CM-values of p-adic Theta-functions

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Stellingen

behorende bij het proefschrift
"CM-values of p-adic Θ-functions"

(i) Let X_6 be the genus zero Shimura curve associated with the indefinite rational quaternion algebra with discriminant 6 and let $j_6 : X_6 \xrightarrow{\sim} \mathbb{P}^1$ denote an isomorphism. Let $\tau_1, \tau_2 \in X_6(\mathbb{C})$ be CM-points with discriminants -19 and -43 respectively and set $\tau_1' = w_3\tau_1$ and $\tau_2' = w_3\tau_2$, where $w_3 : X_6 \to X_6$ denotes the Atkin-Lehner involution associated with the prime 3. Then

$$\mathrm{Nm}\left(\frac{j_6(\tau_1) - j_6(\tau_2)}{j_6(\tau_1') - j_6(\tau_2)}\frac{j_6(\tau_1') - j_6(\tau_2')}{j_6(\tau_1) - j_6(\tau_2')}\right) = \left(\frac{2 \cdot 19}{3 \cdot 29}\right)^{\pm 2}.$$

(ii) Let R be a maximal order in the definite rational quaternion algebra with discriminant 2 and choose a splitting through which it acts on $\mathcal{H}_5$. Let $\tau_1, \tau_2 \in \mathcal{H}_5$ be CM-points with discriminants -43 and -67 respectively with Galois conjugates $\tau_1', \tau_2' \in \mathcal{H}_5$. If $\pi \in R$ denotes a quaternion of norm 5, then

$$\mathrm{Nm}\left(\prod_{\gamma \in R[1/5]_1^\times} \frac{\tau_1 - \gamma\tau_2}{\tau_1 - \gamma\tau_2'}\frac{\tau_1' - \gamma\tau_2'}{\tau_1' - \gamma\tau_2}\frac{\tau_1 - \gamma\pi\tau_2'}{\tau_1 - \gamma\pi\tau_2}\frac{\tau_1' - \gamma\pi\tau_2}{\tau_1' - \gamma\pi\tau_2'}\right) = \left(\frac{3^6}{2^4 \cdot 61}\right)^{\pm 2}.$$

(iii) Let $K_i = \mathbb{Q}(\sqrt{D_i})$ for $i \in \{1, 2\}$ be imaginary quadratic fields with coprime discriminants and embeddings $\alpha_i : K_i \to B$ for a rational quaternion algebra B. Then there exists a unique $F = \mathbb{Q}(\sqrt{D_1 D_2})$-quadratic form $\det_F : B \to F$ with $\mathrm{tr} \circ \det_F = \mathrm{Nm} : B \to \mathbb{Q}$. If $A = \alpha_1(\sqrt{D_1})$ and $B = \alpha_2(\sqrt{D_2})$, then it is given by

$$\det_F(\gamma) = \frac{\mathrm{Nm}(\gamma)}{2} + \frac{\mathrm{tr}(A\gamma B\overline{\gamma})}{4\sqrt{D_1 D_2}}.$$

(iv) Let $\chi : G_F \to \{\pm 1\}$ be the character induced by the unramified field extension L/F, where $L = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$. Let $\eta \in H^1(G_F, \mathbb{Q}_p(\chi))$ be non-trivial, but trivial at one of the two decomposition groups above a prime p that is inert in both K_1 and K_2. If we let ρ_η denote the associated upper-triangular rigidification of $\mathbb{1} \oplus \chi$, then its nearly ordinary deformation functor is representable by a ring $R^{\mathrm{no}}_{\rho_\eta}$. If $\mathbb{T}$ denotes the nilreduction of the completion of the localisation of Hida's cuspidal Hecke algebra at an opposite-sign p-stabilisation of the Eisenstein series $E_{1,\chi}$, then there is an isomorphism $R^{\mathrm{no}}_{\rho_\eta} \xrightarrow{\sim} \mathbb{T}$.

(v) If $\varphi_p : G_F \to \mathbb{Q}_p$ denotes the p-adic cyclotomic character, there is a modular deformation of the form

$$\left(1 + \epsilon \begin{pmatrix} \varphi_p & * \\ 0 & -\varphi_p \end{pmatrix}\right) \rho_\eta$$

that gives rise to an infinitesimal family of p-adic Hilbert modular cuspforms. Let $q = \mathrm{disc}(B)$ and $N = pq$. Further, let $\mathfrak{q}$ denote a prime of F above q and let $\mathcal{D}_F$ denote the different ideal of F. Then the result of taking the derivative of this family with respect to the weight parameter, taking the diagonal restriction with respect to the ideal $\mathcal{D}_F^{-1}\mathfrak{q}$, and applying the ordinary projection operator, is contained in $\mathcal{S}_2(\Gamma_0(N))$. If $N \in \{6, 10, 22\}$, it vanishes, and its Fourier coefficients contain arithmetic information about the CM-values of p-adic Θ-functions.

(vi) The recipe of taking the diagonal restriction of the derivative of a family of modular forms with respect to the weight parameter seems to unite archimedean and non-archimedean theories if one applies a holomorphic projection in the former setting, and an ordinary projection in the latter, suggesting a p-adic treatment in which CM-theory and RM-theory are on more equal footing.

(vii) If $n \geq 2$ is even and R is a ring in which $x^n = x$ for all $x \in R$, then R must be Boolean if and only if $n - 1$ is not divisible by any number from the set $\{2^k - 1 \mid k \in \mathbb{Z}_{\geq 2}\}$, which happens for about 54.83% of all positive even n.

(viii) If A is a finitely generated abelian group and $f : A \to A$ is a group endomorphism such that for all $g \in A$, it holds that $f(g) = k_g g$ for some integer $k_g \in \mathbb{Z}$, then f is given by multiplication by a fixed integer k. This result can be false for A not finitely generated.

(ix) Define the function $f : \mathbb{N} \setminus \{1\} \to \mathbb{N}$ by the rule

$$f(n) = \begin{cases} n^2 - 1 & \text{if } n \text{ is odd;} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

For odd positive integers m, the sequence $m \mapsto f(m) \mapsto f(f(m)) \mapsto \dots$ reaches 1 if and only if $m = 2^t \pm 1$ for some integer $t \geq 2$, or if $m \in \{11, 23, 181\}$.

(x) It is perfectly possible to have a productive working day after waking up at noon.

(xi) Making YouTube videos about puzzle games is a great idea. Telling your colleagues about these videos, however, is not.

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