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E-values for anytime-valid inference with exponential families

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E-Values for Anytime-Valid Inference with Exponential Families

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