CM-values of p-adic Theta-functions

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Summary

In the 1980s, Gross and Zagier studied the differences between singular moduli, which are the CM-values of Klein's j-function. For example,

$$j\left(\frac{1+\sqrt{-43}}{2}\right) - j\left(\frac{1+\sqrt{-163}}{2}\right) = 2^{19} \cdot 3^6 \cdot 5^3 \cdot 7^3 \cdot 37 \cdot 433.$$

They proved that these numbers obey very predictable prime factorisations and that the primes that occur in these expressions are relatively small and inert in both CM-fields in question. These remarkable properties sparked a chain of investigations that fuelled many deep results and crucial ideas across various areas of mathematics over the decades that followed, which continue to inspire mathematicians to this day.

We prove a p-adic version of the work by Gross and Zagier on the differences between singular moduli by proving a set of conjectures by Sofia Giampietro and Henri Darmon, who investigated the factorisation of a rational invariant associated to a pair of CM-points on a genus zero Shimura curve, obtained as the ratio of the CM-values of p-adic Θ-functions. As did Gross and Zagier, we give two proofs; an algebraic proof using CM-theory, and more interestingly, also an analytic proof using p-adic infinitesimal deformations of an Hilbert Eisenstein series. Since there are no explicit formulae for its cuspidal p-adic deformations, we instead compute the Frobenius traces of an appropriate Galois deformation, and show their modularity via an $R = T$ theorem. This approach aims to bridge the gap between classical CM-theory and the more recent p-adic advances in the theory of real multiplication.

Chapter 1 serves to illustrate the context within which this thesis is best viewed and starts by introducing CM-theory from an adèlic perspective. We present the factorisation results by Gross and Zagier on the

differences between singular moduli and illustrate these through various examples. We reinterpret their results in the language of intersection numbers of pairs of embeddings and by means of numerous examples this viewpoint is made very concrete. We sketch both the algebraic and analytic proofs of these factorisation results; especially the analytic proof will serve as a great inspiration for the strategy followed in Chapter 6, albeit in a non-archimedean setting as opposed to an archimedean one.

Thereafter, in Chapter 2, we introduce a certain class of Shimura curves associated with indefinite rational quaternion algebras. After outlining some of their basic properties, we describe the conjectures by Giampietro and Darmon and stress the parallels with the results by Gross and Zagier. We then discuss how the p-adic uniformisation of Shimura curves can be used to approach these conjectures p-adically, and an equivalent non-archimedean statement is posed. We outline the general strategies for the proofs of these key results and we stress the similarities between our approach and recent advancements in the theory of real multiplication, among other recent developments.

In Chapter 3, we describe an approach that mirrors the ideas behind Gross and Zagier's original algebraic proof, exploiting the moduli interpretation of the Shimura curve and the theory of complex multiplication. We appeal to the main result of the PhD thesis of Andrew Phillips, which computes the degree of certain refinements of the moduli stack of false elliptic curves, following ideas of Howard and Yang. Using these results, the proof of Theorem A is rather straightforward.

The weight of this thesis is concentrated in our proof of Theorem B. For this, we follow the general strategy of the main arguments presented in work done by Darmon, Pozzi and Vonk. We study a p-stabilisation of the same Hilbert Eisenstein series as did Gross and Zagier. In this sense, our approach is a true p-adic transposition of their seminal work. This second proof can be divided into three distinct steps, which we will now outline and which will reflect the three chapters that take up the full treatment of this approach.

In Chapter 4, we start by recalling some background on genus theory, a study pioneered by Gauß. We prove the exactness of a key sequence using basic class field theory and group cohomology. Next, we prove the existence of and give various different expressions for a quadratic form refining the norm form on a quaternion algebra, which allows us to describe an algebraic construction that associates to a quaternion a certain ideal in a biquadratic field. Through this, we derive a bijection between quaternions and certain elements in a real quadratic field of fixed

trace. This allows us to rewrite the CM-values of p-adic Θ-functions in a meaningfully different way in Section 6.1.

Since, unlike as in the work of Gross and Zagier, the $\mathbf{q}$-expansion of a cuspidal p-adic family passing through the p-stabilised Hilbert Eisenstein series is not a-priori known, we obtain such a family by deforming a rigidification of the decomposable representation associated with this Eisenstein series. More precisely, we will consider all nearly ordinary deformations, which are all such deformations for which the decomposition groups at both primes above p each fix a distinct line. This approach requires us to prove the modularity of such deformations to justify constructing from this deformation the required family of modular forms.

Therefore, Chapter 5 proves an $R = T$ theorem, a famous instance of which occurred in the proof by Wiles of Fermat's Last Theorem. Here, R denotes the universal nearly ordinary deformation ring, and T denotes Hida's cuspidal nearly ordinary Hecke algebra. Using similar methods as in Pozzi's and Betina's theses, along with many other works, and using fundamental results from Hida, we construct a lift of the rigidified representation to the nearly ordinary Hecke algebra, though some additional care is required to circumvent the difficulties of some cohomology groups being 2-dimensional, rather than 1-dimensional. Comparing the dimensions of the Hecke algebra and the resulting deformation ring, the $R = T$ theorem follows purely by commutative algebra.

Finally, in Chapter 6, we consider one particular nearly ordinary deformation and explicitly compute the infinitesimal family of cuspidal deformations of the Hilbert Eisenstein series that corresponds to it. After taking its diagonal restriction, its derivative with respect to the weight parameter and applying the ordinary projection operator, we argue why the result must vanish identically. Ultimately, we conclude the proof of Theorem B by computing explicitly the Fourier coefficients of the mostly theoretically used ordinary projection and equating the first of these coefficients to zero.

Appendix A provides a brief introduction to various types of modular forms. It assumes the theory of classical modular forms and starts by outlining their automorphic perspective, and we will subsequently generalise this treatment to more general types of modular forms in the sections that follow. First, we change the ground field from $\mathbb{Q}$ to a real quadratic field. Then we combine our treatments and describe an adèlic approach to Hilbert modular forms and Hida families, as we will need them for our methods in Chapter 6, leaving the archimedean world and considering p-adic modular forms instead.