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by

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Abstract. For a fixed partition α, each real number $x \in (0,1)$ can be represented by many different generalised α-Lüroth expansions. Each such expansion produces a sequence $(p_n/q_n)_{n\geq 1}$ of rational approximations to x. We study the corresponding approximation coefficients $(\theta_n(x))_{n\geq 1}$, which are given by

$$\theta_n(x) := q_n\left|x - \frac{p_n}{q_n}\right|.$$

We give the cumulative distribution function and the expected average value of the θ_n, and we identify which generalised α-Lüroth expansion has the best approximation properties. We also analyse the structure of the set $\mathscr{M}_\alpha$ of possible values that the expected average value of θ_n can take, thus answering a question from [J. Barrionuevo et al., Acta Arith. 74 (1996), 311–327].

1. Introduction. *Lüroth expansions*, first introduced by Lüroth [11] in 1883, are expressions for real numbers $x \in [0,1]$ of the form

$$x = \frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots = \sum_{n\geq 1}(d_n - 1)\prod_{i=1}^{n}\frac{1}{d_i(d_i-1)}, \tag{1.1}$$

where $d_n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for each $n \geq 1$. Since the numbers

$$\frac{p_{n,L}}{q_{n,L}} := \sum_{k=1}^{n}(d_k - 1)\prod_{i=1}^{k}\frac{1}{d_i(d_i-1)}, \qquad n \in \mathbb{N},$$

give a sequence of rationals converging to x, one can wonder about the rational approximation properties of Lüroth expansions. The authors of [1] proved that for Lebesgue a.e. $x \in [0,1]$,

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$$\lim_{n\to\infty} -\frac{1}{n}\log\left|x-\frac{p_{n,L}}{q_{n,L}}\right| = \sum_{d=2}^{\infty}\frac{\log(d(d-1))}{d(d-1))}.$$

They further gave a multifractal analysis of the speed of convergence and showed that the range of possible values of this rate is $[\log 2, \infty)$.

Another way to express the quality of the approximations is via the limiting behaviour of the approximation coefficients

$$\theta_n^L(x) := q_{n,L}\left|x-\frac{p_{n,L}}{q_{n,L}}\right|, \quad n\in\mathbb{N},$$

where $q_{n,L} = d_n\prod_{i=1}^{n-1} d_i(d_i-1)$. In particular, for $n=1$ we set $q_{1,L}=d_1$. In [2] it was shown that for Lebesgue a.e. $x\in[0,1]$, $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\theta_n^L(x) = \frac{1}{2}(\zeta(2)-1)$, where ζ is the Riemann zeta function.

In 1990 Kalpazidou, Knopfmacher and Knopfmacher [9] introduced *alternating Lüroth expansions*, where the terms in the summation from (1.1) alternate in sign. It was subsequently proved in [2] that the alternating Lüroth expansions outperform the Lüroth expansions in terms of approximation properties. Since then both Lüroth and alternating Lüroth expansions have been extensively studied and several generalisations have been considered. In [2] a very general set-up is presented, of which the *α-Lüroth expansions* from [12] are a specific case. The properties of such expansions were then further discussed in [10]. A random version of generalised Lüroth expansions was given in [8].

In this article we consider *generalised α-Lüroth expansions*, of which Lüroth expansions and alternating Lüroth expansions are specific examples. We define these generalised α-Lüroth expansions by describing the dynamical algorithms for obtaining them. It was already observed by Jager and De Vroedt [7] that Lüroth expansions can be obtained by dynamically iterating the so-called *Lüroth transformation*, and the same holds for all other generalised Lüroth expansions mentioned above, as shown in [2]. The transformations given below are specific instances of the ones from [2].

To each strictly decreasing sequence $(t_n)_{n\geq 1}\subseteq(0,1]$ satisfying $t_1=1$ and $\lim_{n\to\infty} t_n = 0$ we can associate a countable interval partition $\alpha = \{A_n := (t_{n+1}, t_n] : n\in\mathbb{N}\}\cup\{A_\infty := \{0\}\}$ of $[0,1]$. We call them *α-Lüroth partitions* and use $\mathscr{P}$ to denote the class of such partitions. For each $\alpha\in\mathscr{P}$, set $a_n := t_n - t_{n+1}$, so a_n is the Lebesgue measure of A_n. *Generalised α-Lüroth transformations* $T_\varepsilon = T_{\alpha,\varepsilon} : [0,1]\to[0,1]$, indexed by sequences $\varepsilon = (\varepsilon_n)_{n\geq 1}\in\{0,1\}^{\mathbb{N}}$, are then defined by setting

$$T_\varepsilon(x) = \begin{cases} 0 & \text{if } x\in A_\infty,\\ (x-t_{n+1})/a_n & \text{if } x\in A_n,\ n\neq\infty \text{ and } \varepsilon_n = 0,\\ (t_n - x)/a_n & \text{if } x\in A_n,\ n\neq\infty \text{ and } \varepsilon_n = 1.\end{cases}$$

See Figure 1 for an illustration.

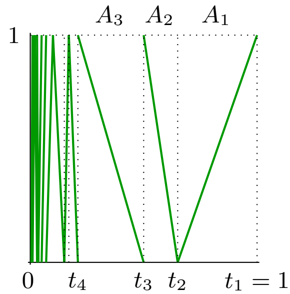


Fig. 1. A transformation $T_{\alpha,\varepsilon}$ with $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$, $\varepsilon_4 = 1$, $\varepsilon_5 = 0$, ...

If we set $\varepsilon_\infty = 0$, then from T_ε we can obtain the *digit sequences* $(d_n)_{n\geq 1}$ and the *sign sequences* $(s_n)_{n\geq 1}$ recursively as follows. For any $x \in [0,1]$ and $n \geq 1$ let $d_n = d_n(x) = k$ and $s_n = s_n(x) = \varepsilon_k$ if $T_\varepsilon^{n-1}(x) \in A_k$. Note that with these definitions we get

$$T_\varepsilon(x) = (-1)^{s_1} \frac{x - t_{d_1+1-s_1}}{a_{d_1}}.$$

Hence, inverting and iterating gives

$$\begin{aligned}
x &= t_{d_1+1-s_1} + (-1)^{s_1} a_{d_1} T_\varepsilon(x) \\
&= t_{d_1+1-s_1} + (-1)^{s_1} a_{d_1} (t_{d_2+1-s_2} + (-1)^{s_2} a_{d_2} T_\varepsilon^2(x)) \\
&= t_{d_1+1-s_1} + (-1)^{s_1} a_{d_1} t_{d_2+1-s_2} + (-1)^{s_1+s_2} a_{d_1} a_{d_2} T_\varepsilon^2(x) \\
&\ \vdots \\
&= t_{d_1+1-s_1} + \cdots + (-1)^{\sum_{i=1}^{n-1} s_i} t_{d_n+1-s_n} \prod_{i=1}^{n-1} a_{d_i} \\
&\quad + (-1)^{\sum_{i=1}^{n} s_i} \prod_{i=1}^{n} a_{d_i} T_\varepsilon^n(x).
\end{aligned} \tag{1.2}$$

Since $T_\varepsilon^n(x) \in [0,1]$ and $a_n \in (0,1)$ for each n, this converges and we get a number expansion of the form

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} t_{d_n+1-s_n} \prod_{i=1}^{n-1} a_{d_i},$$

where $\sum_{i=1}^{0} s_i = 0$ and $\prod_{i=1}^{0} a_{d_i} = 1$. This is called a *generalised α-Lüroth expansion* of x. For $n \in \mathbb{N}$ let

$$\frac{p_n}{q_n} = \frac{p_{n,\alpha,\varepsilon}}{q_{n,\alpha,\varepsilon}}(x) := \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k-1} s_i} t_{d_k+1-s_k} \prod_{i=1}^{k-1} a_{d_i} \tag{1.3}$$

be the *nth approximation* of x. Then we define the *approximation coefficients* θ_n^ε by

$$\theta_n^\varepsilon = \theta_n^{\alpha,\varepsilon}(x) := q_n \left| x - \frac{p_n}{q_n} \right| \quad \text{for all } n \in \mathbb{N}, \tag{1.4}$$

where $q_n = \frac{1}{t_{d_n+1-s_n} \prod_{i=1}^{n-1} a_{d_i}}$ from (1.3). Since we are usually interested in the effect of the sequence ε for a fixed partition α, we will often keep the ε and suppress the α in the notation when no confusion can arise.

For the classical Lüroth partition $\alpha_L \in \mathscr{P}$, given by the sequence $(t_n)_{n\geq 1}$ with $t_n = 1/n$, Barrionuevo et al. [2] gave the cumulative distribution function of the $\theta_n^{\alpha_L,\varepsilon}$.

Theorem 1.1 ([2, Theorem 4]). *For any $\varepsilon \in \{0,1\}^{\mathbb{N}}$, $z \in (0,1]$ and Lebesgue a.e. $x \in [0,1]$ the limit*

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \theta_n^{\alpha_L,\varepsilon}(x) < z\}}{N}$$

exists and equals

$$F_{\alpha_L,\varepsilon}(z) := \sum_{k=2}^{\lfloor 1/z \rfloor + 1 - \varepsilon_{\lfloor 1/z \rfloor}} \frac{z}{k - \varepsilon_{k-1}} + \frac{1}{\lfloor 1/z \rfloor + 1 - \varepsilon_{\lfloor 1/z \rfloor}}.$$

Note that in [2] the intervals in α_L are labelled starting with index 2, so $\alpha_L = \left\{ \left(\frac{1}{n}, \frac{1}{n-1} \right] : n \geq 2 \right\} \cup \{0\}$. Accordingly, the sequences (d_n), (ε_n) and (s_n) are also defined slightly differently. We have adapted the statement of Theorem 1.1 above to fit our notation.

If we write $\bar{0}$ and $\bar{1}$ for the infinite sequences of 0's and 1's, respectively, then Theorem 1.1 in particular gives

$$F_{\alpha_L,\bar{0}}(z) = \sum_{k=2}^{\lfloor 1/z \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor 1/z \rfloor + 1} \quad \text{and} \quad F_{\alpha_L,\bar{1}}(z) = \sum_{k=2}^{\lfloor 1/z \rfloor} \frac{z}{k-1} + \frac{1}{\lfloor 1/z \rfloor}. \tag{1.5}$$

Moreover, it was shown in [2] that $F_{\alpha_L,\bar{0}}(z) \leq F_{\alpha_L,\varepsilon}(z) \leq F_{\alpha_L,\bar{1}}(z)$ for any $\varepsilon \in \{0,1\}^{\mathbb{N}}$.

In this paper we will prove that for any $\alpha \in \mathscr{P}$, $\varepsilon \in \{0,1\}^{\mathbb{N}}$, $z \in (0,1]$ and Lebesgue a.e. $x \in [0,1]$ the limit

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \theta_n^\varepsilon(x) < z\}}{N}$$

exists. Denote this limit by $F_{\alpha,\varepsilon}(z)$ or simply by $F_\alpha(z)$, and let

$$M_\varepsilon = M_{\alpha,\varepsilon} := \int_{[0,1]} (1 - F_{\alpha,\varepsilon})\, d\lambda, \tag{1.6}$$

where we use λ to denote Lebesgue measure. Then (1.6) gives the formula for M_ε which is the expected average value corresponding to the cumulative distribution function F_ε. So for Lebesgue a.e. $x \in [0,1]$, $M_\varepsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i^\varepsilon(x)$

is the average value of the sequence $\{\theta_n^\varepsilon(x)\}_{n=1}^\infty$. Let

$$\mathscr{M} = \mathscr{M}_\alpha := \left\{ M_\varepsilon : \varepsilon \in \{0,1\}^{\mathbb{N}} \right\}$$

be the set of all possible values that M_ε can take. In [2] it was shown that $M_{\alpha_L,\bar{1}} \le M_{\alpha_L,\varepsilon} \le M_{\alpha_L,\bar{0}}$ for each $\varepsilon \in \{0,1\}^{\mathbb{N}}$.

Our first main result generalises Theorem 1.1 to arbitrary partitions $\alpha \in \mathscr{P}$. It also implies that the alternating map $T_{\alpha,\bar{1}}$ always enjoys the best approximation properties and it gives some first information on the structure of $\mathscr{M}_\alpha$.

THEOREM 1.2. *Given a partition $\alpha \in \mathscr{P}$, for any $\varepsilon \in \{0,1\}^{\mathbb{N}}$ and $z \in (0,1]$ and Lebesgue a.e. $x \in [0,1]$ the limit*

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : \theta_n^\varepsilon(x) < z\}}{N}$$

exists and equals

$$F_\varepsilon(z) := \sum_{n\ge1} \min\{a_n, t_{n+1-\varepsilon_n} z\}.$$

Furthermore, $M_{\bar{0}} > M_{\bar{1}}$ and

$$\mathscr{M} = \bigcap_{\kappa=0}^{\infty} \bigcup_{\omega\in\{0,1\}^\kappa} [M_{\omega\bar{1}}, M_{\omega\bar{0}}]. \tag{1.7}$$

After establishing (1.7) we give several results that describe the structure of $\mathscr{M}$ in more detail under various conditions. A partial summary of these results is given in the following theorem.

THEOREM 1.3. *Let $\alpha \in \mathscr{P}$ be such that $\rho := \lim_{n\to\infty} t_{n+1}/t_n \in [0,1]$ exists.*

(i) *If $0 \le \rho < 1/2$, then $\mathscr{M}$ is a Cantor set.*
(ii) *If $1/2 < \rho < 1$, then $\mathscr{M}$ is a finite union of closed intervals.*

Additionally, in case (i) we have expressions for the Hausdorff dimension and packing dimension of $\mathscr{M}$ respectively, and in case (ii) we obtain a condition under which for each interior point M of $\mathscr{M}$ there are uncountably many $\varepsilon \in \{0,1\}^{\mathbb{N}}$ with $M_\varepsilon = M$. Besides Theorem 1.3, we also identify several cases where $\mathscr{M}$ is a *homogeneous Cantor set*.

One particular case that is not covered by Theorem 1.3 is when $\rho = 1$. This includes the classical Lüroth partition α_L, for which the authors of [2] asked whether $\mathscr{M}_{\alpha_L}$ could be a fractal set inside the interval $[M_{\alpha_L,\bar{1}}, M_{\alpha_L,\bar{0}}]$ (see [2, Remark 2, p. 323]). In this article we make some general remarks for $\rho = 1$, and in particular for the Lüroth partition α_L we obtain the following answer.

THEOREM 1.4. *The set $\mathscr{M}_{\alpha_L}$ is the union of eight disjoint closed subintervals of equal length. Furthermore, for any interior point M of $\mathscr{M}_{\alpha_L}$ there exist uncountably many $\varepsilon \in \{0,1\}^{\mathbb{N}}$ such that $M_\varepsilon = M$.*

In case $\lim_{n\to\infty} t_{n+1}/t_n$ exists and equals $1/2$, various outcomes are possible and we discuss some of them below (see Propositions 3.5 and 3.7(iii) for details). In particular, we discuss the partition $\{(2^{-n}, 2^{-(n-1)}]\}_{n\geq 1} \cup \{0\}$ for which $t_{n+1}/t_n = 1/2$ for each n. We also provide some examples where $\lim_{n\to\infty} t_{n+1}/t_n$ does not exist.

The paper is organised as follows. In the next section we recall some basic notation and prove Theorem 1.2. In Section 3 we study the structure of $\mathscr{M}$ in more detail. We obtain various results that together give Theorem 1.3. In that section we also give the fractal dimension of $\mathscr{M}$ in case $\mathscr{M}$ is a Cantor set. In Section 4 we prove Theorem 1.4 and we also present three examples for which $\lim_{n\to\infty} t_{n+1}/t_n$ does not exist.

2. Proof of Theorem 1.2. We first introduce some notation. Let $\{0,1\}^{\mathbb{N}}$ be the set of all infinite sequences $(\varepsilon_i) = \varepsilon_1\varepsilon_2\ldots$ with $\varepsilon_i \in \{0,1\}$ for all $i \geq 1$. By a *word* we mean a finite string of digits over $\{0,1\}$. Denote by $\{0,1\}^*$ the set of all finite words including the empty word ϵ. For any $\omega \in \{0,1\}^*$ we denote by $|\omega|$ the length of ω, where $|\epsilon| = 0$. For any $N \geq 0$, $\{0,1\}^N$ denotes the set of all words of length N. For two words $\omega, \eta \in \{0,1\}^*$ we write $\omega\eta \in \{0,1\}^*$ for their concatenation. Similarly, for a word $\omega = \omega_1 \ldots \omega_n$ and a sequence $\varepsilon = \varepsilon_1\varepsilon_2\ldots$ we use $\omega\varepsilon = \omega_1\ldots\omega_n\varepsilon_1\varepsilon_2\ldots \in \{0,1\}^{\mathbb{N}}$ to denote their concatenation. We use square brackets to denote *cylinder sets*, i.e., for a word $\omega = \omega_1\ldots\omega_k \in \{0,1\}^k$, $k \geq 0$, we write

$$[\omega] = \{\omega\varepsilon : \varepsilon \in \{0,1\}^{\mathbb{N}}\}.$$

On $\{0,1\}^{\mathbb{N}}$ we can define the metric d by setting $d(\varepsilon,\eta) = 0$ if $\varepsilon = \eta$ and

$$d(\varepsilon,\eta) = 2^{-\min\{k\geq 1:\, \varepsilon_k \neq \eta_k\}}$$

otherwise. Cylinder sets are open and closed in the corresponding topology.

Since the generalised α-Lüroth transformations are a particular type of generalised Lüroth transformations from [2], it follows from [2, Theorem 1] that for any $\alpha \in \mathscr{P}$ the maps $T_{\alpha,\varepsilon}$ are measure preserving and ergodic with respect to Lebesgue measure, from which one can deduce among other things that for Lebesgue a.e. $x \in [0,1]$ the sequence $\{T^n_{\alpha,\varepsilon}(x)\}_{n\geq 0}$ is uniformly distributed within the interval $[0,1]$ and that the digits $d_1, d_2, \ldots$ are independent and identically distributed random variables with respect to Lebesgue measure and $\lambda(d_n = k) = a_k$. The next proposition gives the cumulative distribution function of the $\theta_n^{\alpha,\varepsilon}$ by using the above properties.

PROPOSITION 2.1. *Given a partition $\alpha \in \mathscr{P}$, for any $\varepsilon \in \{0,1\}^{\mathbb{N}}$ and $z \in (0,1]$ the limit*

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : \theta_n^\varepsilon(x) < z\}}{N}$$

exists and equals $F_\varepsilon(z)$ for Lebesgue a.e. $x \in [0,1]$, where

$$F_\varepsilon(z) = \sum_{n\ge 1} \min\{a_n, t_{n+1-\varepsilon_n} z\}. \tag{2.1}$$

Proof. For any $x \in [0,1]$ we find by (1.2) that

$$x = t_{d_1+1-s_1} + \cdots + (-1)^{\sum_{i=1}^{n-1} s_i} \prod_{i=1}^{n-1} a_{d_i} t_{d_n+1-s_n} + (-1)^{\sum_{i=1}^{n} s_i} \prod_{i=1}^{n} a_{d_i} T_\varepsilon^n(x)$$

for any $n \ge 1$. Combining this with (1.3) and (1.4) we obtain

$$\theta_n^\varepsilon(x) = q_n \left| x - \frac{p_n}{q_n} \right| = \frac{a_{d_n}}{t_{d_n+1-s_n}} T_\varepsilon^n(x).$$

Note that $\theta_n^\varepsilon(x) < z$ if and only if $T_\varepsilon^n(x) < \frac{t_{d_n+1-s_n}}{a_{d_n}} z$. Since $T_\varepsilon^n(x) \in [0,1]$, this automatically holds for all n such that $\frac{t_{d_n+1-s_n}}{a_{d_n}} z > 1$. We denote

$$N_1 := \left\{ n : \frac{a_n}{t_{n+1-\varepsilon_n}} < z \right\} \quad \text{and} \quad N_2 := \left\{ n : \frac{a_n}{t_{n+1-\varepsilon_n}} \ge z \right\}. \tag{2.2}$$

Then $\theta_n^\varepsilon(x) < z$ for all n with $d_n \in N_1$. On the other hand, for n with $d_n \in N_2$, we have $T_\varepsilon^n(x) < \frac{t_{d_n+1-s_n}}{a_{d_n}} z$ if and only if

$$T_\varepsilon^{n-1}(x) \in A_{d_n} \cap T_\varepsilon^{-1}\left(\left[0, \frac{t_{d_n+1-s_n}}{a_{d_n}} z \right) \right).$$

Hence,

$$\begin{aligned} &\#\{1 \le n \le N : \theta_n^\varepsilon(x) < z\} \\ &= \#\{1 \le n \le N : T_\varepsilon^{n-1}(x) \in A_k,\, k \in N_1\} \\ &\quad + \#\left\{ 1 \le n \le N : T_\varepsilon^{n-1}(x) \in A_k \cap T_\varepsilon^{-1}\left(\left[0, \frac{t_{k+1-\varepsilon_k}}{a_k} z \right) \right),\, k \in N_2 \right\}. \end{aligned} \tag{2.3}$$

Since $\lambda\big(A_k \cap T_\varepsilon^{-1}\big(\big[0, \frac{t_{k+1-\varepsilon_k}}{a_k} z\big)\big)\big) = a_k \cdot \frac{t_{k+1-\varepsilon_k}}{a_k} z = t_{k+1-\varepsilon_k} z$ for any $k \in N_2$, and for Lebesgue a.e. $x \in [0,1]$ the sequence $\{T_\varepsilon^n(x)\}_{n\ge 0}$ is uniformly distributed in $[0,1]$, we see that for those x, the frequencies of both terms of (2.3) exist and equal

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : T_\varepsilon^{n-1}(x) \in A_k,\, k \in N_1\}}{N} = \sum_{k \in N_1} \lambda(A_k) = \sum_{k \in N_1} a_k$$

and

$$\lim_{N\to\infty} \frac{\#\{1\le n\le N : T_\varepsilon^{n-1}(x)\in A_k\cap T_\varepsilon^{-1}\big([0,\frac{t_{k+1-\varepsilon_k}}{a_k}z)\big),\, k\in N_2\}}{N} = \sum_{k\in N_2} t_{k+1-\varepsilon_k} z.$$

This implies that

$$F_\varepsilon(z) = \sum_{n\in N_1} a_n + \sum_{n\in N_2} t_{n+1-\varepsilon_n} z.$$

By (2.2), we have $\min\{a_n, t_{n+1-\varepsilon_n}z\} = a_n$ for $n\in N_1$ and $\min\{a_n, t_{n+1-\varepsilon_n}z\} = t_{n+1-\varepsilon_n}z$ for $n\in N_2$. ∎

For any fixed α, recall from (1.6) that we have defined

$$M_\varepsilon = \int_{[0,1]} (1-F_\varepsilon)\,d\lambda \quad\text{and}\quad \mathscr{M} = \{M_\varepsilon : \varepsilon\in\{0,1\}^{\mathbb{N}}\}.$$

Since for Lebesgue a.e. $x\in[0,1]$, $M_\varepsilon = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \theta_i^\varepsilon(x)$ is the average value of the sequence $\{\theta_n^\varepsilon(x)\}_{n\ge1}$, we would like to describe the approximation properties of the generalised α-Lüroth expansion by M_ε for all $\varepsilon\in\{0,1\}^{\mathbb{N}}$. The next lemma states that for any $\alpha\in\mathscr{P}$ the quality of approximation of the generalised α-Lüroth expansions produced by the function $T_{\bar 1}$ is better than that of $T_{\bar 0}$, thus extending the result from [2, Corollary 2]. In general, for any $\omega\in\{0,1\}^*$, $T_{\omega\bar 1}$ produces the best quality of approximation expansion among all $\varepsilon\in[\omega]$ and $T_{\omega\bar 0}$ produces the worst one, which is important in studying the structure of $\mathscr{M}_\alpha$.

LEMMA 2.2. *Given an $\alpha\in\mathscr{P}$ and a word $\omega\in\{0,1\}^*$, for any $\varepsilon\in\{0,1\}^{\mathbb{N}}$ we have $M_{\omega\bar 1}\le M_{\omega\varepsilon}\le M_{\omega\bar 0}$. In particular, for $\omega=\epsilon$ we have $M_{\bar 1} < M_{\bar 0}$.*

To prove Lemma 2.2 we introduce some notation and give more details about $F_\varepsilon(z)$. Given $\alpha\in\mathscr{P}$, for any $\varepsilon\in\{0,1\}^{\mathbb{N}}$ we denote

$$f_n^{\varepsilon_n}(z) := \min\{a_n, t_{n+1-\varepsilon_n}z\}. \tag{2.4}$$

By (2.1) we have $F_\varepsilon(z) = \sum_{n\ge1} f_n^{\varepsilon_n}(z)$.

Given two sequences $\varepsilon_1,\varepsilon_2\in\{0,1\}^{\mathbb{N}}$ with the first κ entries the same, to compare the approximation quality of T_{ε_1} and T_{ε_2} we only need to consider those entries with index bigger than κ. For all $\kappa\ge0$, $\varepsilon\in\{0,1\}^{\mathbb{N}}$ and $z\in(0,1]$ define

$$N_1^{\varepsilon,\kappa} = N_1^{\varepsilon,\kappa}(z) := \left\{n\ge1 : \frac{a_{\kappa+n}}{t_{\kappa+n+1-\varepsilon_n}} < z\right\},$$
$$N_2^{\varepsilon,\kappa} = N_2^{\varepsilon,\kappa}(z) := \left\{n\ge1 : \frac{a_{\kappa+n}}{t_{\kappa+n+1-\varepsilon_n}} \ge z\right\}.$$

Note that for the sets N_1 and N_2 in (2.2) we have $N_1 = N_1^{\varepsilon,0}$ and $N_2 = N_2^{\varepsilon,0}$. For any $\omega \in \{0,1\}^\kappa$, $\kappa \geq 0$, and $\varepsilon \in \{0,1\}^{\mathbb{N}}$, from the proof of Proposition 2.1 it follows that

$$F_{\omega\varepsilon}(z) = \sum_{n=1}^{\kappa} f_n^{\omega_n}(z) + \sum_{n \in N_1^{\varepsilon,\kappa}} a_{\kappa+n} + \sum_{n \in N_2^{\varepsilon,\kappa}} t_{\kappa+n+1-\varepsilon_n} z.$$

Since the sequence $(t_n)_{n\geq 1}$ is strictly decreasing and

$$\frac{a_n}{t_{n+1-\varepsilon_n}} \geq \frac{a_n}{t_{n+1}},$$

we see that $N_1^{\bar{0},\kappa} \subset N_1^{\varepsilon,\kappa}$ and $N_2^{\varepsilon,\kappa} \subset N_2^{\bar{0},\kappa}$. Since also $N_1^{\varepsilon,\kappa} \setminus N_1^{\bar{0},\kappa} = N_2^{\bar{0},\kappa} \setminus N_2^{\varepsilon,\kappa}$, we obtain

$$\begin{aligned}(2.5)\qquad & F_{\omega\varepsilon}(z) - F_{\omega\bar{0}}(z) \\ &= \sum_{n \in N_1^{\varepsilon,\kappa} \setminus N_1^{\bar{0},\kappa}} a_{\kappa+n} + \sum_{n \in N_2^{\varepsilon,\kappa}} t_{\kappa+n+1-\varepsilon_n} z - \sum_{n \in N_2^{\bar{0},\kappa}} t_{\kappa+n+1} z \\ &= \sum_{n \in N_1^{\varepsilon,\kappa} \setminus N_1^{\bar{0},\kappa}} (a_{\kappa+n} - t_{\kappa+n+1} z) + \sum_{n \in N_2^{\varepsilon,\kappa}} (t_{\kappa+n+1-\varepsilon_n} - t_{\kappa+n+1}) z.\end{aligned}$$

Similarly,

$$\begin{aligned}(2.6)\qquad & F_{\omega\bar{1}}(z) - F_{\omega\varepsilon}(z) \\ &= \sum_{n \in N_1^{\bar{1},\kappa} \setminus N_1^{\varepsilon,\kappa}} (a_{\kappa+n} - t_{\kappa+n+1-\varepsilon_n} z) + \sum_{n \in N_2^{\bar{1},\kappa}} (t_{\kappa+n} - t_{\kappa+n+1-\varepsilon_n}) z.\end{aligned}$$

We will use these two equations to prove Lemma 2.2.

Proof of Lemma 2.2. First we need to prove that

$$M_{\bar{0}} - M_{\bar{1}} = \int_{[0,1]} (1 - F_{\bar{0}})\, d\lambda - \int_{[0,1]} (1 - F_{\bar{1}})\, d\lambda = \int_{[0,1]} (F_{\bar{1}} - F_{\bar{0}})\, d\lambda > 0.$$

By (2.5) with $\kappa = 0$ and $\varepsilon = \bar{1}$, we get

$$F_{\bar{1}}(z) - F_{\bar{0}}(z) = \sum_{n \in N_1^{\bar{1},0} \setminus N_1^{\bar{0},0}} (a_n - t_{n+1} z) + \sum_{n \in N_2^{\bar{1},0}} (t_n - t_{n+1}) z.$$

Since $a_n / t_{n+1} \geq z$ for any $n \in N_1^{\bar{1},0} \setminus N_1^{\bar{0},0}$, for any $z \in (0,1]$ we obtain

$$F_{\bar{1}}(z) - F_{\bar{0}}(z) \geq \sum_{n \in N_2^{\bar{1},0}} a_n z.$$

Set $z_0 = \frac{a_1}{2t_1}$ and fix a $\delta \in \left(0, \frac{a_1}{2t_1}\right)$. Then for any $z \in [z_0 - \delta, z_0 + \delta]$ we have

$1 \in N_2^{\bar{1},0}$, and hence

$$\int_{[0,1]} \sum_{n \in N_2^{\bar{1},0}} (a_n z)\, d\lambda(z) \geq \int_{[z_0-\delta, z_0+\delta]} a_1 z\, d\lambda(z) = 2a_1 z_0 \delta > 0.$$

So we have proved $M_{\bar{0}} > M_{\bar{1}}$.

Fix $\varepsilon \in \{0,1\}^{\mathbb{N}}$, $\kappa \geq 0$ and $\omega \in \{0,1\}^{\kappa}$. As in the proof of $M_{\bar{0}} > M_{\bar{1}}$, to get $M_{\omega\bar{1}} \leq M_{\omega\varepsilon} \leq M_{\omega\bar{0}}$ it is enough to show that $F_{\omega\bar{0}}(z) \leq F_{\omega\varepsilon}(z) \leq F_{\omega\bar{1}}(z)$ for any $z \in (0,1]$. The first inequality follows immediately from (2.5) by noting that $a_{\kappa+n} - t_{\kappa+n+1} z \geq 0$ for all $n \in N_1^{\varepsilon,\kappa} \setminus N_1^{\bar{0},\kappa}$, and $t_{\kappa+n+1-\varepsilon_n} - t_{\kappa+n+1} \in \{0, a_{\kappa+n}\}$ for all n. The second inequality follows similarly from (2.6). ∎

For each word $\omega \in \{0,1\}^*$ set

$$I_\omega := [M_{\omega\bar{1}}, M_{\omega\bar{0}}]. \tag{2.7}$$

From Lemma 2.2 we know that the intervals I_ω are well-defined for each ω and $\lambda(I_\epsilon) > 0$. We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. For any $\varepsilon \in \{0,1\}^{\mathbb{N}}$ by Lemma 2.2 we have $M_\varepsilon \in I_{\varepsilon_1 \ldots \varepsilon_n}$ for each $n \geq 1$. Hence,

$$\mathscr{M} \subseteq \bigcap_{\kappa=0}^{\infty} \bigcup_{\omega \in \{0,1\}^{\kappa}} I_\omega.$$

For the other direction, first note that for any $\kappa \geq 0$ and $\omega \in \{0,1\}^{\kappa}$ we have $I_{\omega 0} = [M_{\omega 0\bar{1}}, M_{\omega\bar{0}}]$, and $M_{\omega 0\bar{1}} \geq M_{\omega\bar{1}}$ by Lemma 2.2. Similarly, $I_{\omega 1} = [M_{\omega\bar{1}}, M_{\omega 1\bar{0}}]$ and $M_{\omega 1\bar{0}} \leq M_{\omega\bar{0}}$. So, for $d = 0,1$ the intervals $I_{\omega d}$ are both subintervals of I_ω, and $I_{\omega 1}$ shares the left endpoint with I_ω and $I_{\omega 0}$ shares the right endpoint with I_ω. Now, let $M \in \bigcap_{\kappa=0}^{\infty} \bigcup_{\omega \in \{0,1\}^{\kappa}} I_\omega$. Then for each $\kappa \geq 0$ there is a collection of words $\omega \in \{0,1\}^{\kappa}$ such that $M \in I_\omega$. Let A_κ be the union of the cylinder sets given by these words. Then A_κ is a compact subset of $\{0,1\}^{\mathbb{N}}$ and $A_{n+1} \subseteq A_n$ for each $n \geq 0$. Hence, $\bigcap_{n=0}^{\infty} A_n$ is a non-empty compact set, which means that there is a sequence $\varepsilon \in \{0,1\}^{\mathbb{N}}$ for which $M = M_\varepsilon$. Thus,

$$\mathscr{M} = \bigcap_{\kappa=0}^{\infty} \bigcup_{\omega \in \{0,1\}^{\kappa}} I_\omega.$$

Combining this with Proposition 2.1 and Lemma 2.2 completes the proof. ∎

3. Structure of $\mathscr{M}$. In this section we will study $\mathscr{M}$ in detail and give the proof of Theorem 1.3. We first compare the size of the subintervals I_ω and I_η for any $\omega, \eta \in \{0,1\}^n$ with $n \geq 0$. The next lemma says that the length of I_ω, and thus the value of $M_{\omega 0\bar{1}} - M_{\omega 1\bar{0}}$, only depends on the length $|\omega|$ of ω.

Lemma 3.1. *Let $\alpha \in \mathscr{P}$ and $n \geq 0$. Then for any $\omega, \eta \in \{0,1\}^n$ we have $\lambda(I_\omega) = \lambda(I_\eta)$ and $M_{\omega 0\bar{1}} - M_{\omega 1\bar{0}} = M_{\eta 0\bar{1}} - M_{\eta 1\bar{0}}$.*

Proof. Take $\omega \in \{0,1\}^n$. By (2.7) we have

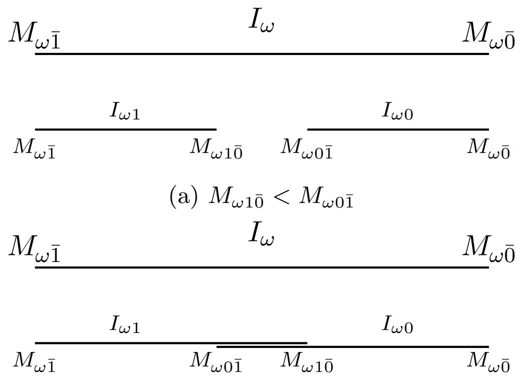
$$\lambda(I_\omega) = M_{\omega\bar{0}} - M_{\omega\bar{1}} = \int\limits_{[0,1]} (F_{\omega\bar{1}} - F_{\omega\bar{0}})\, d\lambda.$$

From (2.5) with $\varepsilon = \bar{1}$ it follows that for any $\omega \in \{0,1\}^n$ with $n \geq 0$, we have

$$F_{\omega\bar{1}}(z) - F_{\omega\bar{0}}(z) = \sum_{n \in N_1^{\bar{1},n} \setminus N_1^{\bar{0},n}} (a_{n+n} - t_{n+n+1} z) + \sum_{n \in N_2^{\bar{1},n}} a_{n+n} z.$$

We see that $F_{\omega\bar{1}} - F_{\omega\bar{0}}$ only depends on $n = |\omega|$, hence so does $\lambda(I_\omega)$. The remaining result follows from $M_{\omega 0\bar{1}} - M_{\omega 1\bar{0}} = \lambda(I_\omega) - \lambda(I_{\omega 1}) - \lambda(I_{\omega 0})$. ∎

By (1.7), $\mathscr{M}$ can be obtained by successively removing a sequence of open intervals from $[M_{\bar{1}}, M_{\bar{0}}]$. For any I_ω with $\omega \in \{0,1\}^*$ we have either $I_{\omega 0} \cap I_{\omega 1} = \emptyset$ or $I_{\omega 0} \cap I_{\omega 1} \neq \emptyset$ (see Figure 2), depending on whether $M_{\omega 1\bar{0}} < M_{\omega 0\bar{1}}$ or $M_{\omega 1\bar{0}} \geq M_{\omega 0\bar{1}}$.



(b) $M_{\omega 1\bar{0}} \geq M_{\omega 0\bar{1}}$

Fig. 2. The interval I_ω and its offsprings $I_{\omega 0}$, $I_{\omega 1}$

Lemma 3.1 implies that for a given $\alpha \in \mathscr{P}$ we can define a function $G : \mathbb{Z}_{\geq 0} \to [0,1]$ by setting

$$\begin{aligned} G(n) := M_{\omega 0\bar{1}} - M_{\omega 1\bar{0}} &= \int\limits_{[0,1]} (F_{\omega 1\bar{0}} - F_{\omega 0\bar{1}})\, d\lambda \\ &= \int\limits_{[0,1]} (f_{n+1}^1 - f_{n+1}^0) d\lambda - \sum_{k \geq n+2} \int\limits_{[0,1]} (f_k^1 - f_k^0)\, d\lambda \end{aligned}$$

for any $\omega \in \{0,1\}^n$. The sign of $G(n)$ will determine the structure of $\mathscr{M}$. This means that in the $(n+1)$th level of the construction of $\mathscr{M}_\alpha$ the basic intervals are disjoint if $G(n) > 0$, and they overlap if $G(n) \le 0$. The following proposition covers the case where the intervals I_ω eventually overlap. Write $I(n) := \lambda(I_\omega)$ for any $\omega \in \{0,1\}^n$.

PROPOSITION 3.2. *Let $\alpha \in \mathscr{P}$ and assume that there is an $N_\alpha \ge 0$ such that $G(n) \le 0$ for all $n \ge N_\alpha$. Then there are finitely many closed intervals $J_1, \dots, J_K$, $K \in \mathbb{N}$, such that $\mathscr{M} = \bigcup_{k=1}^K J_k$. If moreover there exists an $N_\alpha^1 \ge N_\alpha$ such that $\frac{I(n)}{I(n-1)} \in \left(\frac{\sqrt{5}-1}{2}, 1\right)$ for all $n > N_\alpha^1$, then for all interior points M of $\mathscr{M}$ there are uncountably many $\varepsilon \in \{0,1\}^{\mathbb{N}}$ such that $M = M_\varepsilon$.*

Proof. Note that for each $n \ge 0$ the set $\bigcup_{\omega \in \{0,1\}^n} I_\omega$ is a union of at most 2^n closed disjoint intervals. The assumption about N_α implies that $I_\omega = I_{\omega 0} \cup I_{\omega 1}$ for all $\omega \in \{0,1\}^*$ with $|\omega| \ge N_\alpha$, and thus

$$\mathscr{M} = \bigcap_{n=0}^{\infty} \bigcup_{\omega \in \{0,1\}^n} I_\omega = \bigcup_{\omega \in \{0,1\}^{N_\alpha}} I_\omega.$$

This gives the first part of the statement.

To obtain the second part, for each $n > N_\alpha^1$ consider the maps

$$\begin{aligned} T_{n,1} &: \left[0, \frac{I(n)}{I(n-1)}\right] \to [0,1], & x &\mapsto \frac{I(n-1)}{I(n)} x, \\ T_{n,0} &: \left[1 - \frac{I(n)}{I(n-1)}, 1\right] \to [0,1], & x &\mapsto \frac{I(n-1)}{I(n)} x - \frac{I(n-1)}{I(n)} + 1. \end{aligned}$$

For $x \in [0,1]$, let $\varepsilon = (\varepsilon_n)_{n \ge 1} \in \{0,1\}^{\mathbb{N}}$ be such that

$$T_\varepsilon^n(x) := T_{N_\alpha^1+n, \varepsilon_n} \circ \cdots \circ T_{N_\alpha^1+1, \varepsilon_1}(x) \in [0,1] \tag{3.1}$$

for all n. Note that $1 - \frac{I(n)}{I(n-1)} < \frac{I(n)}{I(n-1)}$ since $\frac{I(n)}{I(n-1)} > \frac{1}{2}$ for all $n > N_\alpha^1$. This means the following:

- If $T_\varepsilon^{n-1}(x) \le 1 - \frac{I(N_\alpha^1+n)}{I(N_\alpha^1+n-1)}$, then $\varepsilon_n = 1$.
- If $T_\varepsilon^{n-1}(x) \ge \frac{I(N_\alpha^1+n)}{I(N_\alpha^1+n-1)}$, then $\varepsilon_n = 0$.
- If $T_\varepsilon^{n-1}(x) \in \left[1 - \frac{I(N_\alpha^1+n)}{I(N_\alpha^1+n-1)}, \frac{I(N_\alpha^1+n)}{I(N_\alpha^1+n-1)}\right]$, then ε_n can be 0 or 1.

The condition that $\frac{I(n)}{I(n-1)} \in \left(\frac{\sqrt{5}-1}{2}, 1\right)$ for all $n > N_\alpha^1$ implies that

$$T_{n,1}\left(1 - \frac{I(n)}{I(n-1)}\right) = \frac{I(n-1)}{I(n)} - 1 < \frac{\sqrt{5}-1}{2} < \frac{I(n+1)}{I(n)}.$$

By symmetry we also get $T_{n,0}\left(\frac{I(n)}{I(n-1)}\right) > 1 - \frac{I(n+1)}{I(n)}$ for all $n > N_\alpha^1$. Since $\frac{I(n-1)}{I(n)} > 1$ for all n, this implies that if $T_\varepsilon^{n-1}(x) \notin \left[1 - \frac{I(n)}{I(n-1)}, \frac{I(n)}{I(n-1)}\right] \cup \{0,1\}$, then there is a $k > 1$ such that

$$T_\varepsilon^{n+k-1}(x) \in \left[1 - \frac{I(n+k)}{I(n+k-1)}, \frac{I(n+k)}{I(n+k-1)}\right].$$

From this we can deduce (basically as in [4, Theorem 3]) that for each $x \in (0,1)$ there exist uncountably many sequences $\varepsilon = (\varepsilon_n)_{n\geq 1} \in \{0,1\}^{\mathbb{N}}$ satisfying (3.1).

Now, if we let $M \in I_\omega$ for some $\omega \in \{0,1\}^{N_\alpha^1}$ and M is an interior point of $\mathscr{M}$, then from the above we know that there are uncountably many sequences $\varepsilon \in \{0,1\}^{\mathbb{N}}$ such that $M = M_{\omega\varepsilon}$. This gives the result. ∎

Recall that a *Cantor set* in $\mathbb{R}$ is a non-empty compact set that has neither interior nor isolated points. A Cantor set is called homogeneous if it can be obtained in the following way. Let $(n_k)_{k\geq 1}$ be a sequence of integers greater than 1 and $(c_k)_{k\geq 1}$ be a sequence of real numbers satisfying $0 < n_k c_k < 1$ for all $k \geq 1$. Recursively construct a sequence of intervals as follows. Start with an interval $I \subseteq \mathbb{R}$ and set $I_\epsilon = I$, called the level 0 interval. If for some $k \geq 0$ we have obtained the collection of $\prod_{j=1}^k n_j$ disjoint closed intervals of level k (set $\prod_{j=1}^0 n_j = 1$), then we obtain the collection of level $k+1$ intervals by dividing each level k interval $I_{i_1\ldots i_k}$ into n_{k+1} disjoint intervals of equal length $\prod_{i=1}^{k+1} c_i$, labelled $I_{i_1\ldots i_k 1}, \ldots, I_{i_1\ldots i_k n_{k+1}}$ from left to right, in such a way that $I_{i_1\ldots i_k 1}$ and $I_{i_1\ldots i_k}$ share the common left endpoint, $I_{i_1\ldots i_k n_{k+1}}$ and $I_{i_1\ldots i_k}$ share the common right endpoint, and the distances between $I_{i_1\ldots i_k j}$ and $I_{i_1\ldots i_k (j+1)}$ are the same for all $1 \leq j \leq n_{k+1} - 1$. The set

$$\mathscr{K} = \bigcap_{k\geq 1} \bigcup_{i_1\ldots i_k} I_{i_1\ldots i_k}$$

is called a *homogeneous Cantor set* (see e.g. [6]). The Hausdorff dimension of a homogeneous Cantor set with sequences $(n_k)_{k\geq 1}$ and $(c_k)_{k\geq 1}$ is given by

$$\dim_H(\mathscr{K}) = \liminf_{k\to\infty} \frac{\log(n_1\cdots n_k)}{-\log(c_1\cdots c_k)} \tag{3.2}$$

(see [5, Lemma 2.2]), and its packing dimension is given by

$$\dim_P(\mathscr{K}) = \limsup_{k\to\infty} \frac{\log(n_1\cdots n_{k+1})}{-\log(c_1\cdots c_k) + \log n_{k+1}} \tag{3.3}$$

(see [5, Theorem 3.1]).

The following result concerns the case where $\mathscr{M}$ is a Cantor set.

Proposition 3.3. *Let $\alpha \in \mathscr{P}$. If there is an $N_\alpha \in \mathbb{Z}_{\geq 0}$ such that $G(n) > 0$ for all $n \geq N_\alpha$, then $\mathscr{M}$ is a Cantor set with*

$$\dim_H(\mathscr{M}) = \liminf_{k\to\infty} \frac{k\log 2}{-\log(I(k))}, \quad \dim_P(\mathscr{M}) = \limsup_{k\to\infty} \frac{k\log 2}{-\log(I(k))}. \tag{3.4}$$

In particular, if $N_\alpha = 0$, then $\mathscr{M}$ is a homogeneous Cantor set.

Proof. The set $\mathscr{M}$ is a homogeneous Cantor set for $N_\alpha = 0$, which follows from the construction of $\mathscr{M}$ as described in the proof of Theorem 1.2 (see Figure 2 for an illustration). Note that $n_k = 2$ for all $k \geq 1$ since I_ω has two disjoint subintervals $I_{\omega 0}$ and $I_{\omega 1}$ for all $\omega \in \{0,1\}^*$. The sequence $(c_k)_{k\geq 1}$ is given by $c_k = \frac{I(k)}{I(k-1)}$ for all $k \geq 1$. By (3.2), we have

$$\begin{aligned}\dim_H(\mathscr{M}) &= \liminf_{k\to\infty} \frac{-k\log 2}{\log\left(\frac{I(1)}{I(0)}\cdots\frac{I(k)}{I(k-1)}\right)} = \liminf_{k\to\infty} \frac{k\log 2}{\log(I(0)) - \log(I(k))} \\ &= \liminf_{k\to\infty} \frac{k\log 2}{-\log(I(k))},\end{aligned}$$

where the last equality follows since $\lim_{k\to\infty} I(k) = 0$. The packing dimension follows similarly from (3.3).

For $N_\alpha > 0$, note that for any word $\eta \in \{0,1\}^{N_\alpha}$ the set

$$\bigcap_{n=0}^{\infty} \bigcup_{\omega\in\{0,1\}^n} I_{\eta\omega}$$

is a homogeneous Cantor set. Since a finite union of Cantor sets is again a Cantor set, we see that

$$\mathscr{M} = \bigcup_{\eta\in\{0,1\}^{N_\alpha}} \bigcap_{n=0}^{\infty} \bigcup_{\omega\in\{0,1\}^n} I_{\eta\omega}$$

is also a Cantor set. We get

$$\dim_H(\mathscr{M}) = \max_{\eta\in\{0,1\}^{N_\alpha}} \dim_H\Big(\bigcap_{n=0}^{\infty} \bigcup_{\omega\in\{0,1\}^n} I_{\eta\omega}\Big) = \liminf_{k\to\infty} \frac{k\log 2}{-\log(I(k))}.$$

By the same argument we get the packing dimension of $\mathscr{M}$. ∎

Propositions 3.2 and 3.3 show that to determine the structure of $\mathscr{M}$ it is useful to consider the function G in more detail. Recall that

$$G(n) = \int_{[0,1]} (f^1_{n+1} - f^0_{n+1})\,d\lambda - \sum_{k\geq n+2} \int_{[0,1]} (f^1_k - f^0_k)\,d\lambda.$$

We study the integrals $\int_{[0,1]}(f^1_k - f^0_k)\,d\lambda$ for $k \geq n+1$. By (2.4) we have

$$f^0_k(z) = \begin{cases} a_k & \text{if } a_k/t_{k+1} < z, \\ t_{k+1}z & \text{if } a_k/t_{k+1} \geq z, \end{cases} \qquad f^1_k(z) = \begin{cases} a_k & \text{if } a_k/t_k < z, \\ t_k z & \text{if } a_k/t_k \geq z. \end{cases} \tag{3.5}$$

It is easy to see that

$$0 < \frac{a_k}{t_k} = \frac{a_k}{a_k + t_{k+1}} < 1 \quad \text{and} \quad \frac{a_k}{t_k} < \frac{a_k}{t_{k+1}}.$$

Then we need to compare the values of a_k/t_{k+1} and 1. For those k such that $a_k/t_{k+1} \geq 1$, which is equivalent to $0 < t_{k+1}/t_k \leq 1/2$, by (3.5) we have

$$\begin{aligned}\int_{[0,1]} (f_k^1 - f_k^0)\, d\lambda &= \int_{[0,a_k/t_k]} (t_k - t_{k+1})z\, d\lambda(z) + \int_{[a_k/t_k,1]} (a_k - t_{k+1}z)\, d\lambda(z)\\ &= \frac{a_k^2}{2t_k} - \frac{t_{k+1}a_k^2}{2t_k^2} + a_k - \frac{a_k^2}{t_k} - \frac{t_{k+1}}{2} + \frac{t_{k+1}a_k^2}{2t_k^2}\\ &= a_k - \frac{a_k^2}{2t_k} - \frac{t_{k+1}}{2} = \frac{t_k}{2} - \frac{t_{k+1}}{2} - \frac{t_{k+1}^2}{2t_k}.\end{aligned}$$

Similarly, for those k with $a_k/t_{k+1} < 1$, which implies $1/2 < t_{k+1}/t_k < 1$, we have

$$\begin{aligned}\int_{[0,1]} (f_k^1 - f_k^0)\, d\lambda &= \int_{[0,a_k/t_k]} (t_k - t_{k+1})z\, d\lambda(z) + \int_{[a_k/t_k,a_k/t_{k+1}]} (a_k - t_{k+1}z)\, d\lambda(z)\\ &= \frac{a_k^2}{2t_k} + \frac{a_k^2}{t_{k+1}} - \frac{a_k^2}{t_k} - \frac{a_k^2}{2t_{k+1}} = \frac{a_k^2}{2t_{k+1}} - \frac{a_k^2}{2t_k}\\ &= \frac{t_k^2}{2t_{k+1}} - \frac{t_{k+1}^2}{2t_k} + \frac{3t_{k+1}}{2} - \frac{3t_k}{2}.\end{aligned}$$

To simplify the notation, we set for $k \geq 1$,

$$\begin{aligned}g(k) &:= \int_{[0,1]} (f_k^1 - f_k^0)\, d\lambda\\ &= \begin{cases} \frac{t_k}{2} - \frac{t_{k+1}}{2} - \frac{t_{k+1}^2}{2t_k} & \text{if } 0 \leq \frac{t_{k+1}}{t_k} \leq \frac{1}{2},\\ \frac{t_k^2}{2t_{k+1}} - \frac{t_{k+1}^2}{2t_k} + \frac{3t_{k+1}}{2} - \frac{3t_k}{2} & \text{if } \frac{1}{2} < \frac{t_{k+1}}{t_k} < 1.\end{cases}\end{aligned} \tag{3.6}$$

So, for any given $\alpha \in \mathscr{P}$ we see that for any $n \geq 0$,

$$G(n) = g(n+1) - \sum_{k=n+2}^{\infty} g(k). \tag{3.7}$$

By the above discussion, in the $(n+1)$th level of the construction of $\mathscr{M}_\alpha$ we see gaps as in Figure 2(a) if $G(n) > 0$, and overlaps as in Figure 2(b) if $G(n) \leq 0$. We will use this and (3.7) to describe the structure of $\mathscr{M}_\alpha$ further under some additional restrictions below. For each $n \geq 1$ set

$$\rho_n := t_{n+1}/t_n.$$

According to the two cases of g in (3.6), we will split the discussion of the structure of $\mathscr{M}_\alpha$ into two subsections: (i) $1/2 < \rho < 1$; (ii) $0 < \rho < 1/2$ (see Propositions 3.5 and 3.7 for details).

3.1. $\alpha \in \mathscr{P}$ with $1/2 < \rho_n < 1$ for all large enough n. In this section we assume that $\alpha \in \mathscr{P}_1$, where

$$\mathscr{P}_1 := \{\alpha \in \mathscr{P} : \text{there is an } N_\alpha \geq 0 \text{ such that } 1/2 < \rho_{n+1} < 1 \text{ for all } n \geq N_\alpha\}.$$

By (3.6) and (3.7) we have

$$\begin{aligned}
G(n) &= \frac{t_{n+1}^2}{2t_{n+2}} - \frac{t_{n+2}^2}{2t_{n+1}} + \frac{3t_{n+2}}{2} - \frac{3t_{n+1}}{2} \\
&\quad - \sum_{k=n+2}^{\infty} \left(\frac{t_k^2}{2t_{k+1}} - \frac{t_{k+1}^2}{2t_k} + \frac{3t_{k+1}}{2} - \frac{3t_k}{2} \right) \\
&= \frac{t_{n+1}}{2\rho_{n+1}} - \frac{\rho_{n+1}t_{n+2}}{2} + 3t_{n+2} - \frac{3t_{n+1}}{2} \\
&\quad - \sum_{k=n+2}^{\infty} \frac{t_k}{2\rho_k} + \sum_{k=n+2}^{\infty} \frac{\rho_k t_{k+1}}{2} \\
&= \left(\frac{1}{2\rho_{n+1}} - \frac{\rho_{n+1}^2}{2} + 3\rho_{n+1} - \frac{3}{2} \right) t_{n+1} - \sum_{k=n+2}^{\infty} \left(\frac{1}{2\rho_k} - \frac{\rho_k^2}{2} \right) t_k \\
&= \left(\frac{1}{2\rho_{n+1}} - \frac{\rho_{n+1}^2}{2} + 3\rho_{n+1} - \frac{3}{2} \right. \\
&\quad \left. - \sum_{k=n+2}^{\infty} \left(\frac{1}{2\rho_k} - \frac{\rho_k^2}{2} \right) \prod_{i=n+1}^{k-1} \rho_i \right) t_{n+1}
\end{aligned} \tag{3.8}$$

for all $n \geq N_\alpha$.

We first prove the following technical lemma.

Lemma 3.4. *Let $\alpha \in \mathscr{P}_1$ with*

$$s_{n+1} := \sup_{k > n+1} \rho_k \leq 1 - \frac{7\rho_{n+1}^2}{8\rho_{n+1}^3 + 4} \quad \textit{for all } n \geq N_\alpha.$$

Then $\mathscr{M}$ is a finite union of disjoint closed intervals.

Proof. By Proposition 3.2 we only need to prove $G(n) \leq 0$ for all large enough n. First we will prove that $G(n)$ is always monotonic with respect to ρ_k for all $n \geq N_\alpha$ and $k \geq n+2$. Then we use this in (3.8) to prove that $G(n) \leq 0$ for all large n.

By (3.8) the derivative of $G(n)$ with respect to ρ_k for any $k \geq n+2$ is given by

$$\frac{\partial G}{\partial \rho_k} = \prod_{i=n+1}^{k-1} \rho_i \left(\frac{1}{\rho_k^2} + 2\rho_k - \sum_{j=k+1}^{\infty} \prod_{i=k+1}^{j-1} \rho_i \left(\frac{1}{\rho_j} - \rho_j^2 \right) \right) \frac{t_{n+1}}{2},$$

where $\prod_{i=k+1}^{k} \rho_i = 1$. Since the map $x \mapsto \frac{1}{x} - x^2$ is decreasing on $[1/2, 1]$, we have

$$\begin{aligned}\frac{\partial G}{\partial \rho_k} &> \prod_{i=n+1}^{k-1}\rho_i\Big(\frac{1}{\rho_k^2}+2\rho_k-\frac{7}{4}\sum_{j=k+1}^{\infty}\Big(\prod_{i=k+1}^{j-1}\rho_i\Big)\Big)\frac{t_{n+1}}{2}\\ &\geq \prod_{i=n+1}^{k-1}\rho_i\Big(\frac{1}{\rho_k^2}+2\rho_k-\frac{7}{4}\sum_{i=0}^{\infty}s_k^i\Big)\frac{t_{n+1}}{2}\\ &= \prod_{i=n+1}^{k-1}\rho_i\Big(\frac{1}{\rho_k^2}+2\rho_k-\frac{7}{4(1-s_k)}\Big)\frac{t_{n+1}}{2},\end{aligned}$$

where the second inequality follows from $s_k \geq \rho_n$ for all $n \geq k+1$. By calculation we find that $\frac{\partial G}{\partial \rho_k} > 0$ if $s_k \leq 1-\frac{7\rho_k^2}{8\rho_k^3+4}$, which means that $G(n)$ is increasing as a function of ρ_k. Since k is taken arbitrarily, we obtain $\frac{\partial G}{\partial \rho_k} > 0$ for all $k \geq n+2$. This together with $\rho_k > 1/2$ for all $k \geq n+2$ gives

$$\begin{aligned}G(n) &= \Big(\frac{1}{2\rho_{n+1}}-\frac{\rho_{n+1}^2}{2}+3\rho_{n+1}-\frac{3}{2}-\sum_{k=n+2}^{\infty}\Big(\frac{1}{2\rho_k}-\frac{\rho_k^2}{2}\Big)\prod_{i=n+1}^{k-1}\rho_i\Big)t_{n+1}\\ &\leq \Big(\frac{1}{2\rho_{n+1}}-\frac{\rho_{n+1}^2}{2}+3\rho_{n+1}-\frac{3}{2}-\rho_{n+1}\Big(\sum_{k=n+2}^{\infty}\Big(\frac{1}{2s_{n+1}}-\frac{s_{n+1}^2}{2}\Big)s_{n+1}^{k-n-2}\Big)\Big)\\ &\quad\cdot t_{n+1}\\ &= \Big(\frac{1}{2\rho_{n+1}}-\frac{\rho_{n+1}^2}{2}+3\rho_{n+1}-\frac{3}{2}-\rho_{n+1}\Big(\frac{1}{2s_{n+1}}+\frac{1}{2}+\frac{s_{n+1}}{2}\Big)\Big)t_{n+1} \leq 0,\end{aligned}$$

where the first inequality follows by the increasing property of $G(n)$ with respect to ρ_k for all $k \geq n+2$ and $(s_n)_{n\geq 1}$ is a decreasing sequence. Furthermore, the last inequality follows from

$$\frac{1}{2\rho_{n+1}^2}-\frac{\rho_{n+1}}{2}+3-\frac{3}{2\rho_{n+1}} \leq \frac{1}{2s_{n+1}}+\frac{1}{2}+\frac{s_{n+1}}{2}$$

for all $\rho_{n+1}, s_{n+1} \in [1/2, 1]$. This completes the proof. ∎

In case $(\rho_n)_{n\geq 1}$ converges to a limit $\rho \neq 1$, we have the following result.

PROPOSITION 3.5. *Let $\alpha \in \mathscr{P}_1$. Assume furthermore that $\rho = \lim_{n\to\infty}\rho_n$ exists and $1/2 \leq \rho < 1$. Then $\mathscr{M}_\alpha$ is a finite union of disjoint closed intervals.*

Proof. Similar to the proof of Lemma 3.4, by (3.8) it is enough to show that

$$\frac{1}{2\rho_{n+1}}-\frac{\rho_{n+1}^2}{2}+3\rho_{n+1}-\frac{3}{2}-\sum_{k=n+2}^{\infty}\Big(\frac{1}{2\rho_k}-\frac{\rho_k^2}{2}\Big)\prod_{i=n+1}^{k-1}\rho_i \leq 0$$

for all large enough n.

First we consider $1/2 < \rho < 1$. Since $\lim_{n\to\infty} \rho_n = \rho$, for any $0 < \delta < 1$ there exists an $N_\delta \geq N_\alpha$ such that $\max\{1/2, \rho - \delta\} < \rho_n < \rho + \delta$ for all $n \geq N_\delta$. So, for all $n \geq N_\delta$ we have

$$\begin{aligned}
&\frac{1}{2\rho_{n+1}} - \frac{\rho_{n+1}^2}{2} + 3\rho_{n+1} - \frac{3}{2} - \sum_{k=n+2}^{\infty} \left(\frac{1}{2\rho_k} - \frac{\rho_k^2}{2}\right) \prod_{i=n+1}^{k-1} \rho_i \\
&\qquad < \frac{1}{2(\rho+\delta)} - \frac{(\rho+\delta)^2}{2} + 3(\rho+\delta) - \frac{3}{2} \\
&\qquad\quad - \sum_{k=n+2}^{\infty} \left(\frac{1}{2(\rho+\delta)} - \frac{(\rho+\delta)^2}{2}\right)(\rho-\delta)^{k-n-1} \\
&\qquad = \frac{1}{2(\rho+\delta)} - \frac{(\rho+\delta)^2}{2} + 3(\rho+\delta) - \frac{3}{2} \\
&\qquad\quad - \left(\frac{1}{2(\rho+\delta)} - \frac{(\rho+\delta)^2}{2}\right)\frac{\rho-\delta}{1-(\rho-\delta)} \\
&\qquad = \frac{2\rho^4 - 7\rho^3 + 9\rho^2 - 5\rho + 1}{2(\rho+\delta)(1-\rho+\delta)} \\
&\qquad\quad + \delta \cdot \frac{4\rho^3 - 9\rho^2 + 12\rho - 1 - 4\rho\delta^2 + 3\rho\delta - 2\delta^3 + 5\delta^2 + 3\delta}{2(\rho+\delta)(1-\rho+\delta)},
\end{aligned}$$

where the first inequality follows since $x \mapsto \frac{1}{2x} - \frac{x^2}{2} + 3x - \frac{3}{2}$ is increasing while $x \mapsto \frac{1}{2x} - \frac{x^2}{2}$ is decreasing on $[1/2, 1]$. Note that $2\rho^4 - 7\rho^3 + 9\rho^2 - 5\rho + 1 < 0$ for all $\rho \in (1/2, 1)$. As $\rho \mapsto 4\rho^3 - 9\rho^2 + 12\rho - 1$ is increasing on on $[1/2, 1]$, we have

$$\begin{aligned}
&\frac{4\rho^3 - 9\rho^2 + 12\rho - 1 - 4\rho\delta^2 + 3\rho\delta - 2\delta^3 + 5\delta^2 + 3\delta}{2(\rho+\delta)(1-\rho+\delta)} \\
&\qquad \leq \frac{6 - 4\rho\delta^2 + 3\rho\delta - 2\delta^3 + 5\delta^2 + 3\delta}{2(\rho+\delta)(1-\rho+\delta)} < \frac{6 + 3\rho\delta + 5\delta^2 + 3\delta}{2(\rho+\delta)(1-\rho+\delta)} \\
&\qquad < \frac{17}{2(\rho+\delta)(1-\rho+\delta)}.
\end{aligned}$$

Hence, for a fixed ρ we can choose a sufficiently small $\delta > 0$ such that $2\rho^4 - 7\rho^3 + 9\rho^2 - 5\rho + 1 \leq -17\delta$. Then

$$\begin{aligned}
&\frac{1}{2\rho_{n+1}} - \frac{\rho_{n+1}^2}{2} + 3\rho_{n+1} - \frac{3}{2} - \sum_{k=n+2}^{\infty} \left(\frac{1}{2\rho_k} - \frac{\rho_k^2}{2}\right) \prod_{i=n+1}^{k-1} \rho_i \\
&\qquad < \frac{2\rho^4 - 7\rho^3 + 9\rho^2 - 5\rho + 1}{2(\rho+1)(2-\rho)} + \frac{17\delta}{2(\rho+\delta)(1-\rho+\delta)} \leq 0
\end{aligned}$$

for all $n \geq N_\delta$.

For $\rho = 1/2$ there exists an $N_\alpha^1 \in \mathbb{N}$ such that $1/2 < \rho_{n+1} < 0.55$ for all $n \ge N_\alpha^1$. Then for a fixed $n \ge N_\alpha^1$, we have $s_{n+1} \le 0.55$. Furthermore,

$$1 - \frac{7\rho_{n+1}^2}{8\rho_{n+1}^3 + 4} \ge 1 - \frac{7 \cdot 0.55^2}{8 \cdot 0.55^3 + 4} = 0.6027\ldots > s_{n+1}$$

since $1 - \frac{7\rho_{n+1}^2}{8\rho_{n+1}^3+4}$ is decreasing on $[0.5, 0.55]$. So, by Lemma 3.4 it follows that $G(n) \ge 0$ for all $n \ge N_\alpha^1$. ∎

3.2. $\alpha \in \mathscr{P}$ with $0 < \rho_n \le 1/2$ for all large enough n. We denote

$$\mathscr{P}_2 := \{\alpha \in \mathscr{P} : \text{there exists an } N_\alpha \ge 0 \text{ such that } 0 < \rho_{n+1} \le 1/2 \text{ for all } n \ge N_\alpha\}.$$

Fix an $\alpha \in \mathscr{P}_2$. By (3.6) and (3.7) for $n \ge N_\alpha$ we have

$$\begin{aligned}
G(n) &= \frac{t_{n+1}}{2} - \frac{t_{n+2}}{2} - \frac{t_{n+2}^2}{2t_{n+1}} - \sum_{k=n+2}^{\infty}\left(\frac{t_k}{2} - \frac{t_{k+1}}{2} - \frac{t_{k+1}^2}{2t_k}\right) \\
&= \frac{t_{n+1}}{2} - t_{n+2} - \frac{t_{n+2}^2}{2t_{n+1}} + \sum_{k=n+2}^{\infty}\frac{t_{k+1}^2}{2t_k} \\
&= \left(\frac{1}{2} - \rho_{n+1} - \frac{\rho_{n+1}^2}{2}\right)t_{n+1} + \sum_{k=n+2}^{\infty}\frac{t_{k+1}^2}{2t_k} \\
&= \left(\frac{1}{2} - \rho_{n+1} - \frac{\rho_{n+1}^2}{2} + \sum_{k=n+2}^{\infty}\frac{\rho_k}{2}\prod_{i=n+1}^{k}\rho_i\right)t_{n+1}.
\end{aligned} \tag{3.9}$$

Recall that $s_n = \sup_{k>n}\rho_k$. Set

$$m_n := \inf_{k>n}\rho_k.$$

The following lemma gives sufficient conditions for $G(n) > 0$ and $G(n) \le 0$, respectively.

LEMMA 3.6. *Let $\alpha \in \mathscr{P}_2$ and $n \ge N_\alpha$.*

(i) *If $\rho_{n+1} < \sqrt{2 + \frac{m_n^3}{1-m_n}} - 1$, then $G(n) > 0$.*
(ii) *If $\rho_{n+1} \ge \sqrt{2 + \frac{s_n^3}{1-s_n}} - 1$, then $G(n) \le 0$.*

Proof. By (3.9) and the definition of m_n we have

$$\begin{aligned}
G(n) &\ge \left(\frac{1}{2} - \rho_{n+1} - \frac{\rho_{n+1}^2}{2} + \sum_{k=3}^{\infty}\frac{m_n^k}{2}\right)t_{n+1} \\
&= \left(\frac{1}{2} - \rho_{n+1} - \frac{\rho_{n+1}^2}{2} + \frac{m_n^3}{2(1-m_n)}\right)t_{n+1}
\end{aligned}$$

for all $n \geq N_\alpha$. Hence, $G(n) > 0$ if $\rho_{n+1} < \sqrt{2 + \frac{m_n^3}{1-m_n}} - 1$. Similarly, for all $n \geq N_\alpha$,

$$G(n) \leq \left(\frac{1}{2} - \rho_{n+1} - \frac{\rho_{n+1}^2}{2} + \frac{s_n^3}{2(1-s_n)}\right) t_{n+1},$$

which implies that $G(n) \leq 0$ if $\rho_{n+1} \geq \sqrt{2 + \frac{s_n^3}{1-s_n}} - 1$. ∎

Next we use this lemma to describe the structure of $\mathscr{M}$ with $\alpha \in \mathscr{P}_2$.

PROPOSITION 3.7. *Let $\alpha \in \mathscr{P}_2$.*

(i) *If there is an $N_\alpha^1 \geq N_\alpha$ such that $0 < \rho_{n+1} \leq \sqrt{2} - 1$ for all $n \geq N_\alpha^1$, then $\mathscr{M}$ is a Cantor set. In particular, $\mathscr{M}$ is a homogeneous Cantor set if $N_\alpha^1 = 0$.*
(ii) *If $\rho = \lim_{n\to\infty} \rho_n$ exists and $\rho \in (0, 1/2)$, then $\mathscr{M}$ is a Cantor set.*
(iii) *Assume that* $\lim_{n\to\infty} \rho_n = 1/2$.
   (a) *If there is an $N_\alpha^2 \in \mathbb{N}$ such that $(\rho_n)_{n \geq N_\alpha^2} \subset (0, 1/2)$ is an increasing sequence, then $\mathscr{M}$ is a Cantor set. In particular, if $(\rho_n)_{n\geq 1}$ is an increasing sequence with $\rho_n \neq 1/2$ for any n, then $\mathscr{M}$ is a homogeneous Cantor set.*
   (b) *If there is an $N_\alpha^3 \in \mathbb{N}$ such that $\rho_n = 1/2$ for all $n \geq N_\alpha^3$, then $\mathscr{M}$ is a finite union of closed intervals.*

In all the above cases, if $\mathscr{M}$ is a Cantor set, the Hausdorff and packing dimensions of $\mathscr{M}$ are given by (3.4).

Proof. For (i), from (3.9) we see that

$$G(n) = \left(\frac{1}{2} - \rho_{n+1} - \frac{\rho_{n+1}^2}{2}\right) t_{n+1} + \sum_{k=n+2}^{\infty} \frac{t_{k+1}^2}{2t_k} > 0$$

if $\frac{1}{2} - \rho_{n+1} - \frac{\rho_{n+1}^2}{2} \geq 0$. Note that this is the case for all $\rho_{n+1} \in (0, \sqrt{2}-1]$. If it holds for all $n \geq 0$, we get a homogeneous Cantor set by Proposition 3.3. On the other hand, if it is only true for all large enough n, then we get a Cantor set. The fractal dimensions are then given by (3.4).

For (ii), by the assumption it follows that $m_n \leq \rho < 1/2$ for all $n \geq N_\alpha$, and $\lim_{n\to\infty} m_n = \rho$. The map $f : x \mapsto \sqrt{2 + \frac{x^3}{1-x}} - 1 - x$ is strictly decreasing on $[0, 1/2]$ with $f(1/2) = 0$. Thus $f(\rho) > 0$ for any fixed $\rho \in (0, 1/2)$. Hence, as $\lim_{n\to\infty} \rho_n = \lim_{n\to\infty} m_n = \rho$, there is a large enough $N_\rho \geq N_\alpha$ such that

$$\rho_{n+1} - m_n < f(\rho) \leq f(m_n)$$

for all $n \geq N_\rho$. This implies that $\rho_{n+1} < \sqrt{2 + \frac{m_n^3}{1-m_n}} - 1$ for all $n \geq N_\rho$. Then Lemma 3.6(i) and Proposition 3.3 give the result.

We now turn to (iii). First we assume that $(\rho_n)_{n\geq 1} \subset (0, 1/2)$ is increasing. Since $\rho_n = t_{n+1}/t_n < 1/2$, we have $a_n = t_n - t_{n+1} > t_{n+1}$, and thus $t_n = \sum_{k=n}^{\infty} a_k > \sum_{k=n+1}^{\infty} t_k$. By (3.9) it follows that

$$\begin{aligned} G(n) &> \left(\frac{1}{2} - \rho_n - \frac{\rho_n^2}{2}\right) \sum_{k=n+1}^{\infty} t_k + \sum_{k=n+1}^{\infty} \frac{\rho_k^2 t_k}{2} \\ &= \sum_{k=n+1}^{\infty} \left(\frac{1}{2} - \rho_n - \frac{\rho_n^2}{2} + \frac{\rho_k^2}{2}\right) t_k \\ &> \sum_{k=n+1}^{\infty} \left[\left(\frac{1}{2} - \rho_n\right) - \frac{1}{2}(\rho_k - \rho_n)\right] t_k, \end{aligned}$$

where the last inequality follows from $\rho_n < 1/2$ for all n. Since $(\rho_n)_{n\geq 1}$ is an increasing sequence with $\lim_{n\to\infty} \rho_n = 1/2$, we have $1/2 - \rho_n \geq \rho_k - \rho_n$ for any $k \geq n$. From this and the above inequality we see that $G(n) > 0$ for all $n \geq 0$, which makes $\mathscr{M}$ a homogeneous Cantor set. If $(\rho_n)_{n\geq N_\alpha^2}$ is an increasing sequence, then we see from the previous part that $G(n) > 0$ for all $n \geq N_\alpha^2$, and hence $\mathscr{M}$ is a Cantor set by Proposition 3.3. In both cases, the Hausdorff and packing dimensions of $\mathscr{M}$ are given by (3.4).

Finally, if there is an $N_\alpha^3 \geq 0$ such that $\rho_{n+1} = 1/2$ for all $n \geq N_\alpha^3$, then $G(n) = 0$ for all $n \geq N_\alpha^3$. This implies that for each $n \geq N_\alpha^3$ we get

$$\bigcup_{\omega\in\{0,1\}^n} I_\omega = \bigcup_{\omega\in\{0,1\}^{N_\alpha^3}} I_\omega.$$

So we have proved the result. ∎

Proof of Theorem 1.3. This follows by the combined results of Propositions 3.5 and 3.7. ∎

As an example, let us consider a partition that frequently occurs in connection with generalised Lüroth transformations (see e.g. [3]).

EXAMPLE 3.8. Let $\alpha = \{(1/2^n, 1/2^{n-1}] : n \in \mathbb{N}\} \cup \{0\}$. Then $a_n = 1/2^n$ and $t_n = 1/2^{n-1}$, and hence $\rho_n = 1/2$ for all $n \in \mathbb{N}$. This means that we are in the setting of Proposition 3.7(iii)(b). In fact, we immediately see that $G(n) = 0$ for all $n \geq 0$, so $\mathscr{M}$ is an interval. Next we give the boundary points of $\mathscr{M}$. For any $\varepsilon \in \{0,1\}^{\mathbb{N}}$ and $z \in (0, 1]$, by the proof of Proposition 2.1 we have

$$F_\varepsilon(z) = \sum_{n:\, z\leq 1/2^{\varepsilon_n}} \frac{z}{2^{n-\varepsilon_n}} + \sum_{n:\, z>1/2^{\varepsilon_n}} \frac{1}{2^n}. \tag{3.10}$$

From Lemma 2.2, the right boundary point is $M_{\bar{0}}$ and the left one is $M_{\bar{1}}$.

Then by (1.6) and (3.10) we obtain

$$M_{\bar{0}} = \int_{[0,1]} (1 - F_{\bar{0}})\, d\lambda = \int_{[0,1]} \left(1 - \sum_{n=1}^{\infty} \frac{z}{2^n}\right) d\lambda(z) = 1 - \int_{[0,1]} z\, d\lambda(z) = \frac{1}{2}.$$

On the other hand,

$$F_{\bar{1}}(z) = \begin{cases} \sum_{n=1}^{\infty} \frac{z}{2^{n-1}} & \text{if } z \le 1/2, \\ \sum_{n=1}^{\infty} \frac{1}{2^n} & \text{if } z > 1/2, \end{cases}$$

so that

$$\begin{aligned} M_{\bar{1}} &= \int_{[0,1/2]} \left(1 - \sum_{n=1}^{\infty} \frac{z}{2^{n-1}}\right) d\lambda(z) + \int_{(1/2,1]} \left(1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\right) d\lambda(z) \\ &= \int_{[0,1/2]} (1 - 2z)\, d\lambda(z) = 1/4. \end{aligned}$$

Hence,

$$\mathscr{M} = [M_{\bar{1}}, M_{\bar{0}}] = [1/4, 1/2].$$

Let $M \in \mathscr{M}$. If M is not an endpoint of any interval I_ω with $\omega \in \{0,1\}^*$, then there is a unique ε such that $M = M_\varepsilon$. If M is an endpoint of an interval I_ω, then there are precisely two sequences ε_1 and ε_2 with $M = M_{\varepsilon_1} = M_{\varepsilon_2}$, one ending in $\bar{0}$ and the other one ending in $\bar{1}$.

4. The Lüroth partition and other examples. The results from the previous section do not describe the set $\mathscr{M}$ for all possible partitions $\alpha \in \mathscr{P}$ for which $\rho = \lim_{n\to\infty} \rho_n$ exists. For $\rho = 1/2$, what is not covered is the case where there are infinitely many n such that $\rho_n < 1/2$ and $(\rho_n)_{n\ge1}$ is not increasing.

Another case that is not addressed is when $\lim_{n\to\infty} \rho_n = 1$. Since in this case

$$\lim_{n\to\infty} \left(\frac{1}{2\rho_{n+1}} - \frac{\rho_{n+1}^2}{2} + 3\rho_{n+1}\right) - \frac{3}{2} = \frac{3}{2}, \tag{4.1}$$

we see from (3.8) that the structure of $\mathscr{M}$ depends on the limiting behaviour of

$$L_k := \frac{\left(\frac{1}{2\rho_{k+1}} - \frac{\rho_{k+1}^2}{2}\right) \prod_{i=n+1}^{k} \rho_i}{\left(\frac{1}{2\rho_k} - \frac{\rho_k^2}{2}\right) \prod_{i=n+1}^{k-1} \rho_i} = \frac{\rho_k^2 (1 - \rho_{k+1}^3)}{\rho_{k+1}(1 - \rho_k^3)}, \qquad k \in \mathbb{N}.$$

Set $R := \limsup L_k$ and $r := \liminf L_k$. Then $\sum_{k=1}^{\infty} \left(\frac{1}{2\rho_k} - \frac{\rho_k^2}{2}\right) \prod_{i=n+1}^{k-1} \rho_i$ is divergent if $r > 1$. This would imply that $G(n) \le 0$ for all n large enough, so that $\mathscr{M}$ would become a finite union of closed intervals by Proposition 3.2. On the other hand, $\sum_{k=1}^{\infty} \left(\frac{1}{2\rho_k} - \frac{\rho_k^2}{2}\right) \prod_{i=n+1}^{k-1} \rho_i$ is convergent if $R < 1$, so there exists an $N_R \in \mathbb{N}$ such that

$$\sum_{k=n+2}^{\infty} \left(\frac{1}{2\rho_k} - \frac{\rho_k^2}{2} \right) \prod_{i=n+1}^{k-1} \rho_i \leq \frac{1}{2}$$

for all $n \geq N_R$. By (4.1), there exists an N with

$$\frac{1}{2\rho_{n+1}} - \frac{\rho_{n+1}^2}{2} + 3\rho_{n+1} - \frac{3}{2} \geq 1$$

for all $n \geq N$. Thus for $R < 1$, $G(n) > 0$ for all $n \geq \max\{N_R, N\}$. Proposition 3.3 implies that $\mathscr{M}$ would be a Cantor set in this case. Finally, if $r = R = 1$ we cannot give a general result on the structure of $\mathscr{M}$. Note that the Lüroth partition $\alpha_L = \left\{ \left(\frac{1}{n+1}, \frac{1}{n} \right] \right\}_{n \geq 1} \cup \{0\}$ falls in this last category as then $\rho_n = \frac{n}{n+1}$ for all $n \geq 1$, and thus

$$L_k = \frac{\left(\frac{k}{k+1}\right)^2 \left(1 - \left(\frac{k+1}{k+2}\right)^3\right)}{\frac{k+1}{k+2}\left(1 - \left(\frac{k}{k+1}\right)^3\right)} = \frac{k^2((k+2)^3 - (k+1)^3)}{(k+1)^2((k+1)^3 - k^3)} \to 1$$

as $k \to \infty$. Nonetheless, we have the following result for α_L.

Proposition 4.1. *$\mathscr{M}_{\alpha_L}$ is the union of eight disjoint closed intervals.*

Proof. For α_L we have $\rho_1 = \frac{1}{2}$ and $\rho_n = \frac{n}{n+1} > \frac{1}{2}$ for all $n > 1$. Computing the function g in (3.6) yields $g(1) = \frac{1}{8}$ and

$$g(n) = \frac{n+1}{2n^2} - \frac{n}{2(n+1)^2} - \frac{3}{2}\,\frac{1}{n(n+1)} = \frac{1}{2n^2(n+1)^2}$$

for $n \geq 2$. Using the fact that $\sum_{k \geq 1} \frac{1}{2k^2(k+1)^2} = \frac{\pi^2 - 9}{6}$, we obtain

$$G(0) = g(1) - \sum_{k \geq 2} \frac{1}{2k^2(k+1)^2} = \frac{1}{8} + \frac{9 - \pi^2}{6} + \frac{1}{8} = \frac{21 - 2\pi^2}{12} > 0.$$

And for $n \geq 1$ we get

$$\begin{aligned} G(n) &= \frac{1}{2(n+1)^2(n+2)^2} - \sum_{k \geq n+2} \frac{1}{2k^2(k+1)^2} \\ &= \frac{1}{2(n+1)^2(n+2)^2} - \frac{1}{2(n+2)^2(n+3)^2} - \frac{1}{2(n+3)^2(n+4)^2} \\ &\quad - \sum_{k \geq n+4} \frac{1}{2k^2(k+1)^2} \\ &= \frac{-n^4 - 2n^3 + 27n^2 + 116n + 124}{2(n+1)^2(n+2)^2(n+3)^2(n+4)^2} - \sum_{k \geq n+4} \frac{1}{2k^2(k+1)^2}. \end{aligned}$$

One can show that $-n^4 - 2n^3 + 27n^2 + 116n + 124 < 0$ for all $n \geq 7$. Hence, $G(n) < 0$ for all $n \geq 7$. The values of $G(n)$ for $1 \leq n \leq 6$ can be calculated separately:

$$G(1) = \frac{119 - 12\pi^2}{72} > 0, \qquad G(2) = \frac{237 - 8\pi^2}{144} > 0,$$
$$G(3) = \frac{11843 - 1200\pi^2}{7200} < 0, \qquad G(4) = \frac{5921 - 600\pi^2}{3600} < 0,$$
$$G(5) = \frac{290131 - 29400\pi^2}{176400} < 0, \qquad G(6) = \frac{1160549 - 117600\pi^2}{705600} < 0.$$

From the above calculation and (1.7) we can deduce that

$$\mathscr{M} = \bigcup_{\omega\in\{0,1\}^3} I_\omega,$$

so $\mathscr{M}$ consists of eight closed intervals. ∎

Proof of Theorem 1.4. Proposition 4.1 gives the first part of the theorem. For the second part, take n and $\omega \in \{0,1\}^n$; by Lemma 2.2 and (2.7) we have

$$I(n) = \lambda(I_\omega) = \int_{[0,1]} (F_{\omega\bar{1}} - F_{\omega\bar{0}})\, d\lambda = \sum_{k\geq n+1} \frac{1}{2k^2(k+1)^2}.$$

One can check that

$$\lim_{n\to\infty} \frac{I(n)}{I(n-1)} = 1 - \lim_{n\to\infty} \frac{1}{n^2(n+1)^2}\Big(\sum_{k\geq n} \frac{1}{k^2(k+1)^2}\Big)^{-1} = 1.$$

Thus there exists a large N such that

$$\frac{I(n)}{I(n-1)} \geq \frac{\sqrt{5}-1}{2}$$

for all $n \geq N$. The result then follows from Proposition 3.2. ∎

A geometrical construction of $\mathscr{M}$ with $\alpha = \alpha_L$ is plotted in Figure 3.

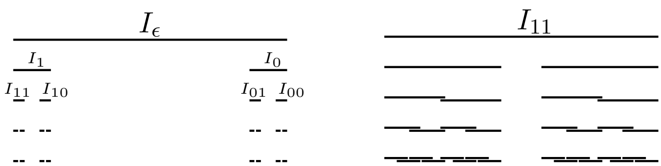


Fig. 3. Left: the first four levels of $\mathscr{M}$. Right: the first four levels of I_{11}.

We finish by giving some examples of partitions for which $\lim_{n\to\infty} \rho_n$ does not exist to illustrate that in this case several different situations can occur.

Example 4.2. Let $\alpha = \{A_n = (t_{n+1}, t_n] : n \in \mathbb{N}\} \cup \{A_\infty = \{0\}\}$ be such that

$$t_n := \begin{cases} \frac{1}{2^m} & \text{if } n = 2m+1 \text{ and } m \geq 0, \\ \frac{3}{5\cdot 2^m} & \text{if } n = 2(m+1) \text{ and } m \geq 0. \end{cases}$$

Then $\rho_n = \frac{t_{n+1}}{t_n} = \frac{3}{5}$ for $n = 2m+1$, and $\rho_n = \frac{t_{n+1}}{t_n} = \frac{5}{6}$ for $n = 2(m+1)$. So

$$\liminf_{n\to\infty} \frac{t_{n+1}}{t_n} = \frac{3}{5} \quad \text{and} \quad \limsup_{n\to\infty} \frac{t_{n+1}}{t_n} = \frac{5}{6}.$$

In other words, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n}$ does not exist. By (3.8) for $n = 2m$ we have

$$\begin{aligned} G(n) &= \left(\frac{5}{6} - \frac{9}{50} + \frac{9}{5} - \frac{3}{2}\right)\frac{1}{2^m} + \left(\frac{25}{72} - \frac{3}{5}\right)\sum_{k=m}^{\infty} \frac{3}{5\cdot 2^k} \\ &\quad + \left(\frac{9}{50} - \frac{5}{6}\right)\sum_{k=m+1}^{\infty} \frac{1}{2^k} \\ &= \left(\frac{9}{5} - \frac{3}{2} + \frac{5}{12} - \frac{18}{25}\right)\frac{1}{2^m} = -\frac{1}{300\cdot 2^m} < 0. \end{aligned}$$

Similarly, for $n = 2m+1$ we find that

$$G(n) = -\frac{4}{75\cdot 2^m} < 0.$$

Hence, in this case $\mathscr{M} = [M_{\bar{1}}, M_{\bar{0}}]$.

EXAMPLE 4.3. Let $\alpha = \{A_n = (t_{n+1}, t_n] : n \in \mathbb{N}\} \cup \{A_\infty = \{0\}\}$, where

$$t_n := \begin{cases} \frac{1}{3^m} & \text{for } n = 2m+1 \text{ and } m \geq 0, \\ \frac{21}{40\cdot 3^m} & \text{for } n = 2(m+1) \text{ and } m \geq 0. \end{cases}$$

Then $\rho_n = \frac{t_{n+1}}{t_n} = \frac{21}{40}$ for $n = 2m+1$, and $\rho_n = \frac{t_{n+1}}{t_n} = \frac{40}{63}$ for $n = 2(m+1)$. So, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n}$ does not exist. By (3.8) for $n = 2m$ we have

$$\begin{aligned} G(n) &= \left(\frac{20}{21} - \frac{21^2}{2\cdot 40^2} + \frac{63}{40} - \frac{3}{2}\right)\frac{1}{3^m} + \left(\frac{40^2}{2\cdot 63^2} - \frac{63}{80}\right)\sum_{k=m}^{\infty} \frac{21}{40\cdot 3^k} \\ &\quad + \left(\frac{21^2}{2\cdot 40^2} - \frac{20}{21}\right)\sum_{k=m+1}^{\infty} \frac{1}{3^k} \\ &= \frac{841}{40320\cdot 3^m} > 0, \end{aligned}$$

while for $n = 2m+1$ we get

$$G(n) = -\frac{12391}{302400\cdot 3^m} < 0.$$

In this case we cannot say what the structure of $\mathscr{M}$ will be, but we do note that even for $\frac{t_{n+1}}{t_n} > \frac{1}{2}$ one can have $G(n) > 0$.

EXAMPLE 4.4. Let $\alpha = \{A_n = (t_{n+1}, t_n] : n \in \mathbb{N}\} \cup \{A_\infty = \{0\}\}$, where

$$t_n := \begin{cases} \frac{1}{4^m} & \text{for } n = 2m+1 \text{ and } m \geq 0, \\ \frac{1}{3\cdot 4^m} & \text{for } n = 2(m+1) \text{ and } m \geq 0. \end{cases}$$

Then $\rho_n = \frac{t_{n+1}}{t_n} = \frac{1}{3}$ for $n = 2m+1$, and $\rho_n = \frac{t_{n+1}}{t_n} = \frac{3}{4}$ for $n = 2(m+1)$. So, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n}$ does not exist. By (3.6) and (3.7) for $n = 2m$ we have

$$\begin{aligned}
G(n) &= \frac{1}{2}\left(1 - \frac{1}{3} - \frac{1}{9}\right)\frac{1}{4^m} - \left(\frac{2}{3} - \frac{9}{32} + \frac{9}{8} - \frac{3}{2}\right)\sum_{k=m}^{\infty}\frac{1}{3\cdot 4^k} \\
&\quad - \frac{1}{2}\left(1 - \frac{1}{3} - \frac{1}{9}\right)\sum_{k=m+1}^{\infty}\frac{1}{4^k} \\
&= \left(\frac{5}{27} - \frac{1}{216}\right)\frac{1}{4^m} > 0,
\end{aligned}$$

while for $n = 2m+1$ we get

$$\begin{aligned}
G(n) &= \left(\frac{2}{3} - \frac{9}{32} + \frac{9}{8} - \frac{3}{2}\right)\frac{1}{3\cdot 4^m} - \frac{1}{2}\left(1 - \frac{1}{3} - \frac{1}{9}\right)\sum_{k=m+1}^{\infty}\frac{1}{4^k} \\
&\quad - \left(\frac{2}{3} - \frac{9}{32} + \frac{9}{8} - \frac{3}{2}\right)\sum_{k=m+1}^{\infty}\frac{1}{3\cdot 4^k} \\
&= \left(\frac{1}{432} - \frac{5}{54}\right)\frac{1}{4^m} < 0.
\end{aligned}$$

Also in this case it is not clear what the structure of $\mathscr{M}$ is.

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