{m}(N)^2 m(N/\tilde{m}(N)) \\ &= o(N^{-2} \ell(N)) = o(\mathbb{P}(X_1 > N)). \end{aligned}$$

By the same argument used in obtaining (2.7), $\mathbb{P}(X_1 + Y > N) = \mathbb{P}(X_1 > N)(1 + o(1))$ and

$$\mathbb{P}(X_1 > N) = \frac{1 + o(1)}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \psi_Y(t) \Psi(t) dt. \quad (2.9)$$

The same choice of h_N gives $\mathbb{P}(S_n > N + h_N) = \mathbb{P}(S_n > N)(1 + o(1))$ and $\mathbb{P}(|Y| > h_N) = o(\mathbb{P}(X_1 > N)) = o(\mathbb{P}(S_n > N))$. Therefore, Lemma 2.1 gives $\mathbb{P}(\tilde{S}_n > N) = \mathbb{P}(S_n > N)(1 + o(1))$. Combining these estimates with (2.6) gives

$$\mathbb{P}(S_n > N) = \frac{1 + o(1)}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \psi_Y(t) \Psi(t)^n dt + o(n\mathbb{P}(X_1 > N)). \quad (2.10)$$

Now we insert the tail (1.6), subtract $\bar{\Phi}(N/a_n)$ and (2.9) n times. This gives

$$\begin{aligned} & \left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\Psi(t)^n - n\Psi(t)}{it} dt - \bar{\Phi}\left(\frac{N}{a_n}\right) \right| \\ &= o\left(n\mathbb{P}(X_1 > N) + \bar{\Phi}\left(\frac{N}{a_n}\right)\right), \end{aligned}$$

as required. For the converse, (1.6) follows directly from the above formula together with (2.9) and (2.10). $\square$

For use below in the setup of dependent random variables (arising in the context of dynamical systems), we record the following consequence of the proof of Proposition 2.2.

Proposition 2.4. *Let $(v^j)_{j \geq 0}$ be a sequence of random variables, not necessarily independent, but identically distributed, on some probability space (Ω, μ) . Suppose that v^j satisfies (1.1) (with v^j instead of X_j and μ instead of $\mathbb{P}$) and write $v_n = \sum_{j=0}^{n-1} v^j$. Assume that v_n satisfies (1.2) with v_n instead of S_n and a_n and b_n as in (1.3) and (1.4).*

Let Y be an L^2 random variable, independent of v_n (for each n), with real-valued, even and C^2 Fourier transform ψ_Y , supported in $[-\epsilon, \epsilon]$ for some^d small $\epsilon > 0$.

Take any $N = N(n)$ such that $a_n = o(N(n))$ and let $g(n) = N(n)^{1+\epsilon_0}$ with $\epsilon_0 > 0$.

Moreover, suppose that there exists $W(N, n)$ so that $n\mathbb{P}(X_1 > N) = O(W(N, n))$ and so that:

(i) *if $\alpha \in (0, 1) \cup (1, 2)$, as $n \rightarrow \infty$,*

$$\left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\mathbb{E}_{\mu}(e^{itv_n}) - n\mathbb{E}_{\mu}(e^{itv^0})}{it} dt \right| = O(W(N, n)); \quad (2.11)$$

(ii) *if $\alpha = 2$, as $n \rightarrow \infty$,*

$$\begin{aligned} & \left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\mathbb{E}_{\mu}(e^{itv_n}) - n\mathbb{E}_{\mu}(e^{itv^0})}{it} dt - \bar{\Phi}\left(\frac{N}{a_n}\right) \right| \\ &= O(W(N, n)) + o\left(\bar{\Phi}\left(\frac{N}{a_n}\right)\right). \end{aligned} \quad (2.12)$$

Then:

(i) *if $\alpha \in (0, 1) \cup (1, 2)$, as $n \rightarrow \infty$,*

$$|\mu(v_n > N) - n\mu(v^0 > N)| = O(W(N, n)); \quad (2.13)$$

^dIn the dynamical setting, ϵ will be fixed at Fact 3.4.

(ii) if $\alpha = 2$, as $n \rightarrow \infty$,

$$|\mu(v_n > N) - n\mu(v^0 > N)| = O(W(N, n)) + o\left(\bar{\Phi}\left(\frac{N}{a_n}\right)\right). \quad (2.14)$$

Proof. First, note that the only place where we required independence inside the proof of Proposition 2.2 was where we translated the content of Theorem 1.1 with $\alpha \neq 1$ in terms of $|\int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))})\psi_Y(t) \frac{\Psi(t)^n - n\Psi(t)}{it} dt|$. As explained below, the same proof (for the converse part) with $\Psi(t)^n$ replaced by $\mathbb{E}_{\mu}(e^{itv_n})$ yields the conclusion of the current proposition.

First, by the same argument as the one used in obtaining (2.3) and (2.4), we have

$$\mu(v_n > N) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \mathbb{E}_{\mu}(e^{itv_n}) dt + o(n\mu(v^0 > N))$$

and $\mu(v^0 > N) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \mathbb{E}_{\mu}(e^{itv^0}) dt$. Also, analogously to (2.5) and (2.6), $\mu(v^0 + Y > N) = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \psi_Y(t) \mathbb{E}_{\mu}(e^{itv^0}) dt$,

$$\begin{aligned} \mu(v_n + Y > N) &= \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \psi_Y(t) \mathbb{E}_{\mu}(e^{itv_n}) dt \\ &\quad + o(n\mu(v^0 > N)) \end{aligned}$$

and

$$\begin{aligned} &|\mu(v_n + Y > N) - n\mu(v^0 > N)| - o(n\mu(v^0 > N)) \\ &= \begin{cases} O(W(N, n)) & \text{if } \alpha \in (0, 1) \cup (1, 2), \\ O(W(N, n)) + o\left(\bar{\Phi}\left(\frac{N}{a_n}\right)\right) & \text{if } \alpha = 2. \end{cases} \end{aligned}$$

It remains to argue that $\mu(v_n + Y > N) = \mu(v_n > N)(1 + o(1))$. If $\alpha \in (0, 1) \cup (1, 2)$, we proceed as in obtaining (2.8), by applying Lemma 2.1 with $Z_n = \tilde{S}_n$, $S_n = \tilde{S}_n - Y$ and $h_N = N^{\frac{\alpha+\delta}{2}} = o(N)$. If $\alpha = 2$, we proceed as in the argument used in obtaining (2.10) with the same choice of h_N as there (and S_n replaced by v_n). $\square$

3. Abstract Setup for Dynamical Systems

Let $T : \Omega \rightarrow \Omega$ be a measure-preserving map on a probability space (Ω, μ) . Let $v : \Omega \rightarrow \mathbb{R}$ be a measurable observable with $\int_{\Omega} v^2 d\mu = \infty$. We fix $\alpha \in (0, 1) \cup (1, 2]$ throughout and assume

$$(H1) \quad \mu(v > x) = p\ell(x)x^{-\alpha}(1 + o(1)), \quad \mu(v \leq -x) = q\ell(x)x^{-\alpha}(1 + o(1)),$$

as $x \rightarrow \infty$, where ℓ is slowly varying and $p + q = 1$.

Recall ℓ and $\hat{\ell}$ from (1.3). Throughout, we shall work with

$$\ell_0 = \begin{cases} \ell & \text{if } \alpha \in (0, 1) \cup (1, 2), \\ \hat{\ell} & \text{if } \alpha = 2. \end{cases} \quad (3.1)$$

Let $R : L^1 \rightarrow L^1$ be the transfer operator for T defined via the formula $\int_{\Omega} Rfgd\mu = \int_{\Omega} fg \circ T d\mu$. Given $t \in \mathbb{R}$, define the perturbed operator $R(t) : L^1 \rightarrow L^1$ by $R(t)f = R(e^{itv}f)$.

We assume that there is a Banach space $\mathcal{B} \subset L^\infty$ containing the constant functions, with norm $\|\cdot\|$ satisfying $|\phi|_\infty \leq \|\phi\|$ for $\phi \in \mathcal{B}$, such that we have the following:

(H2) There exist $\epsilon > 0$, $C > 0$ such that for all $|t|, |h| \in B_\epsilon(0)$:

- (i) $\|R(t)\| \leq C$ for $\alpha \in (0, 1) \cup (1, 2]$ and $\|R'(t)\| \leq C$ for $\alpha \in (1, 2]$;
- (ii) if $\alpha \in (0, 1)$, then $\|R(t+h) - R(t)\| \leq C|h|^\alpha \ell(\frac{1}{|h|})$;
- (iii) if $\alpha \in (1, 2]$, then $\|R(t+h) - R(t) - ihR'(0)\| \leq C|h|(|t|^{\alpha'-1} + |h|^{\alpha'-1})$ for any $\alpha' \in (1, \alpha)$;
- (iv) if $\alpha \in (1, 2]$, then $\|R'(t+h) - R'(t)\| \leq C|h|^{\alpha-1} \ell_0(\frac{1}{|h|})$.

Item (iii) may be not as natural as an estimate on $\|R(t+h) - R(t) - ihR'(t)\|$, but it is what is required in the proofs (it is crucial for the proofs of items (ii) and (iii) of Proposition 3.10), and its validity is checked in Sec. 4. Since $R(0) = R$ and $\mathcal{B}$ contains constant functions, 1 is an eigenvalue of $R(0)$. We assume the following:

(H3) The eigenvalue 1 is simple, and the remainder of the spectrum of $R(0) : \mathcal{B} \rightarrow \mathcal{B}$ is contained in a disk of radius less than 1.

Throughout, write $v_n = \sum_{j=0}^{n-1} v \circ T^j$. It is known, see [2], that under hypotheses (H1)–(H3), distributional convergence $(v_n - b_n)/a_n \rightarrow_d Y_\alpha$ to an α -stable random variable Y_α holds. Here, a_n and b_n are defined in (1.3) and (1.4).

With no loss of generality, when $\alpha \in (1, 2]$ we assume $\int_{\Omega} v d\mu = 0$. Hence, $b_n = 0$ for $\alpha \in (0, 1) \cup (1, 2]$. With this specified, we state the main result in the abstract setup.

Theorem 3.1. *Suppose that (H1)–(H3) hold. Let $\delta > 0$ be arbitrarily small and set*

$$D(x) = \begin{cases} (\log x) \ell(x) x^{-\alpha} & \text{if } \alpha \in (0, 1), \\ x^{-(\alpha-\delta)} & \text{if } \alpha \in (1, 2]. \end{cases}$$

Take $N = N(n)$ such that $N/a_n \rightarrow \infty$. Then, as $n \rightarrow \infty$:

- (i) *if $\alpha \in (0, 1) \cup (1, 2)$, then $|\mu(v_n > N) - n\mu(v > N)| = O(D(N)) + o(n\ell(N)N^{-\alpha})$ and a similar statement holds for $\mu(v_n \leq -N)$;*
- (ii) *if $\alpha = 2$, then $|\mu(v_n > N) - n\mu(v > N)| = O(D(N)) + O(n \log N N^{-2}) + O(n\hat{\ell}(N)N^{-2})$ and a similar statement holds for $\mu(v_n \leq -N)$.*

3.1. Strategy of the proof

In the proof below of Theorem 3.1, we shall proceed by reasoning as in Sec. 2, proving the statement on the right tail. More precisely, as in Sec. 2, we shall work with the inversion formula for distribution functions.

As recalled in Sec. 3.2 below, the relevant characteristic function $\mathbb{E}_\mu(e^{itv_n})$ of the partial sum of dependent quantities $\{v \circ T^j\}_{j \geq 0}$ decomposes into the characteristic function of the partial sum of i.i.d. random variables $\{\hat{v}_j\}_{j \geq 0}$, which we refer to as $\Psi(t)^n$ (see Sec. 3.2.2), and “some other” quantities.

As recalled in Sec. 3.2.2, $\mathbb{E}_\mu(e^{itv}) = \mathbb{E}_\mu(e^{it\hat{v}_j}) = \Psi(t)$ for all t . This together with a fact recorded inside the proof of Proposition 2.2, namely (2.7) (with $X_j = \hat{v}_j$), ensures that given $\epsilon > 0$ small enough,

$$\mu(v > N)(1 + o(1)) = \frac{1}{2\pi} \lim_{n \rightarrow \infty} \int_{-\epsilon}^{\epsilon} \frac{e^{-itN} - e^{-it(N+g(n))}}{it} \psi_Y(t) \Psi(t) dt, \quad (3.2)$$

where Ψ_Y is as in the statement of Proposition 2.2.

By Proposition 2.4 (with $v^j = v \circ T^j$) and (3.2), the conclusion of Theorem 3.1 follows once we show that for some $\epsilon > 0$,

$$\left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\mathbb{E}_\mu(e^{itv_n}) - n\Psi(t)}{it} dt \right| \quad (3.3)$$

is $O(D(N)) + o(n\ell_0(N)N^{-\alpha})$ when $\alpha \in (0, 1) \cup (1, 2)$ and, respectively,

$$\left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\mathbb{E}_\mu(e^{itv_n}) - n\Psi(t)}{it} dt \right| \quad (3.4)$$

is $O(D(N)) + O(n \log NN^{-2}) + O(n\hat{\ell}(N)N^{-2})$ if $\alpha = 2$. Here, we have used that $\bar{\Phi}(N/a_n) \ll (N/a_n)^{-2} \ll n\hat{\ell}(N)N^{-2}$.

Writing $\mathbb{E}_\mu(e^{itv_n}) = \Psi(t)^n + (\mathbb{E}_\mu(e^{itv_n}) - \Psi(t)^n)$ and using the “rewrite” recorded in Proposition 2.2, Eqs. (3.3) and (3.4) follow as soon as we show that

$$\left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\mathbb{E}_\mu(e^{itv_n}) - \Psi(t)^n}{it} dt \right| \quad (3.5)$$

is $O(D(N)) + o(n\ell_0(N)N^{-\alpha})$ when $\alpha \in (0, 1) \cup (1, 2)$ and

$$\left| \int_{-\epsilon}^{\epsilon} (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{\mathbb{E}_\mu(e^{itv_n}) - \Psi(t)^n}{it} dt - \bar{\Phi}(N/a_n) \right| \quad (3.6)$$

is $O(D(N)) + O(n \log NN^{-2}) + O(n\hat{\ell}(N)N^{-2})$ if $\alpha = 2$.

Since $\mathbb{E}_\mu(e^{itv_n}) = \int_{\Omega} R(t)^n 1 d\mu$, we take the next subsection to decompose $\int_{\Omega} R(t)^n 1 d\mu$ into $\Psi(t)^n$ and “good” quantities. The idea is to isolate $\Psi(t)^n$ in the expression of $\int_{\Omega} R(t)^n 1 d\mu$ so as to be able to use the equivalent of (2.1).

The meaning of “good” quantities will become clear in Sec. 3.3 below, where we complete the proof of Theorem 3.1. At this stage, we can point out that the

proof uses a “modulus of continuity” argument (and also a derivative in the case $\alpha \in (1, 2]$). Although this type of argument has some similarity with the ones used in [15], the current arguments are much more delicate. To carry out a “modulus of continuity”-type argument, it is crucial that we obtain enough decay in t in the expression of $\int_{\Omega} R(t)^n 1 d\mu - \Psi(t)^n$. This is necessary to counteract the effect of the division by t in the integrand in (3.5). The details are postponed to Sec. 3.3.

3.2. Decomposing $\int_{\Omega} R(t)^n 1 d\mu$ into $\Psi(t)^n$ and “good” quantities

The main result of this section is Proposition 3.10 stated at the end of this subsection.

We first recall some general facts under (H1)–(H3), some of them being recently clarified in [15, Sec. 3.2].

By (H2) and (H3), there exist $\epsilon > 0$ and a continuous family $\lambda(t)$ of simple eigenvalues of $R(t)$ for $|t| \leq 3\epsilon$ with $\lambda(0) = 1$. The associated spectral projections $P(t)$, $|t| \leq 3\epsilon$, form a continuous family of bounded linear operators on $\mathcal{B}$. Moreover, there exist a continuous family of linear operators $Q(t)$ on $\mathcal{B}$ and the constants $C > 0$, $\delta_0 \in (0, 1)$ such that for $|t| \leq 3\epsilon$,

$$R(t) = \lambda(t)P(t) + Q(t) \quad \text{and} \quad \|Q(t)^n\| \leq C\delta_0^n. \quad (3.7)$$

Let $\zeta(t) = \frac{P(t)1}{\int P(t)1 d\mu}$ be the normalized eigenvector corresponding to $\lambda(t)$.

Lemma 3.2. *Assume (H2). There exists $\epsilon > 0$ such that the properties of $R(t)$ listed in (H2) are inherited by $P(t)$, $Q(t)$, $\lambda(t)$ and $\zeta(t)$ for all $|t|, |t+h| \leq 3\epsilon$.*

Lemma 3.2 holds with a simplified version of (H2)(iii) obtained by moving $ihR'(0)$ to the other side of the $\ll$ -sign; namely, for $\alpha \in (1, 2]$,

$$\|R(t+h) - R(t)\| \ll |h|. \quad (3.8)$$

The full strength of (H2)(iii) is exploited in Lemma 3.3 below.

3.2.1. A consequence of (H2)(iii) for the operators P and Q

Lemma 3.3. *Let $\alpha \in (1, 2]$. There exists $C > 0$ so that the following hold for all $|t|, |t+h| \in B_{3\epsilon}(0)$ and for any $\alpha' \in (1, \alpha)$:*

- (i) $\|P(t+h) - P(t) - ihP'(0)\| \leq C|h|(|t|^{\alpha'-1} + |h|^{\alpha'-1})$. A similar statement holds for $\zeta(t) = \frac{P(t)1}{\int P(t)1 d\mu}$.
- (ii) $\|Q(t+h) - Q(t) - ihQ'(0)\| \leq C|h|(|t|^{\alpha'-1} + |h|^{\alpha'-1})$.

Proof. (i) Recall that $\lambda(t)$ is an isolated eigenvalue in the spectrum of $R(t)$ for every $|t| \leq 3\epsilon$. Hence, for any $\delta \in (0, 3\epsilon)$, $P(t) = \frac{1}{2\pi i} \int_{|\xi-1|=\delta} (\xi - R(t))^{-1} d\xi$ and

thus, $P'(t) = \frac{1}{2\pi i} \int_{|\xi-1|=\delta} (\xi - R(t))^{-1} R'(t) (\xi - R(t))^{-1} d\xi$. With these specified, we see that

$$\begin{aligned} & 2\pi i(P(t+h) - P(t)) \\ &= \int_{|\xi-1|=\delta} (\xi - R(t+h))^{-1} (R(t+h) - R(t)) (\xi - R(t))^{-1} d\xi \\ &= \int_{|\xi-1|=\delta} (\xi - R(t+h))^{-1} (R(t+h) - R(t) - ihR'(0)) (\xi - R(t))^{-1} d\xi \\ &\quad + ih \int_{|\xi-1|=\delta} (\xi - R(t+h))^{-1} R'(0) (\xi - R(t))^{-1} d\xi = I(t, h) + J(t, h). \end{aligned}$$

By (H2)(iii), $\|I(t, h)\| \ll |h|(|t|^{\alpha'-1} + |h|^{\alpha'-1})$, for any $\alpha' \in (1, \alpha)$. Also,

$$\begin{aligned} J(t, h) &= ih \int_{|\xi-1|=\delta} (\xi - R(0))^{-1} R'(0) (\xi - R(t))^{-1} d\xi \\ &\quad + ih \int_{|\xi-1|=\delta} ((\xi - R(t+h))^{-1} - (\xi - R(0))^{-1}) R'(0) (\xi - R(t))^{-1} d\xi \\ &= ih \int_{|\xi-1|=\delta} (\xi - R(0))^{-1} R'(0) (\xi - R(0))^{-1} d\xi \\ &\quad + ih \int_{|\xi-1|=\delta} ((\xi - R(t+h))^{-1} - (\xi - R(0))^{-1}) R'(0) (\xi - R(t))^{-1} d\xi \\ &\quad + ih \int_{|\xi-1|=\delta} (\xi - R(0))^{-1} R'(0) ((\xi - R(t))^{-1} - (\xi - R(0))^{-1}) d\xi \\ &= ih(2\pi i)P'(0) + ih(J_1(t, h) + J_2(t, h)). \end{aligned}$$

By (3.8), i.e. the simplified version of (H2)(iii), we have $\|R(t+h) - R(0)\| \leq \|R(t) - R(0)\| + \|R(t+h) - R(t)\| \ll |t| + |h|$ and thus, $\|J_1(t, h)\|, \|J_2(t, h)\| \ll |h|(|t| + |h|)$.

Altogether, $P(t+h) - P(t) - ihP'(0) = (2\pi i)^{-1}I(t, h) + (2\pi i)^{-1}h(J_1(t, h) + J_2(t, h))$ and each term satisfies the claimed estimate, concluding the proof of the statement on $P(t)$.

The statement on $\zeta(t) = \frac{P(t)1}{\int P(t)1d\mu}$ follows directly from the definition and the statement on $P(t)$.

(ii) We shall use that $Q(t) = R(t)(I - P(t))$ (here, I is the identity operator). So, $Q'(t) = (R(t)(I - P(t)))' = R'(t)(I - P(t)) - R(t)P'(t)$ and thus,

$$Q'(0) = R'(0) - R(0)P'(0) - R'(0)P(0). \quad (3.9)$$

With this specified, we compute that

$$\begin{aligned}
 Q(t+h) - Q(t) &= R(t+h) - R(t) + R(t)P(t) - R(t+h)P(t+h) \\
 &= R(t+h) - R(t) + R(t)(P(t) - P(t+h)) - (R(t+h) - R(t))P(t+h) \\
 &= (R(t+h) - R(t) - ihR'(0)) - R(t)(P(t+h) - P(t) - ihP'(0)) \\
 &\quad - (R(t+h) - R(t) - ihR'(0))P(t+h) \\
 &\quad + ihR'(0) - ihR(t)P'(0) - ihR'(0)P(t+h).
 \end{aligned}$$

Using (H2)(iii) and item (i) above,

$$\begin{aligned}
 &\|Q(t+h) - Q(t) - ih(R'(0) - R(t)P'(0) + R'(0)P(t+h))\| \\
 &\ll |h|(|t|^{\alpha'-1} + |h|^{\alpha'-1}),
 \end{aligned}$$

for any $\alpha' \in (1, \alpha)$. But, by (H2)(iii) and (3.9),

$$\begin{aligned}
 &ih(R'(0) - R(t)P'(0) - R'(0)P(t+h)) \\
 &= ih(R'(0) - R(0)P'(0) - R'(0)P(0)) + D(t, h) \\
 &= ihQ'(0) + D(t, h),
 \end{aligned}$$

where $\|D(t, h)\| \ll |h|(|t| + |h|)$. □

3.2.2. Some properties of λ and details on Ψ

Let $\Psi(t)$ be the characteristic function of the i.i.d. random variables $\hat{v}_j : \Omega \rightarrow \mathbb{R}$ so that $\hat{v}_j$ and $v \circ T^j$ have the same distribution.

Similarly to v , we take $\int_{\Omega} \hat{v}_j d\mu = 0$ (when $\alpha > 1$). Since $\hat{v}_j$ satisfies (H1),

$$1 - \Psi(t) = c_{\alpha} \ell_0 \left(\frac{1}{|t|} \right) |t|^{\alpha} (1 + o(1)) \quad \text{as } t \rightarrow 0. \quad (3.10)$$

Equation (3.10) has been established in [7, 12] (see also [2, Theorem 5.1]). Also, as established in these works (in particular, [2]) and clarified in [15, Lemma 2.1], we have the following.

Fact 3.4. There exist constants $\epsilon, c > 0$, such that

$$|\Psi(t)| \leq \exp \left\{ -c|t|^{\alpha} \ell_0 \left(\frac{1}{|t|} \right) \right\} \quad \text{for all } t \in B_{3\epsilon}(0).$$

Throughout the rest of this paper, we fix ϵ so that both Fact 3.4 and Eq. (3.7) hold.

It is known that there exists $C > 0$ so that for all $|t|, |t+h| \in B_{3\epsilon}(0)$,

$$|\Psi(t+h) - \Psi(t)| \leq \begin{cases} C|h|^\alpha \ell\left(\frac{1}{|h|}\right), & \alpha \in (0, 1), \\ C|h|, & \alpha \in (1, 2]. \end{cases} \quad (3.11)$$

Furthermore, if $\alpha \in (1, 2]$, then Ψ is differentiable and writing Ψ' for its derivative,

$$|\Psi'(t)| \leq C\ell_0 \left(\frac{1}{|t|}\right) |t|^{\alpha-1}, \quad (3.12)$$

for some $C > 0$. For a precise reference for the validity of (3.11) and (3.12), see, for instance, [15, Lemma 2.2].

Lemma 3.5. *Let $V(t) = \lambda(t) - \Psi(t)$. There exist C, C' so that the following hold for all $|t|, |t+h| \in B_{3\epsilon}(0)$:*

(i) *If $\alpha \in (0, 1)$, then $|V(t)| \leq C|t|^{2\alpha} \ell(\frac{1}{|t|})^2$ and $|V(t+h) - V(t)| \leq C'|h|^\alpha \ell(\frac{1}{|h|})|t|^\alpha \ell(\frac{1}{|t|})$.*

If $\alpha \in (1, 2]$, then $|V(t)| \leq C|t|^2$ and $|V(t+h) - V(t)| \leq C'|h|(|t| + |h|)$.

(ii) *If $\alpha \in (1, 2]$, then V is differentiable and writing V' for its derivative, $|V'(t)| \leq C|t|$ and $|V'(t+h) - V'(t)| \leq C|h|^{\alpha'-1} \ell(\frac{1}{|h|})(|t| + |h|) + C'|h|$, for any $\alpha' \in (1, \alpha)$.*

Before proceeding to the proof, we note that Lemma 3.5 together with (3.10) and (3.12) implies that

$$\begin{aligned} 1 - \lambda(t) &= c_\alpha \ell_0 \left(\frac{1}{|t|}\right) |t|^\alpha (1 + o(1)) \quad \text{and for } \alpha \in (1, 2], \\ |\lambda'(t)| &\leq C\ell_0 \left(\frac{1}{|t|}\right) |t|^{\alpha-1}, \end{aligned} \quad (3.13)$$

for some $C > 0$.

Proof. Proof of Lemma 3.5 Write

$$\begin{aligned} \lambda(t) &= \int R(t)\zeta(t)d\mu = \int_\Omega R(t)1d\mu + \int (R(t) - R(0))(\zeta(t) - \zeta(0))d\mu \\ &= \int_\Omega e^{itv}d\mu + \int_\Omega (R(t) - R(0))(\zeta(t) - \zeta(0))d\mu = \Psi(t) + V(t). \end{aligned}$$

This gives the (well-known) decomposition of λ into Ψ and V in the statement of the lemma. We still need to clarify the properties of V .

Since $\mathcal{B} \subset L^\infty$, item (i) for the case $\alpha \in (0, 1)$ follows immediately from (H2)(ii) and Lemma 3.2. The second part of item (i) for the case $\alpha \in (1, 2]$ follows similarly using that by (3.8), $\|R(t+h) - R(0)\| \ll |t| + |h|$.

For item (ii), we first note that

$$V'(t) = \int_{\Omega} (R(t) - R(0))\zeta'(t)d\mu + \int_{\Omega} R'(t)(\zeta(t) - \zeta(0))d\mu.$$

Lemma 3.2 and the fact together with $\|R(t) - R(0)\| \ll |t|$ imply that $|V'(t)| \ll |t|$. Next, compute that

$$\begin{aligned} V'(t+h) - V'(t) &= \int_{\Omega} (R(t+h) - R(t))\zeta'(t+h)d\mu + \int_{\Omega} (R(t) - R(0))(\zeta'(t+h) - \zeta'(t))d\mu \\ &\quad + \int_{\Omega} (R'(t+h) - R'(t))(\zeta(t) - \zeta(0))d\mu + \int_{\Omega} R'(t+h)(\zeta(t+h) - \zeta(t))d\mu \\ &= I_1(t, h) + I_2(t, h) + I_3(t, h) + I_4(t, h). \end{aligned}$$

Recall $\|R'(t)\|, |\zeta'(t)| \ll 1$ and $\|R(t+h) - R(0)\| \ll |t| + |h|$. These together with (H2)(iv) and Lemma 3.2 give that $|I_2(t, h)|, |I_4(t, h)| \ll |h|^{\alpha-1}\ell_0(\frac{1}{|h|})(|t| + |h|)$. Regarding I_1 , compute that

$$\begin{aligned} I_1(t, h) &= ih \int_{\Omega} R'(0)\zeta'(t+h)d\mu + \int_{\Omega} (R(t+h) - R(t) - ihR'(0))\zeta'(t+h)d\mu \\ &= ih \int_{\Omega} R'(0)\zeta'(0)d\mu + ih \int_{\Omega} R'(0)(\zeta'(t+h) - \zeta'(0))d\mu \\ &\quad + \int_{\Omega} (R(t+h) - R(t) - ihR'(0))\zeta'(t+h)d\mu. \end{aligned}$$

Similarly,

$$\begin{aligned} I_4(t, h) &= ih \int_{\Omega} R'(0)\zeta'(0)d\mu + ih \int_{\Omega} (R'(t+h) - R'(0))\zeta'(0)d\mu \\ &\quad + \int_{\Omega} R'(t+h)(\zeta(t+h) - \zeta(t) - ih\zeta'(0))d\mu. \end{aligned}$$

By (H2)(iii), (H2)(iv) and Lemma 3.2,

$$\left| I_1(t, h) - ih \int_{\Omega} R'(0)\zeta'(0)d\mu \right| \ll |h|(|t|^{\alpha'-1} + |h|^{\alpha'-1}),$$

for any $\alpha' \in (1, \alpha)$. By (H2)(iv) (together with Lemma 3.2) and Lemma 3.3(i),

$$\left| I_4(t, h) - ih \int_{\Omega} R'(0)\zeta'(0)d\mu \right| \ll |h|(|t|^{\alpha'-1} + |h|^{\alpha'-1}),$$

for any $\alpha' \in (1, \alpha)$. Hence,

$$|I_1(t, h) + I_4(t, h)| \ll |h| + |h|(|t|^{\alpha'-1} + |h|^{\alpha'-1}),$$

for any $\alpha' \in (1, \alpha)$, ending the proof. $\square$

For use in the proofs to follow, we recall one more property of λ , when $\alpha \in (1, 2]$; this is a refined version of the continuity properties recorded in Lemma 3.2. (This type of estimate appears inside [15, Proof of Theorem 3.2]).

Lemma 3.6. *If $\alpha \in (1, 2]$, then there exists $C > 0$ so that for all $t, t+h \in B_{3\epsilon}(0)$ with $|h| \leq \frac{|t|}{2}$,*

$$|\lambda(t+h) - \lambda(t)| \leq C|h|(|t|^{\alpha-1} + |h|^{\alpha-1})\ell_0\left(\frac{1}{|t|+|h|}\right).$$

A similar statement holds for Ψ .

Proof. By the mean value theorem, $\lambda(t+h) - \lambda(t) = h\lambda'(t^*)$, for some $t^* \in [t, t+h]$ and the conclusion follows. $\square$

3.2.3. *A first decomposition of $\int_{\Omega} R(t)^n d\mu$ into $\Psi(t)^n$ and “good” quantities*

Lemma 3.7. *Let*

$$\hat{V}(t, n) = \int_{\Omega} R(t)^n 1 d\mu - \Psi(t)^n - \int_{\Omega} Q(t)^n 1 d\mu.$$

The following hold for all $n \geq 1$ and all $|t|, |h| \in B_{\epsilon}(0)$ with $|h| \leq |t|/2$:

(i) *If $\alpha \in (0, 1)$, then*

$$|\hat{V}(t, n)| \ll \left(|t|^{\alpha} \ell\left(\frac{1}{|t|}\right) + n|t|^{2\alpha} \ell\left(\frac{1}{|t|}\right)^2\right) |\Psi(t)^n|,$$

$$|\hat{V}(t+h, n) - \hat{V}(t, n)| \ll \left(|h|^{\alpha} \ell\left(\frac{1}{|h|}\right) + n|h|^{\alpha} \ell\left(\frac{1}{|h|}\right) |t|^{\alpha} \ell\left(\frac{1}{|t|}\right)\right) |\Psi(t)^n|.$$

(ii) *If $\alpha \in (1, 2]$, then for any $\alpha' \in (1, \alpha)$,*

$$\left|\hat{V}(t, n) - it\lambda(t)^n \int_{\Omega} P'(0) 1 d\mu\right| \ll (|t|^{\alpha'} + n|t|^2) |\Psi(t)^n|.$$

(iii) *If $\alpha \in (1, 2]$, then for any $\alpha' \in (1, \alpha)$,*

$$\begin{aligned} & \left|\hat{V}(t+h, n) - \hat{V}(t, n) - ih\lambda(t)^n \int_{\Omega} P'(0) 1 d\mu\right| \\ & \ll \left(|h||t|^{\alpha'-1} + n|h||t|^{\alpha-1} \ell_0\left(\frac{1}{|t|}\right)\right) |\Psi(t)^n|. \end{aligned}$$

(iv) *If $\alpha \in (1, 2]$, then $\hat{V}$ is differentiable in t and writing $\hat{V}'(t, n)$ for its derivative,*

$$\begin{aligned} & \left|\hat{V}'(t, n) - \lambda(t)^n \int_{\Omega} P'(0) 1 d\mu\right| \\ & \leq \left(|t|^{\alpha-1} \ell_0\left(\frac{1}{|t|}\right) + n|t| + n^2|t|^{\alpha+1} \ell_0\left(\frac{1}{|t|}\right)\right) |\Psi(t)^n| \end{aligned}$$

and

$$\begin{aligned}
 & \left| \hat{V}'(t+h, n) - \hat{V}'(t, n) - i h n \lambda(t)^{n-1} \lambda'(t) \int_{\Omega} P'(0) 1 d\mu \right| \\
 & \ll |h|^{\alpha-1} \ell_0 \left(\frac{1}{|h|} \right) |\Psi(t)^n| + n \left(|t| |h|^{\alpha'-1} \ell_0 \left(\frac{1}{|h|} \right) + |h| \right) |\Psi(t)^n| \\
 & + n^2 \left(|t|^{2\alpha-1} |h| \ell_0 \left(\frac{1}{|t|} \right)^2 + |t|^2 |h|^{\alpha-1} \ell_0 \left(\frac{1}{|h|} \right) \right. \\
 & \left. + |h| |t|^{\alpha} \ell_0 \left(\frac{1}{|t|} \right) \right) |\Psi(t)^n| + n^3 |t|^{2\alpha} |h| \ell_0 \left(\frac{1}{|t|} \right)^2 |\Psi(t)^n|.
 \end{aligned}$$

Proof. By (3.7),

$$\begin{aligned}
 & \int_{\Omega} R(t)^n 1 d\mu - \int_{\Omega} Q(t)^n 1 d\mu \\
 & = \lambda(t)^n \int_{\Omega} P(t) 1 d\mu \\
 & = \lambda(t)^n \int_{\Omega} P(0) 1 d\mu + \lambda(t)^n \int_{\Omega} (P(t) - P(0)) 1 d\mu \\
 & = \Psi(t)^n + (\lambda(t)^n - \Psi(t)^n) + \lambda(t)^n \int_{\Omega} (P(t) - P(0)) 1 d\mu.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \hat{V}(t, n) &= (\lambda(t)^n - \Psi(t)^n) + \lambda(t)^n \int_{\Omega} (P(t) - P(0)) 1 d\mu \\
 &= A(t, n) + B(t, n).
 \end{aligned} \tag{3.14}$$

Throughout the rest of the proof we will use that by Lemma 3.5, $|\lambda(t)^n| \ll |\Psi(t)^n|$.

Item (i). Note that by Lemma 3.2 and (H2)(ii), $|B(t, n)| \ll |t|^{\alpha} \ell\left(\frac{1}{|t|}\right) |\Psi(t)^n|$. Next, using Lemma 3.2 and (H2)(ii),

$$|B(t+h, n) - B(t, n)| \leq |h|^{\alpha} \ell\left(\frac{1}{|h|}\right) |\Psi(t)^n| + |\lambda(t+h)^n - \lambda(t)^n| |t|^{\alpha} \ell\left(\frac{1}{|t|}\right).$$

Note that $\lambda(t+h)^n - \lambda(t)^n = (\lambda(t+h) - \lambda(t)) \sum_{j=0}^{n-1} \lambda(t+h)^j \lambda(t)^{n-j-1}$. This together with Lemma 3.2 implies that $|\lambda(t+h)^n - \lambda(t)^n| \ll n |h|^{\alpha} \ell\left(\frac{1}{|h|}\right) |\Psi(t)^n|$. Hence,

$$|B(t+h, n) - B(t, n)| \leq |h|^{\alpha} \ell\left(\frac{1}{|h|}\right) \left(n |t|^{\alpha} \ell\left(\frac{1}{|t|}\right) + 1 \right) |\Psi(t)^n|.$$

It remains to deal with $A(t, n)$ in (3.14). Given $V(t) = \lambda(t) - \Psi(t)$ as in Lemma 3.5, $A(t, n) = V(t) \sum_{j=0}^{n-1} \lambda(t)^j \Psi(t)^{n-j-1}$. This together with the properties of V in Lemma 3.5(i) implies that $|A(t, n)| \ll n |t|^{2\alpha} \ell\left(\frac{1}{|t|}\right)^2 |\Psi(t)^n|$. To deal with the

continuity properties of $A(t, n)$, we first compute that

$$\begin{aligned}
 & (\lambda(t+h)^n - \Psi(t+h)^n) - (\lambda(t)^n - \Psi(t)^n) \\
 &= V(t+h) \sum_{j=0}^{n-1} \lambda(t+h)^j \Psi(t+h)^{n-j-1} - V(t) \sum_{j=0}^{n-1} \lambda(t)^j \Psi(t)^{n-j-1} \\
 &= (V(t+h) - V(t)) \sum_{j=0}^{n-1} \lambda(t+h)^j \Psi(t+h)^{n-j-1} \\
 &\quad + V(t) \sum_{j=0}^{n-1} (\lambda(t+h)^j - \lambda(t)^j) \Psi(t+h)^{n-j-1} \\
 &\quad + V(t) \sum_{j=0}^{n-1} \lambda(t)^j (\Psi(t+h)^{n-j-1} - \Psi(t)^{n-j-1}). \tag{3.15}
 \end{aligned}$$

Using (3.15), we compute that

$$|A(t+h, n) - A(t, n)| \ll I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
 I_1 &= |V(t+h) - V(t)| \sum_{j=0}^{n-1} |\lambda(t+h)^j| |\Psi(t+h)^{n-j-1}| \\
 &\ll n|h|^\alpha \ell \left(\frac{1}{|h|} \right) |t|^\alpha \ell \left(\frac{1}{|t|} \right) |\Psi(t+h)^n|, \\
 I_2 &= |V(t)| \sum_{j=0}^{n-1} |\lambda(t+h)^j - \lambda(t)^j| |\Psi(t+h)^{n-j-1}| \\
 &\ll n|h|^\alpha \ell \left(\frac{1}{|h|} \right) |t|^{2\alpha} \ell \left(\frac{1}{|t|} \right)^2 |\Psi(t+h)^n|,
 \end{aligned}$$

where we have used again Lemmas 3.2 and 3.5, and

$$\begin{aligned}
 I_3 &= |V(t)| \sum_{j=0}^{n-1} |\lambda(t)^j| |\Psi(t+h)^{n-j-1} - \Psi(t)^{n-j-1}| \\
 &\ll n|h|^\alpha \ell \left(\frac{1}{|h|} \right) |t|^{2\alpha} \ell \left(\frac{1}{|t|} \right)^2 |\Psi(t)^n|,
 \end{aligned}$$

where we have used (3.11) as well.

Item (ii). We continue from (3.14). First, $|A(t, n)| \ll n|V(t)||\Psi(t)^n|$ and using Lemma 3.5(i) (the statement on V for the case $\alpha \in (1, 2]$), $|A(t, n)| \ll n|t|^2 |\Psi(t)^n|$.

Next, we rewrite $B(t, n)$ in a convenient way,

$$B(t, n) = it\lambda(t)^n \int_{\Omega} P'(0)1d\mu + \lambda(t)^n \int_{\Omega} (P(t) - P(0) - itP'(0))1d\mu.$$

By Lemma 3.3(i), $|\lambda(t)^n \int_{\Omega} (P(t) - P(0) - itP'(0))1d\mu| \ll |t|^{\alpha'} |\Psi(t)^n|$ for any $\alpha' \in (1, \alpha)$, and the conclusion follows.

Item (iii). We start from (3.14). First,

$$\begin{aligned} |A(t+h, n) - A(t, n)| &\ll |\lambda(t+h)^n - \lambda(t)^n| + |\Psi(t+h)^n - \Psi(t)^n| \\ &\ll n|\lambda(t+h) - \lambda(t)| |\lambda(t)^n| + n|\Psi(t+h) - \Psi(t)| |\Psi(t)^n| \\ &\ll n|h||t|^{\alpha-1} \ell_0 \left(\frac{1}{|t|} \right) |\Psi(t)^n|, \end{aligned}$$

where we have used Lemma 3.6.

It remains to deal with B defined in (3.14),

$$\begin{aligned} B(t+h, n) - B(t, n) &= \lambda(t)^n \int_{\Omega} (P(t+h) - P(t))1d\mu + (\lambda(t+h)^n - \lambda(t)^n) \\ &\quad \times \int_{\Omega} (P(t+h) - P(0))1d\mu \\ &= J_1(t, h, n) + J_2(t, h, n). \end{aligned}$$

First, by Lemmas 3.2 and 3.6,

$$|J_2(t, h, n)| \ll n|\lambda(t+h) - \lambda(t)| |\lambda(t)^n| |t+h| \ll n|h||t|^{\alpha} \ell_0 \left(\frac{1}{|t|} \right) |\Psi(t)^n|.$$

Now

$$J_1(t, h, n) = ih\lambda(t)^n \int_{\Omega} P'(0)1d\mu + \lambda(t)^n \int_{\Omega} (P(t+h) - P(t) - ihP'(0))1d\mu.$$

Using Lemma 3.3(i), we obtain that for any $\alpha' \in (1, \alpha)$,

$$\left| J_1(t, h, n) - ih\lambda(t)^n \int_{\Omega} P'(0)1d\mu \right| \ll |h|(|t|^{\alpha'-1} + |h|^{\alpha'-1}) |\Psi(t)^n|,$$

ending the proof of (iii).

Item (iv). Differentiating in (3.14),

$$\begin{aligned} \hat{V}'(t, n) &= n(\lambda(t)^{n-1} \lambda'(t) - \Psi(t)^{n-1} \Psi'(t)) + n\lambda(t)^{n-1} \lambda'(t) \int_{\Omega} (P(t) - P(0))1d\mu \\ &\quad + \lambda(t)^n \int_{\Omega} P'(t)1d\mu = A_0(t, n) + A_1(t, n) + A_2(t, n). \end{aligned} \quad (3.16)$$

Recalling from Lemma 3.5 that $V(t) = \lambda(t) - \Psi(t)$, we compute that

$$\begin{aligned} A_0(t, n) &= n\lambda'(t)(\lambda(t)^{n-1} - \Psi(t)^{n-1}) + n\Psi(t)^{n-1}(\lambda'(t) - \Psi'(t)) \\ &= n\lambda'(t)(\lambda(t)^{n-1} - \Psi(t)^{n-1}) + n\Psi(t)^{n-1}V'(t). \end{aligned} \quad (3.17)$$

It follows from Eq. (3.13) and Lemma 3.5(i) and (ii) that

$$\begin{aligned} |A_0(t, n)| &\ll n|t|^{\alpha-1}\ell_0\left(\frac{1}{|t|}\right)n|V(t)||\Psi(t)^n| + n|V'(t)||\Psi(t)^n| \\ &\ll n^2|t|^{\alpha+1}\ell_0\left(\frac{1}{|t|}\right)|\Psi(t)^n| + n|t||\Psi(t)^n|. \end{aligned}$$

Regarding A_1 , using Eq. (3.13) (statement on the derivative) and Lemma 3.2, we obtain $|A_1(t, n)| \ll n|t|^\alpha\ell_0(\frac{1}{|t|})|\Psi(t)^n|$. For A_2 , write

$$\begin{aligned} A_2(t, n) &= \lambda(t)^n \int_{\Omega} P'(0)1d\mu + \lambda(t)^n \int_{\Omega} (P'(t) - P'(0))1d\mu \\ &= \lambda(t)^n \int_{\Omega} P'(0)1d\mu + A_2^1(t, n). \end{aligned} \quad (3.18)$$

By Lemma 3.2 and (H2)(iv), $|A_2^1(t, n)| \ll |t|^{\alpha-1}\ell_0(\frac{1}{|t|})|\Psi(t)^n|$. Thus,

$$\left| A_2(t, n) - \lambda(t)^n \int_{\Omega} P'(0)1d\mu \right| \ll |t|^{\alpha-1}\ell_0\left(\frac{1}{|t|}\right)|\Psi(t)^n|.$$

Putting together the estimates for A_0 , A_1 and A_2 ,

$$\begin{aligned} &\left| \hat{V}'(t, n) - \lambda(t)^n \int_{\Omega} P'(0)1d\mu \right| \\ &\ll \left(n^2|t|^{\alpha+1}\ell_0\left(\frac{1}{|t|}\right) + n|t| \right) |\Psi(t)^n| + n|t|^\alpha\ell_0\left(\frac{1}{|t|}\right) |\Psi(t)^n| \\ &\quad + |t|^{\alpha-1}\ell_0\left(\frac{1}{|t|}\right) |\Psi(t)^n| \\ &\ll \left(|t|^{\alpha-1}\ell_0\left(\frac{1}{|t|}\right) + n|t| + n^2|t|^{\alpha+1}\ell_0\left(\frac{1}{|t|}\right) \right) |\Psi(t)^n|, \end{aligned}$$

as claimed.

We continue with the continuity properties of $\hat{V}'$ studying each term in (3.14).

Term A_0 in (3.16). Using the expression (3.17), we compute that

$$\begin{aligned}
 & |A_0(t+h, n) - A_0(t, n)| \\
 & \ll n^2 |\lambda'(t+h) - \lambda'(t)| |\Psi(t)^{n-1}| |V(t)| \\
 & \quad + n |\lambda'(t)| |(\lambda(t+h)^{n-1} - \Psi(t+h)^{n-1}) - (\lambda(t)^{n-1} - \Psi(t)^{n-1})| \\
 & \quad + n |\Psi(t+h)^{n-1} - \Psi(t)^{n-1}| |V'(t)| + n |\Psi(t)^{n-1}| |V'(t+h) - V'(t)| \\
 & = I_1(t, h, n) + I_2(t, h, n) + I_3(t, h, n) + I_4(t, h, n).
 \end{aligned}$$

By Lemma 3.5(ii), $I_4(t, h, n) \ll n |h|^{\alpha'-1} \ell_0(\frac{1}{|h|}) (|t| + |h|) |\Psi(t)^{n-1}| + n |h| |\Psi(t)^{n-1}|$.

By Lemma 3.6 (statement on Ψ), Lemma 3.5(ii) and (3.12), $I_3(t, h, n) \ll n^2 |h| |t| \ell_0(\frac{1}{|t|+|h|}) (|t|^{\alpha-1} + |h|^{\alpha-1}) |\Psi(t)^n| \ll n^2 |h| |t| \ell_0(\frac{1}{|t|+|h|}) |t|^{\alpha-1} |\Psi(t)^n|$.

We continue with I_2 . First, recalling Eq. (3.12) and using Lemmas 3.5 and 3.6,

$$\begin{aligned}
 & |(\lambda(t+h)^n - \Psi(t+h)^n) - (\lambda(t)^n - \Psi(t)^n)| \\
 & \ll n |V(t+h) - V(t)| |\Psi(t)^{n-1}| + n^2 |V(t)| |\lambda(t+h) - \lambda(t)| |\Psi(t)^{n-1}| \\
 & \quad + n^2 |V(t)| |\Psi(t+h) - \Psi(t)| |\Psi(t)^{n-1}| \\
 & \ll n |h| (|t| + |h|) |\Psi(t)^{n-1}| + n^2 |h| |t|^2 (|t|^{\alpha-1} + |h|^{\alpha-1}) \\
 & \quad \times \ell_0\left(\frac{1}{|t|+|h|}\right) |\Psi(t)^{n-1}|. \tag{3.19}
 \end{aligned}$$

By Eqs. (3.19) and (3.13),

$$\begin{aligned}
 I_2(t, h, n) & \ll n^2 |h| |t|^\alpha \ell_0\left(\frac{1}{|t|+|h|}\right) |\Psi(t)^{n-1}| \\
 & \quad + n^3 |h| |t|^{2\alpha} \ell_0\left(\frac{1}{|t|+|h|}\right)^2 |\Psi(t)^{n-1}|.
 \end{aligned}$$

Finally, by Lemma 3.2 and (H2)(iv), $|\lambda'(t+h) - \lambda'(t)| \ll h^{\alpha-1} \ell_0(1/|h|)$. This together with Lemma 3.5(i) gives that $I_1(t, h, n) \ll n^2 |h|^{\alpha-1} |t|^2 \ell_0(\frac{1}{|t|+|h|}) |\Psi(t)^n|$. Putting together the estimates for I_4 , I_3 , I_2 and I_1 , we obtain

$$\begin{aligned}
 & |A_0(t+h, n) - A_0(t, n)| \\
 & \ll \left(n |h|^{\alpha-1} \ell_0\left(\frac{1}{|h|}\right) (|t| + |h|) + n |h| \right) |\Psi(t)^{n-1}| \\
 & \quad + \left(n^2 |h|^{\alpha-1} \ell_0\left(\frac{1}{|h|}\right) |t|^2 + n^2 |h| |t|^\alpha \ell_0\left(\frac{1}{|t|+|h|}\right) \right) |\Psi(t)^{n-1}| \\
 & \quad + n^3 |h| |t|^{2\alpha} \ell_0\left(\frac{1}{|t|+|h|}\right)^2 |\Psi(t)^{n-1}|. \tag{3.20}
 \end{aligned}$$

Term A_1 in (3.16). Here we have

$$\begin{aligned} & A_1(t+h, n) - A_1(t, n) \\ &= n\lambda(t)^{n-1}\lambda'(t) \int_{\Omega} (P(t+h) - P(t))1d\mu \\ &\quad + n(\lambda(t+h)^{n-1} - \lambda(t)^{n-1})\lambda'(t) \int_{\Omega} (P(t+h) - P(0))1d\mu \\ &\quad + n\lambda(t+h)^{n-1}(\lambda'(t+h) - \lambda'(t)) \int_{\Omega} (P(t+h) - P(0))1d\mu \\ &= J_1(t, h, n) + J_2(t, h, n) + J_3(t, h, n). \end{aligned}$$

By Lemma 3.2 and arguments similar to the ones used in estimating A_0 above, $|J_2(t, h, n)| \ll n^2|h||t|^{2\alpha-1}\ell_0(\frac{1}{|t|+|h|})^2|\Psi(t)^n|$ and $|J_3(t, h, n)| \ll n|h|^{\alpha-1}\ell_0(\frac{1}{|h|})|t||\Psi(t)^n|$.

Finally, for any $\alpha' \in (1, \alpha)$,

$$\begin{aligned} J_1(t, h, n) &= ih n \lambda(t)^{n-1} \lambda'(t) \int_{\Omega} P'(0)1d\mu + n\lambda(t)^{n-1}\lambda'(t) \int_{\Omega} (P(t+h) \\ &\quad - P(t) - ihP'(0))1d\mu \\ &= ih n \lambda(t)^{n-1} \lambda'(t) \int_{\Omega} P'(0)1d\mu + O(n|h||t|^{\alpha'+\alpha-2}|\Psi(t)^n|), \end{aligned}$$

where in the last line we have used Eq. (3.13) and Lemma 3.3(i).

Altogether,

$$\begin{aligned} & \left| A_1(t+h, n) - A_1(t, n) - ih n \lambda(t)^{n-1} \lambda'(t) \int_{\Omega} P'(0)1d\mu \right| \\ & \ll \left(n|h||t|^{\alpha'+\alpha-2} + n^2|h||t|^{2\alpha-1}\ell_0\left(\frac{1}{|t|+|h|}\right)^2 \right. \\ & \quad \left. + n|h|^{\alpha-1}\ell_0\left(\frac{1}{|h|}\right)|t| \right) |\Psi(t)^n|. \end{aligned} \tag{3.21}$$

Term A_2 in (3.16). Here, we use the definition of A_2^1 in (3.18). By Lemmas 3.2 and 3.6,

$$\begin{aligned} & |A_2^1(t+h, n) - A_2^1(t, n)| \\ & \ll |\lambda(t+h)^n - \lambda(t)^n| \|P'(t)\| + |\lambda(t)^n| \|P'(t+h) - P'(t)\| \\ & \ll n|h||t|^{\alpha-1}\ell_0\left(\frac{1}{|t|+|h|}\right) |\Psi(t)^n| + |h|^{\alpha-1}\ell_0\left(\frac{1}{|h|}\right) |\Psi(t)^n|. \end{aligned}$$

The second statement of item (iv) follows by putting together the estimate on A_2^1 with Eqs. (3.20) and (3.21). More precisely, putting all these together (recorded

in the order A_0 , A_1 and A_2) and recalling (3.21),

$$\begin{aligned}
& \left| \hat{V}'(t+h, n) - \hat{V}'(t, n) - ih n \lambda(t)^{n-1} \lambda'(t) \int_{\Omega} P'(0) 1 d\mu \right| \\
& \ll |h|^{\alpha-1} \ell_0 \left(\frac{1}{|h|} \right) |\Psi(t)^n| \\
& + n \left(|t| |h|^{\alpha'-1} \ell_0 \left(\frac{1}{|h|} \right) + |h| + |t|^{\alpha'+\alpha-2} |h| \ell_0 \left(\frac{1}{|t|} \right) \right) |\Psi(t)^n| \\
& + n^2 \left(|t|^{2\alpha-1} |h| \ell_0 \left(\frac{1}{|t| + |h|} \right)^2 + |t|^2 |h|^{\alpha-1} \ell_0 \left(\frac{1}{|h|} \right) \right. \\
& \left. + |h| |t|^{\alpha} \ell_0 \left(\frac{1}{|t| + |h|} \right) \right) |\Psi(t)^n| + n^3 |t|^{2\alpha} |h| \ell_0 \left(\frac{1}{|t| + |h|} \right)^2 |\Psi(t)^n|.
\end{aligned}$$

Using that $|h| \leq |t|/2$, we can ignore the term $|t|^{\alpha'+\alpha-2} |h| \ell_0(\frac{1}{|t|})$. More precisely, we see that for any $\epsilon > 0$,

$$\begin{aligned}
|t|^{\alpha'+\alpha-2} |h| \ell_0 \left(\frac{1}{|t|} \right) &= |t| |t|^{\alpha'+\alpha-3} |h|^{2-\alpha-\epsilon} \ell_0 \left(\frac{1}{|t|} \right) |h|^{\alpha-1+\epsilon} \\
&\ll |t| |t|^{\alpha'-1-\epsilon} \ell_0 \left(\frac{1}{|t|} \right) |h|^{\alpha-1+\epsilon}.
\end{aligned}$$

Since $\alpha' > 1$ and $\epsilon > 0$ are arbitrarily small, $|t|^{\alpha'-1-\epsilon} \ell_0(\frac{1}{|t|}) = o(1)$ as $|t| \rightarrow 0$. Also, $|h|^{\alpha-1+\epsilon} \ll |h|^{\alpha-1} \ell_0(\frac{1}{|h|})$. Thus,

$$|t|^{\alpha'+\alpha-2} |h| \ell_0 \left(\frac{1}{|t|} \right) \ll |t| |h|^{\alpha-1} \ell_0 \left(\frac{1}{|h|} \right) \ll |t| |h|^{\alpha'-1} \ell_0 \left(\frac{1}{|h|} \right).$$

Item (iv) follows. $\square$

3.2.4. Properties of $Q(t)^n$

Lemma 3.8. *For $n \geq 1$, consider the operator $W(t, n) = Q(t)^n - Q(0)^n$. Let δ_0 be as in (3.7). Let $|t|, |h| \in B_\epsilon(0)$. The following hold for some $\delta_0 < \delta_1 < \delta_2 < 1$ and for some $C', C'' > 0$:*

(i) *If $\alpha \in (0, 1)$, then*

$$\|W(t, n)\| \leq C' \delta_1^n |t|^\alpha \ell \left(\frac{1}{|t|} \right), \quad \|W(t+h, n) - W(t, n)\| \leq C'' \delta_1^n |h|^\alpha \ell \left(\frac{1}{|h|} \right).$$

(ii) *If $\alpha \in (1, 2]$, then for any $\alpha' \in (1, \alpha)$,*

$$\begin{aligned}
& \| (W(t+h, n) - W(t, n) - ihQ(0)^{n-1}Q'(0)) 1 \| \\
& \leq C' \delta_2^n |h| (|t|^{\alpha'-1} + |h|^{\alpha'-1}) + C'' n |h| (|t| + |h|).
\end{aligned}$$

(iii) If $\alpha \in (1, 2]$, then W is differentiable and noticing $W'(t, n) = (Q(t)^n)'$,

$$\|W'(t+h, n) - W'(t, n)\| = \|(Q(t+h)^n)' - (Q(t)^n)'\| \leq C''\delta_1^n |h|^{\alpha-1} \ell_0 \left(\frac{1}{|h|} \right).$$

Proof. Compute that

$$\|W(t, n)\| = \sum_{j=0}^{n-1} \|Q(t)^j\| \|Q(t) - Q(0)\| \|Q(0)^{n-j-1}\|,$$

$$\|W(t+h, n) - W(t, n)\| \leq \sum_{j=0}^{n-1} \|Q(t+h)^j\| \|Q(t+h) - Q(t)\| \|Q(t)^{n-j-1}\|$$

and

$$(Q(t)^n)' = \sum_{j=0}^{n-1} Q(t)^j Q'(t) Q(t)^{n-j-1}.$$

Using these equations, as well as $\|Q(t+h) - Q(t)\| \ll |t| + |h|$ (which follows from (3.8) and $\|Q^n(t)\| \leq C\delta_0^n$ from (3.7), we have

$$\|Q(t+h) - Q(0)\| \ll \delta_0^n (|t| + |h|). \quad (3.22)$$

Items (i) and (iii). They are included in [15, Proof of Lemma 3.7]; the claimed estimates follow directly from the previous three displayed formulas, Eq. (3.7), assumptions (H2)(i) and (H2)(iv), respectively, and Lemma 3.2.

Item (ii). Note that $W(t+h, n) - W(t, n) - ih(Q(0)^n)' = Q(t+h)^n - Q(t)^n - ih(Q(0)^n)'$.

Take $M \in \mathbb{N}$ such that $\|Q^M(t)\| \leq C\delta_0^M \leq \frac{1}{2}$ in (3.7).

We will need a two-step induction argument, first for $n \leq M$, and then for $n > M$.

Step 1: $n \leq M$. Take the induction hypothesis: for any $\alpha' \in (1, \alpha)$, there is $C_{\text{ind}} > 0$ such that

$$\left\| Q(t+h)^n - Q(t)^n - ihQ(0)^{n-1}Q'(0) - ih(Q(0)Q'(0) + Q'(0)Q(0)) \sum_{j=0}^{n-1} Q(0)^j \right\| \ll C_{\text{ind}} n |h| (C^n \delta_1^{n-1} (|t|^{\alpha'-1} + |h|^{\alpha'-1}) + (|t| + |h|)), \quad (3.23)$$

for C from (3.7). We will first prove this statement by induction but only for $1 \leq n \leq M$. Afterward, we explain how to adjust the induction for $n > M$ and with a new uniform constant $\hat{C}_{\text{ind}}$ instead of $C_{\text{ind}}C^n$. It is very likely that the second term $n|h|(|t| + |h|)$ can be improved but this estimate suffices for the proof of the main result.

Since $Q(0)1 = R(0)(I - P(0))1 = 0$, we have $\sum_{j=1}^{n-1} Q(0)^j 1 = 0$ and $n\delta_0^n \leq \delta_1^n$ for some $\delta_1 \in (\delta_0, 1)$, item (ii) follows.

We show that (3.23) holds using an induction argument. A straightforward (but lengthy) calculation verifies the statement for $n = 2$. To see this, compute that

$$\begin{aligned} Q(t+h)^2 - Q(t)^2 &= Q(t+h)(Q(t+h) - Q(t)) + (Q(t+h) - Q(t))Q(t) \\ &= Q(t+h)(Q(t+h) - Q(t) - ihQ'(0)) \\ &\quad + (Q(t+h) - Q(t) - ihQ'(0))Q(t) + ihQ(t+h)Q'(0) + ihQ'(0)Q(t). \end{aligned}$$

So,

$$\begin{aligned} (Q(t+h)^2 - Q(t)^2) - ihQ(0)Q'(0) - ihQ'(0)Q(0) &= Q(t+h)(Q(t+h) - Q(t) - ihQ'(0)) + (Q(t+h) - Q(t) - ihQ'(0))Q(t) \\ &\quad + ih(Q(t+h) - Q(0))Q'(0) + ihQ'(0)(Q(t) - Q(0)). \end{aligned}$$

Using Lemma 3.3(ii), (H2)(iii) and Lemma 3.2,

$$\begin{aligned} &\|Q(t+h)^2 - Q(t)^2 - ihQ(0)Q'(0) - ihQ'(0)Q(0)\| \\ &\ll 2\delta_0|h|(|t|^{\alpha'-1} + |h|^{\alpha'-1}) + |h|(|t| + |h|), \end{aligned}$$

for any $\alpha' \in (1, \alpha)$. This verifies the statement for $n = 2$. To make the assumption (3.23) more clear, we also record for $n = 3$,

$$\begin{aligned} Q(t+h)^3 - Q(t)^3 &= Q(t+h)^2(Q(t+h) - Q(t)) + (Q(t+h)^2 - Q(t)^2)Q(t) \\ &= Q(t+h)^2(Q(t+h) - Q(t) - ihQ'(0)) + (Q(t+h)^2 - Q(t)^2 - ihQ(0)Q'(0) \\ &\quad - ihQ'(0)Q(0))Q(t) + ihQ(t+h)^2Q'(0) + ih(Q(0)Q'(0) + Q'(0)Q(0))Q(t). \end{aligned}$$

So,

$$\begin{aligned} Q(t+h)^3 - Q(t)^3 - ihQ(0)^2Q'(0) + ih(Q(0)Q'(0) + Q'(0)Q(0))Q(0) &= Q(t+h)^2(Q(t+h) - Q(t) - ihQ'(0)) + (Q(t+h)^2 - Q(t)^2 - ihQ(0)Q'(0) \\ &\quad - ihQ'(0)Q(0))Q(t) + ih(Q(t+h)^2 - Q(0)^2)Q'(0) + ih(Q(0)Q'(0) \\ &\quad + Q'(0)Q(0))(Q(t) - Q(0)) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Using the same estimates as for $n = 2$, we have $|I_1| \ll \delta_0^2|h|(|t|^{\alpha'-1} + |h|^{\alpha'-1})$. Using the statement on $n = 2$ and the fact that $\|Q(t)\| \ll \delta_0$, $|I_2| \ll 2\delta_0^2|h|(|t|^{\alpha'-1} + \delta_0|h|^{\alpha'-1}) + |h|(|t| + |h|)$, $|I_3| \ll \delta_0^2|h|(|t| + |h|)$ (using also (3.22)) and $|I_4| \ll |h|(|t| + |h|)$. Thus,

$$|I_1 + I_2 + I_3 + I_4| \ll 3\delta_0^2|h|(|t|^{\alpha'-1} + |h|^{\alpha'-1}) + (1 + \delta_0)|h|(|t| + |h|).$$

Since $1 + \delta_0 < 3$, the statement for $n = 3$ is verified.

The general case $n \geq 1$ follows by induction, assuming the statement that (3.23) holds for n , and we want to prove the hypothesis for $n + 1$. Note that

$$\begin{aligned} & Q(t+h)^{n+1} - Q(t)^{n+1} \\ &= Q(t+h)^n(Q(t+h) - Q(t)) + (Q(t+h)^n - Q(t)^n)Q(t) \\ &= Q(t+h)^n(Q(t+h) - Q(t) - ihQ'(0)) \\ &\quad + \left(Q(t+h)^n - Q(t)^n - ihQ(0)^{n-1}Q'(0) - ih(Q(0)Q'(0) \right. \\ &\quad \left. + Q'(0)Q(0)) \sum_{j=0}^{n-1} Q(0)^j \right) Q(t) + ihQ(t+h)^nQ'(0) + ihQ(0)^{n-1}Q'(0)Q(t) \\ &\quad + ih(Q(0)Q'(0) + Q'(0)Q(0)) \sum_{j=0}^{n-1} Q(0)^j Q(t). \end{aligned}$$

Thus,

$$\begin{aligned} & Q(t+h)^{n+1} - Q(t)^{n+1} - ihQ(0)^nQ'(0) - ih(Q(0)Q'(0) \\ &\quad + Q'(0)Q(0)) \sum_{j=0}^{n-1} Q(0)^j Q(0) \\ &= Q(t+h)^n(Q(t+h) - Q(t) - ihQ'(0)) + \left(Q(t+h)^n - Q(t)^n \right. \\ &\quad \left. - ihQ(0)^{n-1}Q'(0) - ih(Q(0)Q'(0) + Q'(0)Q(0)) \sum_{j=0}^{n-1} Q(0)^j \right) Q(t) \\ &\quad + ih(Q(t+h)^n - Q(0)^n)Q'(0) + ih(Q(0)Q'(0) + Q'(0)Q(0)) \\ &\quad \times \sum_{j=0}^{n-1} Q(0)^j(Q(t) - Q(0)) \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Now, $|J_1| \ll \delta_0^n |h|(|t|^{\alpha'-1} + |h|^{\alpha'-1})$. Using (3.23) and (3.7),

$$|J_2| \leq CC_{\text{ind}} n |h| (C^n \delta_1^{n-1} (|t|^{\alpha'-1} + |h|^{\alpha'-1}) + (|t| + |h|)). \quad (3.24)$$

Using (3.22), $|J_3| \ll \delta_0^1 |h|(|t| + |h|)$. Since $\|Q(t) - Q(0)\| \ll |t|$ and $\|\sum_{j=0}^{n-1} Q(0)^j\| = O(1)$, we get $|J_4| \ll |h|(|t| + |h|)$. The statement on $n + 1$ follows by adding these estimates.

Step 2: $n > M$. Next, we will do the induction in steps of M iterations. That is, we consider

$$\hat{R}(t) := R^M(t) = \lambda^M(t)P(t) + Q^M(t) =: \hat{\lambda}(t)P(t) + \hat{Q}(t),$$

and recall that $\|\hat{Q}\| \leq \frac{1}{2}$ by the choice of M . The reason why the previous induction is of no use for all n is due to the factor C^n , of which we have no control. But using steps of M iterations, C can be replaced by $\frac{1}{2}$, so if we use the induction hypothesis: for any $\alpha' \in (1, \alpha)$, there is $\hat{C}_{\text{ind}} > 0$ such that

$$\left\| \hat{Q}(t+h)^n - \hat{Q}(t)^n - ih\hat{Q}(0)^{n-1}\hat{Q}'(0) - ih(\hat{Q}(0)\hat{Q}'(0) + \hat{Q}'(0)\hat{Q}(0)) \sum_{j=0}^{n-1} \hat{Q}(0)^j \right\| \ll \hat{C}_{\text{ind}} n |h| (\delta_2^{n-1} (|t|^{\alpha'-1} + |h|^{\alpha'-1}) + (|t| + |h|)), \quad (3.25)$$

for $\delta_2 := \delta_1^{1/M}$, the above induction proof works. Apart from M -dependent constants, the estimates of J_1 , J_3 and J_4 still hold. This gives the required result for all $n = n_0 + jM$ and $j \geq 0$ (where the initial step $j = 0$ follows from the first induction for $1 \leq n_0 \leq M$). $\square$

3.2.5. A final form of the decomposition of $\int_{\Omega} R(t)^n 1 d\mu$ into $\Psi(t)^n$ and “good” quantities

To simplify the statement of Lemmas 3.7 and 3.8 further, we record the following facts. We recall that we assume $\int_{\Omega} v d\mu = 0$ (when $\alpha > 1$).

Lemma 3.9. *Let $(Q(0)^n)' = (Q(t)^n)'|_{t=0}$, $n \geq 1$. Then,*

$$\int_{\Omega} P'(0) 1 d\mu = 0 \quad \text{and} \quad \int_{\Omega} (Q(0)^n)' 1 d\mu = \int_{\Omega} Q(0)^{n-1} Q'(0) 1 d\mu = 0.$$

Proof. As in the proof of Lemma 3.3,

$$P'(0) 1 = \frac{1}{2\pi i} \int_{|\xi-1|=\delta} (\xi - R(0))^{-1} R'(0) (\xi - R(0))^{-1} 1 d\xi.$$

But $(\xi - R(0))^{-1} 1 = (\xi - 1)^{-1}$ for all $\xi \neq 1$. Since we also know that $R'(0) 1 = iR(0)v$, using Fubini's theorem we obtain that

$$\begin{aligned} 2\pi i \int_{\Omega} P'(0) 1 d\mu &= i \int_{\Omega} \int_{|\xi-1|=\delta} (\xi - 1)^{-1} (\xi - R(0))^{-1} R(0) v d\xi d\mu \\ &= i \int_{|\xi-1|=\delta} (\xi - 1)^{-1} \sum_{j=0}^{\infty} \xi^{-j-1} \int_{\Omega} R(0)^j R(0) v d\mu d\xi \\ &= i \int_{|\xi-1|=\delta} (\xi - 1)^{-1} \sum_{j=0}^{\infty} \xi^{-j-1} \int_{\Omega} R(0)^{j+1} v d\mu d\xi \\ &= i \int_{|\xi-1|=\delta} (\xi - 1)^{-1} \sum_{j=0}^{\infty} \xi^{-j-1} \int_{\Omega} v d\mu d\xi = 0, \end{aligned}$$

concluding the argument for the first part of the statement.

Next, recall that $(Q(t)^n)' = \sum_{j=0}^{n-1} Q(t)^j Q'(t) Q(t)^{n-j}$. Hence,

$$\int_{\Omega} (Q(0)^n)' 1 d\mu = \int_{\Omega} \sum_{j=0}^{n-1} Q(0)^j Q'(0) Q(0)^{n-j-1} 1 d\mu.$$

Since $Q(0)1 = 0$, $Q(0)^{n-j-1}1 = 0$, for all $j < n-1$. Thus,

$$\int_{\Omega} (Q(0)^n)' 1 d\mu = \int_{\Omega} Q(0)^{n-1} Q'(0) 1 d\mu.$$

Recall from (3.9) that $Q'(0)1 = -R(0)P'(0)1$. As a consequence,

$$\begin{aligned} \int_{\Omega} (Q(0)^n)' 1 d\mu &= - \int_{\Omega} Q(0)^{n-1} R(0) P'(0) 1 d\mu \\ &= - \int_{\Omega} R(0)^{n-1} (I - P(0)) R(0) P'(0) 1 d\mu \\ &= - \int_{\Omega} R(0)^n P'(0) 1 d\mu + \int_{\Omega} R(0)^{n-1} P(0) R(0) P'(0) 1 d\mu \\ &= - \int_{\Omega} P'(0) 1 d\mu + \int_{\Omega} R(0) P'(0) 1 d\mu = 0, \end{aligned}$$

which ends the proof. $\square$

Putting together the last three lemmas, we obtain the following.

Proposition 3.10. *Suppose that (H1)–(H3) hold. Let $U(t, n) = \int_{\Omega} R(t)^n 1 d\mu - \Psi(t)^n$. Then the following hold for all $n \geq 1$ and all $|t|, |h| \in B_{\epsilon}(0)$ with $|h| \leq |t|/2$:*

(i) *For $\alpha \in (0, 1)$,*

$$|U(t, n)| \leq \left(C_0 |t|^{\alpha} \ell \left(\frac{1}{|t|} \right) + n |t|^{2\alpha} \ell \left(\frac{1}{|t|} \right)^2 \right) |\Psi(t)^n|,$$

$$|U(t+h, n) - U(t, n)| \leq \left(|h|^{\alpha} \ell \left(\frac{1}{|h|} \right) + n |h|^{\alpha} \ell \left(\frac{1}{|h|} \right) |t|^{\alpha} \ell \left(\frac{1}{|t|} \right) \right) |\Psi(t)^n|.$$

(ii) *If $\alpha \in (1, 2]$, the following hold for any $\alpha' \in (1, \alpha)$:*

$$|U(t, n)| \leq (|t|^{\alpha'} + n |t|^2) |\Psi(t)^n|$$

and

$$|U(t+h, n) - U(t, n)| \leq |h| |t|^{\alpha'-1} |\Psi(t)^n| + n |h| |t|.$$

(iii) *If $\alpha \in (1, 2]$, then U is differentiable in t and writing $U'(t, n)$ for its derivative,*

$$U'(t, n) \ll \left(|t|^{\alpha-1} \ell_0 \left(\frac{1}{|t|} \right) + n |t| + n^2 |t|^{\alpha+1} \ell_0 \left(\frac{1}{|t|} \right) \right) |\Psi(t)^n|$$

and

$$\begin{aligned}
 & |U'(t+h, n) - U'(t, n)| \\
 & \ll |h|^{\alpha-1} \ell_0 \left(\frac{1}{|h|} \right) |\Psi(t)^n| + n \left(|t||h|^{\alpha'-1} \ell_0 \left(\frac{1}{|h|} \right) + |h| \right) |\Psi(t)^n| \\
 & \quad + n^2 \left(|t|^{2\alpha-1} |h| \ell_0 \left(\frac{1}{|t|} \right)^2 + |t|^2 |h|^{\alpha-1} \ell_0 \left(\frac{1}{|h|} \right) + |h||t|^\alpha \ell_0 \left(\frac{1}{|t|} \right) \right) |\Psi(t)^n| \\
 & \quad + n^3 |t|^{2\alpha} |h| \ell_0 \left(\frac{1}{|t|} \right)^2 |\Psi(t)^n|.
 \end{aligned}$$

Proof. In the notation of Lemmas 3.7 and 3.8,

$$U(t, n) = \int_{\Omega} R(t)^n 1 d\mu - \Psi(t)^n = \hat{V}(t, n) + \int_{\Omega} W(t, n) 1 d\mu,$$

since $W(t, n)1 = Q(t)^n 1 - Q(0)^n 1 = Q(t)^n 1$ (because $Q(0)^n 1 = 0$ for all $n \geq 1$).

Item (i). It follows immediately from Lemmas 3.7(i) and 3.8(i).

Item (ii). By Lemma 3.9,

$$\int_{\Omega} P'(0) 1 d\mu = \int_{\Omega} (Q(0)^n)' 1 d\mu = \int_{\Omega} Q(0)^{n-1} Q'(0) 1 d\mu = 0.$$

Using also Lemma 3.7(ii),

$$|U(t, n)| \ll (|t|^{\alpha'} + n|t|^2) |\Psi(t)^n|,$$

which proves the first statement of item (ii).

We already know that $\int_{\Omega} Q(0)^{n-1} Q'(0) 1 d\mu = 0$. This together with Lemma 3.8(ii) (and recalling $h \leq |t|/2$) implies that

$$\left| \int_{\Omega} (W(t+h, n) - W(t, n)) 1 d\mu \right| \ll \delta_1^n |h| |t|^{\alpha'-1} + n|h||t|.$$

Combining this with Lemma 3.7(iii),

$$\begin{aligned}
 |U(t+h, n) - U(t, n)| & \leq (|h||t|^{\alpha'-1} + n|h||t|^{\alpha-1} \ell_0(1/|t|)) |\Psi(t)^n| + n|h||t| \\
 & \ll |h||t|^{\alpha'-1} |\Psi(t)^n| + n|h||t|,
 \end{aligned}$$

which proves the second statement of item (ii).

Item (iii). It follows immediately from Lemmas 3.7(iv) and 3.8(iii) and the fact that $\int_{\Omega} P'(0) 1 d\mu = 0$. $\square$

3.3. Proof of Theorem 3.1

As already mentioned in the paragraph after the statement of Theorem 3.1, it suffices to show that (3.5) and (3.6) hold. In the notation of Proposition 3.10,

$\mathbb{E}_\mu(e^{itv_n}) = \int_\Omega R(t)^n d\mu = U(t, n) + \Psi(t)^n$ and the LHS of Eq. (3.5), for $\alpha \in (0, 1) \cup (1, 2)$, becomes

$$\left| \int_{-\epsilon}^\epsilon (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{U(t, n)}{it} dt \right| = O(D(N)) + o(n\ell_0(N)N^{-\alpha}), \quad (3.26)$$

where (in the notation of Theorem 3.1),

$$D(N) = \begin{cases} \log N \ell(N) N^{-\alpha} & \text{if } \alpha \in (0, 1), \\ N^{-(\alpha-\delta)} & \text{if } \alpha \in (1, 2]. \end{cases}$$

Similarly, for $\alpha = 2$, Eq. (3.6) becomes

$$\left| \int_{-\epsilon}^\epsilon (e^{-itN} - e^{-it(N+g(n))}) \psi_Y(t) \frac{U(t, n)}{it} dt \right| = O(D(N)) + O(n \log N N^{-2}) + O(n\hat{\ell}(N)N^{-2}). \quad (3.27)$$

In what follows, we show that (3.26) and (3.27) hold by showing that for $\alpha \in (0, 1) \cup (1, 2)$,

$$\left| \int_{-\epsilon}^\epsilon e^{-itN} \psi_Y(t) \frac{U(t, n)}{it} dt \right| = O(D(N)) + o(n\ell_0(N)N^{-\alpha}), \quad (3.28)$$

and for $\alpha = 2$,

$$\left| \int_{-\epsilon}^\epsilon e^{-itN} \psi_Y(t) \frac{U(t, n)}{it} dt \right| = O(D(N)) + O(n \log N N^{-2}) + O(n\hat{\ell}(N)N^{-2}). \quad (3.29)$$

A similar argument shows that for $\alpha \in (0, 1) \cup (1, 2)$,

$$\left| \int_{-\epsilon}^\epsilon e^{-it(N+g(n))} \psi_Y(t) \frac{U(t, n)}{it} dt \right| = O(D(N + g(n))) + o\left(\frac{n\ell_0(N + g(n))}{(N + g(n))^\alpha}\right) + O\left(\frac{n \log(N + g(n))}{(N + g(n))^2}\right),$$

and a corresponding statement for $\alpha = 2$, concluding the proof of Theorem 3.1.

The validity of Eqs. (3.28) and (3.29) will be shown via a “modulus of continuity” argument. To carry out such an argument, it is crucial that we have enough decay in t in the expression of $U(t, n)$ so as to counteract the effect of the division by t in the integrand of (3.28). Proposition 3.10 tracks this type of decay in t (along with finer continuity properties in h) and it is in this sense that we call $U(t, n)$ a “good quantity”.

Before embarking on the final calculations, we need to recall one more technical lemma too.

Lemma 3.11 ([15, Lemma 2.3]). *Let $L : (0, \infty) \rightarrow (0, \infty)$ be a slowly varying function. For all $\beta > 0$, there exists $C > 0$ so that for all $n \geq 1$,*

$$\int_0^\epsilon t^\beta L(1/t) |\Psi(t)^n| dt \leq C \frac{L(a_n)}{a_n^{1+\beta}}.$$

We can now proceed to the following.

Proof of Eq. (3.28). **The case $\alpha \in (0, 1)$.** Set $f(t, n) = \psi_Y(t) \frac{U(t, n)}{t}$ and $K(n, N) = \int_{-\epsilon}^\epsilon e^{-itN} f(t, n) dt$. To be able to apply the modulus of continuity argument, we need to split the domain of integration (as to be able to deal with the difference coming from $\frac{1}{t} - \frac{1}{t - \frac{\pi}{N}}$). For this purpose, write

$$\begin{aligned} K(n, N) &= \int_{-\frac{2\pi}{N}}^{\frac{2\pi}{N}} e^{-itN} f(t, n) dt + \int_{-\epsilon}^{-\frac{2\pi}{N}} e^{-itN} f(t, n) dt + \int_{\frac{2\pi}{N}}^\epsilon e^{-itN} f(t, n) dt \\ &= K_0(n, N) + K^-(n, N) + K^+(n, N). \end{aligned}$$

By Proposition 3.10(i), $|f(t, n)| \ll (n|t|^{2\alpha-1} \ell(\frac{1}{|t|})^2 + |t|^{\alpha-1} \ell(\frac{1}{|t|})) |\Psi(t)^n|$. Thus, we compute (using Karamata's Theorem) that

$$\begin{aligned} |K_0(n, N)| &\ll \int_0^{\frac{2\pi}{N}} t^{\alpha-1} \ell(1/t) dt + n \int_0^{\frac{2\pi}{N}} t^{2\alpha-1} \ell(1/t)^2 dt \\ &\ll N^{-\alpha} \ell(N) + n \ell(N)^2 N^{-2\alpha} = o(n N^{-\alpha} \ell(N)). \end{aligned}$$

We estimate K^+ (via modulus of continuity), the estimate for K^- is similar. With the change of variable $t \rightarrow t - \pi/N$,

$$K^+(n, N) = - \int_{(2\pi+\pi)/N}^{\epsilon+\pi/N} e^{-itN} f\left(t - \frac{\pi}{N}, n\right) dt.$$

Summing the two expressions of K^+ ,

$$\begin{aligned} 2K^+(n, N) &= - \int_{\epsilon}^{\epsilon+\pi/N} e^{-itN} f\left(t - \frac{\pi}{N}, n\right) dt + \int_{\frac{2\pi}{N}}^{\frac{2\pi+\pi}{N}} e^{-itN} f\left(t - \frac{\pi}{N}, n\right) dt \\ &\quad + \int_{\frac{2\pi}{N}}^\epsilon e^{-itN} \left(f(t, n) - f\left(t - \frac{\pi}{N}, n\right)\right) dt = K_1 + K_2 + K_3. \end{aligned}$$

Since the integrands of K_1 and K_2 are bounded, $|K_1|, |K_2| = o(n N^{-\alpha} \ell(N))$. We continue with K_3 , computing that

$$\begin{aligned} |K_3| &\leq \left| \int_{\frac{2\pi}{N}}^\epsilon e^{-itN} \left(\psi_Y(t) - \psi_Y\left(t - \frac{\pi}{N}\right) \right) \frac{U(t, n)}{t} dt \right| \\ &\quad + \left| \int_{\frac{2\pi}{N}}^\epsilon e^{-itN} \psi_Y\left(t - \frac{\pi}{N}\right) \frac{U(t, n) - U\left(t - \frac{\pi}{N}, n\right)}{t} dt \right| \\ &\quad + \left| \int_{\frac{2\pi}{N}}^\epsilon e^{-itN} \psi_Y\left(t - \frac{\pi}{N}\right) U\left(t - \frac{\pi}{N}, n\right) \left(\frac{1}{t} - \frac{1}{t - \frac{\pi}{N}} \right) dt \right| = K_3^1 + K_3^2 + K_3^3. \end{aligned}$$

Since ψ_Y is Lipschitz and $|f(t, n)| \ll (n|t|^{2\alpha-1}\ell(\frac{1}{|t|})^2 + |t|^{\alpha-1}\ell(\frac{1}{|t|}))|\Psi(t)^n|$,

$$\begin{aligned} K_3^1 &\ll \frac{1}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-1} \ell(1/t) dt + \frac{n}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{2\alpha-1} \ell(1/t)^2 dt \\ &\ll \frac{\ell(N)}{N^\alpha} + \frac{n\ell(N)}{N^{2\alpha}} = o(nN^{-\alpha}\ell(N)). \end{aligned}$$

So far, all the terms are as good as desired. The loss in the statement comes from treating the remaining integrals K_3^2 and K_3^3 . By Proposition 3.10(i),

$$|U(t+h, n) - U(t, n)| \ll \left(|h|^\alpha \ell\left(\frac{1}{|h|}\right) + n|h|^\alpha \ell\left(\frac{1}{|h|}\right) |t|^\alpha \ell\left(\frac{1}{|t|}\right) \right) |\Psi(t)^n|.$$

Taking $h = \pi/N$,

$$\begin{aligned} |K_3^2| &\ll \frac{\ell(N)}{N^\alpha} \int_{\frac{2\pi}{N}}^{\epsilon} \frac{1}{t} dt + \frac{n\ell(N)}{N^\alpha} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-1} \ell(1/t) |\Psi(t)^n| dt \\ &\ll \frac{\ell(N) \log N}{N^\alpha} + \frac{\ell(N)}{N^\alpha} n \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-1} \ell(1/t) |\Psi(t)^n| dt. \end{aligned} \quad (3.30)$$

The second term is unproblematic. By Lemma 3.11, with $\beta = \alpha - 1$ and $L = \ell$,

$$n \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-1} \ell(1/t) |\Psi(t)^n| dt \ll na_n^{-\alpha} \ell(a_n) \ll 1.$$

Hence, $K_3^2 \ll \frac{\ell(N) \log N}{N^\alpha}$.

For K_3^3 , we use again that $|U(t, n)/t| \ll n|t|^{2\alpha-1}\ell(\frac{1}{|t|})^2 + |t|^{\alpha-1}\ell(\frac{1}{|t|})$ and that $|\frac{1}{t} - \frac{1}{t-\frac{\pi}{N}}| \ll \frac{1}{N} \frac{1}{t^2}$. So, by Karamata's Theorem,

$$\begin{aligned} |K_3^3| &\ll \frac{1}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-2} \ell(1/t) dt + \frac{n}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{2\alpha-2} \ell(1/t)^2 dt \\ &\ll \frac{1}{N^\alpha} \ell(N) + \frac{n\ell(N)^2}{N^{2\alpha}} \ll N^{-\alpha} \ell(N) + o(nN^{-\alpha} \ell(N)), \end{aligned}$$

ending the argument for the case $\alpha \in (0, 1)$.

The case $\alpha \in (1, 2]$. Set $M(n, N) = \int_{-\epsilon}^{\epsilon} e^{-itN} \psi_Y(t) \frac{U(t, n)}{it} dt$.

Since the boundary terms gives

$$\begin{aligned} M(n, N) &= \frac{1}{iN} \int_{-\epsilon}^{\epsilon} e^{-itN} \psi_Y'(t) \frac{U(t, n)}{it} dt + \frac{1}{iN} \int_{-\epsilon}^{\epsilon} e^{-itN} \psi_Y(t) \frac{U'(t, n)}{it} dt \\ &\quad - \frac{1}{iN} \int_{-\epsilon}^{\epsilon} e^{-itN} \psi_Y(t) \frac{U(t, n)}{t^2} dt \\ &= \frac{1}{iN} M_1(n, N) + \frac{1}{iN} M_2(n, N) + \frac{1}{iN} M_3(n, N). \end{aligned} \quad (3.31)$$

Throughout the rest of the proof, we take α' close to α (i.e. $\alpha - \alpha'$ is positive but as small as we want). By Proposition 3.10(ii), $|U(t, n)| \ll n|t|^2 + |t|^{\alpha'}$.

By Proposition 3.10(iii), $|U'(t, n)| \ll (n|t| + |t|^{\alpha-1}\ell_0(\frac{1}{|t|}) + n^2|t|^{\alpha+1}\ell_0(\frac{1}{|t|}))|\Psi(t)^n|$. Hence, integrating by parts once more,

$$\begin{aligned} \left| \frac{1}{iN} M_1(n, N) \right| &\ll \frac{1}{N^2} \int_0^\epsilon |\psi_Y''(t)| \frac{|U(t, n)|}{t} dt + \frac{1}{N^2} \int_0^\epsilon |\psi_Y'(t)| \frac{|U'(t, n)|}{t} dt \\ &\quad + \frac{1}{N^2} \int_0^\epsilon |\psi_Y'(t)| \frac{|U(t, n)|}{t^2} dt \ll \frac{n}{N^2} = o\left(\frac{n\ell_0(N)}{N^\alpha}\right), \end{aligned} \quad (3.32)$$

where we used that if $\alpha = 2$, we assumed infinite variance, so $\ell_0(N) \rightarrow \infty$. Here, we have also used that ψ_Y is C^2 (with bounded first and second derivatives).

Next, we estimate M_2, M_3 via the modulus of continuity argument similarly to the proof in the case $\alpha \in (0, 1)$ above.

Regarding M_2 ,

$$\begin{aligned} M_2(n, N) &= \int_{-\frac{2\pi}{N}}^{\frac{2\pi}{N}} e^{-itN} \psi_Y(t) \frac{U'(t, n)}{t} dt + \int_{-\epsilon}^{-\frac{2\pi}{N}} e^{-itN} \psi_Y(t) \frac{U'(t, n)}{t} dt \\ &\quad + 7 \int_{\frac{2\pi}{N}}^\epsilon e^{-itN} \psi_Y(t) \frac{U'(t, n)}{t} dt \\ &= M_2^0(n, N) + M_2^-(n, N) + M_2^+(n, N). \end{aligned} \quad (3.33)$$

Next,

$$\begin{aligned} M_2^0(n, N) &\ll n \int_0^{\frac{2\pi}{N}} 1 dt + \int_0^{\frac{2\pi}{N}} t^{\alpha-2} \ell_0\left(\frac{1}{t}\right) dt + n^2 \int_0^{\frac{2\pi}{N}} t^\alpha \ell_0\left(\frac{1}{t}\right) dt \\ &\ll \frac{n}{N} + \frac{\ell_0(N)}{N^{\alpha-1}} + \frac{n^2 \ell_0(N)}{N^{\alpha+1}} = O\left(\frac{n}{N}\right) + o\left(\frac{n\ell_0(N)}{N^{\alpha-1}}\right). \end{aligned}$$

Similarly to the argument used above for the case $\alpha \in (0, 1)$ (in estimating K^+, K^-), it suffices to estimate $M_2^+(n, N)$. Set $g(t, n) = \psi_Y(t) \frac{U'(t, n)}{t}$ and compute that

$$\begin{aligned} 2M_2^+(n, N) &= - \int_\epsilon^{\epsilon + \frac{\pi}{N}} e^{-itN} g\left(t - \frac{\pi}{N}, n\right) dt + \int_{\frac{2\pi}{N}}^{\frac{2\pi + \pi}{N}} e^{-itN} g\left(t - \frac{\pi}{N}, n\right) dt \\ &\quad + \int_{\frac{2\pi}{N}}^\epsilon e^{-itN} \left(g(t, n) - g\left(t - \frac{\pi}{N}, n\right)\right) dt \\ &= M_2^1 + M_2^2 + M_2^3. \end{aligned} \quad (3.34)$$

The estimates for M_2^1 and M_2^2 go similarly to the estimate for $M_2^0(n, N)$ above and give $|M_2^1| = O(\frac{1}{N}) = o(\frac{n\ell_0(N)}{N^{\alpha-1}})$ and $|M_2^2| = o(\frac{n\ell_0(N)}{N^{\alpha-1}})$.

Regarding M_2^3 , recalling that ψ_Y is bounded, we compute that

$$\begin{aligned} |M_2^3| &\ll \int_{\frac{2\pi}{N}}^{\epsilon} \left| \psi_Y(t) - \psi_Y\left(t - \frac{\pi}{N}\right) \right| \frac{|U'(t, n)|}{t} dt \\ &\quad + \int_{\frac{2\pi}{N}}^{\epsilon} \psi_Y\left(t - \frac{\pi}{N}\right) \frac{|U'(t, n) - U'\left(t - \frac{\pi}{N}, n\right)|}{t} dt \\ &\quad + \int_{\frac{2\pi}{N}}^{\epsilon} \psi_Y\left(t - \frac{\pi}{N}\right) \left| U'\left(t - \frac{\pi}{N}, n\right) \right| \left| \frac{1}{t} - \frac{1}{t - \frac{\pi}{N}} \right| dt = L_1 + L_2 + L_3. \end{aligned}$$

Since ψ_Y is Lipschitz, recalling that $|U'(t, n)/t| \ll (n + |t|^{\alpha-2}\ell_0(\frac{1}{|t|}) + n^2|t|^{\alpha}\ell_0(\frac{1}{|t|}))|\Psi(t)^n|$ (and also using Lemma 3.11, with $\beta \in \{\alpha-2, \alpha\}$, and $L = \ell$), we get

$$L_1 \ll \frac{n}{N} \int_{\frac{2\pi}{N}}^{\epsilon} |\Psi(t)^n| dt \ll \frac{n}{a_n} \frac{1}{N}, \quad (3.35)$$

where we have used that $a_n^{\alpha} \sim n\ell_0(a_n)$.

By Proposition 3.10(iii),

$$\begin{aligned} &|U'(t+h, n) - U'(t, n)| \\ &\ll |h|^{\alpha-1}\ell_0\left(\frac{1}{|h|}\right)|\Psi(t)^n| \\ &\quad + n\left(|t||h|^{\alpha'-1}\ell_0\left(\frac{1}{|h|}\right) + |h|\right)|\Psi(t)^n| \\ &\quad + n^2\left(|t|^{2\alpha-1}|h|\ell_0\left(\frac{1}{|t|}\right)^2 + |t|^2|h|^{\alpha-1}\ell_0\left(\frac{1}{|h|}\right) + |h||t|^{\alpha}\ell_0\left(\frac{1}{|t|}\right)\right)|\Psi(t)^n| \\ &\quad + n^3|t|^{2\alpha}|h|\ell_0\left(\frac{1}{|t|}\right)^2|\Psi(t)^n|. \end{aligned}$$

Using this estimate with $h = \pi/N$ and recalling that $|\Psi(t)| \leq e^{-ct^{\alpha}\ell_0(1/t)}$,

$$\begin{aligned} L_2 &\ll \frac{\ell_0(N)}{N^{\alpha-1}} \int_{\frac{2\pi}{N}}^{\epsilon} \frac{1}{t} dt + \frac{n\ell_0(N)}{N^{\alpha'-1}} \int_{\frac{2\pi}{N}}^{\epsilon} |\Psi(t)^n| dt + \frac{n}{N} \int_{\frac{2\pi}{N}}^{\epsilon} \frac{1}{t} dt \\ &\quad + \frac{n^2\ell_0(N)}{N^{\alpha-1}} \int_{\frac{2\pi}{N}}^{\epsilon} t^{2\alpha-2}\ell_0\left(\frac{1}{t}\right)|\Psi(t)^n| dt + \frac{n^2\ell_0(N)}{N^{\alpha-1}} \int_{\frac{2\pi}{N}}^{\epsilon} t|\Psi(t)^n| dt \\ &\quad + \frac{n^2}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-1}\ell_0\left(\frac{1}{t}\right)|\Psi(t)^n| dt + \frac{n^3}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{2\alpha-1}\ell_0\left(\frac{1}{t}\right)^2|\Psi(t)^n| dt. \end{aligned}$$

By Lemma 3.11 with $\beta = \alpha - 1$, and, respectively, $\beta = 2\alpha - 2$ and $L = \ell_0$,

$$\begin{aligned} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-1} \ell_0\left(\frac{1}{t}\right) |\Psi(t)^n| dt &\ll \frac{\ell_0(a_n)}{a_n^{\alpha}} \ll \frac{1}{n}, \\ \int_{\frac{2\pi}{N}}^{\epsilon} t^{2\alpha-2} \ell_0\left(\frac{1}{t}\right) |\Psi(t)^n| dt &\ll \frac{\ell_0(a_n)}{a_n^{2\alpha-1}} \ll \frac{1}{n^{2-1/\alpha-\delta^*}} \quad \text{for any } \delta^* > 0 \end{aligned} \quad (3.36)$$

and using that $a_n^{\alpha} \sim n\ell_0(a_n)$,

$$\int_{\frac{2\pi}{N}}^{\epsilon} t^{2\alpha-1} \ell_0\left(\frac{1}{t}\right)^2 |\Psi(t)^n| dt \ll \frac{\ell_0(a_n)^2}{a_n^{2\alpha}} \ll \frac{1}{n^2}. \quad (3.37)$$

Also, by Lemma 3.11 with $\beta = 1$ and $L = 1$,

$$\int_{\frac{2\pi}{N}}^{\epsilon} t |\Psi(t)^n| dt \ll \frac{1}{a_n^2} \ll \frac{1}{(n\ell_0(a_n))^{2/\alpha}} \ll \frac{1}{n^{2/\alpha-\delta^*}} \quad \text{for any } \delta^* > 0. \quad (3.38)$$

Using the same computation as in (3.35), $\int_{\frac{2\pi}{N}}^{\epsilon} |\Psi(t)^n| dt \ll a_n^{-1}$. These estimates together with (3.36)–(3.38) (after a change of variables $t \rightarrow \sigma/a_n$) imply that

$$\begin{aligned} L_2 &\ll \frac{n\ell_0(N)}{N^{\alpha'-1}} \frac{1}{a_n} + \frac{n \log N}{N} + \frac{\log N \ell_0(N)}{N^{\alpha-1}} \\ &\quad + \frac{n^2 \ell_0(N)}{N^{\alpha-1}} \frac{1}{n^{2-1/\alpha-\delta^*}} + \frac{n^2}{N} \frac{1}{n} + \frac{n^2 \ell_0(N)}{N^{\alpha-1}} \frac{1}{n^{2/\alpha-\delta^*}} + \frac{n^3}{N} \frac{1}{n^2} \\ &= O\left(\frac{\log N \ell_0(N)}{N^{\alpha-1}}\right) + O\left(\frac{n \log N}{N}\right) + o\left(\frac{n\ell_0(N)}{N^{\alpha'-1}}\right). \end{aligned}$$

Here, we have used that $\frac{n^2}{n^{2-1/\alpha-\delta^*}} < n$, $\frac{n^2}{n^{2/\alpha-\delta^*}} < n$ and $a_n^{\alpha} \sim n\ell_0(a_n)$. The argument for estimating L_3 goes similarly to the argument used for K_3^2 in (3.30) (inside the proof for the case $\alpha \in (0, 1)$ above). Recall that $|U'(t, n)| \ll (|t|^{\alpha-1} \ell_0(\frac{1}{|t|}) + n|t| + n^2|t|^{\alpha+1} \ell_0(\frac{1}{|t|})) |\Psi(t)^n|$ and $|\frac{1}{t} - \frac{1}{t-\frac{2\pi}{N}}| \ll \frac{1}{N} \frac{1}{t^2}$. Hence, using Karamata's Theorem,

$$\begin{aligned} L_3 &\ll \frac{1}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-3} \ell_0\left(\frac{1}{t}\right) dt + \frac{n}{N} \int_{\frac{2\pi}{N}}^{\epsilon} \frac{1}{t} dt + \frac{n^2}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha-1} \ell_0\left(\frac{1}{t}\right) |\Psi(t)^n| dt \\ &\ll \frac{\ell_0(N)}{N^{\alpha-1}} + \frac{n \log N}{N} + \frac{n}{N}, \end{aligned}$$

where we have used (3.36). Putting all the above together and recalling (3.34),

$$\left| \frac{1}{iN} M_2(n, N) \right| = O\left(\frac{\log N \ell_0(N)}{N^{\alpha'}}\right) + O\left(\frac{n \log N}{N^2}\right) + o\left(\frac{n\ell_0(N)}{N^{\alpha}}\right). \quad (3.39)$$

It remains to estimate $M_3(n, N)$ in (3.31). We use the modulus of continuity argument again. Similarly to (3.33),

$$\begin{aligned} M_3(n, N) &= \int_{-\frac{2\pi}{N}}^{\frac{2\pi}{N}} e^{-itN} \psi_Y(t) \frac{U(t, n)}{t^2} dt + \int_{-\epsilon}^{-\frac{2\pi}{N}} e^{-itN} \psi_Y(t) \frac{U(t, n)}{t^2} dt \\ &\quad + \int_{\frac{2\pi}{N}}^{\epsilon} e^{-itN} \psi_Y(t) \frac{U(t, n)}{t^2} dt = M_3^0(n, N) + M_3^-(n, N) + M_3^+(n, N). \end{aligned}$$

Recall that by Proposition 3.10(ii), $|U(t, n)| \ll (|t|^{\alpha'} + n|t|^2)|\Psi(t)^n|$, for any $\alpha' \in (1, \alpha)$. An easy computation shows that $|M_3^0(n, N)| = O(N^{1-\alpha'}) + O(nN^{-1}) = O(N^{1-\alpha'}) + o(n\ell_0(N)N^{\alpha-1})$, where we note that if $\alpha = 2$, we assumed infinite variance, so $\ell_0(N) \rightarrow \infty$.

Similarly to the argument used for (3.33), it suffices to estimate $M_3^+(n, N)$. Compute that

$$\begin{aligned} |M_3^+| &\ll \int_{\frac{2\pi}{N}}^{\epsilon} \left| \psi_Y(t) - \psi_Y\left(t - \frac{\pi}{N}\right) \right| \frac{|U(t, n)|}{t^2} dt \\ &\quad + \int_{\frac{2\pi}{N}}^{\epsilon} \psi_Y\left(t - \frac{\pi}{N}\right) \frac{|U(t, n) - U\left(t - \frac{\pi}{N}, n\right)|}{t^2} dt \\ &\quad + \int_{\frac{2\pi}{N}}^{\epsilon} \psi_Y\left(t - \frac{\pi}{N}\right) \left| U\left(t - \frac{\pi}{N}, n\right) \right| \left| \frac{1}{t^2} - \frac{1}{\left(t - \frac{\pi}{N}\right)^2} \right| dt = S_1 + S_2 + S_3. \end{aligned}$$

Clearly, $S_1 \ll nN^{-1} = o\left(\frac{n\ell_0(N)}{N^{\alpha-1}}\right)$.

It remains to estimate S_2 and S_3 . By Proposition 3.10(ii), for any $\alpha' \in (1, \alpha)$,

$$|U(t+h, n) - U(t, n)| \ll |h||t|^{\alpha'-1}|\Psi(t)^n| + n|h||t|.$$

Taking $h = \pi/N$,

$$S_2 \ll \frac{1}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha'-3} dt + \frac{n}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{-1} dt \ll \frac{1}{N^{\alpha'-1}} + \frac{n \log N}{N}.$$

Finally, recall again that by Proposition 3.10(ii), $|U(t, n)| \ll |t|^{\alpha'} + n|t|^2$, for any $\alpha' \in (1, \alpha)$. Hence,

$$S_3 \ll \frac{1}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{\alpha'-3} dt + \frac{n}{N} \int_{\frac{2\pi}{N}}^{\epsilon} t^{-1} dt \ll \frac{1}{N^{\alpha'-1}} + \frac{n \log N}{N}.$$

Thus,

$$|M_3^+| = \begin{cases} O\left(\frac{1}{N^{\alpha'-1}}\right) + o\left(\frac{n\ell_0(N)}{N^{\alpha-1}}\right) & \text{if } \alpha \in (1, 2), \\ O\left(\frac{1}{N^{\alpha'-1}}\right) + O\left(\frac{n(\ell_0(N) + \log N)}{N}\right) & \text{if } \alpha = 2. \end{cases}$$

The integral $|M_3^-|$ can be estimated similarly and gives similar estimates. Altogether,

$$\left| \frac{1}{iN} M_3(n, N) \right| = \begin{cases} O\left(\frac{1}{N^{\alpha'}}\right) + o\left(\frac{n\ell_0(N)}{N^{\alpha}}\right) & \text{if } \alpha \in (1, 2), \\ O\left(\frac{1}{N^{\alpha'}}\right) + O\left(\frac{n(\ell_0(N) + \log N)}{N^2}\right) & \text{if } \alpha = 2. \end{cases}$$

The conclusion follows from the previously displayed equation together with (3.39) and (3.32) (and recalling (3.31)), and also by recalling that $\log N \ell_0(N) \ll N^{\delta_0}$, for any δ_0 arbitrarily small. $\square$

4. Applications

4.1. Gibbs–Markov maps

Roughly speaking, Gibbs–Markov maps are infinite-branch uniformly expanding maps with bounded distortion and big images. We recall the definitions in more detail.

Let (Ω, μ) be a probability space with an at most countable measurable partition $\{\Omega_j\}$, and let $T : \Omega \rightarrow \Omega$ be an ergodic measure-preserving transformation. Define $s(y, y')$ to be the least integer $n \geq 0$ such that $T^n y$ and $T^n y'$ lie in distinct partition elements. Assuming that $s(y, y') = \infty$ if and only if $y = y'$, one obtains that $d_\theta(y, y') = \theta^{s(y, y')}$ is a metric for $\theta \in (0, 1)$.

Let $g = \frac{d\mu}{d\mu \circ T} : \Omega \rightarrow \mathbb{R}$. We say that T is a Gibbs–Markov map if:

- $T(\Omega_j)$ is a union of partition elements and $T|_{\Omega_j} : \Omega_j \rightarrow T(\Omega_j)$ is a measurable bijection for each $j \geq 1$;
- $\inf_j \mu(T(\Omega_j)) > 0$;
- there are constants $C > 0$, $\theta \in (0, 1)$ such that $|\log g(y) - \log g(y')| \leq C d_\theta(y, y')$ for all $y, y' \in \Omega_j$, $j \geq 1$.

For the background on Gibbs–Markov maps, see, for instance, [1, Chap. 4; 2].

Given $f : \Omega \rightarrow \mathbb{R}$, let

$$D_j f = \sup_{y, y' \in \Omega_j, y \neq y'} \frac{|f(y) - f(y')|}{d_\theta(y, y')}, \quad |f|_\theta = \sup_{j \geq 1} D_j f.$$

We let the Banach space $\mathcal{B}_\theta \subset L^\infty$ consist of functions $f : \Omega \rightarrow \mathbb{R}$ such that $|f|_\theta < \infty$ with the norm $\|f\|_\theta = |f|_\infty + |f|_\theta < \infty$.

Proposition 4.1. *Assume T is a mixing Gibbs–Markov map and let $v : \Omega \rightarrow \mathbb{R}$ with $\int_\Omega v^2 d\mu = \infty$ and $|v|_\theta < \infty$. Assume moreover that there exists $C > 0$ so that*

$$|v|_{\Omega_j}|_\theta \leq C \inf_j (|v|_{\Omega_j}|) \quad \text{for all } \Omega_j. \quad (4.1)$$

Fix $\alpha \in (0, 1) \cup (1, 2]$ and assume that the tails of v satisfy (H1). Then, the conditions (H1)–(H3) are satisfied with the Banach space $\mathcal{B} = \mathcal{B}_\theta$.

Proof. Condition (H1) is an assumption on v and condition (H3) is well known (see [1, Chap. 4; 2]). Conditions (H2)(i) and (ii) are again well known and condition (H2)(iv) has been verified in [15, Proof of Proposition 3.10].

It remains to verify (H2)(ii). First note that

$$\|R(t+h) - R(t) - ihR'(0)\| \leq \|R(e^{itv}(e^{ihv} - 1 - ihv))\| + \|R(ihv(e^{itv} - 1))\|.$$

Let $\phi \in \{v(e^{itv} - 1), e^{itv}(e^{ihv} - 1 - ihv)\}$ and recall that ϕ satisfies assumption (4.1). As shown in [14, Proof of Proposition 12.1], under (4.1), there exists $C > 0$ so that

for all $f \in \mathcal{B}_\theta$,

$$\|R(\phi f)\| \leq C\|f\|\|\phi\|_{L^1(\mu)}.$$

Since we know v satisfies (H1), we have for any $\alpha' \in (1, \alpha)$,

$$\|v(e^{itv} - 1)\|_{L^1(\mu)} \ll |t|^{\alpha'-1} \quad \text{and} \quad \|e^{ihv} - 1 - ihv\|_{L^1(\mu)} \ll |h|^{\alpha'}.$$

Thus, $\|R(e^{itv}(e^{ihv} - 1 - ihv))\| \ll |h|^{\alpha'}$ and $\|R(ihv(e^{itv} - 1))\| \ll |h||t|^{\alpha'-1}$. $\square$

4.2. Non-Markov, expanding interval maps (also known as AFU maps)

Let $\Omega = [0, 1]$ with measurable partition $\{I\}$ consisting of open intervals. A map $T : \Omega \rightarrow \Omega$ is called AFU if $T|_I$ is C^2 and strictly monotone for each I , and

- (A) (Adler's condition) $T''/(T')^2$ is bounded on $\bigcup I$.
- (F) (Finite images) The set of images $\{TI\}$ is finite.
- (U) (Uniform expansion) There exists $\rho > 1$ such that $|T'| \geq \rho$ on $\bigcup I$.

Such maps have been introduced in dynamics in [21] (see also [3]). Since AFU maps are not necessarily Markov, Hölder spaces are not preserved by the transfer operator of T and it is standard to consider the space of bounded variation functions. Accordingly, we define the Banach space $\mathcal{B} = \text{BV} \subset L^\infty$ to consist of the functions $f : \Omega \rightarrow \mathbb{R}$ such that $\text{Var}(f) < \infty$ with the norm $\|f\| = |f|_\infty + \text{Var}(f)$, where

$$\text{Var}(f) = \inf_{g \sim f} \sup_{0=y_0 < \dots < y_k=1} \sum_{i=1}^k |g(y_i) - g(y_{i-1})|,$$

where $g \sim f$, if f and g differ on a null set, denotes the variation of the (equivalence class) of f . Also, we let $\text{Var}_I(f)$ denote the variation of f on I .

We suppose that $T : \Omega \rightarrow \Omega$ is topologically mixing. Then, there is a unique absolutely continuous T -invariant probability measure μ , and μ is mixing.

Proposition 4.2. *Assume T is a topologically mixing AFU map and let $v : \Omega \rightarrow \mathbb{R}$ with $\int_\Omega v^2 d\mu = \infty$ and $\sup_I \text{Var}_I(v) < \infty$. Fix $\alpha \in (0, 1) \cup (1, 2]$ and assume that the tails of v satisfy (H1). Then the conditions (H1)–(H3) are satisfied with the Banach space $\mathcal{B} = \text{BV}$.*

Proof. Condition (H1) is an assumption and condition (H3) is well known for mixing AFU maps (see [21]). Apart from (H2)(iii), the remaining items of (H2) have been clarified in [15, Proof of Proposition 3.11].

The verification of (H2)(iii) is similar to the one in Proposition 4.1 with the only change that assumption (4.1) is replaced by $\sup_I \text{Var}_I(v) < \infty$. The assumption $\sup_I \text{Var}_I(v) < \infty$ ensures that $\text{Var}_I(e^{itv}) \ll |t| \text{Var}_I(v) \ll |t|$ and similarly, $\text{Var}_I(e^{ihv}) \ll |h|$. Using this and proceeding as in [15, Proof of Proposition 3.11], we obtain that for $\phi = v(e^{itv} - 1)$, we have that $\|R(\phi f)\| \leq C\|f\|\|\phi\|_{L^1(\mu)}$.

A similar calculation which we sketch below gives the same estimate for $\phi = e^{itv}$ ($e^{ihv} - 1 - ihv$). Compute that $\|R(e^{itv}(e^{ihv} - 1 - ihv)f)\| = S_1 + S_2 + S_3$, where

$$S_1 = \sum_I \mu(I) \sup_I |e^{ihv} - 1 - ihv| \sup_I |f| \ll \|e^{ihv} - 1 - ihv\|_{L^1(\mu)},$$

$$S_2 = \sum_I \mu(I) \sup_I |e^{itv}| (\text{Var}_I(e^{ihv}) + |h| \text{Var}_I(v)) \sup_I |f| \ll |h|,$$

$$S_3 = \sum_I \mu(I) \sup_I |e^{itv}(e^{ihv} - 1 - ihv)| \text{Var}_I(f) \ll \|e^{ihv} - 1 - ihv\|_{L^1(\mu)}.$$

From here onward, the proof goes exactly the same as in Proposition 4.1. $\square$

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