CM-values of p-adic Theta-functions

Daas, M.A.

Citation

Daas, M. A. (2024, October 30). *CM-values of p-adic Theta-functions*. Retrieved from https://hdl.handle.net/1887/4106986

APPENDIX A

Various types of modular forms

This appendix assumes the reader to be familiar with the theory of classical *modular forms*. A comprehensive introduction to classical modular forms can be found in [DS05]. We will start by briefly recalling their automorphic perspective and we will subsequently generalise this treatment to more general types of modular forms in the sections that follow. First, we change the ground field from $\mathbb{Q}$ to a real quadratic field F and then we describe an adèlic approach to Hilbert modular forms. Finally, we will leave the archimedean setting and consider p-adic modular forms instead, briefly introducing Hida families, as we will need them for our methods in Chapter 6.

A.1 Adelic modular forms

In order to motivate some of the later sections, we take a moment to illustrate the way in which the classical modular forms can be reinterpreted in an adèlic language. We closely follow the treatment and notation from [Dei12]. Throughout this section, let

$$\mathrm{SO}(2) := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\};$$

this is the stabiliser of $i \in \mathbb{C}$ under the action of $\mathrm{SL}_2(\mathbb{R})$. Further, for notational convenience, for any commutative ring R we denote

$$G_R := \mathrm{GL}_2(R) \quad \text{and} \quad G_R^1 := \mathrm{SL}_2(R).$$

One may choose a *Haar measure* on the group $G_{\mathbb{R}}^1$ and if $\Gamma \subset G_{\mathbb{R}}^1$ is a discrete subgroup, we obtain a natural Haar measure on the quotient space $\Gamma \setminus G_{\mathbb{R}}^1$. We use this to define the space

$$L^2(\Gamma \setminus G_{\mathbb{R}}^1) := \left\{ f : \Gamma \setminus G_{\mathbb{R}}^1 \to \mathbb{C} \text{ such that } \int_{\Gamma \setminus G_{\mathbb{R}}^1} |f(x)|^2 dx < \infty \right\}.$$

The map

$$\mathbb{C} \to M_2(\mathbb{Q}) : a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

restricts to an isomorphism of groups

$$\{z \in \mathbb{C} : |z| = 1\} =: \mathbb{T} \xrightarrow{\sim} \mathrm{SO}(2).$$

Therefore, for any $k \in \mathbb{Z}$ we can define the character

$$\epsilon_k : \mathrm{SO}(2) \to \mathbb{T} \quad \text{induced by the map} \quad \mathbb{T} \xrightarrow{(-)^k} \mathbb{T}.$$

We may now define the subspace

$$L^2(\Gamma \setminus G^1_{\mathbb{R}})[\epsilon_k] := \{f \in L^2(\Gamma \setminus G^1_{\mathbb{R}}) \mid f(xu) = \epsilon_k(u)f(x),\ \forall u \in \mathrm{SO}(2)\}.$$

Let $|_k$ denote the usual slash operator, and for brevity, we will denote

$$j_\alpha(z) := cz + d \quad \text{for any} \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Let $\mathcal{S}_k(\Gamma)$ denote the space of weight k cuspforms for Γ. The following is Proposition 3.3.4 in [Dei12].

Proposition A.1.1. *Let* $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ *be a congruence subgroup and* $k \in \mathbb{Z}$. *Given* $f \in \mathcal{S}_k(\Gamma)$, *define the map*

$$\phi_f : G^1_{\mathbb{R}} \to \mathbb{C} \quad \textit{through} \quad \phi_f(\alpha) = (f|_k\alpha)(i).$$

This association induces an (isometric) injection

$$\mathcal{S}_k(\Gamma) \hookrightarrow L^2(\Gamma \setminus G^1_{\mathbb{R}})[\epsilon_{-k}].$$

Proof. We content ourselves for now with verifying that indeed it holds that $\phi_f \in L^2(\Gamma \setminus G^1_{\mathbb{R}})[\epsilon_{-k}]$. To this end, we compute for $\gamma \in \Gamma$ that

$$\phi_f(\gamma\alpha) = (f|_k\gamma\alpha)(i) = ((f|_k\gamma)|_k\alpha)(i) = (f|_k\alpha)(i) = \phi_f(\alpha),$$

using the Γ-invariance of f. This shows that $\phi_f : \Gamma \setminus G^1_{\mathbb{R}} \to \mathbb{C}$ is well-defined. Square-integrability follows from the vanishing of f at all cusps, so its remains to verify that ϕ_f lands in the required ϵ_{-k}-eigenspace. Indeed, we compute for $u \in \mathrm{SO}(2)$ that

$$\phi_f(xu) = ((f|_kx)|_ku)(i) = j_u(i)^{-k}(f|_kx)(ui) = \epsilon_{-k}(u)\phi_f(x),$$

where we used that $j_u(i)^{-k} = \epsilon_{-k}(u)$ by definition and that $ui = i$. This completes the proof sketch. □

The proposition above allows us to view classical modular forms as elements of the space $L^2(\Gamma \setminus G^1_{\mathbb{R}})[\epsilon_{-k}]$; in particular is shows that the *weight* is perhaps more naturally viewed as a character, as opposed to a positive integer as the classical theory seems to suggest. In fact, we have a natural eigenspace decomposition

$$L^2(\Gamma \setminus G^1_{\mathbb{R}}) = \bigoplus_{k \in \mathbb{Z}} L^2(\Gamma \setminus G^1_{\mathbb{R}})[\epsilon_k].$$

It is in this sense that studying the space $L^2(\Gamma \setminus G^1_{\mathbb{R}})$ is akin to studying all classical modular forms of all possible weights at once. We can refine the above result if we let $\mathbb{A}$ denote the $\mathbb{Q}$-adèles, $\mathbb{A}^{\mathrm{fin}}$ the finite $\mathbb{Q}$-adèles, and further set

$$Z_{\mathbb{R}} := \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \mid r \in \mathbb{R}^{\times} \right\}.$$

Let $\widehat{\mathbb{Z}}$ denote the ring of profinite integers. The following statement is now purely group-theoretic, see Proposition 7.2.4 in [Dei12].

Lemma A.1.2. *Let Γ be a congruence subgroup and let K_Γ be the closure of Γ in $G_{\widehat{\mathbb{Z}}}$. Then the map*

$$\Gamma \setminus G^1_{\mathbb{R}} \xrightarrow{\sim} G_{\mathbb{Q}} \setminus G^1_{\mathbb{A}} \,/K_\Gamma$$

given by $\Gamma x \mapsto G_{\mathbb{Q}}(1,x)K_\Gamma$ is a $G^1_{\mathbb{R}}$-equivariant isomorphism, where we wrote $(1,x) \in G_{\mathbb{A}^{\mathrm{fin}}} \times G_{\mathbb{R}} = G_{\mathbb{A}}$.

This has the following consequence, which allows us to view classical modular forms as adèlic objects, using Proposition A.1.1. This is Theorem 7.2.5 in [Dei12].

Theorem A.1.3. *Let Γ be a congruence subgroup and let K_Γ be the closure of Γ in $G_{\widehat{\mathbb{Z}}}$. Then restriction induces a (unitary) $G^1_{\mathbb{R}}$-equivariant isomorphism*

$$L^2(G_{\mathbb{Q}} Z_{\mathbb{R}} \setminus G_{\mathbb{A}} \,/K_\Gamma) \xrightarrow{\sim} L^2(\Gamma \setminus G^1_{\mathbb{R}}).$$

The elements from $L^2(G_{\mathbb{Q}} Z_{\mathbb{R}} \setminus G_{\mathbb{A}})$ are called *automorphic forms.* Through the arrows

$$\mathcal{S}_k(\Gamma) \hookrightarrow L^2(\Gamma \setminus G^1_{\mathbb{R}})[\epsilon_{-k}] \hookrightarrow L^2(\Gamma \setminus G^1_{\mathbb{R}}) \xrightarrow{\sim} L^2(G_{\mathbb{Q}} Z_{\mathbb{R}} \setminus G_{\mathbb{A}} \,/K_\Gamma),$$

we have now realised classical modular forms as special cases of the more general concept of an automorphic form. The class of automorphic forms is more broad however, for it also includes objects like Maaß-waveforms, as explained in Section 2.8 in [Dei12]. We opt to not dive deeper into this theory here, however, thus concluding this section.

A.2 Classical Hilbert modular forms

Throughout this section, we fix an embedding $F \subset \mathbb{R}$, and for $\nu \in F$, we let $\nu' = \sigma(\nu) \in F$ denote its Galois conjugate. It is our aim to sketch a theory of modular forms, but now with the base field F instead of $\mathbb{Q}$. We follow the notation from Chapter 2 in [DDP11].

Definition A.2.1. For $z = (z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ and $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$, define for $a, b \in F$ the quantity

$$(az + b)^{\mathbf{k}} := (az_1 + b)^{k_1}(a'z_2 + b')^{k_2}.$$

We will also write

$$\mathrm{GL}_2^+(F) := \{A \in \mathrm{GL}_2(F) \mid \det(A) \gg 0\}.$$

Definition A.2.2. Let $\gamma \in \mathrm{GL}_2^+(F)$ and $z = (z_1, z_2) \in \mathcal{H} \times \mathcal{H}$. We set

$$\gamma \cdot z := \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'}\right) \quad \text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A fundamental domain for this group action, roughly speaking, is the union of $h_F := \#\mathrm{Pic}(F)$ different open subsets of the product $\mathcal{H} \times \mathcal{H}$. This explains in part why the class numbers h_F and, more precisely, $h_F^+ := \#\mathrm{Pic}(F)^+$, will appear centrally in the theory.

Definition A.2.3. The action of $\gamma \in \mathrm{GL}_2^+(F)$ defined above induces the *weight* $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ action on the space of functions $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$.

$$f|_{\mathbf{k}}\gamma(z) := \det(\gamma)^{k_1/2}\det(\gamma')^{k_2/2}(cz + d)^{-\mathbf{k}}f(\gamma z).$$

Definition A.2.4. For each class $\lambda \in \mathrm{Pic}(F)^+$, we fix a representative ideal $t_\lambda \subset \mathcal{O}_F$ with $[t_\lambda] = \lambda$. We use it to define the groups Γ_λ as

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F) : a, d \in \mathcal{O}_F,\ b \in t_\lambda^{-1}\mathcal{D}_F^{-1},\ c \in t_\lambda\mathcal{D}_F,\ ad - bc \in \mathcal{O}_F^\times\right\}.$$

Definition A.2.5. A weight $\mathbf{k} \in \mathbb{Z}^2$ *classical Hilbert modular form* is an h_F^+-tuple of holomorphic functions

$$f_\lambda : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

where $\lambda \in \mathrm{Pic}(F)^+$, such that $f_\lambda|_{\mathbf{k}}\gamma = f_\lambda$ for all $\gamma \in \Gamma_\lambda$.

This means that f_λ for any $\lambda \in \mathrm{Pic}(F)^+$ has a Fourier expansion

$$f_\lambda(z_1, z_2) = a_\lambda(0) + \sum_{\nu \in t_\lambda^+} a_\lambda(\nu)e^{2\pi i(z_1\nu + z_2\nu')}.$$

Definition A.2.6. The coefficients $a_\lambda(\nu)$ are called the *unnormalised Fourier coefficients* of f. They may be normalised to eradicate the initial choices of the representative ideals t_λ by setting for each integral ideal $\mathfrak{m} \subset \mathcal{O}_F$,

$$a(\mathfrak{m}, f) := a_\lambda(\nu)\mathrm{Nm}(t_\lambda)^{-(k_1+k_2)/2}\nu^{(k_1-k_2)/2},$$

where $\lambda = -[\mathfrak{m}] \in \mathrm{Pic}(F)^+$ and $\nu \in t_\lambda^+$ is such that $\mathfrak{m}t_\lambda = (\nu)$.

Lemma A.2.7. *The normalised Fourier coefficients* $a(\mathfrak{m}, f)$ *are well-defined for any* $\mathfrak{m} \subset \mathcal{O}_F$.

Proof. The only choice we make during their construction is that of the element $\nu \in t_\lambda^+$ generating the ideal $\mathfrak{m}t_\lambda$. Let $\epsilon \in \mathcal{O}_F^{\times,+}$ denote a totally positive unit and let $\mu = \epsilon\nu$ be any other such choice. Then we must show that $a_\lambda(\nu) = a_\lambda(\mu)\epsilon^{(k_1-k_2)/2}$. Indeed, we compute that

$$f(z_1, z_2) = f|_{\mathbf{k}} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} (z_1, z_2) = \epsilon^{k_1/2}(\epsilon')^{k_2/2} f(\epsilon z_1, \epsilon' z_2).$$

Now note that $\epsilon^{k_1/2}(\epsilon')^{k_2/2} = \epsilon^{(k_1-k_2)/2}$, as $\epsilon \gg 0$ implies that $\epsilon\epsilon' = \mathrm{Nm}(\epsilon) = 1$. Now comparing the coefficient in front of $e^{2\pi i(z_1\mu + z_2\mu')} = e^{2\pi i(z_1\epsilon\nu + z_2\epsilon'\nu')}$ on both sides yields the claimed equality. ☐

Remark A.2.8. The observant reader might have noticed the striking absence of any conditions regarding holomorphicity at the cusps in Definition A.2.5. Rather remarkably, such conditions are automatically satisfied for holomorphic functions satisfying the transformation properties given above. This is called *Koecher's principle* and can be found for instance as Theorem 1.20 in [Bru08].

To introduce the most vital example for the purposes of this thesis, we must first define the complex L-function

$$L(\chi, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a}) \mathrm{Nm}(\mathfrak{a})^{-s},$$

which converges as soon as $\mathrm{Re}(s) > 1$ and allows a holomorphic continuation to all of $\mathbb{C}$. The following is a special case of the much more general Proposition 2.1 in [DDP11].

Proposition A.2.9. *There exists a parallel weight* $\mathbf{k} = \mathbf{1} = (1, 1)$ *Hilbert modular form* $E_{1,\chi}$ *with normalised Fourier coefficients*

$$a_\lambda(0, E_{1,\chi}) = \frac{1 + \chi(\lambda)}{4} L(\chi, 0),$$

and for $\mathfrak{m} \subset \mathcal{O}_F$,

$$a(\mathfrak{m}, E_{1,\chi}) = \sum_{I | \mathfrak{m}} \chi(I).$$

Using the following lemma, we may rewrite these coefficients in a more concrete way.

Lemma A.2.10. *For any ideal $\mathfrak{m} \subset \mathcal{O}_F$, it holds that*

$$\sum_{I|\mathfrak{m}} \chi(I) = \rho(\mathfrak{m}) \geq 0.$$

Proof. Because both χ and ρ are multiplicative, we can rewrite

$$\sum_{I|\mathfrak{m}} \chi(I) = \prod_{\mathfrak{r}^k \| \mathfrak{m}} \sum_{i=0}^{k} \chi(\mathfrak{r}) \quad \text{and} \quad \rho(\mathfrak{m}) = \prod_{\mathfrak{r}^k \| \mathfrak{m}} \rho(\mathfrak{r}^k).$$

We thus complete the proof after showing that for every prime $\mathfrak{r} \subset \mathcal{O}_F$,

$$\sum_{i=0}^{k} \chi(\mathfrak{r}) = \rho(\mathfrak{r}^k) = \begin{cases} k+1 & \text{if } \chi(\mathfrak{r}) = 1; \\ 1 & \text{if } \chi(\mathfrak{r}) = -1 \text{ and } k \text{ is even;} \\ 0 & \text{if } \chi(\mathfrak{r}) = -1 \text{ and } k \text{ is odd.} \end{cases}$$

This is easily verified for the leftmost term, and counting ideals is easy after recalling that $\chi(\mathfrak{r})$ measures the splitting behaviour of $\mathfrak{r}$ in L/F. □

Corollary A.2.11. *Suppose that $\chi(\mathfrak{m}) = -1$. Then*

$$\rho(\mathfrak{m}) = 0, \quad \textit{and as such,} \quad (E_{1,\chi})_{[\mathfrak{m}]} = 0.$$

Proof. It is clear from Proposition A.2.9 that the constant term vanishes in this case, so consider now the higher Fourier coefficients.

Using the same product expansion as in the proof of Lemma A.2.10, it follows that the result must vanish if there exists some prime $\mathfrak{r}$ of $\mathcal{O}_F$ with $\chi(\mathfrak{r}) = -1$ dividing $\mathfrak{m}$ an odd number of times. Again by multiplicativity of χ, this is ensured if $\chi(\mathfrak{m}) = -1$, as we assume.

Alternatively, Proposition 4.1.5 in Chapter 4 shows that the image of the norm map Nm_F^L is contained in the kernel of χ, hence in particular $\mathfrak{m}$ is not a norm. □

Since χ is totally odd, it holds that $\chi(\mathcal{D}_F) = -1$; see also Proposition 4.0.3. Therefore, it follows from the above that

$$(E_{1,\chi})_{[\mathcal{D}_F^{-1}]} = 0.$$

This fact has historically come to be known as *Hecke's sign error*, as after missing a single minus sign in an otherwise impressive calculation, Hecke once mistakenly concluded the incorrect *non-vanishing* of the Hilbert Eisenstein series above. However, as unfortunate as this may have been for Hecke at the time, this fact was later exploited by Gross and Zagier in [GZ84], and lies at the foundation of their analytic proof of Theorem 1.2.1, as described in Section 1.6. To understand that method, we will need to introduce the following concept.

Definition A.2.12. The *diagonal restriction* of a weight $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ Hilbert modular form $f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is defined as the map

$$\Delta f : \mathcal{H} \to \mathbb{C} \quad \text{given by} \quad \Delta f(z) = f(z, z);$$

one checks that this must be a classical modular form of weight $k_1 + k_2$.

From a more algebraic perspective, we may try to describe this operation on the level of q-expansions. To this end, we remark that setting $z_1 = z_2 = z$ in the expression $e^{2\pi i (z_1 \nu + z_2 \nu')}$ yields $e^{2\pi i z \operatorname{tr}(\nu)} = \mathbf{q}^{\operatorname{tr}(\nu)}$, where $\mathbf{q} = e^{2\pi i z}$ as usual. In other words, the diagonal restriction collects elements $\nu \in F$ of fixed trace and together these constitute the Fourier coefficients of the classical modular form.

Concretely, one chooses some fractional F-ideal $\mathfrak{m}$ with respect to which to compute the diagonal restriction, and one obtains for the Fourier coefficients

$$a_n(\Delta_{\mathfrak{m}} f) = \sum_{\substack{\nu \in \mathfrak{m}^+ \\ \operatorname{tr}(\nu) = n}} a(\nu \mathfrak{m}^{-1}, f).$$

For those primes $\mathfrak{l} \subset \mathcal{O}_F$ that satisfy $[\mathfrak{l}] = [0] \in \operatorname{Pic}(F)^+$, we may write $\mathfrak{l} = (\varpi)$ for some $\varpi \in \mathcal{O}_F^+$. Let $B \subset \Gamma_\lambda$ be a set of representatives with

$$\Gamma_\lambda \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} \Gamma_\lambda = \bigsqcup_{\gamma \in B} \Gamma_\lambda \gamma.$$

As explained in [MS15], one may then define the Hecke operator $T_{\mathfrak{l}}$ as

$$T_{\mathfrak{l}} f_\lambda := \varpi^{k_1/2-1} (\varpi')^{k_2/2-1} \sum_{\gamma \in B} f|_{\mathbf{k}} \gamma.$$

One can check that this definition is independent of the choice of ϖ. This definition displays strong similarities to its classical counterpart. One may further define diamond operators $\langle \mathfrak{l} \rangle$ to recursively set

$$T_{\mathfrak{l}^n} := T_{\mathfrak{l}^{n-1}} T_{\mathfrak{l}} - \langle \mathfrak{l} \rangle \varpi^{k_1-1} (\varpi')^{k_2-1} T_{\mathfrak{l}^{n-2}}.$$

However, the failure of the triviality of the narrow class group prevents us from using the above definition to define operators $T_{\mathfrak{l}^n}$ for each prime ideal $\mathfrak{l}$. This suggests that perhaps the theory is more easily set up in an adèlic or local language, as opposed to the global language considered hitherto. Together with the results from Section A.1, this more than justifies the ideas contained in the Section A.4, which, along with the section thereafter, forms the technical heart of this appendix.

A.3 p-stabilisations

We recall the notion of a p-stabilisation for a classical modular form.

Definition A.3.1. Let $f \in \mathcal{S}_k(\Gamma_0(N))$ for some positive integer N and let $p \nmid N$ be a prime number. Define its *Hecke polynomial* as

$$X^2 - a_p(f)X + p^{k-1} =: (X - \alpha)(X - \beta);$$

this is the characteristic polynomial of Frobenius at p for the Galois representation associated with the modular form f. We say that f is *regular* at p whenever $\alpha \neq \beta$.

From now on, we let $p \nmid N$ be such a regular prime and for any $f \in \mathcal{S}(\Gamma_0(N))$ we define

$$f_\alpha(z) := f(z) - \beta f(pz) \quad \text{and} \quad f_\beta(z) := f(z) - \alpha f(pz).$$

It is clear that these are both elements of $\mathcal{S}(\Gamma_0(Np))$ and they are called the two *p-stabilisations* of f. The following lemma explains this.

Lemma A.3.2. *Let $f \in \mathcal{S}(\Gamma_0(N))$ be a normalised eigenform. Then both f_α and f_β are normalised eigenforms of level Np, with U_p-eigenvalues α and β respectively.*

Proof. It is easy to check that both f_α and f_β remain T_ℓ-eigenvectors for each prime $\ell \neq p$ with the same eigenvalue. We thus reduce to analysing the operators T_p on $\mathcal{S}(\Gamma_0(N))$ and U_p on $\mathcal{S}(\Gamma_0(Np))$.

By symmetry, it suffices to show that $U_p f_\alpha = \alpha f_\alpha$. We prove this on the level of $\mathbf{q}$-expansions. On the left hand side, the n-th Fourier coefficient is equal to

$$a_{np}(f_\alpha) = a_{np}(f) - \beta a_n(f)$$

whereas on the right hand side, it is equal to

$$\alpha a_n(f) - \alpha\beta a_{n/p}(f).$$

To prove equality, it thus suffices to show that

$$a_{np}(f) = (\alpha + \beta)a_n(f) - \alpha\beta a_{n/p}(f) = a_p(f)a_n(f) - p^{k-1}a_{n/p}.$$

Now use that f is a normalised eigenform and as such, coprime coefficients are multiplicative. Writing $n = p^k m$ with $p \nmid m$, we divide out by $a_m(f)$ to reduce to showing that

$$a_{p^{k+1}} = a_p(f)a_{p^k}(f) - p^{k-1}a_{p^{k-1}};$$

this is trivial for $k = 0$ and for positive k again ensured by the fact that f is a normalised eigenform and the well known recursive definition of the Hecke operators for prime powers. $\square$

A p-stabilised modular form can be viewed as a p-adic modular form, see Section A.5. We mimic Definition A.3.1 and consider various ways to *p-stabilise* the object $E_{1,\chi}$. Let $V_{\mathfrak{p}_1}$ and $V_{\mathfrak{p}_2}$ be the level raising operators, denoted as $|_{\mathfrak{p}_i}$ in Section 2.6 in [DK20]. On the normalised Fourier coefficients of the Hilbert modular form f, these operators act through the rules

$$a_\lambda(0, V_{\mathfrak{p}_i} f) = a_{\lambda\mathfrak{p}_i}(0, f) \quad \text{and} \quad a(\mathfrak{m}, V_{\mathfrak{p}_i} f) = \begin{cases} a(\mathfrak{m}/\mathfrak{p}_i, f) & \text{if } \mathfrak{p}_i \mid \mathfrak{m}; \\ 0 & \text{otherwise.} \end{cases}$$

These operators refine the classical level raising operation $f(z) \mapsto f(pz)$ in the sense that for a Hilbert modular form f, it holds that

$$V_{\mathfrak{p}_1} V_{\mathfrak{p}_2} f(z_1, z_2) = V_{\mathfrak{p}_2} V_{\mathfrak{p}_1} f(z_1, z_2) = f(pz_1, pz_2).$$

We now specialise to the form $E_{1,\chi}$, whose associated Galois representation of G_F is given by $\rho = \mathbb{1} \oplus \chi$. For $i \in \{1, 2\}$ its Hecke polynomial, which is the characteristic polynomial of Frobenius, is then given by

$$X^2 + [\mathbb{1}(\mathfrak{p}_i) + \chi(\mathfrak{p}_i)]X + \mathbb{1}(\mathfrak{p}_i)\chi(\mathfrak{p}_i) = X^2 - 1,$$

as $\chi(\mathfrak{p}_i) = -1$. There are therefore four possible ways for us to *p-stabilise* the modular form $E_{1,\chi}$, given by

$$(1 \pm V_{\mathfrak{p}_1})(1 \pm V_{\mathfrak{p}_2}) E_{1,\chi}.$$

For the purposes of this thesis, we choose the stabilisation

$$E_{1,\chi}^{(p)} := (1 - V_{\mathfrak{p}_1})(1 + V_{\mathfrak{p}_2}) E_{1,\chi}$$

and fix it throughout this thesis. For a discussion explaining this choice conceptually, see the start of Section 5.4. However, we do note here that our choice for opposite signs for this p-stabilisation has the following consequence.

Lemma A.3.3. *The form $E_{1,\chi}^{(p)}$ is a p-adic Hilbert cusp form.*

Proof. It is noted above, and on page 461 of [DDP11] and Equation 10 in [DK20] it is shown, that the constant term of the result of applying level-raising operator at a certain class $\lambda \in \mathrm{Pic}(F)^+$ is the constant term at the class $\lambda\mathfrak{p}_i$. With Proposition 2.1 in [DDP11], or equivalently our Proposition A.2.9, we find that the constant term of $E_{1,\chi}^{(p)}$ at any class λ equals

$$\Big[\big(1+\chi(\lambda)\big) - \big(1-\chi(\lambda)\big) + \big(1-\chi(\lambda)\big) - \big(1+\chi(\lambda)\big)\Big]\frac{L_F(\chi,0)}{4} = 0.$$

This shows that $E_{1,\chi}^{(p)}$ vanishes at all the cusps lying over ∞, and therefore it must be a p-adic cusp form. □

The following makes the form $E_{1,\chi}^{(p)}$ a bit more explicit.

Proposition A.3.4. *The normalised Fourier coefficients of the form $E_{1,\chi}^{(p)}$ are given by*

$$a\left(\mathfrak{m}, E_{1,\chi}^{(p)}\right) = (-1)^{v_{\mathfrak{p}_1}(\mathfrak{m})}\rho(\widetilde{\mathfrak{m}}),$$

where $\widetilde{\mathfrak{m}}$ denotes the p-depletion *of $\mathfrak{m} \subset \mathcal{O}_F$, obtained from $\mathfrak{m}$ by removing all factors of primes of F above p.*

Proof. By the rules above, we find for any nonzero integral ideal $\mathfrak{m} \subset \mathcal{O}_F$,

$$a(\mathfrak{m}, E_{1,\chi}^{(p)}) = a(\mathfrak{m}, E_{1,\chi}) - a(\mathfrak{m}/\mathfrak{p}_1, E_{1,\chi}) + a(\mathfrak{m}/\mathfrak{p}_2, E_{1,\chi}) - a(\mathfrak{m}/p, E_{1,\chi}),$$

where we adopt the convention $a(\mathfrak{n}, E_{1,\chi}) = 0$ if $\mathfrak{n}$ is a non-integral fractional ideal of F. Lemma A.2.9 and Lemma A.2.10 now combine to yield

$$a(\mathfrak{m}, E_{1,\chi}^{(p)}) = \rho(\mathfrak{m}) - \rho(\mathfrak{m}/\mathfrak{p}_1) + \rho(\mathfrak{m}/\mathfrak{p}_2) - \rho(\mathfrak{m}/p)$$

For the function ρ to be non-zero, the proof of Corollary A.2.11 shows that both the number of factors of both $\mathfrak{p}_i$ for $i \in \{1,2\}$ must be even. Therefore, at most one of these four terms can be non-zero. The sign of this term is determined by the parity of $v_{\mathfrak{p}_1}(\mathfrak{m})$. Finally, an even number of factors of $\mathfrak{p}_i$ for $i \in \{1,2\}$ does not change the outcome of the ρ-function; the result follows. □

Remark A.3.5. Suppose that the ideal $\mathfrak{m}$ is coprime to p, satisfies $\chi(\mathfrak{m}) = -1$ and is stable under the non-trivial automorphism σ of $F/\mathbb{Q}$. We then claim that the diagonal restriction of $E_{1,\chi}^{(p)}$ with respect to the ideal $\mathfrak{m}$ vanishes identically. This would follow if we can show that $a\left(\nu\mathfrak{m}, E_{1,\chi}^{(p)}\right) = -a\left(\nu'\mathfrak{m}, E_{1,\chi}^{(p)}\right)$ for all $\nu \in \mathfrak{m}^+$, for then the contributions of ν and $\nu' \in \sigma(\mathfrak{m}) = \mathfrak{m}$ to the appropriate Fourier coefficient would cancel out. To see this, first note that $\rho(\widetilde{\nu\mathfrak{m}^{-1}}) = \rho(\widetilde{\nu'\mathfrak{m}^{-1}})$ because these ideals are Galois conjugates. Second, the contribution of ν can only be non-zero if $\chi(\widetilde{\nu\mathfrak{m}^{-1}}) = 1$ by the proof of Lemma A.2.11, so the total number of factors of primes of F above p inside ν must be odd. Therefore $(-1)^{v_{\mathfrak{p}_1}(\nu)} = -(-1)^{v_{\mathfrak{p}_2}(\nu)} = -(-1)^{v_{\mathfrak{p}_1}(\nu')}$. We conclude using Proposition A.3.4.

Remark A.3.6. From the remark above, we conclude that the diagonal restriction of $E_{1,\chi}^{(p)}$ with respect to the ideal $\mathcal{D}_F^{-1}$ vanishes. However, it does *not* show that the diagonal restriction of $E_{1,\chi}^{(p)}$ with respect to the ideal $\mathcal{D}_F^{-1}\mathfrak{q}_1$ vanishes, and indeed it generally will not. The importance and subtleties caused by this observation are explained at the start of Section 5.4 and in the proof of Proposition 6.3.1.

A.4 Adelic Hilbert modular forms

We aim to redo most of the previous sections in an adèlic language from the automorphic perspective outlined in Section A.1. This theory was developed in the second half of the 20th century, and later refined by Hida over the scope of many papers, including but not limited to [Hid89b, Hid91, Hid89a]. These works also discuss p-adic modular forms, which will be the focus of the next section. We refer to Hida's work for a comprehensive introduction; here we only recall some definitions and focus on what we will need for our purposes. We shall closely adhere to the notation used in Section 2.1 in Fornea's thesis [For19].

We define the algebraic group

$$G_F := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F},$$

where $\mathrm{Res}_{F/\mathbb{Q}}$ denotes the Weil-restriction of scalars. In view of Proposition A.1.1, we set $\mathbf{i} := (i, i) \in \mathcal{H} \times \mathcal{H}$ and let C_∞^+ be the stabiliser of $\mathbf{i}$ under the natural action of the group $G_F(\mathbb{R})^+$, which we recall denotes the subgroup of $G_F(\mathbb{R})$ of all matrices with totally positive determinant.

Further, we let $K \subset G_F(\mathbb{A}_F^{\mathrm{fin}})$ be a compact open subgroup, which will be the *level* of the modular object we will define momentarily. Note the similarities here with the case of $\mathbb{Q}$: the ring of finite adèles $\mathbb{A}_F^{\mathrm{fin}}$ can be written as $F\widehat{\mathcal{O}_F}$. The ring $\widehat{\mathcal{O}_F}$ may be compared to $\widehat{\mathbb{Z}}$ and the closure $K_\Gamma \subset \mathrm{SL}_2(\widehat{\mathbb{Z}})$ of a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ describing the level of a classical modular form will indeed be a compact open subgroup. This is the group that appears in Theorem A.1.3.

Let $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ and $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^2$ be such that $\mathbf{k} - 2\mathbf{w} = m\mathbf{1}$ for some integer m, where $\mathbf{1} = (1,1)$.

Definition A.4.1. A *Hilbert modular form* of weight $(\mathbf{k}, \mathbf{w})$ is a function $f : G_F(\mathbb{A}_F) \to \mathbb{C}$ satisfying the following properties:

- For any $\alpha \in G_F(\mathbb{Q})$ and $u \in K \cdot C_\infty^+$, it holds that $f(\alpha x u) = f(x) j_{\mathbf{k},\mathbf{w}}(u_\infty, \mathbf{i})^{-1}$, where $u_\infty \in G_F(\mathbb{R})$. Here we wrote for
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ that } j_{\mathbf{k},\mathbf{w}}(A, z) := \det(A)^{-\mathbf{w}}(cz+d)^{\mathbf{k}},$$
adopting the notation $\det(A)^{-\mathbf{w}}$ and $(cz+d)^{\mathbf{k}}$ from Section A.2.
- For every $x \in G_F(\mathbb{A}_F^{\mathrm{fin}})$, define the function $f_x : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ given by $f_x(z) := f(xu_\infty) j_{\mathbf{k},\mathbf{w}}(u_\infty, \mathbf{i})$, where $u_\infty \in G_F(\mathbb{R})^+$ is such that $u_\infty \mathbf{i} = z \in \mathcal{H} \times \mathcal{H}$. Then f_x is holomorphic.

If in addition for all $x \in G_F(\mathbb{A})$ and all additive measures on $F \setminus \mathbb{A}_F$ it holds that
$$\int_{F\setminus\mathbb{A}_F} f\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x\right) da = 0,$$
then we say that f is a *cuspform*.

We denote the space of cuspforms of weight $(\mathbf{k}, \mathbf{w})$ and level $K \subset G_F(\mathbb{A}_F^{\mathrm{fin}})$ by $\mathcal{S}_{\mathbf{k},\mathbf{w}}(K)$. Let $\mathfrak{m} \subset \mathcal{O}_F$ be an integral ideal. We then define the subgroups

$$U_0(\mathfrak{m}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_F(\widehat{\mathbb{Z}}) \mid c \in \mathfrak{m}\widehat{\mathcal{O}}_F \right\};$$
$$V_1(\mathfrak{m}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{m}) \mid d - 1 \in \mathfrak{m}\widehat{\mathcal{O}}_F \right\};$$
$$V_{1,1}(\mathfrak{m}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_1(\mathfrak{m}) \mid a - 1 \in \mathfrak{m}\widehat{\mathcal{O}}_F \right\};$$
$$U(\mathfrak{m}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_{1,1}(\mathfrak{m}) \mid b \in \mathfrak{m}\widehat{\mathcal{O}}_F \right\}.$$

Definition A.4.2. Let p be a rational prime and $\mathfrak{m} \subset \mathcal{O}_F$ an ideal. Suppose that p and $\mathfrak{m}$ are coprime and let $K \subset G_F(\widehat{\mathbb{Z}})$ be a compact open subgroup satisfying $V_1(\mathfrak{m}) \subset K \subset U_0(\mathfrak{m})$. We then define for any positive integer n the subgroups

$$K(p^n) := K \cap V_{1,1}(p^n \mathcal{O}_F) \subset K.$$

We now describe the unique adèlic $\mathbf{q}$-expansion associated with Hilbert modular forms as defined above. Let $d_F \in \mathbb{A}_F^{\mathrm{fin},\times}$ be fixed such that $d_F \mathcal{O}_F = \mathcal{D}_F$ and let H_F denote the Hilbert class field of F. By the Principal Ideal Theorem, for each ideal $\mathfrak{a} \subset \mathcal{O}_F$, the ideal $\mathfrak{a}\mathcal{O}_{H_F}$ is principal. For each prime ideal $\mathfrak{r} \subset \mathcal{O}_F$, we may thus choose a generator $\{\mathfrak{r}\} \in \mathcal{O}_{H_F}$ of this ideal and we extend this definition multiplicatively to define $\{\mathfrak{a}\}$ for any ideal $\mathfrak{a} \subset \mathcal{O}_F$.

Let $\mathrm{Pic}(F, \mathfrak{m})^+$ denote the narrow ray class group of F with modulus $\mathfrak{m} \subset \mathcal{O}_F$. For each $\lambda \in \mathrm{Pic}(F, \mathfrak{m})^+$, we may fix $a_\lambda \in \mathbb{A}_F^{\mathrm{fin},\times}$ such that we have the decomposition

$$\mathbb{A}_F^\times = \bigsqcup_{\lambda \in \mathrm{Pic}(F,\mathfrak{m})^+} F^\times a_\lambda \det(V_{1,1}(\mathfrak{m})) F_\infty^{\times,+}.$$

By construction, we will have $[\mathfrak{a}_\lambda] := [a_\lambda \mathcal{O}_F] = \lambda \in \mathrm{Pic}(F)^+$. This induces a similar decomposition

$$G_F(\mathbb{A}) = \bigsqcup_{\lambda \in \mathrm{Pic}(F,\mathfrak{m})^+} G_F(\mathbb{Q}) t_\lambda V_{1,1}(\mathfrak{m}) G_F(\mathbb{R})^+, \text{ where } t_\lambda := \begin{pmatrix} a_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

We may decompose any $z \in \mathcal{H} \times \mathcal{H}$ as $z = x_\infty + \mathbf{i} y_\infty$ for uniquely determined $x_\infty \in \mathbb{R} \times \mathbb{R}$ and $y_\infty \in \mathbb{R}^+ \times \mathbb{R}^+$. Now let $f \in \mathcal{S}_{\mathbf{k},\mathbf{w}}(V_{1,1}(\mathfrak{m}))$ and define for any $\lambda \in \mathrm{Pic}(F, \mathfrak{m})^+$ the map

$$f_\lambda : \mathcal{H} \times \mathcal{H} \to \mathbb{C} : z \mapsto y_\infty^{-\mathbf{w}} f\left(t_\lambda \begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right).$$

Using the theory outlined in Section A.2, one shows that this function admits a Fourier expansion of the form

$$f_\lambda(z) = \sum_{\nu \in (\mathfrak{a}_\lambda \mathcal{D}_F^{-1})^+} a(\nu, f_\lambda) e^{2\pi i (\nu z_1 + \nu' z_2)}.$$

We are now ready to define the adèlic and p-adic $\mathbf{q}$-expansions of f. One may deduce from the decomposition above that any $y \in \mathbb{A}_F^{\mathrm{fin},\times} F_\infty^{\times,+}$

can be decomposed as $y = \nu a_\lambda^{-1} d_F u$ for some $\nu \in F^{\times,+}$ and $u \in \det(U(\mathfrak{m}))F_\infty^{\times,+}$. We then define the functions

$$a(-,f) : \mathbb{A}_F^{\mathrm{fin},\times} F_\infty^{\times,+} \to \mathbb{C} \quad \text{and} \quad a_p(-,f) : \mathbb{A}_F^{\mathrm{fin},\times} F_\infty^{\times,+} \to \overline{\mathbb{Q}}_p$$

to be zero outside of $\widehat{\mathcal{O}}_F F_\infty^{\times,+}$, and otherwise through the formulas

$$a(y,f) := a(\nu, f_\lambda)\{y^{\mathbf{w}-\mathbf{1}}\}\nu^{\mathbf{1}-\mathbf{w}}|a_i|_{\mathbb{A}_F}$$

and

$$a_p(y,f) := a(\nu, f_\lambda) y_p^{\mathbf{w}-\mathbf{1}} \nu^{\mathbf{1}-\mathbf{w}} a_{\lambda,p}^{\mathbf{1}} |a_\lambda^\infty|_{\mathbb{A}_F}.$$

If we now define the maps

$$e_F : \mathbb{C} \times \mathbb{C} : (z_1, z_2) \mapsto \exp(2\pi i(z_1 + z_2))$$

and

$$\chi_F : \mathbb{A}_F/F \to \mathbb{C}^\times : x \mapsto e_F(x_\infty),$$

the following is Theorem 1.1 in [Hid91].

Theorem A.4.3. *Each holomorphic Hilbert cuspform $f \in \mathcal{S}_{\mathbf{k},\mathbf{w}}(V_{1,1}(\mathfrak{m}))$ admits an adèlic $\mathbf{q}$-expansion of the form*

$$f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y|_{\mathbb{A}_F} \sum_{\nu \in F^+} a(\nu y d_F, f)\{(\nu y d_F)^{\mathbf{1}-\mathbf{w}}\}(\nu y_\infty)^{\mathbf{w}-\mathbf{1}} e_F(\mathbf{i}\nu y_\infty)\chi_F(\nu x),$$

where $y \in \mathbb{A}_F^{\mathrm{fin},\times} F_\infty^{\times,+}$ and $x \in \mathbb{A}_F^\times$.

We now define Hecke operators acting on these objects, following Section 2.1.2 of [For19]. These act on the space $\mathcal{S}_{\mathbf{k},\mathbf{w}}(K)$, where $V_{1,1}(\mathfrak{m}) \subset K \subset U_0(\mathfrak{m})$ for some ideal $\mathfrak{m} \subset \mathcal{O}_F$. Choose a prime $\mathfrak{p}$ above p in F. Similar to before, we may for each ideal $\mathfrak{a} \subset \mathcal{O}_F$ define an element $\{\mathfrak{a}\}_p \in \mathcal{O}_{F,\mathfrak{p}}$ such that $\mathfrak{a}\mathcal{O}_{F,\mathfrak{p}} = \{\mathfrak{a}\}_p \mathcal{O}_{F,\mathfrak{p}}$. We will assume that $\{\mathfrak{a}\}_p = 1$ whenever $\mathfrak{a}$ is prime to $p\mathcal{O}_F$. This notion can be extended to all adèles by considering the ideal that it generates.

Let $\mathfrak{l} \subset \mathcal{O}_F$ be a prime ideal not dividing $\mathfrak{m}$ and let $\varpi \in \mathcal{O}_{F,\mathfrak{l}} \subset \mathbb{A}_F^\times$ be a local uniformiser. Then we define for $f \in \mathcal{S}_{\mathbf{k},\mathbf{w}}(K)$,

$$T_\mathfrak{l} f(x) := \{\varpi\}_p^{\mathbf{w}-\mathbf{1}} \sum_{b \in B} f(xb),$$

where $B \subset G_F(\mathbb{A})$ is such that

$$V_{1,1}(\mathfrak{m}) \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} V_{1,1}(\mathfrak{m}) = \bigsqcup_{b \in B} V_{1,1}(\mathfrak{m}) b.$$

In addition, we define $\langle\mathfrak{l}\rangle f(x) = f(x\varpi)$. If $\mathfrak{l} \mid \mathfrak{m}$, we use the same definition, but now we will denote the resulting operators by U_{ϖ} and $\langle\varpi\rangle$ respectively, as they will depend on the precise choice of uniformiser ϖ at $\mathfrak{l}$. To capture this effect, for a local unit $a \in \prod_{\mathfrak{l}|\mathfrak{m}} \mathcal{O}_{F,\mathfrak{l}}^{\times}$, we define the operator $T(a,1)$ using the same definition above, but with ϖ replaced by a. Finally, we may extend the above definitions to define the operators $T_{\mathfrak{l}^n}$ and U_{ϖ^n} using the usual recursive relations.

To each element from $\mathbb{A}_F^{\times}$, we may now associate one or more Hecke operators as follows. Let $y \in \widehat{\mathcal{O}}_F \cap \mathbb{A}_F^{\times}$. Then we can write

$$y = a \prod_{\mathfrak{l}\subset\mathcal{O}_F} \varpi_{\mathfrak{l}}^{e(\mathfrak{l})} u, \quad \text{where} \quad a \in \prod_{\mathfrak{l}|\mathfrak{m}} \mathcal{O}_{F,\mathfrak{l}}^{\times} \quad \text{and} \quad u \in \det V_{1,1}(\mathfrak{m}).$$

If we let $\mathfrak{n} \subset \mathcal{O}_F$ denote the ideal locally generated by the $\varpi_{\mathfrak{l}}^{e(\mathfrak{l})}$, when we define the Hecke operator

$$T_y := T(a,1) T_{\mathfrak{n}} \prod_{\mathfrak{l}|\mathfrak{m}} U_{\varpi_{\mathfrak{l}}}^{e(\mathfrak{l})}.$$

This associates with each adèle $y \in \widehat{\mathcal{O}}_F \cap \mathbb{A}_F^{\times}$ uniquely a Hecke operator. The $\mathcal{O}_{F,\mathfrak{p}}$-subalgebra of $\mathrm{End}(\mathcal{S}_{\mathbf{k},\mathbf{w}}(K))$ generated by the operators described above is denoted $h_{\mathbf{k},\mathbf{w}}(K, \mathcal{O}_{F,\mathfrak{p}})$; the *Hecke algebra.* The following result can be shown to hold by construction.

Proposition A.4.4. *Let $f \in \mathcal{S}_{\mathbf{k},\mathbf{w}}(K)$ be a Hilbert modular eigenform, normalised such that $a(1, f) = 1$. Then the T_y-eigenvalue of f is given by the Fourier coefficient $a(y, f)$.*

A.5 p-adic modular forms and Hida families

In this thesis, we will not set up the theory of p-adic modular forms in great detail. For a very clear introduction to the topic and the notion of overconvergent modular forms, consult for example [Von21]. We will however recall a few concepts in the setting over $\mathbb{Q}$ that will illustrate our definitions in the setting from Section A.4 above.

Definition A.5.1. A p-adic modular form for $\mathrm{SL}_2(\mathbb{Z})$ is a power series $f(\mathbf{q}) \in \mathbb{Q}_p[\![\mathbf{q}]\!]$ with the property that there is a sequence $f_i \in \mathcal{M}_{k_i}(\mathrm{SL}_2(\mathbb{Z}))$ with rational Fourier coefficients such that

$$\lim_{i\to\infty} \mathrm{inf}_{n\in\mathbb{N}}\big(v_p(f(\mathbf{q}) - f_i(\mathbf{q}))\big) = \infty.$$

In other words, a p-adic modular form can be viewed as the p-adic limit of $\mathbf{q}$-expansions of classical modular forms. One can show that the weights k_i of the classical modular forms f_i must also converge to a limit p-adically, thus establishing a well-defined notion of the *weight* $k = \lim k_i$ of the p-adic modular form defined above. The reader should find it easy to generalise this notion to more general congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with level $p \nmid N$.

Classical modular forms of weight $2k$ for Γ are in bijection with the global sections of the kth tensor power of the sheaf of holomorphic differentials on the modular curve X_Γ through an elementary argument. For $\Gamma = \Gamma_1(N)$, the curve $X_1(N)(\mathbb{C}) = \Gamma_1(N)\backslash\mathcal{H}$ parametrizes elliptic curves together with an N-torsion point. Let A_p denote the *Hasse-invariant* of an elliptic curve. Then one may define for $0 \leq r \leq 1$ the subsets

$$X_r(\mathbb{C}_p) := \{x \in X_1(N)(\mathbb{C}_p) \mid v_p(A_p(x)) \leq r\}.$$

Then $X_1(\mathbb{C}_p)^{\mathrm{ord}} := X_0(\mathbb{C}_p)$ denotes the *ordinary locus* of $X_1(N)(\mathbb{C}_p)$, corresponding to all elliptic curves with ordinary reduction at p. It was shown by Katz that p-adic modular forms as defined above biject with the holomorphic differentials on $X_1(\mathbb{C}_p)^{\mathrm{ord}}$. Those modular forms that extend to holomorphic differentials on $X_r(\mathbb{C}_p)$ for some $r > 0$ are called (r-) *overconvergent*.

Definition A.5.2. An overconvergent modular form is called *ordinary* if it is a U_p-eigenvector with eigenvalue a p-adic unit. Recall that, if $f = \sum a_n \mathbf{q}^n$ is a(n overconvergent) modular cuspform, then

$$U_p f := \sum_{n=1}^{\infty} a_{pn} \mathbf{q}^n.$$

Definition A.5.3. The *ordinary projection* operator is defined as

$$e^{\mathrm{ord}} := \lim_{n\to\infty} U_p^{n!}.$$

One checks that the image of e^{ord} is indeed always an ordinary overconvergent modular form, and

$$a_m(e^{\mathrm{ord}} f) = \lim_{n\to\infty} a_{m\cdot p^{n!}}(f).$$

We finally aim to generalise these notions to Hida's adèlic setting over the real quadratic field F. We will follow Section 2.2 in [For19]. Recall the notation from Section A.4 and let $V_{1,1}(\mathfrak{m}) \subset K \subset U_0(\mathfrak{m})$ be a compact open subgroup. We mimic the idea of p-adic Hilbert modular forms being the limit of classical Hilbert modular forms, by first setting

$$\mathcal{S}_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}}) := \varinjlim_{n} S_{\mathbf{k},\mathbf{w}}(K(p^n), \mathcal{O}_{F,\mathfrak{p}}),$$

which is a module over the p-adic Hecke algebra

$$h_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}}) := \varprojlim_{n} h_{\mathbf{k},\mathbf{w}}(K(p^n), \mathcal{O}_{F,\mathfrak{p}}).$$

This algebra contains operators of the form

$$\mathbf{T}_y := \varprojlim_{n} = T_y\{y\}^{\mathbf{1}-\mathbf{w}} y_p^{\mathbf{w}-\mathbf{1}}.$$

As before, the supremum of the p-adic valuation of the $a_p(-, f)$ defines a p-adic norm on the space $\mathcal{S}_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}})$. The object of interest is now defined as its completion;

$$\overline{\mathcal{S}}_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}}),$$

much like Definition A.5.1.

Definition A.5.4. Let the *nearly ordinary* Hecke algebra be the largest subalgebra $h^{\text{n.o.}}_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}}) \subset h_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}})$ in which $\mathbf{T}_p$ is a p-adic unit. Define the (nearly) ordinary projection operator by $e^{\text{ord}} := \lim_{n\to\infty} \mathbf{T}_p^{n!}$ and we let the space of nearly ordinary p-adic cuspforms be

$$\overline{\mathcal{S}}^{\text{n.o.}}_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}}) := e^{\text{ord}}\overline{\mathcal{S}}_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}}).$$

We will need the following result later to prove a modularity theorem, which is Theorem 2.4 in [Hid89b].

Proposition A.5.5. *The nearly ordinary Hecke algebra defined above,* $h^{\text{n.o.}}_{\mathbf{k},\mathbf{w}}(K(p^\infty), \mathcal{O}_{F,\mathfrak{p}})$*, is finite and torsion-free over a ring* $\Lambda \cong \mathcal{O}_{F,\mathfrak{p}}[\![W_F]\!]$*, where there is an isomorphism* $W_F \cong \mathbb{Z}_p^3$*.*

In formulating the above result, we implicitly used that Leopoldt's defect vanishes for F, which is known as $F/\mathbb{Q}$ is abelian. This concludes our brief mathematical promenade that took us from the well-known theory of classical modular forms to the depths of Hida's adèlic description of Hilbert modular forms; a language that we use in Chapter 6.

Bibliography

[BC91] Jean-François Boutot and Henri Carayol. Uniformisation p-adique des courbes de Shimura: les théorèmes de Cerednik et de Drinfeld. *Astérisque*, (196-97):45–158, 1991.

[BC06] Joel Bellaïche and Gaetan Chenevier. Lissité de la courbe de Hecke de aux points Eisenstein critiques. *Journal of the Institute of Mathematics of Jussieu*, 5(2):333–349, 2006.

[BCG23] Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia. Elliptic units for complex cubic fields. *arXiv preprint arXiv:2311.04110*, 2023.

[BD16] Joël Bellaïche and Mladen Dimitrov. On the eigencurve at classical weight 1 points. *Duke Mathematical Journal*, 165(2):245 – 266, 2016.

[BDP22] Adel Betina, Mladen Dimitrov, and Alice Pozzi. On the failure of Gorensteinness at weight 1 Eisenstein points of the eigencurve. *American Journal of Mathematics*, 144(1):227–265, 2022.

[BDS20] Adel Betina, Mladen Dimitrov, and Sheng-Chi Shih. Eisenstein points on the Hilbert cuspidal eigenvariety. *preprint*, 2020.

[Bet20] Adel Betina. Ramification of the eigencurve at classical RM points. *Canadian Journal of Mathematics*, 72(1):57–88, 2020.

[Bru08] Jan Hendrik Bruinier. Hilbert modular forms and their applications. In *The 1-2-3 of modular forms*, pages 105–179. Springer, 2008.

[Čer76] Ivan V. Čerednik. Uniformization of algebraic curves by discrete arithmetic subgroups of with compact quotients. *Mathematics of the USSR-Sbornik*, 29(1):55, 1976.

[Cla03] Pete L. Clark. *Rational points on Atkin-Lehner quotients of Shimura curves.* Harvard University, 2003.

[Col96] Robert F. Coleman. Classical and overconvergent modular forms. *Inventiones mathematicae*, 124:215–241, 1996.

[CSS13] Gary Cornell, Joseph H. Silverman, and Glenn Stevens. *Modular forms and Fermat's last theorem.* Springer Science & Business Media, 2013.

[Daa23] Michael A. Daas. CM-values of p-adic Θ-functions. *arXiv preprint arXiv:2309.17251*, 2023.

[DDP11] Samit Dasgupta, Henri Darmon, and Robert Pollack. Hilbert modular forms and the Gross-Stark conjecture. *Annals of mathematics*, pages 439–484, 2011.

[Dei12] Anton Deitmar. *Automorphic forms.* Springer Science & Business Media, 2012.

[Del71] Pierre Deligne. Travaux de shimura. In *Séminaire Bourbaki vol. 1970/71 Exposés 382–399*, pages 123–165. Springer, 1971.

[DGL23] Henri Darmon, Lennart Gehrmann, and Michael Lipnowski. Rigid meromorphic cocycles for orthogonal groups. 2023.

[DK20] Samit Dasgupta and Mahesh Kakde. On constant terms of Eisenstein series. *arXiv preprint arXiv:2010.00650*, 2020.

[DK23a] Samit Dasgupta and Mahesh Kakde. Brumer-Stark Units and Explicit Class Field Theory. *Duke Mathematical Journal*, 2023.

[DK23b] Samit Dasgupta and Mahesh Kakde. On the Brumer-Stark conjecture. *Annals of Mathematics*, 197(1):289–388, 2023.

[DKV18] Samit Dasgupta, Mahesh Kakde, and Kevin Ventullo. On the Gross-Stark Conjecture. *Annals of Mathematics*, 188(3):833–870, 2018.

[DPV21] Henri Darmon, Alice Pozzi, and Jan Vonk. Diagonal restrictions of p-adic Eisenstein families. *Mathematische Annalen*, 379(1):503–548, 2021.

[DPV23] Henri Darmon, Alice Pozzi, and Jan Vonk. The values of the Dedekind-Rademacher cocycle at real multiplication points. *J. Eur. Math. Soc.*, 2023.

[Dri76] Vladimir G. Drinfeld. Coverings of p-adic symmetric domains. *Funkcional. Anal. i Priložen*, 10:29–40, 1976.

[DS05] Fred Diamond and Jerry Michael Shurman. *A first course in modular forms*, volume 228. Springer, 2005.

[DT08] Henri Darmon and Gonzalo Tornaría. Stark-Heegner points and the Shimura correspondence. *Compositio Mathematica*, 144(5):1155–1175, 2008.

[DV21] Henri Darmon and Jan Vonk. Singular moduli for real quadratic fields: a rigid analytic approach. *Duke Mathematical Journal*, 170(1):23–93, 2021.

[Elk98] Noam D. Elkies. Shimura curve computations. In *International Algorithmic Number Theory Symposium*, pages 1–47. Springer, 1998.

[Err11] Eric Errthum. Singular moduli of Shimura curves. *Canadian Journal of Mathematics*, 63(4):826–861, 2011.

[For19] Michele Fornea. On twisted triple products and the arithmetic of elliptic curves. 2019.

[GD22] Sofia Giampietro and Henri Darmon. A p-adic approach to singular moduli on Shimura curves. *Involve, a Journal of Mathematics*, 15(2):345–365, 2022.

[Geh20] Lennart Gehrmann. On quaternionic rigid meromorphic cocycles. *Mathematical Research Letters*, 2020.

[GKZ87] Benedict H. Gross, Winfried Kohnen, and Don B. Zagier. Heegner points and derivatives of L-series. II. *Mathematische Annalen*, 278:497–562, 1987.

[GMX21] Xavier Guitart, Marc Masdeu, and Xavier Xarles. A quaternionic construction of p-adic singular moduli. *Research in the Mathematical Sciences*, 8:1–20, 2021.

[Gro86] Benedict H. Gross. Local heights on curves. In *Arithmetic geometry*, pages 327–339. Springer, 1986.

[Gro87] Benedict H. Gross. Heights and the special values of L-series. In *Conference Proceedings of the CMS*, volume 7, 1987.

[GvdP06] Lothar Gerritzen and Marius van der Put. *Schottky groups and Mumford curves*, volume 817. Springer, 2006.

[GZ84] Benedict H. Gross and Don B. Zagier. On singular moduli. *Journal für die reine und angewandte Mathematik*, 355:191–220, 1984.

[GZ86] Benedict H. Gross and Don B. Zagier. Heegner points and derivatives of L-series. *Inventiones mathematicae*, 84:225–320, 1986.

[Hid89a] Haruzo Hida. Nearly ordinary hecke algebras and Galois representations of several variables. *Algebraic Analysis, Geometry and Number Theory*, pages 115–134, 1989.

[Hid89b] Haruzo Hida. On Nearly Ordinary Hecke Algebras for GL(2) over Totally Real Fields. In *Algebraic Number Theory—in honor of K. Iwasawa*, volume 17, pages 139–170. Mathematical Society of Japan, 1989.

[Hid91] Haruzo Hida. On p-adic L-functions of GL(2)×GL(2) over totally real fields. In *Annales de l'institut Fourier*, volume 41, pages 311–391, 1991.

[HY12] Benjamin J. Howard and Tonghai Yang. Singular moduli refined. *arXiv: Number Theory*, 2012.

[KRY04] Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. Derivatives of Eisenstein series and Faltings heights. *Compositio Mathematica*, 140(4):887–951, 2004.

[LN19] Matteo Longo and Marc-Hubert Nicole. The p-adic variation of the Gross-Kohnen-Zagier theorem. *Forum Mathematicum*, 31, 06 2019.

[Lu11] Wei Lu. Introduction to Local and Global Euler Characteristic Formulas. *arXiv preprint arXiv:1112.4373*, 2011.

[Maz89] Barry Mazur. Deforming Galois representations. In *Galois Groups over $\mathbb{Q}$: Proceedings of a Workshop Held March 23–27, 1987*, pages 385–437. Springer, 1989.

[Mor70] Yasuo Morita. Ihara's conjectures and moduli space of abelian varieties. *Master's Thesis, Tokyo*, 1970.

[MS15] Richard A. Moy and Joel Specter. There exist non-CM Hilbert modular forms of partial weight 1. *International Mathematics Research Notices*, 2015(24):13047–13061, 2015.

[Ogg83] Andrew P. Ogg. Real points on shimura curves. *Arithmetic and Geometry: Papers Dedicated to I.R. Shafarevich on the Occasion of His Sixtieth Birthday Volume I Arithmetic*, pages 277–307, 1983.

[Pad23] Oana Padurariu. Shimura curves admitting a smooth plane model. *arXiv preprint arXiv:2310.11767*, 2023.

[Phi15] Andrew Phillips. *Moduli of CM false elliptic curves.* PhD thesis, Boston College, 2015.

[Poz19] Alice Pozzi. *The eigencurve at weight one Eisenstein points.* PhD thesis, McGill University, 2019.

[Ric21] James Rickards. *Intersections of closed geodesics on Shimura curves.* PhD thesis, McGill University, 2021.

[Rom13] Matthieu Romagny. Models of curves. In *Arithmetic and geometry around Galois theory*, pages 149–170. Springer, 2013.

[Sch09] Jarad Schofer. *Borcherds forms and generalizations of singular moduli.* Walter de Gruyter GmbH & Co. KG, 2009.

[Shi67] Goro Shimura. Construction of class fields and zeta functions of algebraic curves. *Annals of Mathematics*, pages 58–159, 1967.

[Sil94] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151. Springer Science & Business Media, 1994.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106. Springer Science & Business Media, 2009.

[vdP92] Marius van der Put. Discrete groups, Mumford curves and theta functions. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 1, pages 399–438, 1992.

[Vis89] Angelo Vistoli. Intersection theory on algebraic stacks and on their moduli spaces. *Inventiones mathematicae*, 97(3):613–670, 1989.

[Voi09] John Voight. Shimura curve computations. *Arithmetic geometry*, 8:103–113, 2009.

[Voi21] John Voight. *Quaternion algebras*. Springer Nature, 2021.

[Von21] Jan Vonk. Overconvergent modular forms and their explicit arithmetic. *Bulletin of the American Mathematical Society*, 58(3):313–356, 2021.

[Wer96] Annette Werner. Local heights on Mumford curves. *Mathematische Annalen*, 306:819–831, 1996.

[Wil70] Kenneth S. Williams. Integers of biquadratic fields. *Canadian Mathematical Bulletin*, 13(4):519–526, 1970.

[YZZ13] Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang. *The Gross-Zagier Formula on Shimura Curves:(AMS-184)*, volume 184. Princeton University Press, 2013.

[Zha01] Shouwu Zhang. Heights of Heegner points on Shimura curves. *Annals of mathematics*, 153(1):27–147, 2001.