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Generic vanishing, 1-forms, and topology of Albanese maps

Yajnaseni Dutta¹ · Feng Hao² · Yongqiang Liu³

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Abstract

Given a bounded constructible complex of sheaves $\mathcal{F}$ on a complex Abelian variety, we prove an equality relating the cohomology jump loci of $\mathcal{F}$ and its singular support. As an application, we identify two subsets of the set of holomorphic 1-forms with zeros on a complex smooth projective irregular variety X ; one from Green-Lazarsfeld's cohomology jump loci and one from the Kashiwara's estimates for singular supports. This result is related to Kotschick's conjecture about the equivalence between the existence of nowhere vanishing global holomorphic 1-forms and the existence of a fibre bundle structure over the circle. Our results give a conjecturally equivalent formulation using singular support, which is equivalent to a criterion involving cohomology jump loci proposed by Schreieder. As another application, we reprove a recent result proved by Schreieder and Yang; namely if X has simple Albanese variety and admits a fibre bundle structure over the circle, then the Albanese morphism cohomologically behaves like a smooth morphism with respect to integer coefficients. In a related direction, we address the question whether the set of 1-forms that vanish somewhere is a finite union of linear subspaces of $H^0(X, \Omega_X^1)$. We show that this is indeed the case for forms admitting zero locus of codimension 1.

Keywords Holomorphic one-form · Cohomology jump loci · Constructible sheaves · Singular support · Linearity · Abelian variety

Mathematics Subject Classification Primary 32Q55 · 32S60 · 14K12 · 14C30; Secondary 14F40

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1 Introduction

We study the topology of the Albanese map via constructible complexes on Abelian varieties. The latter has been recently explored extensively by Schnell [33] using generalised Fourier–Mukai transforms and the language of holonomic D -modules. His results vastly extended the foundational results in [1, 13, 14, 36] about cohomology jump loci of rank one local systems, as well as their incarnations in the moduli of line bundles with connections or rank 1 higgs bundles on smooth irregular projective varieties. In particular, Schnell proved a structure theorem for cohomology jump loci for any bounded constructible complex of sheaves on complex Abelian varieties. To state Schnell’s results, we first recall the definition of cohomology jump loci.

Let $D_c^b(A)$ denote the category of bounded constructible complex of $\mathbb{C}$ -sheaves on a complex Abelian variety A . Set $\text{Char}^0(A) := \text{Hom}(H_1(A, \mathbb{Z}), \mathbb{C}^*)$, which is the moduli space of rank one $\mathbb{C}$ -local systems on A .

Definition 1.1 The i -th cohomology jump loci of $\mathcal{F} \in D_c^b(A)$ is defined to be the set

$$\mathcal{V}^i(A, \mathcal{F}) := \{\rho \in \text{Char}^0(A) \mid H^i(A, \mathcal{F} \otimes \mathbb{C}_\rho) \neq 0\},$$

where $\mathbb{C}_\rho$ denotes the rank 1 local system associated to $\rho \in \text{Char}^0(A)$. Furthermore, we write $\mathcal{V}(A, \mathcal{F}) := \bigcup_i \mathcal{V}^i(A, \mathcal{F})$.

Schnell’s structure theorem states that $\mathcal{V}^i(A, \mathcal{F})$ is finite union of translated subtori of $\text{Char}^0(A)$ [33]. In this paper, we prove an equality relating the cohomology jump loci $\mathcal{V}(A, \mathcal{F})$ and the singular support of $\mathcal{F}$ on complex Abelian varieties. Moreover, we show that the projection of the singular support of $\mathcal{F}$ is linear, which is compatible with Schnell’s structure theorem on cohomology jump loci, see Remark 1.3.

The $(1, 0)$ -piece of the tangent space $\text{TC}_\rho(\text{Char}^0(A))$ at a character $\rho \in \text{Char}^0(A)$ is $H^0(A, \Omega_A^1)$. Let $\mathcal{T}(A, \mathcal{F})$ denote the union of the $(1, 0)$ -part of the tangent space to the irreducible components of the subvariety $\mathcal{V}(A, \mathcal{F})$.

Theorem 1.2 Let A be a complex Abelian variety. For any $\mathcal{F} \in D_c^b(A)$, we have the equality

$$\pi(\text{SS}(\mathcal{F})) = \mathcal{T}(A, \mathcal{F}), \quad (1)$$

where $\text{SS}(\cdot)$ denotes the singular support (see Definition 2.7) of constructible complexes in the cotangent space $T^*A \simeq A \times H^0(A, \Omega_A^1)$ and $\pi: T^*A \rightarrow H^0(A, \Omega_A^1)$ is the natural projection. In particular, $\pi(\text{SS}(\mathcal{F}))$ is a finite union of linear subspaces of the vector space $H^0(A, \Omega_A^1)$.

Remark 1.3 The linearity part of the theorem could also follow from the structure theorem for $\mathcal{V}(A, \mathcal{F})$ proved by Schnell [33, Theorem 7.3], once the equality (1) is provided. Here we give a direct proof of the linearity property of $\pi(\text{SS}(\mathcal{F}))$ without using Schnell’s results (see Proposition 3.2).

Remark 1.4 Due to Riemann–Hilbert correspondence, Theorem 1.2 also holds for regular holonomic D -module complexes.

The key technique we use is to relate the two sides of the equality in Theorem 1.2 via the Euler characteristic formula given by the Kashiwara index theorem (see Theorem 2.6). In fact, Theorem 1.2 should be viewed as a modified version of Kashiwara index theorem on complex Abelian varieties, since $\text{SS}(\mathcal{F})$ records a piece of information about characteristic

cycles of $\mathcal{F}$ and $\mathcal{V}(A, \mathcal{F})$ records a piece of information about the Euler characteristic number of $\mathcal{F}$.

As an application of Theorem 1.2, we have the following result proved recently by Schreieder and Yang [35, Corollary 3.4]. This prompted us to prove Theorem 1.2 in arbitrary characteristics for simple abelian varieties (see Proposition 3.7) and obtain their result as a corollary. We thank the referee for encouraging us to generalize our main theorem in this direction.

Corollary 1.5 *Let $f: X \rightarrow A$ be a morphism from a smooth complex projective variety to a simple Abelian variety. If there exists a $\mathcal{C}^\infty$ -fibre bundle structure $p_X: X \rightarrow S^1$ such that $p_X^*(d\theta) \in f^*H^1(A, \mathbb{R})$, where θ is a coordinate of the circle, then f is a $\mathbb{Z}$ -homology fiber bundle. Moreover, for any algebraically closed field coefficient $\mathbb{K}$, $\pi(\text{SS}(\mathbb{R}f_*\mathbb{K}_X)) = \{0\}$.*

Remark 1.6 When the assumption on the simplicity of A is dropped in the Corollary above, our theorem more generally gives information on the topological structure of $\text{SS}(\mathbb{R}f_*\mathbb{C}_X)$. See Corollary 3.8 for more details.

As another application of Theorem 1.2, we have the following

Theorem 1.7 *Let X be a smooth projective variety with $a: X \rightarrow A_X$ its Albanese morphism. Under the linear isomorphism $H^0(X, \Omega_X^1) \cong H^0(A_X, \Omega_{A_X}^1)$ one can identify*

$$\bigcup_{\tau} \pi(\text{SS}(\mathbb{R}(a \circ \tau)_*\mathbb{C}_{X'})) = \bigcup_{\tau} \{\omega \in H^0(X, \Omega_X^1) \mid (H^\bullet(X', \mathbb{C}), \wedge \tau^*\omega) \text{ is not exact}\}, \quad (2)$$

where both unions are running over all possible finite étale cover $\tau: X' \rightarrow X$.

The study of the above theorem is motivated by the following conjecture, which was posed by Kotschick [23, Question 15] and Schreieder [34].

Conjecture 1.8 *Let X be a smooth complex projective variety. Then the following three statements are equivalent:*

- (1) X admits a global holomorphic 1-form without zeros.
- (2) X admits a $\mathcal{C}^\infty$ real closed 1-form which has no zeros, or equivalently X admits a $\mathcal{C}^\infty$ -fibre bundle structure over the circle [38, Theorem 1].
- (3) There exists $\omega \in H^0(X, \Omega_X^1)$ such that for all finite étale morphism $\tau: X' \rightarrow X$, the complex $(H^\bullet(X', \mathbb{C}), \wedge \tau^*\omega)$ is exact.

Theorem 1.7 then gives the fourth criterion, which is equivalent to (3) in Conjecture 1.8:

- (4) There exists $\omega \in H^0(X, \Omega_X^1) \setminus \bigcup_{\tau} \pi(\text{SS}(\mathbb{R}(a \circ \tau)_*\mathbb{C}_{X'}))$.

Remark 1.9 All the 1-forms involved in Theorem 1.7 are contained in $W(X)$, the collection of holomorphic one forms on X with zeros. By a result of Green and Lazarsfeld [14] we know that the set on the right side of (2) is contained in $W(X)$. On the other hand, it follows from Kashiwara's estimate (see Theorem 2.8) that the left side is also contained in $W(X)$. Hence we pose the following question: Is it true that

$$\overline{\bigcup_{\tau} \pi(\text{SS}(\mathbb{R}(a \circ \tau)_*\mathbb{C}_{X'}))} = \overline{\bigcup_{\mathbb{L}} \pi(\text{SS}(\mathbb{R}a_*\mathbb{L}))} = W(X)? \quad (3)$$

Here $\overline{}$ denotes the Zariski closure and the second union is running over all possible local systems $\mathbb{L}$ on X .

The answer is yes for varieties of dimension less than or equal to 3. In this situation, in fact, using Theorem 1.7, [34, Corollary 3.1] and [16, Theorem 1.4] we have

$$\bigcup_v \pi(\mathrm{SS}(\mathbb{R}(a \circ v)_* \mathbb{C}_{X'})) = W(X),$$

with the union running over all possible finite Abelian étale covers $v: X' \rightarrow X$.

1.1 linearity of 1-forms admitting zeros

By Theorem 1.2, $\pi(\mathrm{SS}(\mathbb{R}(a \circ \tau)_* \mathbb{C}_{X'}))$ is a linear subspace of the vector space $H^0(X, \Omega_X^1)$. From the point of view of Theorem 1.7 and Remark 1.9, one may wonder whether the set $W(X)$ is also a finite union of vector subspaces of $H^0(X, \Omega_X^1)$. Such a statement is true for the set of global holomorphic tangent vector fields with zeros due to the work of Carrell and Lieberman [5].

More specifically, consider

$$W^i(X) = \{\omega \in H^0(X, \Omega_X^1) \mid \mathrm{codim}_X Z(\omega) \leq i\},$$

where $Z(\omega)$ is the zero set of ω . Green and Lazarsfeld showed [13] that $W^i(X)$ contains the $(1, 0)$ -piece of the tangent cone of the cohomology jump loci of X up to degree i . Note that the cohomology jump loci of X are finite union of torsion translated sub-tori. We ask the following

Question 1.10 *Are $W^i(X)$ linear for every degree i , i.e. a finite union of vector subspaces of the vector space $H^0(X, \Omega_X^1)$?*

We answer the question positively for $W^1(X)$.

Theorem 1.11 (see Theorem 4.4) *Let X be a smooth projective variety of dimension n . Then $W^1(X)$ is linear.*

This follows from a result of Spurr [37, Theorem 2]; whenever a 1-form ω vanish along a divisor E , one has either E is rigid in the sense that $E^2 \cdot H^{n-2} < 0$ with respect to some polarisation H on X , or ω comes from a curve. We generalise this statement in the setting of a pair (see Theorem 5.1) and prove the linearity statement for logarithmic 1-forms admitting codimension one zeros.

Theorem 1.12 (see Theorem 4.7) *Let (X, D) be a pair with X a smooth projective variety and D a simple normal crossing divisor of X . Then the following set is linear*

$$W^1(X, D) := \{\omega \in H^0(X, \Omega_X^1(\log D)) \mid \mathrm{codim}_X Z(\omega) \leq 1\}.$$

Convention

In this paper, all complex of sheaves and perverse sheaves are defined with complex coefficients except in Sects. 2.2 and 3.2. All the varieties are complex quasi-projective varieties. Unless specified otherwise by 1-forms on a smooth projective variety, we mean global holomorphic 1-forms.

2 Preliminaries

2.1 1-forms and associated local systems

The results in this subsection should be well known to the experts and we include it here due to the lack of references.

Let X be a smooth projective variety. Consider the identity component of the character variety $\text{Char}(X) := \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$ given by

$$\text{Char}^0(X) := \text{Hom}(H_1(X, \mathbb{Z})/\text{torsion}, \mathbb{C}^*).$$

The i -th cohomology jump loci $\mathcal{V}^i(X, \mathcal{F}) \subset \text{Char}^0(X)$ for $\mathcal{F} \in D_c^b(X)$ is defined in a similar way as in Definition 1.1. As in the introduction $\mathcal{V}(X, \mathcal{F}) = \bigcup_i \mathcal{V}^i(X, \mathcal{F})$. The corresponding tangent cone $\mathcal{T}(X, \mathcal{F}) \subset H^0(X, \Omega_X^1)$ is also defined similarly as was done before Theorem 1.2. More precisely,

$$\mathcal{T}(X, \mathcal{F}) := H^0(X, \Omega_X^1) \cap \left(\bigcup_{\rho} \text{TC}_{\rho} \mathcal{V}(X, \mathcal{F}) \right),$$

where the union is running over representative points from irreducible components of $\mathcal{V}(X, \mathcal{F})$ and $\text{TC}_{\rho} \mathcal{V}(X, \mathcal{F}) \subseteq H^1(X, \mathbb{C})$ denotes the tangent cone at ρ . When $\mathcal{F} = \mathbb{C}_X$, we simply write $\mathcal{T}(X) := \mathcal{T}(X, \mathbb{C}_X)$.

Given a 1-form $\omega \in H^0(X, \Omega_X^1)$, the kernel $L(\omega)$ of $\mathcal{O}_X \xrightarrow{d+\wedge\omega} \Omega_X^1$ is a rank 1 local system resolved by the de Rham complex (see the proof of [34, Lemma 2.2])

$$\mathcal{K}^{\bullet}(\omega) := [\mathcal{O}_X \xrightarrow{d+\wedge\omega} \Omega_X^1 \longrightarrow \cdots \xrightarrow{d+\wedge\omega} \Omega_X^{n-1} \xrightarrow{d+\wedge\omega} \Omega_X^n], \quad (4)$$

and hence $H^k(X, L(\omega)) = \mathbb{H}^k(X, (\Omega_X^{\bullet}, d+\wedge\omega))$. What's more, the corresponding line bundle $\mathcal{L}_{\omega} := L(\omega) \otimes_{\mathbb{C}} \mathcal{O}_X$ is isomorphic to $\mathcal{O}_X$. Hence we have the following short exact sequence

$$\begin{aligned} 0 \rightarrow H^0(X, \Omega_X^1) &\rightarrow \text{Char}^0(X) \rightarrow \text{Pic}^0(X) \rightarrow 0, \\ \omega &\mapsto L(\omega); L \mapsto L \otimes_{\mathbb{C}} \mathcal{O}_X, \end{aligned} \quad (5)$$

In order to obtain a Kodaira–Nakano-type vanishing theorem for degree zero line bundles Green–Lazarsfeld [13] considered the following spectral sequence associated to the complex (4)

$$E_1^{p,q}(\omega) := H^p(X, \Omega_X^q) \Rightarrow \mathbb{H}^{p+q}(X, (\Omega_X^{\bullet}, d+\wedge\omega)),$$

with differential $d_1 = \wedge\omega : E_1^{p,q}(\omega) \rightarrow E_1^{p,q+1}(\omega)$ induced by $d+\wedge\omega$ in complex (4). Using this spectral sequence, [13, Proposition 3.4] shows that if there is a holomorphic 1-form ω whose zero locus $Z(\omega)$ has codimension $\geq k$, then the sequence

$$\cdots \rightarrow H^p(X, \Omega_X^{q-1}) \xrightarrow{\wedge\omega} H^p(X, \Omega_X^q) \xrightarrow{\wedge\omega} H^p(X, \Omega_X^{q+1}) \rightarrow \cdots$$

is exact for all $p+q < k$. Putting these together by the Hodge decomposition for $H^i(X, \mathbb{C})$ we get

$$(H^{\bullet}(X, \mathbb{C}), \wedge\omega) := [\cdots \rightarrow H^{i-1}(X, \mathbb{C}) \xrightarrow{\wedge\omega} H^i(X, \mathbb{C}) \xrightarrow{\wedge\omega} H^{i+1}(X, \mathbb{C}) \rightarrow \cdots] \quad (6)$$

is exact for all $i < k$. This prompts the following

Definition 2.1 (Holomorphic resonant varieties) The k -th *holomorphic resonant variety* of X is defined as

$$\mathcal{R}^k(X) := \{\omega \in H^0(X, \Omega_X^1) \mid H^k(H^\bullet(X, \mathbb{C}), \wedge\omega) \neq 0\}$$

and we set $\mathcal{R}(X) = \bigcup_k \mathcal{R}^k(X)$. We will refer to the sequence $(H^\bullet(X, \mathbb{C}), \wedge\omega)$ above as the *resonance sequence*.

More generally, both $\mathcal{K}^\bullet(\omega)$ and $\mathcal{R}^k(X)$ can be twisted by unitary local systems. Recall that the space of unitary local systems is defined to be

$$\text{Char}^0(X)^u := \text{Hom}(H_1(X, \mathbb{Z})/\text{torsion}, U(1)).$$

For any unitary character $\eta \in \text{Char}^0(X)^u$, the corresponding local system $\mathbb{C}_\eta$ corresponds to a degree 0 line bundle $\mathcal{L}_\eta := \mathbb{C}_\eta \otimes_{\mathbb{C}} \mathcal{O}_X$. In fact, this gives a one-to-one correspondence between $\text{Char}^0(X)^u$ and $\text{Pic}^0(X)$.

Definition 2.2 (Generalised holomorphic resonant variety) Given a local system $\mathbb{C}_\eta$ associated to a unitary character η , the k -th *generalised holomorphic resonant variety* associated to η is defined as

$$\mathcal{R}^k(X, \mathbb{C}_\eta) := \{\omega \in H^0(X, \Omega_X^1) \mid H^k(H^\bullet(X, \mathbb{C}_\eta), \wedge\omega) \neq 0\}$$

Also, we set¹

$$\mathcal{R}_{\text{dol}}(X) := \bigcup_{k, \eta \text{ unitary}} \mathcal{R}^k(X, \mathbb{C}_\eta).$$

As noted in the introduction, another way to understand $\mathcal{R}_{\text{dol}}(X)$ is via the tangent cone of the cohomology jump loci $\mathcal{V}(X)$. We have the following lemma due to [13, Proposition 3.7, Remark on p. 404], which directly generalises the so-called *tangent-cone equality*

$$H^0(X, \Omega_X^1) \cap \text{TC}_1 \mathcal{V}(X, \mathbb{C}_X) = \mathcal{R}(X). \quad (7)$$

Lemma 2.3 *With the notation as above, we have*

$$H^0(X, \Omega_X^1) \cap \text{TC}_\eta(\mathcal{V}^k(X, \mathbb{C}_X)) = \mathcal{R}^k(X, \mathbb{C}_\eta).$$

Remark 2.4 A more precise version of the above lemma for anti-holomorphic 1-forms can be found in [3, Theorem 1.3] due to Budur-Wang.

The following corollary follows directly from Lemma 2.3.

Corollary 2.5 *Let X be a smooth projective variety. Then we have $\mathcal{T}(X) = \mathcal{R}_{\text{dol}}(X)$.*

2.2 Constructible complexes of sheaves and perverse sheaves

Let $D_c^b(X)$ denote the derived category of bounded constructible complex of sheaves on X with coefficients in a field $\mathbb{K}$. *Perverse sheaves* on X are, roughly speaking, a generalisation of local systems, i.e. locally constant sheaves. We refer the readers to [17, Chapter 8] and [6, Sect. 5] for definitions and a comprehensive background on this topic.

¹ Here we only write $\wedge\omega$ in $(H^\bullet(X, \mathbb{C}_\eta), \wedge\omega)$, since d acts trivially on $H^\bullet(X, \mathbb{C}_\eta)$.

Given $\mathcal{F} \in D_c^b(X)$, its *characteristic cycle* $\text{CC}(\mathcal{F})$ is a finite $\mathbb{Z}$ -linear combination of irreducible conic Lagrangian cycles $T_Z^*X := \overline{T_{Z_i}^{\text{reg}} X}$ in T^*X over certain irreducible closed subvarieties $Z_i \subseteq X$

$$\text{CC}(\mathcal{F}) = \sum_i n_{Z_i} T_{Z_i}^* X.$$

Here $T_{Z_i}^{\text{reg}} X$ is the conormal bundle of the regular locus Z_i^{reg} of Z_i in X . For the definition when $\mathcal{F} \in D_c^b(X, \mathbb{K})$, for a field $\mathbb{K}$ in any characteristic, see e.g. [29, Definition 3.34].

The Euler characteristic of $\mathcal{F}$ satisfy the following Kashiwara's index theorem, see [19] for complex coefficients and also [29, Theorem 3.38, Example 3.39, 3.40] for any field coefficients.

Theorem 2.6 *For $\mathcal{F} \in D_c^b(X)$ on a smooth projective variety X , we have*

$$\chi(X, \mathcal{F}) = \text{CC}(\mathcal{F}) \cdot T_X^* X = \sum_i n_{Z_i} (T_{Z_i}^* X \cdot T_X^* X)$$

where the dot denotes intersections of cycles in the complex manifold T^*X .

When $\mathcal{F}$ happens to be a perverse sheaf $\mathcal{P}$ on X , $n_{Z_i} \geq 0$ (see [6, Corollary 5.2.24] or [29, Definition 3.34] for any characteristic) and its singular support of $\mathcal{P}$ is defined as

$$\text{SS}(\mathcal{P}) := \bigcup_{n_{Z_i} > 0} T_{Z_i}^* X.$$

Then we define the singular support of the bounded complex of constructible sheaves as follows.

Definition 2.7 For a constructible complex $\mathcal{F} \in D_c^b(X)$ and any integer i , the singular support of $\mathcal{F}$ is defined as

$$\text{SS}(\mathcal{F}) := \bigcup_i \text{SS}({}^p\mathcal{H}^i(\mathcal{F})),$$

where ${}^p\mathcal{H}^i(\mathcal{F})$ is the i -th perverse cohomology of $\mathcal{F}$.

Similar definition has also been used in [21, Exercise X.6] and [17, p. 373].

Given $f: X \rightarrow A$ a morphism from a smooth projective variety X to an Abelian variety A , Kashiwara's estimate for the behaviour of the singular support produces a breeding ground for 1-forms with zeros. This estimate was exploited in [30] to show that all 1-forms admit zeros on smooth projective varieties of general type. We recall it in our current framework. Up to our knowledge, Kashiwara's estimate are only proved for complex coefficients.

Theorem 2.8 (Kashiwara's estimate [20, Theorem 4.2]) *Given $f: X \rightarrow A$, consider the following commutative diagram*

$$\begin{array}{ccccccc} T^*X & \xleftarrow{df} & f^*T^*A & \xrightarrow{f \times \text{id}} & T^*A & \xrightarrow{\pi} & H^0(A, \Omega_A^1) \\ & \searrow & \downarrow & & \downarrow & & \\ & & X & \xrightarrow{f} & A & & \end{array} \quad (8)$$

Then for any complex local system $\mathbb{L}$ on X

$$\text{SS}(\mathbb{R}f_*\mathbb{L}) \subseteq (f \times \text{id})(df^{-1}(0_X)),$$

where 0_X denotes the zero section T_X^*X of T^*X .

Note that $\pi(f \times \text{id})(df^{-1}(0_X)) = W(X) \cap H^0(A, \Omega_A^1)$ under a suitable identification. We use this result frequently as follows:

$$\pi(\text{SS}(\mathbb{R}f_*\mathbb{L})) \subseteq W(X) \cap H^0(A, \Omega_A^1) \quad (9)$$

Finally we recall some special properties exhibited by the cohomology jump loci of perverse sheaves on Abelian varieties.

Theorem 2.9 ([27, Theorem 4.3, Corollary 1.3, Corollary 4.8], [11, Corollary 1.4]) *Let $\mathcal{P}$ be a perverse sheaf with coefficients in fields of any characteristic on a complex Abelian variety A with $\dim A = g$. The cohomology jump loci of $\mathcal{P}$ satisfy the following*

(1) *Propagation property:*

$$\mathcal{V}^{-g}(A, \mathcal{P}) \subseteq \cdots \subseteq \mathcal{V}^{-1}(A, \mathcal{P}) \subseteq \mathcal{V}^0(A, \mathcal{P}) \supseteq \mathcal{V}^1(A, \mathcal{P}) \supseteq \cdots \supseteq \mathcal{V}^g(A, \mathcal{P}).$$

Furthermore, $\mathcal{V}^i(A, \mathcal{P}) = \emptyset$, if $i \notin [-g, g]$.

(2) *Signed Euler characteristic property:* $\chi(A, \mathcal{P}) \geq 0$. Moreover, the equality holds if and only if $\mathcal{V}^0(A, \mathcal{P}) \neq \text{Char}(A)$.

(3) *For a general $\rho \in \text{Char}^0(A)$, $H^i(A, \mathcal{P} \otimes \mathbb{L}_\rho) = 0$ for all $i \neq 0$, where $\mathbb{L}_\rho$ is the rank 1 local system associated to ρ .*

The Statement (3) in positive characteristic above is originally due to [2, Theorem 1.1].

3 Constructible complex of sheaves on Abelian varieties

3.1 Linearity and comparison

In this subsection, we prove that the set of 1-forms supported on the conormal sheaf of a subvariety of an Abelian variety is linear. As a consequence we obtain that the set of 1-forms associated to the singular support of any constructible complex $\mathcal{F}$ is linear. All constructible sheaves and complexes considered in this subsection are over the complex number.

Proposition 3.1 *Let A be an Abelian variety. For any $\mathcal{F} \in D_C^b(A)$, $\pi(\text{SS}(\mathcal{F}))$ is linear in $H^0(A, \Omega_A^1)$.*

This proposition directly follows from the following

Proposition 3.2 *Let A be an Abelian variety and Z be a proper irreducible subvariety of A . Then the following are equivalent*

- (1) *Z is not fibred by tori and $\dim Z > 0$.*
- (2) *General holomorphic 1-form $\omega \in H^0(A, \Omega_A^1)$ restricted to Z^{reg} , i.e. $\omega|_{Z^{\text{reg}}}$ admits isolated zeros on the smooth locus Z^{reg} .*

*In particular, let $B \subseteq A$ be the largest (in dimensional sense) Abelian subvariety such that Z is fibred by B , we have $\pi(T_Z^*A) = H^0(C, \Omega_C^1)$ identified as a vector subspace of the vector space $H^0(A, \Omega_A^1)$. Here C denotes the quotient Abelian variety A/B .*

Remark 3.3 When Z is smooth, this result is well-known (see e.g. [25, Proposition 6.3.10.] when A is simple; it follows from [30] when A is not simple). Hacon and Kovács showed this under the additional assumption that A is simple [15, Proposition 3.1]. In fact, our proof follows from a close inspection of Hacon and Kovács' argument. After this draft was written we also noticed that the result is stated in the preprint [40, Theorem 1] with a different argument.

Proof of Proposition 3.2 (2) $\Rightarrow$ (1): Suppose Z is fibred by a Abelian subvariety B and $\dim Z > 0$. Let $C := A/B$ and $Y = \varphi(Z)$ under the projection $\varphi: A \rightarrow C$. Considering the isogeny $\tau: B \times C \rightarrow A$, we obtain $\tau^{-1}(Z) = B \times Y$. Then the non-trivial 1-forms coming from B do not vanish on the smooth locus of $\tau^{-1}(Z)$, hence general 1-forms on A do not vanish on the smooth locus of Z , which contradicts the assumption (2).

(1) $\Rightarrow$ (2): Denote $d = \dim Z$ and $g = \dim A$. If $d = 0$ it is trivial, so we assume $d > 0$. Let N be the normal bundle of Z^{reg} in A . Associated to the surjection

$$T_A|_{Z^{\text{reg}}} \rightarrow N,$$

there is the following chain of maps

$$\varphi := (\mathbb{P}(N) \rightarrow Z^{\text{reg}} \times \mathbb{P}(T_0 A) \rightarrow \mathbb{P}(T_0 A) = \mathbb{P}^{g-1}).$$

It suffices to show that φ is dominant. Denote by $p: \mathbb{P}(N) \rightarrow Z^{\text{reg}}$ the projective bundle map. Given a point $s \in S := \varphi(\mathbb{P}(N))$, we can associate a hyperplane $H_s \subset T_0 A$. Then p induces an isomorphism

$$p: \varphi^{-1}(s) \xrightarrow{\sim} \{z \in Z^{\text{reg}} | T_z Z^{\text{reg}} \subset H_s\}. \quad (10)$$

If $\dim S < g - 1$, for general $s \in S$, $Z_s := p(\varphi^{-1}(s))$ has dimension $g - 1 - \dim S$. Let B denote the Abelian subvariety generated by Z_s in A . Note that B does not depend on general s , since A only contains countably many Abelian subvarieties. Also, $\dim B > g - 1 - \dim S$. Indeed, (1) implies that Z_s cannot itself be an Abelian variety. By (10), $H_s \supset T_0 B$ for general $s \in S$. Thus $\dim T_0 B \leq g - 1 - \dim S$, which gives the contradiction. Hence φ is quasi-finite dominant morphism.

For the second part, if Z is not fibred by tori, $\pi(T_Z^* A) = H^0(A, \Omega_A^1)$. When Z is fibred by some tori B , let C and $\varphi: A \rightarrow C$ be as in the beginning of the proof. Let $\varphi^*: H^0(C, \Omega_C^1) \rightarrow H^0(A, \Omega_A^1)$ be the induced injective morphism. We have

$$\pi(T_Z^* A) = \varphi^*(H^0(C, \Omega_C^1)).$$

Hence $\pi(T_Z^* A)$ is linear.

Proof of Proposition 3.1 The claim follows from Proposition 3.2 since $\text{SS}(\mathcal{F})$ is a finite union of conormal sheaves along various subvarieties of A .

A consequence of the proposition above is the following

Corollary 3.4 *If X admits a finite morphism $f: X \rightarrow A$ to its image, then $W(X) \cap H^0(A, \Omega_A^1)$ (under suitable identification induced by f) is linear. In particular, if the Albanese morphism is finite to its image, then $W(X)$ is linear.*

Proof By [26, Proposition 3.9 (2)] and [32, Proposition 3.3], $(f \times \text{id})(df^{-1}(0_X))$ in the diagram (8) is Lagrangian, i.e. it is a finite union of conormal sheaves along various subvarieties of A . Then the corollary follows from Proposition 3.2.

Proof of Theorem 1.2 The proof is divided into 2 steps.

Step 1: We first prove the case where $\mathcal{F} = \mathcal{P}$ is a perverse sheaf on A .

Note that for a short exact sequences of perverse sheaves on A

$$0 \rightarrow \mathcal{P}' \rightarrow \mathcal{P} \rightarrow \mathcal{P}'' \rightarrow 0,$$

we have $\mathcal{V}(A, \mathcal{P}) = \mathcal{V}(A, \mathcal{P}') \cup \mathcal{V}(A, \mathcal{P}'')$ and $\text{SS}(\mathcal{P}) = \text{SS}(\mathcal{P}') \cup \text{SS}(\mathcal{P}'')$. Since perverse sheaves admit Jordan–Holder type filtration with simple perverse sheaves as quotients, it is

enough to deal with the case of simple perverse sheaves. From now on let us assume that $\mathcal{P}$ is a simple perverse sheaf on A . First note that by the propagation property in Theorem 2.9 (1), we have $\mathcal{V}(A, \mathcal{P}) = \mathcal{V}^0(A, \mathcal{P})$. According to Theorem 2.9 (2), the argument can be split in two cases:

Case I: $\chi(A, \mathcal{P}) > 0$. In this case Theorem 2.9 (2) shows that $\mathcal{V}^0(A, \mathcal{P}) = \text{Char}(A)$, hence $\mathcal{T}(A, \mathcal{P}) = H^0(A, \Omega_A^1)$. On the other hand by the Kashiwara index Theorem 2.6 we have

$$\chi(A, \mathcal{P}) = \text{CC}(\mathcal{P}) \cdot T_A^* A.$$

Note that if $Z \subset A$ is fibred by an Abelian subvariety, $(T_Z^* A \cdot T_A^* A) = 0$. Therefore, $\text{SS}(\mathcal{P})$ must contain a subvariety $Z \subset A$ such that Z is not fibred by tori. By Proposition 3.2, we conclude that $\pi(T_Z^* A) = H^0(A, \Omega_A^1)$ and the desired equality follows.

Case II: $\chi(A, \mathcal{P}) = 0$. As in [41, Main Theorem and Lemma 6] we have

$$\mathcal{P} \otimes \mathbb{C}_\rho \simeq \varphi^* \mathcal{P}_C [\dim A - \dim C]$$

with notations from before. Since $\chi(C, \mathcal{P}_C) > 0$, by Kashiwara's index theorem (see also Lemma 3.6 in v3 of this article on arXiv) there exists a component $T_Z^* A \subset \text{SS}(\mathcal{P})$ such that $\varphi(Z) \subset C$ is not fibred by tori. Since $\mathcal{P}$ and $\mathcal{P} \otimes \mathbb{C}_\rho$ have the same singular support, by Proposition 3.2 we conclude that

$$\pi(\text{SS}(\mathcal{P})) = \varphi^* H^0(C, \Omega_C^1). \quad (11)$$

From Case I above we have

$$\mathcal{T}(C, \mathcal{P}_C) = H^0(C, \Omega_C^1).$$

On the other hand, it follows from [27, Theorem 5.5] that

$$\mathcal{V}^0(A, \mathcal{P}) = \rho^{-1} \cdot \varphi^* (\mathcal{V}^0(C, \mathcal{P}_C)) \quad (12)$$

where $\varphi^*: \text{Char}(C) \rightarrow \text{Char}(A)$ is given by the induced representation. Putting (11) and (12) together, the desired equality follows.

Step 2: In general for any $\mathcal{F} \in D_c^b(A)$, [27, Proposition 6.11] shows that

$$\mathcal{V}(A, \mathcal{F}) = \bigcup_i \mathcal{V}^0(A, {}^p \mathcal{H}^i(\mathcal{F})),$$

where ${}^p \mathcal{H}^i(\mathcal{F})$ is the i -th perverse cohomology of $\mathcal{F}$. On the other hand, $\text{SS}(\mathcal{F}) = \bigcup_i \text{SS}({}^p \mathcal{H}^i(\mathcal{F}))$ by Definition 2.7. Then the claim follows.

We are now ready to prove Theorem 1.7. Let us first introduce a general version for any morphism $f: X \rightarrow A$ from a smooth projective variety X to an Abelian variety A . Let

$$\mathcal{R}_{\text{dol}}(X, f) := \bigcup_{\eta \in \text{Char}(A)^u} \{\omega \in H^0(A, \Omega_A^1) \mid (H^\bullet(X, f^* \mathbb{C}_\eta), \wedge f^* \omega) \text{ not exact}\},$$

Theorem 3.5 *With the above hypothesis and notations, we have*

$$\pi(\text{SS}(\mathbb{R} f_* \mathbb{C}_X)) = \mathcal{T}(A, \mathbb{R} f_* \mathbb{C}_X) = \mathcal{R}_{\text{dol}}(X, f),$$

which is a finite union of vector subspaces of $H^0(A, \Omega_A^1)$.

Proof The first equality follows from Theorem 1.2. The second equality follows from Proposition 2.3. Finally the statement about linearity follows from Proposition 3.1.

Proof of Theorem 1.7 Given any finite étale cover $\tau: X' \rightarrow X$, by Theorem 1.2 we have

$$\pi(\mathrm{SS}(\mathbb{R}(a \circ \tau)_* \mathbb{C}_X)) = \mathcal{T}(A_X, \mathbb{R}(a \circ \tau)_* \mathbb{C}_X).$$

Given any such finite étale cover $\tau: X' \rightarrow X$, by the lemma below there exists a finite étale cover $\sigma: \tilde{X}' \rightarrow X'$ such that

$$\mathcal{T}(A_X, \mathbb{R}(a \circ \tau)_* \mathbb{C}_X) = \{\omega \in V | (H^\bullet(\tilde{X}', \mathbb{C}), \wedge(\sigma \circ \tau)^* \omega) \text{ is not exact}\} \quad (13)$$

Then the claim follows by taking unions over all possible finite étale covers on both sides.

Lemma 3.6 Consider $f: X \rightarrow A$ to be a morphism from a smooth projective variety X to an Abelian variety A . Then there exists a finite étale Abelian cover $\sigma: \tilde{X} \rightarrow X$ such that

$$\mathcal{T}(A_X, \mathbb{R}f_* \mathbb{C}_X) = \{\omega \in H^0(A, \Omega_A^1) | (H^\bullet(\tilde{X}, \mathbb{C}), \wedge(f \circ \sigma)^* \omega) \text{ is not exact}\}$$

Proof Recall that $\mathcal{V}(A, \mathbb{R}f_* \mathbb{C}_X)$ has finitely many irreducible components, say $\{S_1, \dots, S_k\}$ and every irreducible component S_j is a torsion translated subtori of $\mathrm{Char}^0(A)$ [33]. Then there exists a finite Abelian cover $\sigma: \tilde{X} \rightarrow X$ such that $\bigoplus_{i,j} f^* \mathbb{C}_{\rho_i^j} = \mathbb{R}\sigma_* \mathbb{C}_{\tilde{X}}$ for $i = 1, \dots, k$ and finitely many powers j for each ρ_i .

Now, for any $\rho \in \mathrm{Char}^0(X)$, by the projection formula we have

$$H^*(\tilde{X}, \sigma^* \mathbb{C}_\rho) \cong H^*(A, \mathbb{R}\sigma_* \mathbb{C}_{\tilde{X}} \otimes \mathbb{C}_\rho). \quad (14)$$

In particular, every component $\rho_i^{-1} \cdot S_i$ contains the constant sheaf $\mathbb{C}_A$. Furthermore, for any j we have $\mathcal{V}(\mathbb{R}f_* \mathbb{C}_X \otimes \mathbb{C}_\rho) = \rho^{1-j} \cdot \mathcal{V}(\mathbb{R}f_* \mathbb{C}_X \otimes \mathbb{C}_{\rho^j})$. Hence

$$\mathcal{T}(A, \mathbb{R}f_* \mathbb{C}_X) = \mathrm{TC}_1 \mathcal{V}(A, \mathbb{R}(f \circ \sigma)_* \mathbb{C}_{\tilde{X}}).$$

Then the claim follows by the tangent-cone equality (7).

3.2 Proof of Corollary 1.5

So far all the results in this section are about bounded constructible complexes of sheaves with complex coefficients. In this subsection, we prove when A is a simple Abelian variety Theorem 1.2 holds for any algebraic closed field coefficients. As most of the definitions and tools used in the proof of Theorem 1.2 works over $\mathbb{K}$ (see Sect. 2.2), the main difference here is in how we reduce the argument (see Step 2) from bounded complexes of constructible sheaves to perverse sheaves. In characteristic 0, we resorted to [28] for this.

Proposition 3.7 Fix any algebraically closed field $\mathbb{K}$. Let $\mathcal{F}$ be a bounded constructible complex of sheaves with field coefficients $\mathbb{K}$ on a complex simple Abelian variety A . Then we have

- either $\mathcal{V}(A, \mathcal{F}) = \mathrm{Char}^0(A, \mathbb{K})$ and $\pi(\mathrm{SS}(\mathcal{F})) = H^0(A, \Omega_A^1) = \mathcal{T}(A, \mathcal{F})$
- or $\mathcal{V}(A, \mathcal{F}) \neq \mathrm{Char}^0(A, \mathbb{K})$ and $\pi(\mathrm{SS}(\mathcal{F})) = \{0\} = \mathcal{T}(A, \mathcal{F})$.

Here $\mathrm{Char}^0(A, \mathbb{K}) = \mathrm{Hom}(H_1(A, \mathbb{Z}), \mathbb{K}^*)$ is the moduli space of rank one $\mathbb{K}$ -local system on A .

Proof The proof is divided into 2 steps.

Step 1: We first prove the case where $\mathcal{F} = \mathcal{P}$ is a perverse sheaf on A . Note that $\mathcal{V}(A, \mathcal{P}) = \mathcal{V}^0(A, \mathcal{P})$ due to the propagation property, see e.g. [27, Theorem 4.7]. Since A is simple, then there are two possibilities as follows

- Either $\mathcal{V}^0(A, \mathcal{P}) = \text{Char}^0(A, \mathbb{K})$ and $\mathcal{T}(A, \mathcal{P}) = H^0(A, \Omega_A^1)$.

In this case, by Theorem 2.9(3) we know that $\chi(A, \mathcal{P}) \neq 0$. Hence, by the Kashiwara index Theorem 2.6 we have

$$\chi(A, \mathcal{P}) = \text{CC}(\mathcal{P}) \cdot T_A^* A \neq 0.$$

Therefore, $\text{SS}(\mathcal{P})$ must contain a subvariety $Z \subset A$ such that Z is not fibred by tori. By Proposition 3.2, we conclude that $\pi(T_Z^* A) = H^0(A, \Omega_A^1)$ and the claim follows.

- Or, $\mathcal{V}^0(A, \mathcal{P}) \neq \text{Char}^0(A, \mathbb{K})$ and hence $\chi(A, \mathcal{P}) = 0$. By [24, Proposition 10.1] (the proof for simple abelian varieties works over any characteristic; alternatively see the arxiv version 1 of [28]), $\mathcal{P}$ is a shifted local system. Since $\mathbb{K}$ is algebraically closed, any $\mathbb{K}$ -local system on A is a extensions of rank one local systems. Hence $\mathcal{V}(A, \mathcal{P})$ are just finitely many points in this case and $\mathcal{T}(A, \mathcal{P}) = \{0\}$.

On the other hand, since $\mathcal{P}$ is a shifted local system we have that $\text{SS}(\mathcal{P}) = T_A^* A$, hence $\pi(T_A^* A) = \{0\}$.

Step 2: More generally for any bounded complex of constructible sheaves $\mathcal{F}$ and any rank one $\mathbb{K}$ -local system $\mathbb{L} \in \text{Char}^0(A, \mathbb{K})$, we consider the perverse cohomology spectral sequence

$$E_2^{i,j} = H^i(A, {}^p\mathcal{H}^j(\mathcal{F}) \otimes_{\mathbb{K}} \mathbb{L}) \Rightarrow H^{i+j}(A, \mathcal{F} \otimes_{\mathbb{K}} \mathbb{L}).$$

Observe that $\bigcup_j \mathcal{V}^0(A, {}^p\mathcal{H}^j(\mathcal{F})) \supseteq \mathcal{V}(A, \mathcal{F})$.

Since ${}^p\mathcal{H}^j(\mathcal{F})$ are perverse sheaves, with Step 1 at our disposal we proceed as follows:

- Either $\mathcal{V}(A, \mathcal{F}) = \text{Char}^0(A, \mathbb{K})$. In this case, $\mathcal{V}^0(A, {}^p\mathcal{H}^j(\mathcal{F})) = \text{Char}^0(A, \mathbb{K})$ for some j . Then $\pi(\text{SS}({}^p\mathcal{H}^j(\mathcal{F}))) = H^0(A, \Omega_A^1)$, and hence $\pi(\text{SS}(\mathcal{F})) = H^0(A, \Omega_A^1)$ by definition.
- Or $\mathcal{V}(A, \mathcal{F}) \neq \text{Char}^0(A, \mathbb{K})$. In this case, we claim that $\mathcal{V}^0(A, {}^p\mathcal{H}^j(\mathcal{F})) \neq \text{Char}^0(A, \mathbb{K})$ for all j . Indeed, by choosing $\mathbb{L} \in \text{Char}^0(A, \mathbb{K})$ generically, by Theorem 2.9 (3) we have $H^i(A, {}^p\mathcal{H}^j(\mathcal{F}) \otimes_{\mathbb{K}} \mathbb{L}) = 0$ for any $i \neq 0$. So the spectral sequence degenerates at the second page. It shows that $H^j(A, \mathcal{F} \otimes_{\mathbb{K}} \mathbb{L}) \neq 0$ for generic $\mathbb{L} \in \text{Char}^0(A, \mathbb{K})$, giving $\mathcal{V}(A, \mathcal{F}) = \text{Char}^0(A, \mathbb{K})$, a contradiction.

Then by Step 1, $\pi(\text{SS}({}^p\mathcal{H}^j(\mathcal{F}))) = \{0\}$, and hence $\pi(\text{SS}(\mathcal{F})) = \{0\}$ by definition. On the other hand, we claim that in this case, in fact, one has $\mathcal{V}(A, \mathcal{F}) = \bigcup_j \mathcal{V}^0(A, {}^p\mathcal{H}^j(\mathcal{F}))$. To see this, for any $\mathbb{L} \in \bigcup_j \mathcal{V}^0(A, {}^p\mathcal{H}^j(\mathcal{F}))$, let j' be the lowest degree such that $\mathbb{L} \in \mathcal{V}^0(A, {}^p\mathcal{H}^{j'}(\mathcal{F}))$. Set $\dim_{\mathbb{C}} A = d$. Since ${}^p\mathcal{H}^{j'}(\mathcal{F})$ is a local system, it must be that ${}^p\mathcal{H}^{j'}(\mathcal{F}) \otimes_{\mathbb{K}} \mathbb{L}[-d]$ contains the constant sheaf as a sub-local system. In particular,

$$E_2^{-d,j'} = H^{-d}(A, {}^p\mathcal{H}^{j'}(\mathcal{F}) \otimes_{\mathbb{K}} \mathbb{L}) \neq 0.$$

Meanwhile, for any $i < -d$, because of [27, Theorem 4.7(i)], or for any $j < j'$, because of the choice of j' and the propagation property in Theorem 2.9 (1), we have $E_2^{i,j} = 0$. Therefore, the above spectral sequence satisfies

$$E_{\infty}^{-d,j'} = E_2^{-d,j'} \neq 0.$$

Hence $H^{-d+j'}(A, \mathcal{F} \otimes_{\mathbb{K}} \mathbb{L}) \neq 0$ and $\mathbb{L} \in \mathcal{V}(A, \mathcal{F})$. Now $\mathcal{F}$ being non-zero implies that $\mathcal{V}(A, \mathcal{F})$ are just finitely many points. The claim follows.

Now we are ready to prove Corollary 1.5.

Proof of Corollary 1.5 Following the proof of [35, Proposition 3.1], up to perturbing $p_X^*(d\theta)$ slightly and multiplying by a suitable integer, we reduce to the case where $p_X^*(d\theta) \in f^*H^1(A, \mathbb{Z})$.

As observed in the proof of [35, Lemma 3.3], we only need to prove that $\mathbb{R}f_*\mathbb{K}_X$ is locally constant for any algebraically closed field $\mathbb{K}$. Then it follows from Qin–Wang’s result [31, Proposition 5.4] (see also [35, Proposition 3.1]) that $\chi(A, {}^p\mathcal{H}^j\mathbb{R}f_*\mathbb{K}_X) = 0$ or equivalently $\mathcal{V}^0(A, {}^p\mathcal{H}^j\mathbb{R}f_*\mathbb{K}_X) \neq \text{Char}^0(A, \mathbb{K})$ for any j . Their proof is based on the observation that the eigenvalues on the cohomology of the fibers induced by the monodromy action of the circle bundle is a finite set. Since A is simple, Proposition 3.7 implies that $\pi(\text{SS}(\mathbb{R}f_*\mathbb{K}_X)) = \{0\}$ and hence ${}^p\mathcal{H}^j\mathbb{R}f_*\mathbb{K}_X[-j] = R^j f_*\mathbb{K}_X$ are local systems for all j .

When A is not necessarily simple, assuming $\mathbb{K} = \mathbb{C}$ we prove a slightly stronger version.

Corollary 3.8 *With the same assumptions and notations as in Corollary 1.5, without the simplicity of A , one has for every irreducible component T_Z^*A of $\text{SS}(\mathbb{R}f_*\mathbb{C}_X)$ the sub-variety Z is not of general type.*

Proof Assume that there exists an irreducible component T_Z^*A of $\text{SS}(\mathbb{R}f_*\mathbb{C}_X)$, where Z is of general type. By [39, Theorem 3.10], Z is not fibred by any sub-Abelian variety of A . Hence $\pi(T_Z^*A) = H^0(A, \Omega_A^1)$ by Proposition 3.2. In particular, Theorem 1.7 implies that $\mathcal{V}(A, \mathbb{R}f_*\mathbb{C}_X) = \text{Char}^0(A, \mathbb{C})$. On the other hand, the same proof as in Corollary 1.5 shows that one can find rank one $\mathbb{C}$ -local system $\mathbb{C}_\rho$ on A such that

$$H^*(A, \mathbb{R}f_*\mathbb{C}_X \otimes \mathbb{C}_\rho) = 0$$

for all degrees. Hence $\mathcal{V}(A, \mathbb{R}f_*\mathbb{C}_X) \neq \text{Char}^0(A, \mathbb{C})$, which gives a contradiction.

4 (Logarithmic) 1-forms with codimension one zeros

4.1 Arapura’s result about cohomology jump loci

Let X be a smooth projective variety with a simple normal crossing divisor D . Set $U = X - D$. Note that the space of logarithmic 1-forms $H^0(X, \Omega_X^1(\log D))$ does not depend on the choice of the good compactification of U . Similar to the projective case, one can define [4]

$$W^i(X, D) := \{\omega \in H^0(X, \Omega_X^1(\log D)) \mid \text{codim}_X Z(\omega) \leq i\},$$

where $Z(\omega)$ is the zero locus of ω . By Chevalley’s upper-semicontinuity theorem, $W^i(X, D)$ are all algebraic sets.

Define²

$$\Sigma^1(U) := \{\rho \in \text{Char}(U) \mid H^1(U, \mathbb{C}_\rho) \neq 0\}.$$

Arapura’s work gives a geometric interpretation of the set $\Sigma^1(U)$. We briefly outline it here. An algebraic morphism $f: U \rightarrow C$ from U to a smooth curve C is called an *orbifold map*, if f is surjective, has connected generic fibre, and one of the following condition holds:

- $\chi(C) < 0$
- $\chi(C) = 0$ and f has at least one multiple fibre.

² Compare this notion to $\mathcal{V}^i(U, \mathbb{C}_U)$. In this section we will use $\text{Char}(U)$ instead of $\text{Char}^0(U)$. Note that when $H_1(U, \mathbb{Z})$ has no torsion, $\Sigma^i(U) = \mathcal{V}^i(U, \mathbb{C}_U)$.

Roughly speaking, Arapura [1] (also see [7, Corollary 5.4, Corollary 5.8]) showed that every positive dimensional component of $\Sigma^1(U)$ arises from some orbifold map. More precisely, an orbifold map induces an injection: $f^*: H^1(C, \mathbb{C}) \rightarrow H^1(U, \mathbb{C})$. Arapura's work implies that

$$\bigcup_{\rho} \mathrm{TC}_{\rho} \Sigma^1(U) = \bigcup_f \mathrm{Im} f^*,$$

where the first union is running over representative points from irreducible components of $\Sigma^1(U)$, $\mathrm{TC}_{\rho} \Sigma^1(U) \subseteq H^1(U, \mathbb{C})$ denotes the tangent cone at ρ , and the second union runs over all possible orbifold maps for U . In particular, there are at most finitely many equivalent orbifold maps for a fixed U [1, Theorem 1.6]. So the second union is indeed a finite union. We define

$$\mathcal{T}_{\Sigma}^1(U) := H^0(X, \Omega_X^1(\log D)) \cap \left(\bigcup_{\rho} \mathrm{TC}_{\rho} \Sigma^1(U) \right) = H^0(X, \Omega_X^1(\log D)) \cap \left(\bigcup_f \mathrm{Im} f^* \right).$$

In particular, $\mathcal{T}_{\Sigma}^1(U)$ is a finite union of vector subspaces.

Remark 4.1 (1) Consider the following construction given in [9, Example 1.11], which shows that $\mathcal{T}_{\Sigma}^1(X)$ indeed captures more information than $\mathcal{T}^1(X)$ in general.

Let C_1 be a higher genus curve that admits a degree 2 finite morphism to an elliptic curve E , and let C_2 be an elliptic curve. Consider σ_1 to be the involution action such that $C_1/\sigma_1 \simeq E$ and σ_2 induces an isogeny $C_2 \rightarrow C_2/\sigma_2$. Then their example is given by $X := C_1 \times C_2/\sim$, where the $\sim$ is a diagonal action induced by σ_1 and σ_2 . In this case, one can compute that

$$\mathcal{T}^1(X) = \mathcal{R}_{\mathrm{dol}}(X) = \{0\}$$

but one can check that $W(X) = f^* H^0(E, \Omega_E^1)$ for the natural map $f: X \rightarrow E$ and

$$\mathcal{T}_{\Sigma}^1(X) = \mathcal{T}^1(A_X, (a \circ \tau)_* \mathbb{C}_{X'}) = H^0(E, \Omega_E^1),$$

where τ is the étale covering $X' := C_1 \times C_2 \rightarrow X$.

(2) In general, one should not expect $\mathcal{T}_{\Sigma}^1(X) = W^1(X)$. For instance, let X be a complex Abelian surface and Y be the blowup of X along a point. Then we take Z to be the blowup of Y along a point in the exceptional divisor. Then $\mathcal{T}_{\Sigma}^1(X) = \mathcal{T}_{\Sigma}^1(Z)$, but $W^1(X) \subsetneq W^1(Z)$.

4.2 Projective case

The observation of Arapura discussed above allows us to turn the piece of $W^1(X)$ that traditionally arises from cohomology jump loci into a set arising out of orbifold maps. In fact we will see in Theorem 4.4 that holomorphic 1-form in $W^1(X) \setminus \mathcal{T}_{\Sigma}^1(X)$ vanishes along some negative divisors. See Theorem 4.7 for its quasi-projective incarnation.

Definition 4.2 Let X be a smooth projective variety of dimension n with a fixed ample line bundle H and an integral divisor E on X . We say that E is H -negative if $E^2 \cdot H^{n-2} < 0$, where

$$E^2 \cdot H^{n-2} = \int_X c_1^2(E) \wedge c_1^{n-2}(H).$$

Similarly, E is called H -trivial if $E^2 \cdot H^{n-2} = 0$.

We need the following a more precise version of [37, Theorem 2].

Lemma 4.3 *Let E be an integral divisor in X . Suppose there exists a holomorphic 1-form ω such that $E \subseteq Z(\omega)$. Then the following statements are true:*

(1) *When E is H -trivial, there exists an orbifold map $f: X \rightarrow C$ such that $\omega = f^*\eta$ for some $\eta \in H^0(C, \Omega_C^1)$ and E is the only component of the fibre of f containing E .*

(2) *the sign of the intersection number $E^2 \cdot H^{n-2}$ does not depend on the choice H .*

Proof For (1), by [37, Theorem 2], we only need to show that E is the unique component in the fibre containing it. To this end, let E' be the union of components not supported on E in the scheme-theoretic fibre containing E and let $aE + E'$ denote the fibre class for some positive integer a . Since f has connected fibres, $E \cdot E' \cdot H^{n-2} > 0$. On the other hand since E' is contained in a fibre, we have $(aE + E') \cdot E' \cdot H^{n-2} = 0$. Then from $(aE + E')^2 \cdot H^{n-2} = 0$, we get $E^2 \cdot H^{n-2} < 0$, which is a contradiction.

To see (2), note that if for any ample class H , E is H -nonnegative, i.e., $E^2 \cdot H^{n-2} \geq 0$, by [37, Theorem 2] it must be H -trivial and then by (1), we know that E is the unique component of the fibre of f . Therefore for any other ample class H' , E must be H' -trivial. As a consequence, if E is H -negative, it is H' -negative for any other ample class H' .

We denote

$$W_{\text{neg}}(X) := \{\omega \in H^0(X, \Omega_X^1) \mid \exists \text{ some negative integral divisor } E \text{ such that } E \subset Z(\omega)\}.$$

Then we have the following result.

Theorem 4.4 *Let X be a smooth projective variety of dimension n . With the above notations, we have*

$$W^1(X) = \mathcal{T}_{\Sigma}^1(X) \cup W_{\text{neg}}(X).$$

In particular, $W^1(X)$ is linear.

Remark 4.5 (1) Theorem 4.4 complements the result of Green–Lazarsfeld [13] which ensures the linearity of $\mathcal{T}_{\Sigma}^1(X) \subset W^1(X)$. As noted in Remark 4.1 (2) this is often a proper subset.

(2) The two pieces $\mathcal{T}_{\Sigma}^1(X)$ and $W_{\text{neg}}(X)$ may overlap. For example let $f: S \rightarrow C$ be a morphism from a smooth projective surface S to a smooth projective curve C with genus $g(C) \geq 2$. Take a 1-form $\omega \in H^0(C, \Omega_C^1)$ which has a zero at $p \in C$. Let X be the blow-up of S along a point in $f^{-1}(p)$. The exceptional curve E has negative self-intersection. Consider the natural morphism $f': X \rightarrow C$, then $(f')^*\omega \in \mathcal{T}_{\Sigma}^1(X) \cap W_{\text{neg}}(X)$.

Lemma 4.6 *Let X be a smooth projective variety of dimension n with an ample divisor H . Then there are at most countably many H -negative divisors in X .*

Proof Let E be any H -negative divisor. Let S be a general complete intersection surface by the hyperplanes in $|mH|$ for $m \gg 0$. Then $E \cap S$ is a negative curve. Since there are at most countably many negative curves on S , the claim follows.

Proof of Theorem 4.4 We assume $n > 1$. For any 1-form $\omega \in W^1(X)$, there is an integral divisor $E \subset X$ such that $E \subseteq Z(\omega)$. By [37, Theorem 2] we have either $\omega \in W_{\text{neg}}$, or there exists an orbifold map $f: X \rightarrow C$ with genus $g(C) > 0$ and $\omega = f^*\eta$ for some $\eta \in H^0(C, \Omega_C^1)$. In the latter case by Lemma 4.3 we know that E is the only component of a fibre. Since $E \subseteq Z(f^*\eta)$ and E is the whole fibre, either f has a multiple fibre and $g(C) = 1$, or $g(C) > 1$ and $\eta(f(E)) = 0$. Hence $\omega \in \mathcal{T}_{\Sigma}^1(X)$. Notice that $\mathcal{T}_{\Sigma}^1(X) \subseteq W^1(X)$ and hence the first part of the theorem follows.

For the second part, note that $W^1(X)$ is an algebraic set. Since $\mathcal{T}_{\Sigma}^1(X)$ is linear and W_{neg} is a union of at most countably many linear subspaces in $H^0(X, \Omega_X^1)$ by Lemmas 4.6 and 4.3, $W^1(X)$ is also linear.

4.3 Quasi-projective case

Let X be a smooth projective variety with a simple normal crossing divisor $D = \sum_{j=1}^r D_j$. Set $U = X - D$. Similar to the projective case, we denote

$$W_{\text{neg}}(X, D) := \{\omega \in H^0(X, \Omega_X^1(\log D)) \mid E \subseteq Z(\omega) \text{ for some negative integral divisor } E\}.$$

Theorem 4.7 *With the above notations, we have*

$$W^1(X, D) = \mathcal{T}_{\Sigma}^1(U) \cup W_{\text{neg}}(X, D).$$

In particular $W^1(X, D)$ is linear in $H^0(X, \Omega_X^1(\log D))$.

The proof of Theorem 4.7 follows that of Theorem 4.4 closely with Theorem 5.1 in appendix, which is a generalisation of Spurr [37, Theorem 2] for pairs.

Remark 4.8 In [4], Budur, Wang and Yoon identified a linear piece of $W^1(X, D)$; namely

$$(\mathbf{R}^1 \cup \mathbf{R}_{2n-1}) \cap H^0(X, \Omega_X^1(\log D)) \subseteq W^1(X, D).$$

Here we use the same notations as in their paper. Note that $\mathbf{R}^1 \cap H^0(X, \Omega_X^1(\log D))$ coincides with $\mathcal{T}_{\Sigma}^1(U)$. But it is not clear to us how $\mathbf{R}_{2n-1} \cap H^0(X, \Omega_X^1(\log D))$ is connected to $W_{\text{neg}}(X, D)$.

Dimca in [8] define the first logarithmic resonance variety

$$\mathcal{LR}_1(U) := \{\omega \in H^0(X, \Omega_X^1(\log D)) \mid H^1(H^0(X, \Omega_X^{\bullet}(\log D)), \wedge \omega) \neq 0\}$$

In particular, [8, Proposition 4.5] implies that $\mathcal{LR}_1(U) = \bigcup_f \text{Im } f^*$, where the union runs over all possible orbifold maps $f : U \rightarrow C$ with $\chi(C) < 0$ and C not being a once-punctured elliptic curve. Hence $\mathcal{LR}_1(U) \subseteq \mathcal{T}_{\Sigma}^1(U)$.

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Appendix A. Logarithmic generalization of A Theorem of Spurr

We show the following generalization of Spurr [37, Theorem 2] for pairs.

Theorem 5.1 *Let (X, D) be a pair with X a smooth projective variety of dimension n and D a simple normal crossing divisor on X . Let H be an ample divisor on X . If (X, D) carries a nonzero logarithmic 1-form $\omega \in H^0(X, \Omega_X^1(\log D))$ such that there exists an integral divisor E with $E^2 \cdot H^{n-2} \geq 0$, $E \subseteq Z(\omega)$, and $E \not\subseteq D$, then there is a surjective morphism $f: X - D \rightarrow C$ to a smooth quasi-projective curve $C = \bar{C} - B$ (where $\bar{C}$ is a smooth completion of C and B can be empty) with*

- (1) $\chi(C) \leq 0$ and $\omega = f^*\eta$ for some $\eta \in H^0(\bar{C}, \Omega_{\bar{C}}^1(\log B))$.
- (2) f has connected generic fibres.
- (3) $E^2 \cdot H^{n-2} = 0$.
- (4) If $\chi(C) = 0$, then f has at least one multiple fibre.

To prepare for the proof, recall the following construction of Albanese varieties, which can be found in [18] (see also [12] for a survey).

Let X be a smooth projective variety of dimension with a simple normal crossing divisor D . Pick a basis $\{\theta_1, \dots, \theta_q\}$ for $H^0(X, \Omega_X^1)$ and $\{\omega_1, \dots, \omega_r\} \in H^0(X, \Omega_X^1(\log D))$ such that $\{\theta_1, \dots, \theta_q, \omega_1, \dots, \omega_r\}$ is a basis of $H^0(X, \Omega_X^1(\log D))$. Pick a basis $\{\gamma_1, \dots, \gamma_{2q}\}$ for the free part of $H_1(X, \mathbb{Z})$ and a basis $\{\delta_1, \dots, \delta_r\}$ for the free part of $\ker\{H_1(U, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})\}$. Then we have the following periods as a semi-lattice for $H^0(X, \Omega_X^1(\log D))^\vee$

$$\Lambda = \sum_{i=1}^{2q} \mathbb{Z} \left(\int_{\gamma_i} \theta_1, \dots, \int_{\gamma_i} \omega_r \right) + \sum_{j=1}^r \mathbb{Z} \left(\int_{\delta_j} \theta_1, \dots, \int_{\delta_j} \omega_r \right).$$

The Albanese variety is then defined as the semi-Abelian variety $A_U = \frac{H^0(X, \Omega_X^1(\log D))^\vee}{\Lambda}$ and the Albanese map $a_U: U \rightarrow A_U$ is given by

$$a_U(x) = \left[\sum_{i=1}^q \left(\int_p^x \theta_i \right) \theta_i^* + \sum_{j=1}^r \left(\int_p^x \omega_j \right) \omega_j^* \right] / \Lambda,$$

where $p \in U$ is a chosen base-point and θ_i^*, ω_j^* are the dual bases in $H^0(X, \Omega_X^1(\log D))^\vee$.

Proof of Theorem 5.1 First we claim that if $E \cap D_i \neq \emptyset$ for some component D_i of D , then ω has no pole along D_i . In other words, $\omega \in H^0(X, \Omega_X^1(\log D - D_i))$. To see the claim consider the following diagram

$$\begin{array}{ccccc} H^0(X, \Omega_X^1(\log D - D_i) \otimes \mathcal{O}_X(-E)) & \longrightarrow & H^0(X, \Omega_X^1(\log D - D_i)) & \longrightarrow & H^0(\Omega_X^1(\log D - D_i)|_E) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(X, \Omega_X^1(\log D) \otimes \mathcal{O}_X(-E)) & \longrightarrow & H^0(X, \Omega_X^1(\log D)) & \longrightarrow & H^0(X, \Omega_X^1(\log D)|_E) \end{array} \quad (15)$$

The left vertical arrow is an isomorphism. Indeed, it is injective and the cokernel is contained in $H^0(D_i, \mathcal{O}_{D_i}(-E))$ [10, 2.3 properties] which is zero since $E|_{D_i}$ is an effective divisor. Therefore any $\omega \in H^0(X, \Omega_X^1(\log D))$ such that $E \subseteq Z(\omega)$ must come from $H^0(X, \Omega_X^1(\log D - D_i))$. Hence the claim.

Now it suffices to deal with the case when $D \cap E = \emptyset$. Indeed, let $D = D' + D''$ such that E intersects each component of D' and $E \cap D'' = \emptyset$, by the reduction step we can construct

an orbifold map from $f: X \setminus D'' \rightarrow C$ satisfying the desired properties. Then the restriction $f|_U: U \rightarrow C$ is also an orbifold map and satisfies the same properties. In what follows we assume $D \cap E = \emptyset$.

We may assume that ω is not everywhere holomorphic, otherwise we are done by [37, Theorem 1]. Let $\phi: N \rightarrow E$ be the normalisation map and $\varphi: N \rightarrow U$ be the composition map (this makes sense since $E \cap D = \emptyset$). Consider the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & U \\ \downarrow a_N & & \downarrow a_U \\ A_N & \xrightarrow{\psi} & A_U, \end{array}$$

where a_N and a_U are Albanese maps and the base points are chosen in an appropriate way such that ψ is a group homomorphism.

Now we consider the transpose of the pullback map

$$\varphi^\vee: H^0(N, \Omega_N^1)^\vee \rightarrow H^0(X, \Omega_X^1(\log D))^\vee,$$

which induces the morphism ψ . Without any loss of generality we assume $\omega_1 = \omega$ with the notations introduced shortly before the proof. Hence by hypothesis $\varphi^*(\omega_1) = 0$. Let z_1 be the coefficient coordinate of $\omega_1^* \in H^0(X, \Omega_X^1(\log D))^\vee$. We get $\psi(A_N)$ is contained in $\{z_1 = 0\}/\Lambda$. We define $\beta: U \rightarrow T := A_U/\psi(A_N)$ as the composition of a_U and the quotient $A_U \rightarrow T$. Since $a_U(E) \subset \psi(A_N)$, E is contracted by β .

Notice that β is not the constant map. We claim that $\dim \beta(U) = 1$. Otherwise, replacing U by an intersection of general hyperplane sections coming from the very ample linear system $|mH|$ for some $m \gg 0$ we may assume $\dim U = \dim \beta(U) = 2$. In this case $\beta: U \rightarrow \beta(U)$ is a generically finite surjective morphism. Projectivising and resolving indeterminacy we get a generically finite morphism $\bar{\beta}: \bar{U} \rightarrow \bar{\beta}(U)$ where $\bar{U}$ is smooth and projective. Note that $E \cap D = \emptyset$. Hence $E^2 < 0$ in $\bar{U}$ (See e.g., [22, Theorem 10.1]), which contradicts the assumption $E^2 \geq 0$.

Taking the Stein factorisation of $\bar{\beta}$, we get the following commutative diagram:

$$\begin{array}{ccc} \bar{U} & \xrightarrow{f} & \bar{C} \\ & \searrow \bar{\beta} & \downarrow \\ & & \bar{\beta}(U), \end{array}$$

where $\bar{C}$ is the smooth curve defined by the Stein factorization and $C := f(U)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} U & \xrightarrow{f|_U} & C & \longrightarrow & \beta(U) \\ \downarrow & & \downarrow & & \downarrow \\ A_U & \xrightarrow{\psi_f} & A_C & \longrightarrow & T \end{array}$$

Note that the holomorphic 1-form dz_1 on T pulls back to logarithmic 1-form ω . Therefore to see (1), it suffices to show that T is isogenous to A_C . First all the horizontal maps in the diagram are surjective. In fact we only need to show ψ_f is surjective. Since $f|_U: U \rightarrow C$ is surjective and has connected generic fibres, the induced map on the first homology groups

$H_1(U, \mathbb{Z}) \rightarrow H_1(C, \mathbb{Z})$ is surjective and the surjectivity of ψ_f follows. Since f comes from the Stein factorisation and β contracts E , so does f . By choosing appropriate base points, we get that $\psi(A_N)$ is contained in the kernel of ψ_f . Hence A_C is isogenous to T . To see (3), notice that E is contained in a fibre of f and $E^2 \cdot H^{n-2} \geq 0$. Hence $E^2 \cdot H^{n-2} = 0$. Finally for (4), when $\chi(C) = 0$, notice that for any non-zero $\eta \in H^0(\overline{C}, \Omega_{\overline{C}}^1(\log B))$, it has no zeros. Then E has to be a multiple fibre.

Proof of Theorem 4.7 We assume $n > 1$. For any 1-form $\omega \in W^1(X, D)$, pick an integral divisor $E \subset X$ such that $E \subseteq Z(\omega)$. We may assume that $\omega \notin W_{\text{neg}}(X, D)$ so that $E^2 \cdot H^{n-2} \geq 0$. If E is not a component of D , then $\omega \in \mathcal{T}_{\Sigma}^1(U)$ by Theorem 5.1. Otherwise, say $E = D_1$ a component of D . Then we have an injective map [10, 2.3 Properties (c)]

$$H^0(X, \Omega_X^1(\log D)(-D_1)) \hookrightarrow H^0(X, \Omega_X^1(\log D - D_1)).$$

In particular, $\omega \in H^0(X, \Omega_X^1(\log D - D_1))$. Set $U_1 = X - \bigcup_{j \neq 1} D_j$. By Theorem 5.1, we have an orbifold map $f_1: U_1 \rightarrow C_1$ such that $\omega \in f_1^* H^0(\overline{C}_1, \Omega_{\overline{C}_1}^1(\log B_1))$, where $B_1 = \overline{C}_1 - C_1$. Note that $f := f_1|_U: U \rightarrow C$ is also an orbifold map, where C is the image of U . Furthermore we know that $f_1^* H^0(\overline{C}_1, \Omega_{\overline{C}_1}^1(\log B_1))$ is contained in $f^* H^0(\overline{C}, \Omega_{\overline{C}}^1(\log B))$, where $\overline{C} = \overline{C}_1$ and $B = \overline{C} - C$. It implies that $\omega \in f^* H^0(\overline{C}, \Omega_{\overline{C}}^1(\log B))$, i.e. $\omega \in \mathcal{T}_{\Sigma}^1(U)$. The first part follows.

To see the linearity, we notice that there are at most countably many negative divisors. Indeed, similar to the projective case the sign of the intersection of $E^2 \cdot H^{n-2}$ does not depend on the choice of H when $E \subset Z(\omega)$ for some $\omega \in H^0(X, \Omega_X^1(\log D))$. This can be seen using the same argument as in the proof of Lemma 4.3. Then the proof follows the arguments in Theorem 4.4 verbatim.

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