CM-values of p-adic Theta-functions

Daas, M.A.

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CHAPTER 6

The p-adic analytic proof

The aim of this final chapter is to carry out **Step 3** of the proof of Theorem B as outlined in Section 2.4 by following in rough lines the following steps. First, using the results from Chapter 4, we rewrite the defining formula for the cross ratio of CM-values of the Θ-function into a more convenient form. Next, we will consider one particular infinitesimal deformation of the rigidified Galois representation ρ_η defined in Section 5.2 and supported by the main result from Chapter 5, we compute the morphism from the Hecke algebra that corresponds to it. From this, we determine the Fourier coefficients of the infinitesimal cuspidal family of p-adic modular forms centered around $E_{1,\chi}^{(p)}$ that corresponds to it. Finally, we take its diagonal restriction, take its derivative with respect to the weight parameter and apply the ordinary projection operator. We argue that the first Fourier coefficient of the result of this computation must vanish, and comparing its explicit description to 0 then will yield Theorem B. This approach mimics the strategies used in [DPV21, DPV23]. For notational convenience, thoughout this chapter, we will for $\nu \in F$ denote $\nu' := \sigma(\nu) \in F$; its Galois conjugate.

6.1 Rewriting the Θ-series

Recall that our main object of interest was defined as

$$\Theta(D_1, D_2) := \prod_{[c_1],[c_2]} \frac{\Theta\big([c_1]\cdot\tau_1, [c_1]\cdot\tau_1'; [c_2]\cdot\tau_2\big)}{\Theta\big([c_1]\cdot\tau_1, [c_1]\cdot\tau_1'; [c_2]\cdot\tau_2'\big)}$$

where by $[c_i]\cdot\tau_i$ for $[c_i] \in \mathrm{Pic}(K_i)$ we mean the fixed point in $\mathcal{H}_p$ for the embedding $[c_i]\cdot\alpha_i$. We remark here that the point $[c_i]\cdot\tau_i$ is only well defined up to multiplication by $R_q^\times$, but this does not change the value of the Θ-functions we are computing, since

$$\frac{\Theta(\tau_1, \tau_1'; \tau_2)}{\Theta(\tau_1, \tau_1'; \tau_2')} = \prod_{b \in R_q[1/p]_1^\times} [\tau_1, \tau_1', b\tau_2, b\tau_2'],$$

where we defined the cross-ratio as

$$[\tau_1, \tau_1', b\tau_2, b\tau_2'] = \frac{(\tau_1 - b\tau_2)(\tau_1' - b\tau_2')}{(\tau_1 - b\tau_2')(\tau_1' - b\tau_2)}.$$

Let $\overline{\alpha_1}$ denote the embedding α_1 precomposed with the non-trivial automorphism of K_i. Recall Section 4.4, in which we associated to every pair of embeddings $\mathcal{O}_i \to R_q$ for $i \in \{1, 2\}$ an F-quadratic form refining the quaternion norm on B_q. If $\det_F$ denotes that form for the pair (α_1, α_2), let $\det'_F$ denote that form for the pair $(\overline{\alpha_1}, \alpha_2)$. These two forms relate to each other through the following lemma.

Lemma 6.1.1. *The functions* $\det_F$ *and* $\det'_F$ *are* $\mathrm{Gal}(F/\mathbb{Q})$*-conjugates.*

Proof. This is immediate from Lemma 4.4.4. Indeed, there it was shown that the bilinear form associated with $\det_F$ was given by

$$B_q \times B_q \to F : (\gamma_1, \gamma_2) \mapsto \frac{\mathrm{tr}(\gamma_1 \overline{\gamma_2})}{2} + \frac{\mathrm{tr}(A\gamma_1 B \overline{\gamma_2})}{2\sqrt{D}},$$

where $A = \alpha_1(\sqrt{D_1})$ and $B = \alpha_2(\sqrt{D_2})$. The effect of changing to $\overline{\alpha_1}$ is to replace A by $-A$, which clearly yields the Galois conjugate bilinear form in the formula above. □

Lemma 6.1.2. *For any* $b \in B_q$*, we have*

$$[\tau_1, \tau_1', b\tau_2, b\tau_2'] = -\frac{\det_F(b)}{\det'_F(b)}.$$

Proof. Recall from Theorem 4.4.9 that

$$\det_F(b) = \mathrm{Nm}(b) \frac{(\tau_1 - b\tau_2)(\tau_1' - b\tau_2')}{(\tau_1 - \tau_1')(b\tau_2 - b\tau_2')}.$$

Similarly, using Lemma 6.1.1 and the final claim of Proposition 4.4.8, to obtain the value of $\det'_F(b)$ we swap τ_1 and τ_1' to find

$$\det'_F(b) = \mathrm{Nm}(b) \frac{(\tau_1' - b\tau_2)(\tau_1 - b\tau_2')}{(\tau_1' - \tau_1)(b\tau_2 - b\tau_2')},$$

Since the denominators agree up to a sign, the result follows from dividing these two expressions. □

The following lemma will be used implicitly and without mention throughout the remainder of this section.

Lemma 6.1.3. *The maximal order* R_q *is stable under its involution.*

Proof. Let $b \in R_q$ be arbitrary. Then $\mathbb{Z}[b] \subset R_q$ must be a finitely generated $\mathbb{Z}$-module since R_q is. But $\mathbb{Z}[b]$ is a subring of a quadratic number field, and it is well known that such rings are finitely generated if and only if b is integral over $\mathbb{Z}$. In particular, its minimal polynomial in $\mathbb{Z}[X]$ must be monic. This minimal polynomial is given by $X^2 - \mathrm{tr}(b)X + \mathrm{Nm}(b) = 0$. It follows that in particular, $\mathrm{tr}(b)$ must be an integer and since $1 \in R_q$, it follows that also $\overline{b} = \mathrm{tr}(b) - b \in R_q$. □

We are now ready for the main result of this section.

Proposition 6.1.4. *The following equality holds true:*

$$\frac{2}{w_1 w_2}\log_p \Theta(D_1,D_2) = \lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{2n}}} \log_p\left(\frac{\nu}{\nu'}\right)\cdot\rho(\nu\mathfrak{q}_1^{-1}\mathcal{D}_F).$$

Proof. We apply Lemma 6.1.2 to the expression defining $\Theta(D_1,D_2)$, ignoring the sign by pairing each quaternion with its negative. We obtain

$$\Theta(D_1,D_2) = \prod_{[c_1],[c_2]}\prod_{b\in R_q[1/p]_1^\times}\frac{\det_F[c_1,c_2](b)}{\det'_F[c_1,c_2](b)},$$

where the first product is taken over all $[c_i]\in\mathrm{Pic}(K_i)$ for $i\in\{1,2\}$. Now let $b\in R_q[1/p]_1^\times$. In other words, $\mathrm{Nm}(b)=1$ and there exists some *minimal* $k\geq 0$ such that $B := p^k b\in R_q$. This association induces a bijection

$$R_q[1/p]_1^\times \to \bigsqcup_{k=0}^{\infty}\left\{B\in R_q \mid p\nmid B,\ \mathrm{Nm}(B)=p^{2k}\right\}.$$

We define for any $n\geq 0$ the set

$$R_q(n) := \left\{B\in R_q \mid\ \mathrm{Nm}(B)=p^{2n}\right\}.$$

Now we observe the association $B\mapsto p^{n-k}B$ induces a bijection

$$\bigsqcup_{k=0}^{n}\left\{B\in R_q \mid p\nmid B,\ \mathrm{Nm}(B)=p^{2k}\right\} \xrightarrow{\sim} R_q(n).$$

By $\mathbb{Z}$-linearity, we find that

$$\frac{\det_F[c_1,c_2](p^{n-k}b)}{\det'_F[c_1,c_2](p^{n-k}b)} = \frac{\det_F[c_1,c_2](b)}{\det'_F[c_1,c_2](b)}.$$

As such, we may use the above bijections to write

$$\Theta(D_1,D_2) = \lim_{n\to\infty}\prod_{[c_1],[c_2]}\prod_{b\in R_q(n)}\frac{\det_F[c_1,c_2](b)}{\det'_F[c_1,c_2](b)}.$$

We now change the way in which we view these finite products over which we take the limit. Instead of ranging over all $b\in R_q(n)$ and recording its associated $\det_F[c_1, c_2]$-value, we will range over each possible $\det_F[c_1, c_2]$-value and record how often it is reached by some $b \in R_q(n)$. Recalling that $b \in R_q(n)$ means that $\mathrm{Tr}(\det[c_1, c_2]_F(b)) = \mathrm{Nm}(b) = p^{2n}$ and that the values of $\det_F[c_1, c_2]$ are always totally positive, we may rewrite the above to

$$\Theta(D_1, D_2) = \lim_{n\to\infty} \prod_{\substack{\nu \gg 0,\\ \mathrm{tr}(\nu)=p^{2n}}} \left(\frac{\nu}{\nu'}\right)^{\#\{(b,[c_1],[c_2])\,|\,\det_F[c_1,c_2](b)=\nu\}},$$

where $(b, [c_1], [c_2]) \in R_q(n)\times\mathrm{Pic}(K_1)\times\mathrm{Pic}(K_2)$. Now we invoke Theorem 4.0.2, using that $\#\left(\mathcal{O}_1^\times\mathcal{O}_2^\times\right) = w_1w_2/2$, to find that

$$\#\left\{(b, [c_1], [c_2]) \mid \det_F[c_1, c_2](b) = \nu\right\} = \frac{w_1w_2}{2}\rho(\nu\mathfrak{q}_1^{-1}\mathcal{D}_F),$$

where $\mathfrak{q}_1$ is the reflex prime induced by the two embeddings α_1 and α_2 and ρ counts the number of integral ideals of L with given norm in F. Applying the p-adic logarithm now yields the result. $\square$

We continue this section by examining the closely associated quantity $\Theta_p(D_1, D_2)$. Recall that $\pi \in R_q$ denoted a quaternion with $\mathrm{Nm}(\pi) = p$. Then we have the following lemma.

Lemma 6.1.5. *Right multiplication by π induces a bijection between*

$$R_q[1/p]_1^\times := \left\{b \in R_q[1/p]^\times \mid \mathrm{Nm}(b) = 1\right\}$$

and the set

$$R_q[1/p]_p^\times := \left\{b \in R_q[1/p]^\times \mid \mathrm{Nm}(b) = p\right\}.$$

Proof. Is it clear that this map is well-defined and injective. For surjectivity, we remark that if $b \in R_q[1/p]^\times$ satisfies $\mathrm{Nm}(b) = p$, then

$$(b\overline{\pi}/p)\cdot\pi = b(\overline{\pi}\pi)/p = b,$$

with $b\overline{\pi}/p \in R_q[1/p]_1^\times$, completing the proof. $\square$

Proposition 6.1.6. *The following equality holds true:*

$$\log_p \Theta_p(D_1, D_2) = \frac{w_1w_2}{2} \lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+\\ \mathrm{tr}(\nu)=p^{2n+1}}} \log_p\left(\frac{\nu}{\nu'}\right)\cdot\rho(\nu\mathfrak{q}_1^{-1}\mathcal{D}_F),$$

Proof. Using Lemma 6.1.5 above, we rewrite

$$\begin{aligned}\frac{\Theta(\tau_1,\tau_1';\pi\tau_2)}{\Theta(\tau_1,\tau_1';\pi\tau_2')} &= \prod_{b\in R_q[1/p]_1^\times} \frac{\big(\tau_1 - b(\pi\tau_2)\big)\big(\tau_1' - b(\pi\tau_2')\big)}{\big(\tau_1 - b(\pi\tau_2')\big)\big(\tau_1' - b(\pi\tau_2)\big)}\\ &= \prod_{b\in R_q[1/p]_1^\times} \frac{\big(\tau_1 - (b\pi)\tau_2\big)\big(\tau_1' - (b\pi)\tau_2'\big)}{\big(\tau_1 - (b\pi)\tau_2'\big)\big(\tau_1' - (b\pi)\tau_2\big)}\\ &= \prod_{b\in R_q[1/p]_p^\times} \frac{(\tau_1 - b\tau_2)(\tau_1' - b\tau_2')}{(\tau_1 - b\tau_2')(\tau_1' - b\tau_2)}.\end{aligned}$$

One may now repeat the proof of Proposition 6.1.4 above verbatim to arrive at the desired conclusion. ☐

We conclude this section by proving that the formula from Theorem B correctly counts the factors of p on both sides of the equation. This is necessary to do separately, as our application of $\log_p$ forces us to forfeit any information regarding the number of factors of p on both sides, by virtue of $\log_p(p) = 0$. Therefore, we are forced to count these factors by hand first. We need a small elementary lemma.

Lemma 6.1.7. *Let $\nu \in \mathcal{D}_F^{-1}$ be such that $\mathrm{tr}(\nu) = p^n$ for some positive integer n and suppose further that $v_{\mathfrak{p}_1}(\nu) \neq v_{\mathfrak{p}_2}(\nu)$ or that $v_p(\mathrm{Nm}(\nu))$ is odd. Then $p^n \mid \nu$.*

Proof. Let us write

$$\nu\sqrt{D} = \frac{x + p^n\sqrt{D}}{2}, \quad \text{so that} \quad \mathrm{Nm}(\nu\sqrt{D}) = \frac{x^2 - p^{2n}D}{4}.$$

Write $x = p^k y$ where $p \nmid y$. If $k \geq n$, we are done. If not, we find that

$$\nu\sqrt{D} = p^k\frac{y + p^{n-k}\sqrt{D}}{2} \quad \text{and} \quad \mathrm{Nm}(\nu\sqrt{D}) = p^{2k}\frac{y^2 - p^{2n-2k}D}{4}.$$

If $v_{\mathfrak{p}_1}(\nu) \neq v_{\mathfrak{p}_2}(\nu)$, then the fraction $(y + p^{n-k}\sqrt{D})/2$ must still contain a factor of $\mathfrak{p}_i$ for some $i \in \{1, 2\}$. As such, its norm must still be divisible by p. If $v_p(\mathrm{Nm}(\nu))$ is odd, the same conclusion follows. But this means that $p \mid y$; this is a contradiction. ☐

Proposition 6.1.8. *It holds that*

$$\frac{\pm 2}{w_1w_2} v_p\left(\frac{\Theta(D_1,D_2)}{\Theta_p(D_1,D_2)}\right) = \sum_{\substack{x^2<D\\ x^2\equiv D \bmod 4N}} \delta(x) v_p\left(F\left(\frac{D-x^2}{4N}\right)\right).$$

Proof. The proof of Theorem 6.1.4 showed that, before applying $\log_p$,

$$\Theta(D_1, D_2)^{\frac{\pm 2}{w_1 w_2}} = \lim_{n\to\infty} \prod_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \operatorname{tr}(\nu) = p^{2n}}} \left(\frac{\nu}{\nu'}\right)^{\rho(\nu \mathfrak{q}_1^{-1}\mathcal{D}_F)}$$

and similarly for $\Theta_p(D_1, D_2)$ as a result of Proposition 6.1.6. We claim that the $\mathfrak{p}_1$-adic valuation of each term in the limit is constant. Indeed, only terms with $v_{\mathfrak{p}_1}(\nu) \neq v_{\mathfrak{p}_1}(\nu') = v_{\mathfrak{p}_2}(\nu)$ can contribute to the $\mathfrak{p}_1$-adic valuation of this expression. By Lemma 6.1.7, this means that only those ν lifted from trace 1 can contribute. This contribution is independent of n because $\rho(p^{2n}\nu\mathfrak{q}_1^{-1}\mathcal{D}_F) = \rho(\nu\mathfrak{q}_1^{-1}\mathcal{D}_F)$. We then conclude that

$$\begin{aligned}\frac{\pm 2}{w_1 w_2} v_{\mathfrak{p}_1}\left(\Theta(D_1, D_2)\right) &= \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \operatorname{tr}(\nu) = 1}} \rho(\nu\mathfrak{q}_1^{-1}\mathcal{D}_F)\big(v_{\mathfrak{p}_1}(\nu) - v_{\mathfrak{p}_1}(\nu')\big) \\ &= \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{p}_1\mathfrak{q}_1)^+ \\ \operatorname{tr}(\nu) = 1}} \rho(\nu\mathfrak{q}_1^{-1}\mathcal{D}_F) v_{\mathfrak{p}_1}(\nu) - \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{p}_1\mathfrak{q}_2)^+ \\ \operatorname{tr}(\nu) = 1}} \rho(\nu\mathfrak{q}_2^{-1}\mathcal{D}_F) v_{\mathfrak{p}_1}(\nu).\end{aligned}$$

Note that for all contributing terms, we must have that $v_{\mathfrak{p}_1}(\nu)$ is even; indeed, otherwise $\mathfrak{p}_1$ is a special prime of $\nu\mathfrak{q}_1^{-1}\mathcal{D}_F$ and as such, its value under ρ would vanish.

On the other hand, we remark that either $v_{\mathfrak{p}_1}(\nu) = 0$ or $v_{\mathfrak{p}_2}(\nu) = 0$ for any $\nu \in \mathcal{D}_F^{-1}$ of trace 1. As such, it follows that $\rho(p^{2n+1}\nu\mathfrak{q}_1^{-1}\mathcal{D}_F) = 0$ for any $\nu \in \mathcal{D}_F^{-1}$ of trace 1; indeed, at least one of the $\mathfrak{p}_i$ for $i \in \{1, 2\}$ must be a special prime. Following the argument above, we conclude that $v_{\mathfrak{p}_1}(\Theta_p(D_1, D_2)) = 0$. For the $\mathfrak{p}_2$-adic valuation, an analogous argument applies.

By definition of the F-function, its value is only a power of p if the prime p divides the quantity $\operatorname{Nm}(\nu\sqrt{D})/N = (D - x^2)/4N$ an odd number of times. In other words, the corresponding element $\nu \in \mathcal{D}_F^{-1}$ of trace 1 must contain an even number of factors of one of the $\mathfrak{p}_i$ for $i \in \{1, 2\}$. These are the same $\nu \in \mathcal{D}_F^{-1}$ of trace 1 that we found earlier to contribute to the valuation of $\Theta(D_1, D_2)$. The agreement between the exponent in the definition of the F-value and the function ρ has been shown before in the proof of Theorem 3.2.5; the alternating sum matches up all terms and equality has been established. □

6.2 Extracting a_ν from $\tilde{\rho}$

In this section we will compute the p-adic modular form that is associated with a particular nearly ordinary deformation as considered in Chapter 5, of which we have proved its modularity in Theorem 5.0.1. We record its existence in the following lemma.

Lemma 6.2.1. *There exists a deformation $\widetilde{\rho}: G_F \to \mathbb{Q}_p[\epsilon]$ of the form*

$$\widetilde{\rho} = \left(1 + \epsilon \begin{pmatrix} \phi_p & b \\ 0 & -\phi_p \end{pmatrix}\right) \begin{pmatrix} 1 & \chi\eta \\ 0 & \chi \end{pmatrix},$$

where $\phi_p \in \mathrm{Hom}(G_F, \mathbb{Q}_p)$ denotes the p-adic cyclotomic character from in Proposition 5.1.5 and where $b : G_F \to \mathbb{Q}_p$ satisfies $b|_{G_{\mathfrak{p}_2}} = 0$.

Proof. First note that the vanishing of the lower-left entry forces both diagonal entries to be elements from $\mathrm{Hom}(G_F, \mathbb{Q}_p) = \langle \phi_p \rangle$ as a result of Lemma 5.3.3. We may choose them freely, as the proof of Proposition 5.3.5 shows that the map

$$H^1(G_F, W_1) \to H^1(G_F, \mathbb{Q}_p \oplus \mathbb{Q}_p)$$

is surjective. Finally, one verifies that we may always adjust the top-right entry by an element from $H^1(G_F, \mathbb{Q}_p(\chi))$. Since its restriction to $G_{\mathfrak{p}_2}$ is always in $H^1(G_{\mathfrak{p}_2}, \mathbb{Q}_p(\chi))$ as η vanishes there as a result of Lemma 5.2.1, by the isomorphism from Corollary 5.1.6 we can in particular make this restriction vanish. This completes the proof. ☐

By construction, the lines $\langle e_1 \rangle$ and $\langle e_2 \rangle$ are now fixed by $G_{\mathfrak{p}_1}$ and $G_{\mathfrak{p}_2}$ respectively under the action of $\widetilde{\rho}$. It follows that the quotient characters δ_i from Theorem 5.4.1 for this deformation are given by

$$\delta_1 = \chi - \chi\phi_p\epsilon \quad \text{and} \quad \delta_2 = 1 + \phi_p\epsilon.$$

Since $\det(\tilde{\rho})$ is constant, after identifying $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \cong \mathcal{O}_{F_{\mathfrak{p}_1}}^\times \times \mathcal{O}_{F_{\mathfrak{p}_2}}^\times \cong I_{\mathfrak{p}_1} \times I_{\mathfrak{p}_2}$, it is a general fact that the weight character of the modular form associated with this Galois representation is given by $\delta_1 \times \delta_2$. In particular, the weight character for the diagonal restriction is the map

$$(\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \xrightarrow{\Delta} (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \cong \mathcal{O}_{F_{\mathfrak{p}_1}}^\times \times \mathcal{O}_{F_{\mathfrak{p}_2}}^\times \to \mathbb{Q}_p[\epsilon],$$

where Δ here denotes the diagonal embedding. Using that χ is trivial on the inertia subgroups $I_{\mathfrak{p}_i} \subset G_{\mathfrak{p}_i}$ for $i \in \{1, 2\}$, as the extension L/F is unramified, we trace through the maps to find

$$x \mapsto (x, x) \mapsto \delta_1(x)\delta_2(x) = \left(1 - \log_p(x)\epsilon\right)\left(1 + \log_p(x)\epsilon\right) = 1.$$

In particular, the diagonal restriction is of *constant* weight. This shows that the above deformation describes an infinitesimal family of modular forms in the so-called *anti-parallel* weight direction. We will use this in the proof of Proposition 6.3.1 to allow us to conclude that the derivative with respect to the weight parameter of our family is in fact a modular object, even though the specialisation of this family does *not* vanish.

By its universal property, the deformation $\widetilde{\rho}$ comes from some map $R^{\mathrm{no}}_{\rho_\eta} \to \mathbb{Q}_p[\epsilon]$. By Theorem 5.0.1, the isomorphism yields a unique associated map $\varphi : \mathbb{T} \to \mathbb{Q}_p[\epsilon]$. The majority of this section will be dedicated to computing this morphism.

Recall that $\mathbb{T}$ is generated by the operators $T_{\mathfrak{l}}$ and $\langle \mathfrak{l} \rangle$ for all primes $\mathfrak{l}$ of F coprime to p, and the operators U_{π_1} and U_{π_2} where $\pi_1, \pi_2 \in \mathbb{A}_F^\times$ are local uniformisers at $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively.

Theorem 5.4.1 dictates how we should deduce the images of the various Hecke operators from the associated representation. Indeed, it shows that $\varphi(T_{\mathfrak{l}})$ should correspond to the trace of $\widetilde{\rho}(\mathrm{Frob}_{\mathfrak{l}})$, whereas $\langle \mathfrak{l} \rangle \mathrm{Nm}(\mathfrak{l})$ should correspond to its determinant. Finally, the images of the operators U_{π_i} are deduced from the values $\delta_i(\pi_i)$ for $i \in \{1, 2\}$. We will now determine these quantities in this order.

Lemma 6.2.2. *For any $\tau \in G_F$ it holds that*

$$\mathrm{Tr}(\tilde{\rho}(\tau)) = 1 + \chi(\tau) + (1 - \chi(\tau))\phi_p(\tau)\epsilon.$$

Proof. By writing out the definition of $\widetilde{\rho}(\tau)$, we find that

$$\mathrm{Tr}(\tilde{\rho}(\tau)) = 1 + \phi_p(\tau)\epsilon + \chi(\tau)(1 - \phi_p(\tau)\epsilon).$$

Rewriting then yields the result. □

Lemma 6.2.3. *Let $\varphi : \mathbb{T} \to \mathbb{Q}_p[\epsilon]$ be the morphism induced by the deformation $\tilde{\rho}$ and let $\mathfrak{l} \nmid p$ be a prime ideal of F. Then*

$$\varphi(T_{\mathfrak{l}}) = \begin{cases} 2 & \text{if } \chi(\mathfrak{l}) = 1; \\ 2\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon & \text{if } \chi(\mathfrak{l}) = -1. \end{cases}$$

Proof. We split cases, using that $\varphi(T_{\mathfrak{l}}) = \mathrm{Tr}(\tilde{\rho}(\mathrm{Frob}_{\mathfrak{l}}))$ as long as $\mathfrak{l} \nmid p$. If $\chi(\mathrm{Frob}_{\mathfrak{l}}) = 1$, then Lemma 6.2.2 yields the claim.

If $\chi(\mathrm{Frob}_{\mathfrak{l}}) = -1$, however, Lemma 6.2.2 then yields that

$$\mathrm{Tr}(\tilde{\rho}(\mathrm{Frob}_{\mathfrak{l}})) = 2\phi_p(\mathrm{Frob}_{\mathfrak{l}})\epsilon.$$

We complete the proof by remarking that by definition, $\mathrm{Frob}_{\mathfrak{l}}$ will raise the topological generator ζ_{p^∞} of the extension $F(\zeta_{p^\infty})/F$ to the power $\mathrm{Nm}(\mathfrak{l})$, so the result follows by tracing through the definition of ϕ_p. □

We continue with the images of the diamond operators.

Lemma 6.2.4. *For any prime ideal* $\mathfrak{l} \nmid p$ *of* F*, it holds that*

$$\varphi(\langle \mathfrak{l} \rangle \mathrm{Nm}(\mathfrak{l})) = \chi(\mathrm{Frob}_{\mathfrak{l}}).$$

Proof. Recall that the image of $\mathrm{Frob}_{\mathfrak{l}}$ has determinant $\varphi(\langle \mathfrak{l} \rangle \mathrm{Nm}(\mathfrak{l}))$. In our case, the determinant is kept constant, so this is $\chi(\mathrm{Frob}_{\mathfrak{l}})$. □

Finally, we determine the images of the operators U_{π_i} for π_i a uniformiser of $\mathcal{O}_{F_{\mathfrak{p}_i}}$ for $i \in \{1, 2\}$.

Lemma 6.2.5. *Let* π_i *for* $i \in \{1, 2\}$ *be a uniformiser of* $\mathcal{O}_{F_{\mathfrak{p}_i}}$*. Then*

$$\varphi(U_{\pi_1}) = -1 + \log_p(\pi_1)\epsilon \quad \textit{and} \quad \varphi(U_{\pi_2}) = 1 + \log_p(\pi_2)\epsilon.$$

Proof. Since the images of the operators U_{π_i} for $i \in \{1, 2\}$ agree with the images of the elements π_i under the map δ_i, suppressing the local reciprocity map, this follows from combining the earlier found expressions

$$\delta_1 = \chi - \chi\phi_p\epsilon \quad \text{and} \quad \delta_2 = 1 + \phi_p\epsilon$$

with the definition of the map ϕ_p, with the p-adic logarithm extended to all of $\mathbb{Q}_p^\times$ through the Iwasawa branch, as $\chi(\pi_i) = -1$ because the primes $\mathfrak{p}_i$ for $i \in \{1, 2\}$ are inert in the extension L/F. □

Proposition 6.2.6. *Let* $\varphi : \mathbb{T} \to \mathbb{Q}_p[\epsilon]$ *be the morphism induced by the deformation* $\tilde{\rho}$*, let* $\mathfrak{l} \nmid p$ *be a prime ideal of* F *and* $n \geq 0$ *an integer. Then*

$$\varphi(T_{\mathfrak{l}^n}) = \begin{cases} n+1 & \textit{if } \chi(\mathfrak{l}) = 1; \\ (n+1)\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon & \textit{if } \chi(\mathfrak{l}) = -1 \textit{ and } n \textit{ is odd;} \\ 1 & \textit{if } \chi(\mathfrak{l}) = -1 \textit{ and } n \textit{ is even.} \end{cases}$$

Further, for π_i *for* $i \in \{1, 2\}$ *a uniformiser of* $\mathcal{O}_{F_{\mathfrak{p}_i}}$*, it holds that*

$$\begin{aligned} \varphi(U_{\pi_1^n}) &= (-1)^n(1 - n\log_p(\pi_1)\epsilon); \\ \varphi(U_{\pi_2^n}) &= 1 + n\log_p(\pi_2)\epsilon. \end{aligned}$$

Proof. We remind the reader of the essential recursion relation

$$T_{\mathfrak{l}^{n+1}} = T_{\mathfrak{l}^n}T_{\mathfrak{l}} - \langle \mathfrak{l} \rangle \mathrm{Nm}(\mathfrak{l})T_{\mathfrak{l}^{n-1}}$$

for $\mathfrak{l} \nmid p$, whereas $U_{\pi^n} = U_\pi^n$ for the adèles above p.

- If $\chi(\mathrm{Frob}_\mathfrak{l}) = 1$, then $\mathrm{Tr}(\tilde{\rho}(\mathrm{Frob}_\mathfrak{l})) = 2$ by Lemma 6.2.3. Further using Lemma 6.2.4, we obtain the recursion
$$T(n+1) = 2T(n) - T(n-1) \quad \text{with} \quad T(0) = 1, \quad T(1) = 2.$$
This is easily solved and yields $T(n) = n+1$ for all $n \geq 0$.

- If on the other hand $\chi(\mathfrak{l}) = -1$, then $\mathrm{Tr}(\tilde{\rho}(\mathrm{Frob}_\mathfrak{l})) = 2\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon$ by Lemma 6.2.3. Again by Lemma 6.2.4, we obtain the recursion
$$L(n+1) = 2\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon \cdot L(n) + L(n-1)$$
with $L(0) = 1$ and $L(1) = 2\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon$. One deduces
$$L(2n) = 1 \quad \text{and} \quad L(2n-1) = 2n\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon \quad \text{for all } n \geq 1.$$

- In the final case, we raise the expressions from Lemma 6.2.5 to the appropriate exponent to find
$$\varphi(U_{\pi_1^n}) = (-1 + \log_p(\pi_1)\epsilon)^n = (-1)^n(1 - n\log_p(\pi_1)\epsilon).$$

Similarly, we obtain $\varphi(U_{\pi_2^n})$, completing the proof. □

Remark 6.2.7. By considering the quotient of two uniformisers at a given place, it is not difficult to convince oneself that for any idèles α_1, α_2 supported only at $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively, it holds that
$$\varphi(U_{\alpha_1}) = (-1)^{v_{\mathfrak{p}_1}(\alpha_1)}(1 - \log_p(\alpha_1)\epsilon) \quad \text{and} \quad \varphi(U_{\alpha_2}) = 1 + \log_p(\alpha_2)\epsilon.$$
This extends the formulae for π_1^n and π_2^n above.

Corollary 6.2.8. *Let $\mathfrak{l} \nmid p$ be a prime ideal of F and let $n \in \mathbb{N}$. Recalling $\rho(I) := \#\{J \subset \mathcal{O}_L \mid \mathrm{Nm}_F^L(J) = I\}$ for $I \subset \mathcal{O}_F$, we have that*
$$\varphi(T_{\mathfrak{l}^n}) = \rho(\mathfrak{l}^n) + \frac{1}{2}(n+1)\big(1 - \chi(\mathfrak{l}^n)\big)\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon.$$

Proof. Indeed, we have seen before in Chapter 3 that
$$\rho(\mathfrak{l}^n) = \begin{cases} n+1 & \text{if } \chi(\mathfrak{l}) = 1; \\ 0 & \text{if } \chi(\mathfrak{l}) = -1 \text{ and } n \text{ is odd;} \\ 1 & \text{if } \chi(\mathfrak{l}) = -1 \text{ and } n \text{ is even.} \end{cases}$$
These quantities match the integral parts of $\varphi(T_{\mathfrak{l}^n})$ from Proposition 6.2.6 above. As for the infinitesimal part, we remark that we get no contribution if and only if $\chi(\mathfrak{l}^n) = 1$, and as such, the expression $1-\chi(\mathfrak{l}^n)$ is twice the indicator function for the case $\chi(\mathfrak{l}) = -1$ and n is odd. Combining these two parts yields the corollary. □

The appearance of the ρ-function in the integral part should not be surprising, as this simply corresponds to the Eisenstein series we are deforming; compare with Proposition A.2.9 and Lemma A.2.10.

Because we have well-defined operators $T_{\mathfrak{l}^n}$ for any integral ideal $\mathfrak{l} \nmid p$, we may by multiplying these expressions together extend this to a well-defined notion of an operator T_J for any integral ideal $J \subset \mathcal{O}_F$ coprime to p. Using this notion, we arrive at the following Corollary from the above computations.

Corollary 6.2.9. *Let $J \subset \mathcal{O}_F$ be any ideal coprime to p. Then*

$$\varphi(T_J) = \rho(J) + \frac{1}{2} \sum_{\mathfrak{l}^n \| J} \Big((n+1)\big(1 - \chi(\mathfrak{l}^n)\big)\rho(J/\mathfrak{l}^n) \Big) \log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon.$$

Proof. By definition

$$T_J = \prod_{\mathfrak{l}^n \| J} T_{\mathfrak{l}^n}.$$

One may write out, keeping in mind that $\epsilon^2 = 0$, that

$$\begin{aligned}
\varphi(T_J) &= \prod_{\mathfrak{l}^n \| J} \Big(\rho(\mathfrak{l}^n) + \frac{1}{2}(n+1)\big(1 - \chi(\mathfrak{l}^n)\big) \log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon \Big) \\
&= \prod_{\mathfrak{l}^n \| J} \rho(\mathfrak{l}^n) + \frac{1}{2} \sum_{\mathfrak{l}^n \| J} (n+1)\big(1 - \chi(\mathfrak{l}^n)\big) \log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon \prod_{\mathfrak{t}^m \| J/\mathfrak{l}^n} \rho(\mathfrak{t}^m).
\end{aligned}$$

The corollary now follows from the multiplicativity of ρ. □

For any integral ideal J coprime to p, we define $\mathcal{F}(J) \in \mathbb{N}$ by

$$\log_p(\mathcal{F}(J)) := \frac{1}{2} \sum_{\mathfrak{l}^n \| J} \Big((n+1)\big(1 - \chi(\mathfrak{l}^n)\big)\rho(J/\mathfrak{l}^n) \Big) \log_p(\mathrm{Nm}(\mathfrak{l})).$$

This quantity has the following crucial property. As in Chapter 3, we will refer to those prime powers $\mathfrak{l}^n \| J$ with $\chi(\mathfrak{l}^n) = -1$, as the *special primes* of the ideal J. Note that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ can also be special primes, if we relax the condition that J be coprime to p, as we will soon do.

Proposition 6.2.10. *Let $J \subset \mathcal{O}_F$ be any integral ideal coprime to p. Then*

$$\varphi(T_J) = \rho(J) + \log_p(\mathcal{F}(J))\epsilon.$$

In addition, $\mathcal{F}(J)$ is a power of a single rational prime. If J is a primitive ideal, then it even holds that $\mathcal{F}(J) = F(\mathrm{Nm}(J))^2$, where F is as in our main results, Theorem A and Theorem B.

Proof. The first claim follows from Corollary 6.2.9 and the definition of $\mathcal{F}(J)$. For the second, we note that the only summands in the expression defining $\mathcal{F}(J)$ that could possibly contribute are those for the special primes of J. If there are no such primes, then $\mathcal{F}(J) = 1$. If there is more than special prime, one of which being $\mathfrak{l}^n$, then its contribution will also vanish. Namely, in that case it holds that $\rho(J/\mathfrak{l}^n) = 0$, as the existence of a special prime obstructs an ideal from being a norm from the field L. We conclude that $\mathcal{F}(J) = 1$ in that case too. Only the case in which there is a unique special prime remains, proving that $\mathcal{F}(J)$ is a power of the underlying rational prime ℓ, as claimed. Finally, our primitivity assumption forces all primes dividing J to split in $F/\mathbb{Q}$ and to all lie above different rational primes, and as such, the prime factorisation of J in F matches the prime factorisation of its norm in $\mathbb{Q}$. From the proof of Proposition 3.2.5 in Chapter 3, it follows that the exact exponent of ℓ occuring in the expression of Giampietro's F can also be written as $(n+1)\rho(J/\mathfrak{l}^n)/2$, proving the claimed equality. □

By Theorem 5.0.1, the morphism $\varphi : \mathbb{T} \to \mathbb{Q}_p[\epsilon]$ computed above will correspond to a cuspidal family of p-adic Hilbert modular forms, which we will denote by $E_{1,\chi}^{(p)}(\epsilon)$. In the next section, we will be interested in the first derivative with respect to ϵ of the diagonal restriction of this family with respect to the ideal $t_\lambda = \mathcal{D}_F^{-1}\mathfrak{q}_1$. To simplify notation, we will thus set $a_\nu := a_p(\mathcal{D}_F\mathfrak{q}_1^{-1}\nu)$, as these will be the relevant coefficients to compute the desired diagonal restriction, see in Appendix A.2.

For notational convenience, for any ideal $J \subset \mathcal{O}_F$, we let $\widetilde{J}$ denote its *p-deprivation*; in other words, $\widetilde{J}$ denotes the ideal J with all factors of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ removed. Finally, we introduce the notation J_ν for the ideal $\nu\mathcal{D}_F\mathfrak{q}_1^{-1}$. We now state the main result of this section.

Theorem 6.2.11. *For any $\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+$, it holds that*

$$a_\nu = (-1)^{v_{\mathfrak{p}_1}(\nu)}\big(\rho(\widetilde{J_\nu}) + \log_p(\mathcal{F}(\widetilde{J_\nu}))\epsilon - \rho(\widetilde{J_\nu})\log_p(\nu/\nu')\epsilon\big).$$

Proof. Using the same argument as in Theorem 3.13 in [DPV23], we may use the relations proved by Hida for classical forms also in this setting. By the results of Section 1 in [Hid91] and Proposition A.4.4, for any $\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+$, we must consider the idèle $\alpha = \nu d\varpi_{\mathfrak{q}_1}^{-1}$, where $\varpi_{\mathfrak{q}_1}$ is any idèle that equals 1 everywhere away from $\mathfrak{q}_1$, where it is a uniformiser, and where $d \in \mathbb{A}_F^\times$ is such that it generates the ideal $\mathcal{D}_F$ and is trivial at p. Let $\widetilde{\nu}$ denote the idèle that is equal to ν everywhere away from p, where it is equal to 1. Then $\nu = \widetilde{\nu}\nu_{\mathfrak{p}_1}\nu_{\mathfrak{p}_2}$ and we note that α generates the ideal J_ν, so that $\widetilde{J_\nu}$ is generated by the idèle $\tilde{\nu} d \varpi_{\mathfrak{q}_1}^{-1}$. Using Remark 6.2.7, we compute that

$$\begin{aligned}
\varphi(T_\alpha) &= \varphi(T_{\tilde{\nu} d \varpi_{\mathfrak{q}_1}^{-1}}) \varphi(U_{\nu_{\mathfrak{p}_1}}) \varphi(U_{\nu_{\mathfrak{p}_2}}) \\
&= \varphi(T_{\widetilde{J_\nu}}) \cdot (-1)^{v_{\mathfrak{p}_1}(\nu_{\mathfrak{p}_1})} (1 - \log_p(\nu_{\mathfrak{p}_1})\epsilon) \cdot (1 + \log_p(\nu_{\mathfrak{p}_2})\epsilon) \\
&= (-1)^{v_{\mathfrak{p}_1}(\nu)} \big(\rho(\widetilde{J_\nu}) + \log_p(\mathcal{F}(\widetilde{J_\nu}))\epsilon\big)(1 - \log_p(\nu_{\mathfrak{p}_1})\epsilon)(1 + \log_p(\nu'_{\mathfrak{p}_1})\epsilon) \\
&= (-1)^{v_{\mathfrak{p}_1}(\nu)} \big(\rho(\widetilde{J_\nu}) + \log_p(\mathcal{F}(\widetilde{J_\nu}))\epsilon\big) \big(1 - \log_p(\nu_{\mathfrak{p}_1}/\nu'_{\mathfrak{p}_1})\epsilon\big) \\
&= (-1)^{v_{\mathfrak{p}_1}(\nu)} \big(\rho(\widetilde{J_\nu}) + \log_p(\mathcal{F}(\widetilde{J_\nu}))\epsilon - \rho(\widetilde{J_\nu}) \log_p(\nu/\nu')\epsilon\big);
\end{aligned}$$

this is the statement of the theorem. We used here that $v_{\mathfrak{p}_2}$ can be identified with $v'_{\mathfrak{p}_1}$, since under the isomorphism $\mathcal{O}_F \otimes \mathbb{Z}_p \cong \mathcal{O}_{F_{\mathfrak{p}_1}} \times \mathcal{O}_{F_{\mathfrak{p}_2}}$, the element ν is sent to (ν, ν'). ☐

6.3 Proof of Theorem B

In Theorem 6.2.11, we computed the relevant Fourier coefficients for the diagonal restriction with respect to the ideal $\mathcal{D}_F^{-1}\mathfrak{q}_1$ of the cuspidal family of p-adic Hilbert modular forms $E_{1,\chi}^{(p)}(\epsilon)$ associated with the morphism $\varphi : \mathbb{T} \to \mathbb{Q}_p[\epsilon]$ induced by the deformation $\widetilde{\rho}$ defined in Lemma 6.2.1. Taking its derivative with respect to the parameter ϵ amounts to considering only the ϵ-part. Let Δ denote the diagonal restriction operator for the ideal $\mathcal{D}_F^{-1}\mathfrak{q}_1$. As $E_{1,\chi}^{(p)}(\epsilon)$ is p-adically a cusp form by Lemma A.3.3, we have

$$\Delta \frac{d}{d\epsilon} E_{1,\chi}^{(p)}(\epsilon) = \sum_{n=1}^{\infty} \Big(\sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu) = n}} \frac{d}{d\epsilon} a_\nu \Big) \mathbf{q}^n.$$

By Theorem 6.2.11 and the equation above, this yields for any $n \in \mathbb{N}$,

$$a_n \left(\Delta \frac{d}{d\epsilon} E_{1,\chi}^{(p)}(\epsilon) \right) = \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu) = n}} (-1)^{v_{\mathfrak{p}_1}(\nu)} \big(\log_p \mathcal{F}(\widetilde{J_\nu}) - \rho(\widetilde{J_\nu}) \log_p(\nu/\nu') \big).$$

Proposition 6.3.1. *The object $\Delta \frac{d}{d\epsilon} E_{1,\chi}^{(p)}(\epsilon)$ is an overconvergent p-adic modular form of weight 2. Its ordinary projection $e^{\mathrm{ord}} \left(\Delta \frac{d}{d\epsilon} E_{1,\chi}^{(p)}(\epsilon) \right)$ is a classical modular form in $\mathcal{S}_2(\Gamma_0(N))$.*

Proof. We have seen before that the weight character for $\Delta E_{1,\chi}^{(p)}(\epsilon)$ is constant, and because for the $\epsilon = 0$-specialisation its weight is $1 + 1 = 2$, the result will be of constant weight 2. Its specialisation does not vanish, but by subtracting a constant family, Lemma 2.1 in [DPV21] yields that its derivative is still an overconvergent p-adic modular form of weight 2. By Coleman's Classicality Theorem, which can be found as Theorem 6.1 in [Col96], its ordinary projection is of slope $0 < 1$ and hence classical. Further, it is a cusp form because $E^{(p)}_{1,\chi}$ is a p-adic cusp form by Lemma A.3.3. For the level, since we used the ideal $\mathcal{D}_F\mathfrak{q}_1^{-1}$, the tame level of our diagonal restriction will be q. The level of its ordinary projection is then obtained by multiplying its tame level by p. Combining all of this, we obtain an object in $\mathcal{S}_2(\Gamma_0(N))$, as claimed. □

If we apply the operator e^{ord} from Appendix A.5, we obtain

$$\begin{aligned} a_1\left(e^{\mathrm{ord}}\left(\frac{d}{d\epsilon}\Delta E^{(p)}_{1,\chi}(\epsilon)\right)\right) &= \lim_{n\to\infty} a_{p^{n!}}\left(\frac{d}{d\epsilon}\Delta E^{(p)}_{1,\chi}(\epsilon)\right) \\ &= \lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{n!}}} (-1)^{v_{\mathfrak{p}_1}(\nu)}\big(\log_p \mathcal{F}(\widetilde{J_\nu}) - \rho(\widetilde{J_\nu})\log_p(\nu/\nu')\big). \end{aligned}$$

We analyse the two terms occuring in this expression separately. Define

$$A := \lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{n!}}} (-1)^{v_{\mathfrak{p}_1}(\nu)}\rho(\widetilde{J_\nu})\log_p(\nu/\nu')$$

and

$$B := \lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{n!}}} (-1)^{v_{\mathfrak{p}_1}(\nu)}\log_p(\mathcal{F}(\widetilde{J_\nu})).$$

For the sake of brevity, we extend the definition of v_p to F by setting it equal to $v_{\mathfrak{p}_1} \times v_{\mathfrak{p}_2}$ there; this will save some space in what is to come.

Proposition 6.3.2. *It holds that*

$$A = \frac{2}{w_1w_2}\log_p\left(\Theta(D_1, D_2)\right) - \frac{2}{w_1w_2}\log_p\left(\Theta_p(D_1, D_2)\right).$$

Proof. We note that since $\chi(J_\nu) = \chi(\mathcal{D}_F)\chi(\mathfrak{q}_1) = (-1)^2 = 1$, the parities of $v_{\mathfrak{p}_1}(J_\nu)$ and $v_{\mathfrak{p}_2}(J_\nu)$ being different would imply that $\chi(\widetilde{J_\nu}) = -1$, and as such, $\rho(\widetilde{J_\nu}) = 0$ by Corollary A.2.11. Hence we may write

$$A = \lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{n!} \\ v_p(\nu)\equiv(0,0)}} \rho(\widetilde{J_\nu})\log_p(\nu/\nu') - \lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{n!} \\ v_p(\nu)\equiv(1,1)}} \rho(\widetilde{J_\nu})\log_p(\nu/\nu'),$$

both congruences being modulo 2. For the first term, one may observe that $\rho(\widetilde{J_\nu}) = \rho(J_\nu)$. In fact, $\rho(J_\nu) = 0$ unless $v_p(\nu) \equiv (0,0) \mod 2$, and as a result, we may even write the first term as

$$\lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{n!}}} \rho(J_\nu)\log_p(\nu/\nu') = \frac{2}{w_1 w_2}\log_p\left(\Theta(D_1, D_2)\right),$$

where we appealed to Proposition 6.1.4 and the fact that the limit exists when taken over all $\mathrm{tr}(\nu) = p^{2n}$; as such, the limit taken over $\mathrm{tr}(\nu) = p^{n!}$ exists too and equals the same value, as $n!$ is even for $n \geq 2$. For the second term, one may observe that $p \mid \nu$, and as such, we may make the substitution $\nu \leftrightarrow p\nu$, further using that $\rho(\widetilde{J_\nu}) = \rho(J_{p\nu})$, to obtain

$$\lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{n!} \\ v_p(\nu)\equiv(1,1)}} \rho(\widetilde{J_\nu})\log_p(\nu/\nu') = \lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=p^{n!-1}}} \rho(J_\nu)\log_p(\nu/\nu'),$$

where we were allowed to omit the bottom subscript for the same reason as before. We conclude by Proposition 6.1.6. ☐

Proposition 6.3.3. *It holds that*

$$B = \sum_{\mathrm{Nm}(\mathfrak{a})=N} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}a)^+ \\ \mathrm{tr}(\nu)=1}} \delta(\mathfrak{a})\log_p(F(\mathrm{Nm}(J_\nu)/p)).$$

Proof. First note that, by Lemma 6.2.10, it holds that $\mathcal{F}(\widetilde{J_\nu}) = 1$ as soon as $\chi(\widetilde{J_\nu}) = 1$, because this implies that the number of special primes is even, and thus in particular not one. Since $\chi(J_\nu) = 1$, it follows that one of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ must be special to get a non-zero contribution to the sum. Since ν contains the full p-part of J_ν, this implies that $v_p(\mathrm{Nm}(\nu))$ must be odd. By Lemma 6.1.7, it follows that $p^{n!} \mid \nu$ if $\mathrm{tr}(\nu) = p^{n!}$. It follows that all contributing summands to the n-th term in the limit are lifted from those ν of unit trace. In fact, since $\widetilde{J_{p^{n!}\nu}} = \widetilde{J_\nu}$ and $v_{\mathfrak{p}_1}(J_{p^{n!}\nu}) \equiv v_{\mathfrak{p}_1}(J_\nu) \mod 2$ for $n \geq 2$, each summand induced by some ν of unit trace is independent of n. The limit thus equals its first term;

$$B = \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=1 \\ v_p(\mathrm{Nm}(\nu))\text{ odd}}} (-1)^{v_{\mathfrak{p}_1}(J_\nu)}\log_p(\mathcal{F}(\widetilde{J_\nu})).$$

Note that the ideal $\widetilde{J_\nu}$ is always primitive by Lemma 3.2.1. As it is prime to p by definition, it follows from Proposition 6.2.10 that $\mathcal{F}(\widetilde{J_\nu}) = F(\mathrm{Nm}(\widetilde{J_\nu}))^2$ in all cases. Because the number of factors of p in $\mathrm{Nm}(J_\nu)$ is odd for all the terms contributing to the sum, $\mathrm{Nm}(J_\nu)/p$ will have an even number, as does $\mathrm{Nm}(\widetilde{J_\nu})$. As a result, the F-values of these integers must be the same. We conclude that

$$B = 2 \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=1 \\ v_p(\mathrm{Nm}(\nu)) \text{ odd}}} (-1)^{v_{\mathfrak{p}_1}(J_\nu)} \log_p(F(\mathrm{Nm}(J_\nu)/p)).$$

Note that we may omit any congruence conditions on $v_p(\mathrm{Nm}(\nu))$, as in the case of the number of factors of p dividing $\mathrm{Nm}(J_\nu)$ being even, dividing by p makes p a special prime of $\mathrm{Nm}(J_\nu)/p$. As such, its F-value must be a power of p, of which the p-adic logarithm vanishes. Since contributing ν must contain a factor of $\mathfrak{p}_1$ or $\mathfrak{p}_1'$, we have proved that

$$B = 2 \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{p}_1\mathfrak{q}_1)^+ \\ \text{and } \nu \in (\mathcal{D}_F^{-1}\mathfrak{p}_2\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=1}} (-1)^{v_{\mathfrak{p}_1}(J_\nu)} \log_p(F(\mathrm{Nm}(J_\nu)/p)).$$

Adding in those $\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_2)^+$ is the same as adding a term for the Galois-conjugate of every $\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+$. For every non-zero term in the sum, we have that $v_{\mathfrak{p}_1}(J_{\nu'}) \not\equiv v_{\mathfrak{p}_1}(J_\nu) \mod 2$ and $\mathrm{Nm}(J_{\nu'}) = \mathrm{Nm}(J_\nu)$. In other words, the summands for ν and ν' would agree up to a sign. Finally, $\delta(\mathfrak{a})$ measures this sign together with $(-1)^{v_{\mathfrak{p}_1}(\nu)}$. □

To complete the proof, we will only need one more result about the ordinary projection that we have now computed. Namely, we will show that its first Fourier coefficient must always vanish.

Lemma 6.3.4. *The form* $e^{\mathrm{ord}}\left(\Delta \frac{d}{d\epsilon} E_{1,\chi}^{(p)}(\epsilon)\right)$ *is a* U_p*-eigenvector with eigenvalue* -1.

Proof. By definition of the U_p operator, we should check that $a_{pm} = -a_m$ for all $m \in \mathbb{N}$. The coefficient for a_m can be split up into the two terms

$$\lim_{n\to\infty} \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=m\cdot p^{n!}}} (-1)^{v_{\mathfrak{p}_1}(\nu)} \rho(\widetilde{J_\nu}) \log_p(\nu/\nu')$$

and

$$\lim_{n\to\infty} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q}_1)^+ \\ \mathrm{tr}(\nu)=m\cdot p^{n!}}} (-1)^{v_{\mathfrak{p}_1}(\nu)} \log_p(\mathcal{F}(\widetilde{J_\nu})).$$

For the former, one may trace through the proof of Proposition 6.3.2 to find that the limit is only dependent on the parity of the numbers $v_p(m) + n!$ for $n \geq 2$, the difference being a sign when we move from m to pm, caused by the factor $(-1)^{v_{\mathfrak{p}_1}(\nu)}$. For the latter, the proof of Proposition 6.3.3 reduces the contributions to those ν of trace $m/p^{v_p(m)}$, once again the only difference being the sign $(-1)^{v_{\mathfrak{p}_1}(\nu)}$ dependent on the parity of $v_p(m)$. We leave the details to the reader. □

Corollary 6.3.5. *For $N \in \{6, 10, 22\}$, it holds that*

$$e^{\mathrm{ord}}\left(\Delta \frac{d}{d\epsilon} E^{(p)}_{1,\chi}(\epsilon)\right) = 0.$$

Proof. For $N \in \{6, 10\}$, one checks that $\mathcal{S}_2(\Gamma_0(N)) = 0$ so the result follows from Proposition 6.3.1. It remains to analyse the case that $N = 22$, when the space $\mathcal{S}_2(\Gamma_0(22))$ is 2-dimensional and spanned by two oldforms. Recall that we assumed $q \in \{2, 3, 5, 7, 13\}$, so we need only check the case that $p = 11$ and $q = 2$. It is easy to check that the operator U_{11} acts trivially on $\mathcal{S}_2(\Gamma_0(22))$, so by Lemma 6.3.4, the result follows. □

Proof. (of Theorem B) We conclude from Corollary 6.3.5 that $A + B = 0$. Using Proposition 6.3.2 and Proposition 6.3.3, this shows that

$$\frac{2}{w_1 w_2} \log_p\left(\frac{\Theta(D_1, D_2)}{\Theta_p(D_1, D_2)}\right) = \sum_{\mathrm{Nm}(a)=N} \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}a)^+ \\ \mathrm{tr}(\nu)=1}} \delta(a) \log_p(F(\mathrm{Nm}(J_\nu)/p)).$$

Finally, for $\nu = (x + \sqrt{D})/2\sqrt{D}$, it holds that

$$\mathrm{Nm}(J_\nu)/p = \frac{D - x^2}{4N}.$$

This proves Theorem B up to an element from $\pm p^{\mathbb{Z}}$, as this is the kernel of $\log_p$. We conclude by Proposition 6.1.8 that the factors of p on both sides match up too, which reduces our ambiguity to a mere sign. This is permitted by Theorem B and concludes the proof. □