CM-values of p-adic Theta-functions

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CHAPTER 5

Deformation theory

The aim of this chapter is to carry out **Step 2** of the proof of Theorem B as explained in Section 2.4. We remind the reader that our stategy is ultimately to perform certain operations on a cuspidal p-adic family of modular forms passing through the appropriate p-stabilisation $E_{1,\chi}^{(p)}$ of the Hilbert Eisenstein series $E_{1,\chi}$. However, there are no easy ways to obtain such a family. Our approach is therefore to obtain this by deforming the Galois representation $\rho = \mathbb{1} \oplus \chi$ associated with the Eisenstein series $E_{1,\chi}^{(p)}$ instead, and compute from this the associated family of modular forms that we need to carry out our proof.

However, this approach requires us to show that these deformations are in fact modular; that is, that these Galois representations actually come from modular forms. This is done by proving a so-called $R = T$ theorem, where R typically denotes a certain universal deformation ring, and T a certain Hecke algebra. Proving such a theorem is the main focus of this chapter.

Similar results have been obtained in Pozzi's thesis [Poz19] for the ground field $\mathbb{Q}$ and in the case that the prime p is *irregular*, as opposed to the *regular* case as we are in here, since $\chi(\mathfrak{p}_i) \neq 1$ for $i \in \{1, 2\}$. For other closely related works in these directions, see [BDP22, BD16, BDS20, BC06, DKV18]. With small adjustments, our $R = T$ theorem can also be deduced from the results of [Bet20], where very similar methods to the ones we will employ throughout this chapter were used.

We start by recalling some results from class field theory and use these to compute various Galois cohomology groups, that will play a key role throughout. For representability reasons that we will explain in Section 5.2, we should rigidify the decomposable representation $\rho = \mathbb{1} \oplus \chi$ before attempting to deform it. This rigidified representation is denoted ρ_η. We will consider a class of deformations that are called *nearly ordinary* and show that it admits a universal deformation ring $R_{\rho_\eta}^{\mathrm{no}}$. Next, we construct from a key result of Hida in [Hid89a] a Galois representation onto Hida's *nearly ordinary cuspidal Hecke algebra* $\mathbb{T}$ localised at the Eisenstein series $E_{1,\chi}^{(p)}$ from Appendix A.2. We show that it must be a lift of ρ_η, so that by universality, this will yield a map $\mathcal{T} : R_{\rho_\eta}^{\mathrm{no}} \to \mathbb{T}$, where $\mathbb{T}$ denotes the appropriate Hecke algebra. Using various results from commutative algebra, we can then show the following.

Theorem 5.0.1. *The map $\mathcal{T} : R_{\rho_\eta}^{\mathrm{no}} \to \mathbb{T}$ is an isomorphism.*

Throughout this chapter, we let $\mathbb{Q}_p := \mathbb{Q}_p(\mathbb{1})$ denote the Galois module with trivial G_F-action, whereas $\mathbb{Q}_p(\chi)$ denotes the same module but with the action of G_F through the character $\chi : G_F \to \{\pm 1\}$.

Furthermore, for any G_F-module A, we let $Z^1(G_F, A)$ denote the group of *continuous* 1-cocycles for G_F with values in A. All cohomology groups in this chapter will be derived from group cohomology and will as such be quotients of the groups $Z^1(G_F, A)$.

Finally, throughout this chapter, we will denote

$$G := \mathrm{Gal}(L/F) = \langle \sigma_F \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

and to simplify notation, we write for $i \in \{1, 2\}$ the decomposition group

$$G_{\mathfrak{p}_i} := G_{F_{\mathfrak{p}_i}} \cong \mathrm{Gal}(\overline{F}_{\mathfrak{p}_i}/F_{\mathfrak{p}_i}) \quad \text{inside } G_F.$$

5.1 Some Galois cohomology

Our main tool to compute Galois cohomology groups comes from the following result, which relies on global class field theory. It can be found as Lemma 1.1.1 in [Poz19] or as Lemma 3.2 in [DPV23].

Proposition 5.1.1. *We have the following exact sequence:*

$$0 \to \mathrm{Hom}(G_L^{\mathrm{ab}}, \mathbb{Q}_p) \to \prod_{v|p} \mathrm{Hom}(L_v^\times, \mathbb{Q}_p) \to \mathrm{Hom}(\mathcal{O}_L[1/p]^\times, \mathbb{Q}_p).$$

We opt to omit its proof, for it is standard, yet somewhat lengthy. However, we do stress that the proof of this result uses the fact that L is totally imaginary, allowing us to easily identify the connected component of the identity inside the idele class group. Therefore, we may not use this proposition with the field L replaced by either F or $\mathbb{Q}$. The next result aims to connect the cohomology groups for G_F to those for G_L.

Lemma 5.1.2. *Restriction to $G_L \subset G_F$ yields isomorphisms*

$$H^1(G_F, \mathbb{Q}_p) \cong \mathrm{Hom}(G_L, \mathbb{Q}_p)^G;$$
$$H^1(G_F, \mathbb{Q}_p(\chi)) \cong \mathrm{Hom}(G_L, \mathbb{Q}_p(\chi))^G.$$

Proof. Note that the action of G_L on $\mathbb{Q}_p(\chi)$ is trivial. We may then use the inflation-restriction sequence to obtain that

$$0 \to H^1(G, \mathbb{Q}_p) \to \mathrm{Hom}(G_F, \mathbb{Q}_p) \to \mathrm{Hom}(G_L, \mathbb{Q}_p)^G \to H^2(G, \mathbb{Q}_p)$$

is exact, and similarly that also

$$0 \to H^1(G, \mathbb{Q}_p(\chi)) \to H^1(G_F, \mathbb{Q}_p(\chi)) \to \mathrm{Hom}(G_L, \mathbb{Q}_p(\chi))^G \to H^2(G, \mathbb{Q}_p(\chi))$$

is exact. To prove the lemma, we thus reduce to showing that the groups $H^1(G,\mathbb{Q}_p)$, $H^2(G,\mathbb{Q}_p)$, $H^1(G,\mathbb{Q}_p(\chi))$ and $H^2(G,\mathbb{Q}_p(\chi))$ are all trivial. Using that G is cyclic, we identify these spaces with

$$H^1(G,-) = \frac{\ker(\mathcal{N})}{\mathrm{im}(\Delta)} \quad \text{and} \quad H^2(G,-) = \frac{\ker(\Delta)}{\mathrm{im}(\mathcal{N})},$$

where $\Delta = 1 - \sigma_F$ and $\mathcal{N} = 1 + \sigma_F$ in the group ring $\mathbb{Z}[G]$. In both cases, one of these maps is zero, whereas the other is multiplication by 2, which is a bijective map on $\mathbb{Q}_p$. All claims follow. □

We now identify the spaces on the right hand side of Lemma 5.1.2. Any finite dimensional G-representation V can be decomposed as a finite sum of copies of $\mathbb{1}$ and χ. Henceforth, for $\varphi \in \{1,\chi\}$, we will let V^φ denote the sum of all φ-eigenspaces inside V.

Lemma 5.1.3. *For $\varphi \in \{\mathbb{1},\chi\}$, it holds that*

$$\mathrm{Hom}(G_L,\mathbb{Q}_p(\varphi))^G \cong \mathrm{Hom}(G_L,\mathbb{Q}_p)^\varphi.$$

Proof. Recall that the action of some $g \in G_F$ on some homomorphism $f \in \mathrm{Hom}(G_L,\mathbb{Q}_p(\varphi))$ is defined as $g \cdot f : x \mapsto \varphi(g)f(gxg^{-1})$. To be invariant under this action, we thus require that $\varphi(g)f(gxg^{-1}) = f(x)$ for all $x \in G_L$. On the other hand, the action of some $g \in G_F$ on some homomorphism $f \in \mathrm{Hom}(G_L,\mathbb{Q}_p)$ is defined as $g \cdot f : x \mapsto f(gxg^{-1})$. So, to be in the φ-eigenspace, we thus require that $\varphi(g)f(gxg^{-1}) = f(x)$ for all $x \in G_L$. These two conditions agree. □

Lemma 5.1.4. *It holds that*

$$\dim\left(\mathrm{Hom}(G_L,\mathbb{Q}_p)^{\mathbb{1}}\right) = 1 \quad \textit{and} \quad \dim\left(\mathrm{Hom}(G_L,\mathbb{Q}_p)^\chi\right) = 2.$$

Proof. For $i \in \{1,2\}$, the space $\mathrm{Hom}(L_{\mathfrak{p}_i}^\times,\mathbb{Q}_p)$ is 3-dimensional and spanned by $\mathrm{ord}_{\mathfrak{p}_i}$, the p-adic logarithm $\log_p$ and its composition with the non-trivial element from $\mathrm{Gal}(L_{\mathfrak{p}_i}/\mathbb{Q}_p)$. Since the primes above p in F are inert in L, the action of G fixes the former, but interchanges the latter two basis elements. It follows that

$$\dim\left(\mathrm{Hom}(L_{\mathfrak{p}_1}^\times \times L_{\mathfrak{p}_2}^\times,\mathbb{Q}_p)^{\mathbb{1}}\right) = 4 \quad \text{and} \quad \dim\left(\mathrm{Hom}(L_{\mathfrak{p}_1}^\times \times L_{\mathfrak{p}_2}^\times,\mathbb{Q}_p)^\chi\right) = 2.$$

For the rightmost term, we observe that the image of the norm map

$$\mathcal{O}_L[1/p]^\times \to \mathcal{O}_F[1/p]^\times$$

has finite index. Indeed, we know that the map $\mathcal{O}_L^\times \to \mathcal{O}_F^\times$ has index at most 2, and since the primes above p in F are inert in L, the claim follows. After tensoring with $\mathbb{Q}_p$, these Galois modules are therefore isomorphic. Therefore, using Dirichlet's unit theorem, we find that

$$\dim\left(\mathrm{Hom}(\mathcal{O}_L[1/p]^\times, \mathbb{Q}_p)^{\mathbb{1}}\right) = 3 \quad \text{and} \quad \mathrm{Hom}(\mathcal{O}_L[1/p]^\times, \mathbb{Q}_p)^\chi = 0.$$

We now consider the short exact sequence from Proposition 5.1.1:

$$0 \to \mathrm{Hom}(G_L, \mathbb{Q}_p) \to \mathrm{Hom}(L_{\mathfrak{p}_1}^\times \times L_{\mathfrak{p}_2}^\times, \mathbb{Q}_p) \to \mathrm{Hom}(\mathcal{O}_L[1/p]^\times, \mathbb{Q}_p).$$

All of these spaces are finite dimensional G-representations. As such, they can be decomposed into a direct sum of $\mathbb{1}$ and χ-eigenspaces. First, we take $\mathbb{1}$-eigenspaces to find

$$0 \to \mathrm{Hom}(G_L, \mathbb{Q}_p)^{\mathbb{1}} \to \mathrm{Hom}(L_{\mathfrak{p}_1}^\times \times L_{\mathfrak{p}_2}^\times, \mathbb{Q}_p)^{\mathbb{1}} \to \mathrm{Hom}(\mathcal{O}_L[1/p]^\times, \mathbb{Q}_p)^{\mathbb{1}}.$$

Surjectivity on the right is equivalent with Leopoldt's conjecture for L. Since $L/\mathbb{Q}$ is abelian, and as such, this conjecture is known and the claim about the dimension now follows. Next, taking χ-eigenspaces, we obtain a sequence

$$0 \to \mathrm{Hom}(G_L, \mathbb{Q}_p)^\chi \to \mathrm{Hom}(L_{\mathfrak{p}_1}^\times \times L_{\mathfrak{p}_2}^\times, \mathbb{Q}_p)^\chi \to \mathrm{Hom}(\mathcal{O}_L[1/p]^\times, \mathbb{Q}_p)^\chi = 0,$$

and the proof is complete by our dimension computations above. □

Combining all the above, we obtain the following two core results.

Corollary 5.1.5. *The group* $\mathrm{Hom}(G_F, \mathbb{Q}_p)$ *is 1-dimensional and spanned by the p-adic cyclotomic character:*

$$\phi_p : G_F \to \mathrm{Gal}(F(\zeta_p^\infty)/F) \cong \mathbb{Z}_p^\times \xrightarrow{\log_p} \mathbb{Q}_p.$$

Proof. Combining Lemma 5.1.2 with Lemma 5.1.3, we obtain an isomorphism $\mathrm{Hom}(G_F, \mathbb{Q}_p) \cong \mathrm{Hom}(G_L, \mathbb{Q}_p)^{\mathbb{1}}$ and by Lemma 5.1.4, this space is 1-dimensional, as claimed. As it contains the p-adic cyclotomic character as defined above, which is clearly non-zero, this character must span the space, completing the proof. □

Corollary 5.1.6. *For* $i \in \{1, 2\}$*, the groups* $H^1(G_{\mathfrak{p}_i}, \mathbb{Q}_p(\chi))$ *are both 1-dimensional, and restriction gives an isomorphism*

$$H^1(G_F, \mathbb{Q}_p(\chi)) \cong H^1(G_{\mathfrak{p}_1}, \mathbb{Q}_p(\chi)) \oplus H^1(G_{\mathfrak{p}_2}, \mathbb{Q}_p(\chi)).$$

Proof. Using analogous arguments to those in the proofs of Lemma 5.1.2 and Lemma 5.1.3, one can show that

$$H^1(G_{\mathfrak{p}_i}, \mathbb{Q}_p(\chi)) \cong \mathrm{Hom}(G_{L_{\mathfrak{p}_i}}, \mathbb{Q}_p(\chi)) \cong \mathrm{Hom}(G_{L_{\mathfrak{p}_i}}, \mathbb{Q}_p)^{\chi}.$$

In the proof of Lemma 5.1.4, we showed that the natural map

$$\mathrm{Hom}(G_L, \mathbb{Q}_p)^{\chi} \xrightarrow{\sim} \mathrm{Hom}(L_{\mathfrak{p}_1}^{\times} \times L_{\mathfrak{p}_2}^{\times}, \mathbb{Q}_p)^{\chi}$$

is an isomorphism. Another application of Lemma 5.1.2 and Lemma 5.1.3 now yields the claimed isomorphism. The dimension of the spaces $\mathrm{Hom}(G_{L_{\mathfrak{p}_i}}, \mathbb{Q}_p)^{\chi}$ were also computed as part of the proof of Lemma 5.1.4, and indeed both of these spaces contributed 1 dimension to the total 2-dimensional space $\mathrm{Hom}(G_L, \mathbb{Q}_p)^{\chi}$, completing the proof. □

To conclude this section, we will record one more result about the G_F-modules $\mathbb{Q}_p$ and $\mathbb{Q}_p(\chi)$ that we will need later.

Lemma 5.1.7. *It holds that*

$$H^2(G_F, \mathbb{Q}_p) = 0 \quad \textit{and} \quad H^2(G_F, \mathbb{Q}_p(\chi)) = 0.$$

Proof. We use the global Euler characteristic formula, also used by Pozzi in [Poz19] in the proof of Lemma 1.5.2; see [Lu11]. Now, since F has two real places, we compute that

$$\begin{aligned}\dim\, H^2(G_F, \mathbb{Q}_p) &= \dim\, H^1(G_F, \mathbb{Q}_p) - \dim\, H^0(G_F, \mathbb{Q}_p) \\ &\quad + 2\,\dim\, H^0(G_{\mathbb{R}}, \mathbb{Q}_p) - 2\,\dim\, \mathbb{Q}_p \\ &= 1 - 1 + 2 - 2 = 0,\end{aligned}$$

where we used Corollary 5.1.5 and Corollary 5.1.6. Similarly, using Proposition 5.1.4, we compute that

$$\begin{aligned}\dim\, H^2(G_F, \mathbb{Q}_p(\chi)) &= \dim\, H^1(G_F, \mathbb{Q}_p(\chi)) - \dim\, H^0(G_F, \mathbb{Q}_p(\chi)) \\ &\quad + 2\,\dim\, H^0(G_{\mathbb{R}}, \mathbb{Q}_p(\chi)) - 2\,\dim\, \mathbb{Q}_p(\chi) \\ &= 2 - 0 + 0 - 2 = 0,\end{aligned}$$

because complex conjugation in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts non-trivially through χ, as the field L is not totally real. □

5.2 Nearly ordinary deformation rings

We now start our journey into deformation theory. Similar treatments in various settings can be found in Pozzi's thesis [Poz19] and the works [BDP22, BD16, BDS20, BC06, DKV18].

In this section we start our study of the infinitesimal deformations of the representation $\rho = \mathbb{1} \oplus \chi$. The following lemma connects these deformations to the results from the previous section. From now on, $\mathbb{Q}_p[\epsilon]$ denotes the ring of dual numbers over $\mathbb{Q}_p$. Therefore, $\epsilon^2 = 0$.

Lemma 5.2.1. *Let $\tilde{\rho} : G_F \to \mathrm{GL}_2(\mathbb{Q}_p[\epsilon])$ be a group homomorphism that reduces to $\rho = \mathbb{1} \oplus \chi$ after composition with the natural map $\mathbb{Q}_p[\epsilon] \to \mathbb{Q}_p$. Let $a, b, c, d : G_F \to \mathbb{Q}_p$ be those functions such that*

$$\tilde{\rho}(x) = \left(1 + \epsilon \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}\right) \cdot \rho(x)$$

for all $x \in G_F$. Then these functions must respectively satisfy

$$a, d \in \mathrm{Hom}(G_F, \mathbb{Q}_p), \quad \textit{and} \quad b, c \in Z^1(G_F, \mathbb{Q}_p(\chi)).$$

Proof. This will follow from the condition that $\tilde{\rho}$ is a group homomorphism. Indeed, one may write out that for any $x, y \in G_F$,

$$\tilde{\rho}(xy) = \begin{pmatrix} 1 + a(xy)\epsilon & \chi(xy)b(xy)\epsilon \\ c(xy)\epsilon & \chi(xy)(1 + d(xy)\epsilon) \end{pmatrix}.$$

On the other hand, we expand the product $\tilde{\rho}(x)\tilde{\rho}(y)$ to obtain

$$\begin{pmatrix} 1 + (a(x) + a(y))\epsilon & \chi(xy)(\chi(x)b(y) + b(x))\epsilon \\ (c(x) + \chi(x)c(y))\epsilon & \chi(xy)(1 + (d(x) + d(y))\epsilon) \end{pmatrix}.$$

Comparing coefficients, we find that

$$\begin{aligned} a(xy) = a(x) + a(y), \quad b(xy) = b(x) + \chi(x)b(y), \\ c(xy) = c(x) + \chi(x)c(y), \quad d(xy) = d(x) + d(y). \end{aligned}$$

This is our lemma. □

However, in order to obtain a good theory of deformations, it is essential to work with a residually indecomposable representation, as becomes appararent from Proposition 1 in Mazur's [Maz89]. We have many ways to rigidify the reducible representation $\rho = \mathbb{1} \oplus \chi$, some of which are classified in the following proposition.

Proposition 5.2.2. *For any $\eta \in Z^1(G_F, \mathbb{Q}_p(\chi))$, the representation*

$$\rho_\eta : G_F \to \mathrm{GL}_2(\mathbb{Q}_p) : \tau \mapsto \begin{pmatrix} 1 & \chi(\tau)\eta(\tau) \\ 0 & \chi(\tau) \end{pmatrix}$$

has no non-scalar endomorphisms if and only if $\eta \neq 0 \in H^1(G_F, \mathbb{Q}_p(\chi))$.

Proof. We first verify that ρ_η is a group homomorphism:

$$\begin{pmatrix} 1 & \chi(\sigma\tau)\eta(\sigma\tau) \\ 0 & \chi(\sigma\tau) \end{pmatrix} = \begin{pmatrix} 1 & \chi(\sigma)\eta(\sigma) \\ 0 & \chi(\sigma) \end{pmatrix} \begin{pmatrix} 1 & \chi(\tau)\eta(\tau) \\ 0 & \chi(\tau) \end{pmatrix}.$$

Comparing the top-right entries, we obtain equality if and only if

$$\eta(\sigma\tau) = \eta(\sigma) + \chi(\sigma)\eta(\tau) \iff \eta \in Z^1(G_F, \mathbb{Q}_p(\chi)).$$

To show the claim about endomorphisms, it suffices to show that the centraliser of the image of ρ_η consists of solely scalar matrices. To this end, we must investigate when

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \chi(\tau)\eta(\tau) \\ 0 & \chi(\tau) \end{pmatrix} = \begin{pmatrix} 1 & \chi(\tau)\eta(\tau) \\ 0 & \chi(\tau) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some fixed $a, b, c, d \in \mathbb{Q}_p$. Comparing the top-left entry yields the relation that $a = a + c\chi(\tau)\eta(\tau)$. Hence choosing τ such that $\eta(\tau) \neq 0$ forces $c = 0$. The relation above is then satisfied if and only if the top-right entries match up, yielding

$$a\chi(\tau)\eta(\tau) + b\chi(\tau) = b + d\chi(\tau)\eta(\tau) \iff (a - d)\eta(\tau) = b(\chi(\tau) - 1).$$

If $a = d$, then the above forces $b = 0$ and as such the matrix is scalar. If $a \neq d$, then η is a scalar multiple of the map $\tau \mapsto \chi(\tau) - 1$. This means that η is a coboundary, completing the proof. □

Definition 5.2.3. Let $\mathcal{C}_{\mathbb{Q}_p}$ denote the category of local complete Noetherian $\mathbb{Q}_p$-algebras with residue field $\mathbb{Q}_p$. For any $A \in \mathcal{C}_{\mathbb{Q}_p}$, we let $\mathfrak{m}_A$ denote its maximal ideal. Given any object $(A, \mathfrak{m}_A)$ of $\mathcal{C}_{\mathbb{Q}_p}$, a *lift* of ρ_η to A is a representation $\rho : G_F \to \mathrm{GL}_2(A)$ that reduces to ρ_η after composing with the natural map $\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbb{Q}_p)$ induced by the natural map $A \mapsto A/\mathfrak{m}_A \cong \mathbb{Q}_p$. We say that two lifts are equivalent if they are conjugate by a matrix in the kernel of the map $\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbb{Q}_p)$ above. A *deformation* of ρ_η to A is an equivalence class of lifts of ρ_η to A.

Definition 5.2.4. We define the functor $D_{\rho_\eta} : \mathcal{C}_{\mathbb{Q}_p} \to \mathbf{Set}$ by sending any $(A, \mathfrak{m}_A) \in \mathcal{C}_{\mathbb{Q}_p}$ to the set of deformations of ρ_η to A. The *tangent space* to such a deformation functor is defined as $D_{\rho_\eta}(\mathbb{Q}_p[\epsilon])$.

Proposition 5.2.5. *If $\eta \neq 0 \in H^1(G_F, \mathbb{Q}_p(\chi))$, the functor D_{ρ_η} is represented by a universal deformation ring R_{ρ_η}.*

Proof. This is done analogously to the proof of Proposition 1.2.6 in [Poz19], where the burden is reduced to showing the absence of non-scalar endomorphisms and finite dimensionality of the tangent spaces, by virtue of Proposition 1 in [Maz89]. The former is taken care of by Proposition 5.2.2, and for the latter we may bound the dimension of the tangent space of the functor D_{ρ_η} by the dimension of the tangent space of its semisimplification; $D_{\rho_0}(\mathbb{Q}_p[\epsilon])$. The structure of this tangent space has already been established in Lemma 5.2.1; we leave it to the reader to check that two lifts are equivalent if and only if all four matrix entries are cohomologous. By Corollary 5.1.5 and Corollary 5.1.6, the dimension of this tangent space is $1+2+2+1=6$; in particular, it is finite. □

From now on, we assume that $\eta \neq 0 \in H^1(G_F, \mathbb{Q}_p(\chi))$ to ensure that the results from Proposition 5.2.2 and Proposition 5.2.5 apply. Next, we introduce the notion of being *nearly ordinary*. In fact, to proceed, we must additionally require that $\eta|_{G_{\mathfrak{p}_2}} = 0$. The reason as to why is immediate from the following observation; the line $\langle e_2 \rangle$ is only fixed by $G_{\mathfrak{p}_2}$ if this condition on η is satisfied. Note that this fixes η uniquely up to scalar multiplication, as a result of Corollary 5.1.6.

Definition 5.2.6. Consider triples (ρ, L_1, L_2) where ρ is a lift of ρ_η to some $(A, \mathfrak{m}_A) \in \mathcal{C}_{\mathbb{Q}_p}$ and L_i is a free direct summand of A^2 such that L_i lifts the line $\langle e_i \rangle$ of $\mathbb{Q}_p^2$ for $i \in \{1, 2\}$. We say that two such triples (ρ, L_1, L_2) and (ρ', L_1', L_2') are equivalent if for some $g \in \ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbb{Q}_p))$ it holds that $\rho = g\rho' g^{-1}$ and $L_i = gL_i'$ for $i \in \{1, 2\}$. We let $D^{\mathrm{fil}}_{\rho_\eta}$ be the functor sending an object $(A, \mathfrak{m}_A)$ to the set of equivalence classes of triples (ρ, L_1, L_2) as defined above.

Proposition 5.2.7. *The functor $D^{\mathrm{fil}}_{\rho_\eta}$ is represented by $R_{\rho_\eta}[\![X, Y]\!]$.*

Proof. This mildly generalises Lemma 1.3.2 in [Poz19]. We define a bijection

$$\mathrm{Hom}(R_{\rho_\eta}[\![X, Y]\!], A) \to D^{\mathrm{fil}}_{\rho_\eta}(A)$$

by sending some map $f : R_{\rho_\eta}[\![X, Y]\!] \to A$ to the representation

$$G_F \to \mathrm{GL}_2(R_{\rho_\eta}) \to \mathrm{GL}_2(R_{\rho_\eta}[\![X, Y]\!]) \to \mathrm{GL}_2(A)$$

induced by the universal representation and the map f itself, together with the lines $L_{f,1} = \langle e_1 + f(X)e_2 \rangle \subset A^2$ and $L_{f,2} = \langle f(Y)e_1 + e_2 \rangle \subset A^2$. Because f is a morphism of local rings, $f(X), f(Y) \in \mathfrak{m}_A$ and thus after composing with the quotient map $A/\mathfrak{m}_A \cong \mathbb{Q}_p$, the line $L_{f,i}$ will reduce to $\langle e_i \rangle$ for $i \in \{1,2\}$, as desired. This shows that the map is well defined.

For surjectivity, we consider an arbitrary triple (ρ_A, L_1, L_2). By the universal property of R_{ρ_η}, possibly after conjugation with an element from $\ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbb{Q}_p))$, there exists a map $R_{\rho_\eta} \to A$ such that ρ_A is obtained from the universal representation after composing with this map. It remains to choose an appropriate image of X. To this end, we note that $L_1 = \langle a \cdot e_1 + b \cdot e_2 \rangle \subset A^2$ lifts $\langle e_1 \rangle$ by definition. After composing with the quotient map $A/\mathfrak{m}_A \to \mathbb{Q}_p$, this shows that $a \notin \mathfrak{m}_A$, or equivalently $a \in A^\times$, and that $b \in \mathfrak{m}_A$. We then map X to $a^{-1}b$, so that indeed $\langle e_1 + a^{-1}be_2 \rangle = \langle ae_1 + be_2 \rangle = L_A$. Very similarly, we construct the appropriate image for Y, showing surjectivity.

For injectivity, we must consider two morphisms $f, f' : R_{\rho_\eta}[\![X,Y]\!] \to A$ that have the same images $(\rho, L_1, L_2) = (\rho', L_1', L_2')$ under the above association. In particular, this means that they give rise to the same deformation, and hence by the universal property of R_{ρ_η}, there exists a unique map $R_{\rho_\eta} \to A$ that induces them from the universal deformation. However, by construction, these two deformations are induced by the two maps $R_{\rho_\eta} \to R_{\rho_\eta}[\![X,Y]\!] \to A$ induced by f and f'. These maps must thus coincide; in other words, f and f' must agree when restricted to R_{ρ_η}. This means that ρ and ρ' are not only equivalent, they are in fact equal. Note that $\rho = g\rho g^{-1}$ for some $g \in \ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbb{Q}_p))$ would entail finding an endomorphism for ρ; as we have shown before, this forces g to be scalar. Hence $L_1 = gL_1' = L_1'$, so $\langle e_1 + f(X)e_2 \rangle = \langle e_1 + f'(X)e_2 \rangle$, and so it follows that $f(X) = f'(X)$. Very similarly, we find $f(Y) = f'(Y)$, completing the proof. □

Definition 5.2.8. Let $D^{\mathrm{no}}_{\rho_\eta} : \mathcal{C}_{\mathbb{Q}_p} \to \mathbf{Set}$ be the subfunctor of $D^{\mathrm{fil}}_{\rho_\eta}$ sending an object $(A, \mathfrak{m}_A) \in \mathcal{C}_{\mathbb{Q}_p}$ to the equivalence class of triples (ρ, L_1, L_2) as above with the properties that the line L_1 is $G_{\mathfrak{p}_1}$-stable and the line L_2 is $G_{\mathfrak{p}_2}$-stable. We call such deformations *nearly ordinary*, and these induce for $i \in \{1,2\}$ characters $\vartheta_i : G_{\mathfrak{p}_i} \to (A/L_i)^\times \cong A^\times$, called the associated *quotient characters*. Note that, since the line L_i lifts $\langle e_i \rangle$ for $i \in \{1,2\}$, it holds that $\vartheta_1 \equiv \chi \mod \mathfrak{m}_A$ and $\vartheta_2 \equiv \mathbb{1} \mod \mathfrak{m}_A$.

This practice of keeping track of the lines fixed by the decomposition groups is necessary in case the prime defining the decomposition group is *irregular*, because the line fixed by this group need not be unique. For us, $\chi(\mathfrak{p}_i) \neq 1$ for $i \in \{1,2\}$, so we are in the *regular* case, and therefore the fixed line will be unique, as Proposition 5.3.8 below will show. However, there is no harm in keeping track of this line at first in the regular case too, and we will do so in this section.

Proposition 5.2.9. *The functor $D^{\mathrm{no}}_{\rho_\eta}$ is represented by a universal deformation ring $R^{\mathrm{no}}_{\rho_\eta}$.*

Proof. This resembles part of Proposition 1.3.5(i) in [Poz19], but we repeat the argument here. The idea is to find an ideal $I \subset R_{\rho_\eta}[\![X, Y]\!]$ such that in the bijection

$$\mathrm{Hom}(R_{\rho_\eta}[\![X, Y]\!], A) \to D^{\mathrm{fil}}_{\rho_\eta}(A),$$

the image of an element on the left is contained in the subset $D^{\mathrm{no}}_{\rho_\eta}(A)$ if and only if it factors through $R_{\rho_\eta}[\![X, Y]\!]/I$. This would yield a bijection

$$\mathrm{Hom}(R_{\rho_\eta}[\![X, Y]\!]/I, A) \to D^{\mathrm{no}}_{\rho_\eta}(A),$$

establishing the desired conclusion $R^{\mathrm{no}}_{\rho_\eta} \cong R_{\rho_\eta}[\![X, Y]\!]/I$. It remains to identify I. Let us consider a representative for the universal deformation

$$\rho^{\mathrm{univ}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and investigate when its universal line $L^{\mathrm{univ}} = \langle e_1 + X e_2 \rangle$ is stable under the action of $G_{\mathfrak{p}_1}$ via ρ^{univ}. By changing bases, this happens when

$$\begin{pmatrix} 1 & 0 \\ -X & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} = \begin{pmatrix} a + bX & b \\ c + (d-a)X - bX^2 & d - bX \end{pmatrix}$$

fixes the line $\langle e_1 \rangle$ on $G_{\mathfrak{p}_1}$. This is easy to read off; it happens when

$$c(\sigma) + (d(\sigma) - a(\sigma))X - b(\sigma)X^2 \quad \text{vanishes for all } \sigma \in G_{F_{\mathfrak{p}_1}}.$$

Let $I_1 \subset R_{\rho_\eta}[\![X, Y]\!]$ be the ideal generated by all the elements above. Then for any A, using linearity of the map $f : R_{\rho_\eta}[\![X, Y]\!] \to A$ and all conditions involved, the line L_1 is fixed by $G_{\mathfrak{p}_1}$ if and only if $f(I_1) = 0$; in other words, when f factors through the quotient ring $R_{\rho_\eta}[\![X, Y]\!]/I_1$.

Similarly, we define an ideal $I_2 \subset R_{\rho_\eta}[\![X, Y]\!]$ with the property that $G_{\mathfrak{p}_2}$ fixes the line L_2 when f factors through the quotient ring $R_{\rho_\eta}[\![X, Y]\!]/I_2$. It is now easy to see that we may set $I = I_1 + I_2$ to complete the proof. □

5.3 Computing tangent spaces

Recall that the tangent space to a deformation functor $D : \mathcal{C}_{\mathbb{Q}_p} \to \mathrm{Set}$ is defined as its value on the ring of dual numbers $\mathbb{Q}_p[\epsilon] \in \mathcal{C}_{\mathbb{Q}_p}$. Henceforth, we will also use the notation

$$t_D := D(\mathbb{Q}_p[\epsilon]).$$

Lemma 5.3.1. *Let V be the free $\mathbb{Q}_p$-module of rank 2 with its G_F-action given by ρ_η. Let t_{ρ_η} be the tangent space to D_{ρ_η}. Then t_{ρ_η} is isomorphic to $H^1(G_F, \mathrm{End}(V))$.*

Proof. This is Proposition 1 on page 284 of [CSS13], which provides us with the explicit bijection

$$H^1(G_F, \mathrm{End}(V)) \to t_{\rho_\eta}$$

by mapping Θ to the lift $(1 + \epsilon\Theta)\rho_\eta \in D_{\rho_\eta}(\mathbb{Q}_p[\epsilon])$. Checking this is just a calculation and we leave it to the curious reader to verify. □

We stress here that the action of G_F on $\mathrm{End}(V)$ is given by $gM = \rho_\eta(g)^{-1} M \rho_\eta(g)$ for any $g \in G_F$ and $M \in \mathrm{End}(V) \cong M_2(\mathbb{Q}_p)$. This G_F-module is also often written as $\mathrm{ad}(\rho_\eta)$, the *adjoint representation*. We will do so henceforth.

Lemma 5.3.2. *There is a well-defined map of G_F-modules*

$$\varphi_1 : \mathrm{ad}(\rho_\eta) \to \mathbb{Q}_p(\chi) : \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto z.$$

Proof. Since this is clearly a group homomorphism, it suffices to verify the Galois-equivariance. To this end, we compute that

$$\begin{aligned}\begin{pmatrix} 1 & -\eta \\ 0 & \chi \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & \chi\eta \\ 0 & \chi \end{pmatrix} &= \begin{pmatrix} 1 & -\eta \\ 0 & \chi \end{pmatrix}\begin{pmatrix} x & x\chi\eta + y\chi \\ z & z\chi\eta + w\chi \end{pmatrix} \\ &= \begin{pmatrix} x - z\eta & x\chi\eta + y\chi - z\chi\eta^2 - w\chi\eta \\ z\chi & z\eta + w \end{pmatrix}.\end{aligned}$$

This shows that conjugation indeed induces multiplication by χ on the bottom-left entry, completing the proof. □

Now let $W_1 = \ker(\varphi_1)$; in other words, we have a short exact sequence

$$0 \to W_1 \to \mathrm{ad}(\rho_\eta) \to \mathbb{Q}_p(\chi) \to 0.$$

We proceed to refine W_1 slightly.

Lemma 5.3.3. *There is a well-defined map of G_F-modules*

$$\varphi_2 : W_1 \to \mathbb{Q}_p \oplus \mathbb{Q}_p : \begin{pmatrix} x & y \\ 0 & w \end{pmatrix} \mapsto (x, w).$$

Proof. Once again, we are left to verify Galois equivariance. This becomes apparent from the following specialisation of the computation from the proof of Lemma 5.3.2 above;

$$\begin{pmatrix} 1 & -\eta \\ 0 & \chi \end{pmatrix} \begin{pmatrix} x & y \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & \chi\eta \\ 0 & \chi \end{pmatrix} = \begin{pmatrix} x & x\chi\eta + y\chi - w\chi\eta \\ 0 & w \end{pmatrix},$$

obtained by setting $z = 0$. Indeed, we see that x and w remain untouched by the Galois action, as desired. □

We now define $W_2 = \ker(\varphi_2)$, so that we have a short exact sequence

$$0 \to W_2 \to W_1 \to \mathbb{Q}_p \oplus \mathbb{Q}_p \to 0.$$

The following result completes the filtration and allows us to start computing cohomology groups using long exact sequences.

Lemma 5.3.4. *There is a well-defined map of G_F-modules*

$$W_2 \to \mathbb{Q}_p(\chi) : \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mapsto y.$$

Proof. Indeed, we copy the computation from before once more to find

$$\begin{pmatrix} 1 & -\eta \\ 0 & \chi \end{pmatrix} \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \chi\eta \\ 0 & \chi \end{pmatrix} = \begin{pmatrix} 1 & -\eta \\ 0 & \chi \end{pmatrix} \begin{pmatrix} 0 & y\chi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y\chi \\ 0 & 0 \end{pmatrix},$$

which completes the proof. □

Proposition 5.3.5. *The group $H^1(G_F, W_1)$ is 3-dimensional. On the other hand, it holds that $H^2(G_F, W_1) = 0$.*

Proof. Since $\chi \neq \mathbb{1}$, it is clear that

$$H^0(G_F, \mathbb{Q}_p(\chi)) = 0 \quad \text{and} \quad H^0(G_F, \mathbb{Q}_p \oplus \mathbb{Q}_p) = \mathbb{Q}_p \oplus \mathbb{Q}_p.$$

Further, one may observe that

$$\begin{aligned} H^0(G_F, W_1) = W_1^{G_F} &= \{M \in W_1 \mid \rho^{-1} M \rho = M\} \\ &= W_1 \cap \{M \in M_2(\mathbb{Q}_p) \mid M\rho = \rho M\} = \langle \mathrm{id} \rangle \cong \mathbb{Q}_p, \end{aligned}$$

since in the proof of Proposition 5.2.2, we showed that only scalar matrices commute with the image of ρ_η for $\eta \neq 0 \in H^1(G_F, \mathbb{Q}_p(\chi))$. Finally, we recall Lemma 5.1.7, which states that $H^2(G_F, \mathbb{Q}_p(\chi)) = 0$. Combining all of this, the long exact sequence associated with the short exact sequence defining W_2 becomes

$$0 \to \mathbb{Q}_p \to \mathbb{Q}_p \oplus \mathbb{Q}_p \to H^1(G_F, \mathbb{Q}_p(\chi)) \to H^1(G_F, W_1) \to H^1(G_F, \mathbb{Q}_p \oplus \mathbb{Q}_p) \to 0.$$

We now find that

$$\dim\, H^1(G_F, W_1) + 2 = 1 + \dim\, H^1(G_F, \mathbb{Q}_p(\chi)) + \dim\, H^1(G_F, \mathbb{Q}_p \oplus \mathbb{Q}_p).$$

Applying Corollary 5.1.5 and Corollary 5.1.6, we conclude that

$$\dim\, (H^1(G_F, W_1)) = 1 + 2 + 2 - 2 = 3,$$

completing the proof of the first claim. For the second, we look slightly further along in the long exact sequence, to find

$$\ldots \to H^2(G_F, \mathbb{Q}_p(\chi)) \to H^2(G_F, W_1) \to H^2(G_F, \mathbb{Q}_p \oplus \mathbb{Q}_p) \to \ldots$$

Now again, by Lemma 5.1.7, the first and last terms here must vanish, and as such, so will $H^2(G_F, W_1)$. □

Theorem 5.3.6. *The tangent space t_{ρ_η} is 5-dimensional.*

Proof. Using the isomorphism

$$t_{\rho_\eta} \cong H^1(G_F, \mathrm{ad}(\rho_\eta))$$

from Lemma 5.3.1, we reduce to computing the dimension of the latter cohomology group. We now use the long exact sequence associated with the short exact sequence defining W_1. Recalling that

$$H^0(G_F, \mathbb{Q}_p(\chi)) = 0 \quad \text{and} \quad H^2(G_F, W_1) = 0$$

by Proposition 5.3.5 above, we conclude that part of this sequence reads

$$0 \to H^1(G_F, W_1) \to H^1(G_F, \mathrm{ad}(\rho_\eta)) \to H^1(G_F, \mathbb{Q}_p(\chi)) \to 0.$$

In particular, by Corollary 5.1.6 and Proposition 5.3.5, we find that

$$\dim\, H^1(G_F, \mathrm{ad}(\rho_\eta)) = \dim\, H^1(G_F, W_1) + \dim\, H^1(G_F, \mathbb{Q}_p(\chi)) = 5,$$

completing the proof. □

With this in hand, we will next concern ourselves with the tangent spaces associated with the two other deformation functors we introduced;

$$t^{\mathrm{fil}}_{\rho_\eta} := D^{\mathrm{fil}}_{\rho_\eta}(\mathbb{Q}_p[\epsilon]) \quad \text{and} \quad t^{\mathrm{no}}_{\rho_\eta} := D^{\mathrm{no}}_{\rho_\eta}(\mathbb{Q}_p[\epsilon]).$$

Lemma 5.3.7. *The space* $t^{\mathrm{fil}}_{\rho_\eta}$ *is isomorphic to* $H^1(G_F, \mathrm{ad}(\rho_\eta)) \oplus \mathbb{Q}_p^2$.

Proof. By its definition and using Proposition 5.2.7, we have

$$t^{\mathrm{fil}}_{\rho_\eta} = \mathrm{Hom}(R^{\mathrm{fil}}_{\rho_\eta}, \mathbb{Q}_p[\epsilon]) = \mathrm{Hom}(R_{\rho_\eta}[\![X,Y]\!], \mathbb{Q}_p[\epsilon]).$$

Now observe that we may choose the images of X and Y arbitrarily and independently in the maximal ideal $\mathbb{Q}_p\epsilon$ and can thus be specified with two additional numbers. We then obtain the isomorphism

$$\mathrm{Hom}(R_{\rho_\eta}[\![X,Y]\!], \mathbb{Q}_p[\epsilon]) \cong \mathrm{Hom}(R_{\rho_\eta}, \mathbb{Q}_p[\epsilon]) \oplus \mathbb{Q}_p^2.$$

Finally, using Lemma 5.3.1, we find that

$$\mathrm{Hom}(R_{\rho_\eta}, \mathbb{Q}_p[\epsilon]) \cong D_{\rho_\eta}(\mathbb{Q}_p[\epsilon]) \cong t_{\rho_\eta} \cong H^1(G_F, \mathrm{ad}(\rho_\eta)).$$

This proves the result. □

Proposition 5.3.8. *A triple* $(\Theta, \lambda_1, \lambda_2) \in H^1(G_F, \mathrm{ad}(\rho_\eta)) \oplus \mathbb{Q}_p^2$ *corresponds to a nearly ordinary deformation of* ρ_η *if and only if*

$$c|_{G_{\mathfrak{p}_1}} = \lambda_1(1-\chi) \quad \textit{and} \quad \chi b|_{G_{\mathfrak{p}_2}} = \lambda_2(\chi - 1), \quad \textit{where} \quad \Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proof. By its definition and using Proposition 5.2.9, we have

$$t^{\mathrm{no}}_{\rho_\eta} = \mathrm{Hom}(R^{\mathrm{no}}_{\rho_\eta}, \mathbb{Q}_p[\epsilon]) = \mathrm{Hom}(R_{\rho_\eta}[\![X,Y]\!]/I, \mathbb{Q}_p[\epsilon]),$$

where $I = I_1 + I_2$ as in the proof of Proposition 5.2.9. In other words, the tangent space $t^{\mathrm{no}}_{\rho_\eta}$ is identified with those elements from $\mathrm{Hom}(R_{\rho_\eta}[\![X,Y]\!], \mathbb{Q}_p[\epsilon])$ for which all images of expressions generating the ideal I vanish inside $\mathbb{Q}_p[\epsilon]$. If

$$\rho^{\mathrm{univ}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then the ideal I is generated by the expressions

$$\begin{aligned} \gamma(x) + (\delta(x) - \alpha(x))X - \beta(x)X^2 & \quad \text{for all } x \in G_{\mathfrak{p}_1}; \\ \beta(y) + (\alpha(y) - \delta(y))Y - \gamma(y)Y^2 & \quad \text{for all } y \in G_{\mathfrak{p}_2}. \end{aligned}$$

Now let $\varphi \in \mathrm{Hom}(R_{\rho_\eta}[\![X,Y]\!], \mathbb{Q}_p[\epsilon])$. By Lemma 5.3.7 above, there is a unique triple $(\Theta, \lambda_1, \lambda_2) \in H^1(G_F, \mathrm{ad}(\rho_\eta)) \oplus \mathbb{Q}_p^2$ such that

$$\varphi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (1 + \epsilon\Theta)\rho_\eta = \begin{pmatrix} 1 + a\epsilon & \chi\eta + \chi[\eta a + b]\epsilon \\ c\epsilon & \chi + \chi[\eta c + d]\epsilon \end{pmatrix}$$

and such that $\varphi(X) = \lambda_1\epsilon$ and $\varphi(Y) = \lambda_2\epsilon$. We can now compute the constraints posed by the vanishing on I, using that $\epsilon^2 = 0$ to simplify our expressions, to be

$$c = \lambda_1(1-\chi) \quad \text{on } G_{\mathfrak{p}_1} \quad \text{and} \quad \chi\eta + \chi[\eta a + b]\epsilon + \lambda_2(1-\chi)\epsilon = 0 \quad \text{on } G_{\mathfrak{p}_2}.$$

However, η is chosen to be trivial on $G_{\mathfrak{p}_2}$, and as such, we obtain the two expressions that we were to prove. □

Lemma 5.3.9. *There is a well-defined surjective homomorphism*

$$f : H^1(G_F, \mathrm{ad}(\rho_\eta)) \to H^1(G_{\mathfrak{p}_1}, \mathbb{Q}_p(\chi)) \oplus H^1(G_{\mathfrak{p}_2}, \mathbb{Q}_p(\chi))$$

given by, adopting the usual notation for the components of Θ*,*

$$(\Theta, \lambda_1, \lambda_2) \mapsto \big(c|_{G_{\mathfrak{p}_1}}, b|_{G_{\mathfrak{p}_2}}\big).$$

Proof. The proof of Theorem 5.3.6 showed the existence of a surjective map

$$H^1(G_F, \mathrm{ad}(\rho_\eta)) \to H^1(G_F, \mathbb{Q}_p(\chi))$$

which is easily seen to be given by mapping $\Theta \in H^1(G_F, \mathrm{ad}(\rho_\eta))$ to $c \in H^1(G_F, \mathbb{Q}_p(\chi))$. On the other hand, when restricting to $G_{\mathfrak{p}_2}$, by virtue of $\eta|_{G_{\mathfrak{p}_2}} = 0$, the representation ρ_η reduces to $\mathbb{1} \oplus \chi$. Using Lemma 5.2.1, it follows that also $b|_{G_{\mathfrak{p}_2}} \in H^1(G_{\mathfrak{p}_2}, \mathbb{Q}_p(\chi))$. Combining these two maps yields f. For its surjectivity, it suffices to show that f its not trivial on the second component. However, the map restricted to the submodule $H^1(G_F, W_2)$ of $H^1(G_F, \mathrm{ad}(\rho_\eta))$ is clearly surjective, completing the proof. □

Proposition 5.3.10. *The tangent space* $t^{\mathrm{no}}_{\rho_\eta}$ *is in bijection with* $\ker(f)$.

Proof. For $i \in \{1, 2\}$, fix $x_i \in G_{\mathfrak{p}_i} \setminus G_L$. Given $\Theta \in \ker(f)$, we may construct a nearly ordinary triple $(\Theta, \lambda_1, \lambda_2)$ using the association

$$\Theta \mapsto \big(\Theta, c(x_1)/2, b(x_2)/2\big).$$

This is in fact nearly ordinary, because by virtue of Θ being in the kernel of f, the cocycles $c|_{\mathfrak{p}_1}$ and $b|_{\mathfrak{p}_2}$ are coboundaries and as such, are given by $c|_{\mathfrak{p}_1} = \mu_1(1-\chi)$ and $\chi b|_{\mathfrak{p}_2} = \mu_2(\chi - 1)$ for certain $\mu_1, \mu_2 \in \mathbb{Q}_p$. Using that $\chi(x_i) = -1$ by construction, evaluating yields that $\mu_1 = c(x_1)/2$ and $\mu_2 = b(x_2)/2$ are uniquely determined. Comparing with Proposition 5.3.8, this shows that the triple is indeed nearly ordinary. Conversely, to any nearly ordinary triple we may associate its first component Θ, since by Proposition 5.3.8, for nearly ordinary triples, $c|_{\mathfrak{p}_1}$ and $b|_{\mathfrak{p}_2}$ must be coboundaries, and thus Θ must in fact be inside $\ker(f)$. Since these operations are evidently inverse, this establishes the proposition. □

Corollary 5.3.11. *The tangent space $t^{\text{no}}_{\rho_\eta}$ is 3-dimensional.*

Proof. Using Lemma 5.3.9 and Proposition 5.3.10 above, we have identified this tangent space as the kernel of the surjective map

$$f : t_{\rho_\eta} \to H^1(G_{\mathfrak{p}_1}, \mathbb{Q}_p(\chi)) \oplus H^1(G_{\mathfrak{p}_2}, \mathbb{Q}_p(\chi))$$

Using Theorem 5.3.6 and Corollary 5.1.6, we may now conclude that

$$\dim\, t^{\text{no}}_{\rho_\eta} = 5 - 2 = 3;$$

as claimed. □

5.4 Representations on Hecke algebras

Let $\mathbb{T}^{\text{no}} := h^{\text{n.o.}}_{\mathbf{1},\mathbf{0}}(K(p^\infty), \mathbb{Q}_p)$ denote the nearly ordinary Hecke algebra as defined in [Hid89a] and sketched in Appendix A.5. We recall that this inherently adèlic object is generated by Hecke operators $T_\mathfrak{l}$ for prime ideals $\mathfrak{l} \subset \mathcal{O}_F$ with $\mathfrak{l} \nmid p$, and operators U_{π_i} for $i \in \{1, 2\}$ as π_i ranges over the uniformisers of $\mathcal{O}_{F,\mathfrak{p}_i}$. Let $E_{1,\chi}$ denote the parallel weight $(1, 1)$ Hilbert Eisenstein series as defined in Appendix A.2 and let

$$f := (1 - V_{\mathfrak{p}_1})(1 + V_{\mathfrak{p}_2})E_{1,\chi} = E^{(p)}_{1,\chi}$$

be the appropriate p-stabilisation that is core to all our arguments. A priori there are four ways to p-stabilise $E_{1,\chi}$, corresponding with the signs $(+, +)$, $(+, -)$, $(-, +)$ and $(-, -)$. The choices $(-, +)$ and $(+, -)$ yield equivalent theories, but with the roles of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ reversed. However, the other two choices will not work for our purposes for two reasons. First and foremost, from the proof of Lemma A.3.3, one easily deduces that these two other choices do not yield a p-adic cusp form, and therefore we have no hope of deforming it as one.

The second reason is more subtle, and it will manifest itself throughout this section. If we choose the same sign twice, then associated with this p-stabilisation is a nearly ordinary deformation problem in which we would insist both $G_{\mathfrak{p}_1}$ and $G_{\mathfrak{p}_2}$ to fix a lift of *the same* line among $\langle e_1 \rangle$ or $\langle e_2 \rangle$. This is because the sign of the p-stabilisation is connected to the quotient character of the associated nearly ordinary Galois representation we are deforming. This character is χ for the line $\langle e_1 \rangle$ and $\mathbb{1}$ for $\langle e_2 \rangle$. Evaluated at $\mathfrak{p}_i$ for $i \in \{1, 2\}$, these characters yield -1 and 1 respectively. In this sense, our choice of p-stabilisation matches our Definition 5.2.8 of a nearly ordinary deformation.

There are again two reasons for why it is important to work with a deformation problem in which the stable lines for the decomposition groups are lifts of different lines. The first is that it enables us to carry out the strategy that we will follow in this section; in particular, in the proof of Proposition 5.4.10 we will use that the nearly ordinary condition affects the lower left entry for one of the decomposition groups, and the upper right entry for the other, cutting out a 1-dimensional subspace of the 2-dimensional space $H^1(G_F, \mathbb{Q}_p(\chi))$. Without this information, it is not obvious how to proceed.

The second reason for this importance is again rather subtle and will only manifest itself in Chapter 6. To prove Theorem B, we will have to take the diagonal restriction of an infinitesimal cuspidal p-adic family of modular forms with respect to the ideal $\mathcal{D}_F^{-1}\mathfrak{q}_1$. However, as explained in Remark A.3.6, its $\epsilon = 0$ specialisation will *not* vanish. In view of Lemma 2.1 in [DPV21], which requires this vanishing to ensure that the derivative of such a family is still an overconvergent modular form, this fact might jeopardise our strategy. However, we remedy this issue by choosing in Section 6.2 a deformation in the so-called *anti-parallel* weight direction, which ensures that the weight of the diagonal restriction of our family remains constant. Then we may still use Lemma 2.1 in [DPV21] to conclude that our derivative is an overconvergent modular form by subtracting a constant family; we use this trick in the proof of Proposition 6.3.1. The weight character for nearly ordinary modular forms can be computed from the two quotient characters. If we had been working with a deformation problem in which both decomposition groups would have fixed lines lifting the same line, then the two quotient characters would have been equal and we would not have had the freedom required to write down a deformation with the properties that we need.

The modular form $f = E_{1,\chi}^{(p)}$ defines a morphism $\mathbb{T}^{\mathrm{no}} \to \mathbb{Q}_p$ by sending a Hecke operator to its f-eigenvalue. Let $\mathbb{T}$ be the nilreduction of the completion of the localisation of $\mathbb{T}^{\mathrm{no}}$ at the prime ideal $\mathfrak{m}_f$ given by the kernel of this morphism. Let $\mathbb{K}$ be its ring of fractions, which is a product of fields. Then Hida proved the following in [Hid89a].

Theorem 5.4.1. *There exists a unique semisimple Galois representation* $\pi : G_F \to \mathrm{GL}_2(\mathbb{K})$ *with the following properties:*

- π *is continuous, odd and unramified outside* p*;*
- *For each prime* $\mathfrak{l} \nmid p$*, it holds that*

$$\det(1 - \pi(\mathrm{Frob}_{\mathfrak{l}})X) = 1 - T_{\mathfrak{l}}X + \langle \mathfrak{l} \rangle \mathrm{Nm}(\mathfrak{l})X^2;$$

- *For $i \in \{1,2\}$ there exist characters $\epsilon_i, \delta_i : G_{\mathfrak{p}_i} \to \mathbb{T}^\times$ such that, up to equivalence, when restricted to $G_{\mathfrak{p}_i}$, the representation π is of the form*

$$\pi(\sigma) = \begin{pmatrix} \epsilon_i(\sigma) & * \\ 0 & \delta_i(\sigma) \end{pmatrix} \quad \text{for} \quad \sigma \in G_{\mathfrak{p}_i}.$$

- *If we identify $G_{\mathfrak{p}_i}^{\mathrm{ab}} \cong F_{\mathfrak{p}_i}^\times$ through the arithmetic local reciprocity map, then we have the identity $\delta_i(x) = U_x$ for all $x \in F_{\mathfrak{p}_i}^\times$.*

To obtain from this a deformation of ρ_η, we are to refine this representation in two ways. First, we must find a stable lattice inside $\mathbb{K}^2$ so that we obtain a representation $G_F \to \mathrm{GL}_2(\mathbb{T})$ instead, which we may then reduce modulo its maximal ideal $\mathfrak{m}_f$. Secondly, we must insist that this reduction equals ρ_η. Let us achieve these two results in succession. The former will turn out to consume most of the work. The techniques used in this section reflect strongly those employed in Section 4 of [DKV18], but again see also the works [Poz19, BDP22, BD16, BDS20, BC06].

Lemma 5.4.2. *There exist an element $\gamma \in G_F$ and a basis of $\{e_1, e_2\}$ of $\mathbb{K}^2$ such that $\chi(\gamma) = -1$ and in addition,*

$$\pi(\gamma) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

where $\lambda_1 \equiv 1 \mod \mathfrak{m}_f$ and $\lambda_2 \equiv -1 \mod \mathfrak{m}_f$, and such that the unique lines fixed by the subgroup $G_{\mathfrak{p}_i}$ for $i \in \{1,2\}$, can be written as $\langle e_1 + y_i e_2 \rangle$ where $y_i \in \mathbb{K}^\times$.

Proof. This is the content of Lemma 4.3, Equation 64 in Section 4.2 and Lemma 4.6 in [DKV18]. The fact that there must be a unique fixed line comes down to the fact that π is irreducible, and the statement about the lines being in general position is not deep considering $\mathbb{K}$ is a product of fields, and as such, almost all lines are of that form. □

In any basis as in the lemma above, write

$$\pi(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$$

for certain functions $a, b, c, d : G_F \to \mathbb{K}$.

Lemma 5.4.3. *The functions a and d take values in $\mathbb{T}$. In fact, it holds that $a \equiv \mathbb{1} \mod \mathfrak{m}_f$ and $d \equiv \chi \mod \mathfrak{m}_f$.*

Proof. For any prime $\mathfrak{l} \nmid p$, using Theorem 5.4.1, we have that

$$a(\mathrm{Frob}_{\mathfrak{l}}) + d(\mathrm{Frob}_{\mathfrak{l}}) = \mathrm{Tr}(\pi(\mathrm{Frob}_{\mathfrak{l}})) = T_{\mathfrak{l}} \in \mathbb{T}.$$

In other words, the continuous map $\mathrm{Tr}(\pi) : G_F \to \mathbb{K}$ takes on integral values for every element $\mathrm{Frob}_{\mathfrak{l}}$ for $\mathfrak{l} \nmid p$. By Chebotarev's Density Theorem, these Galois elements are dense inside G_F, and as such, by continuity the result follows for all of G_F. We may then observe that also

$$\lambda_1 a(\mathrm{Frob}_{\mathfrak{l}}) + \lambda_2 d(\mathrm{Frob}_{\mathfrak{l}}) = \mathrm{Tr}(\pi(\gamma\mathrm{Frob}_{\mathfrak{l}})) \in \mathbb{T}.$$

Combining these two expressions and using that $\lambda_1 - \lambda_2 \in \mathbb{T}^\times$ then yields that a and d both must have integral image themselves, completing the proof of the first claim. To see the second, we observe that $T_{\mathfrak{l}} \equiv 1 + \chi(\mathfrak{l}) \mod \mathfrak{m}_f$ because $f = E_{1,\chi}^{(p)}$ has eigenvalue $1 + \chi(\mathfrak{l})$ for the operator $T_{\mathfrak{l}}$. Again, by continuity, this implies that for any $\sigma \in G_F$, it holds that

$$a(\sigma) + d(\sigma) = \mathrm{Tr}(\pi(\sigma)) \equiv 1 + \chi(\sigma) \mod \mathfrak{m}_f$$

and as such, it also holds that

$$a(\sigma) - d(\sigma) \equiv \lambda_1 a(\sigma) + \lambda_2 d(\sigma) = \mathrm{Tr}(\pi(\gamma\sigma)) \equiv 1 - \chi(\sigma) \mod \mathfrak{m}_f.$$

Combining these two equations shows the claim from the lemma. □

Lemma 5.4.4. *For any $\sigma, \tau \in G_F$, it holds that $b(\sigma)c(\tau) \in \mathfrak{m}_f$.*

Proof. This follows from the fact that π is a homomorphism;

$$\begin{pmatrix} a(\sigma\tau) & b(\sigma\tau) \\ c(\sigma\tau) & d(\sigma\tau) \end{pmatrix} = \pi(\sigma\tau) = \pi(\sigma)\pi(\tau) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}.$$

Comparing the top left entry then yields the equality

$$a(\sigma\tau) = a(\sigma)a(\tau) + b(\sigma)c(\tau).$$

By the above, all values of a must be contained in $1 + \mathfrak{m}_f\mathbb{T}$, and as such, the expression $b(\sigma)c(\tau)$ must be inside of $\mathfrak{m}_f$ for all $\sigma, \tau \in G_F$. □

Definition 5.4.5. Let B denote the $\mathbb{T}$-submodule of $\mathbb{K}$ generated by all elements of the form $b(\sigma)$ for $\sigma \in G_F$, and C the analogous submodule using the elements $c(\sigma)$.

Lemma 5.4.6. *There are well-defined injective maps*

$$j_B : \mathrm{Hom}_{\mathbb{T}}(B/\mathfrak{m}_f B, \mathbb{T}/\mathfrak{m}_f) \to H^1(G_F, \mathbb{Q}_p(\chi)) : \phi \mapsto \chi \cdot (\phi \circ b);$$
$$j_C : \mathrm{Hom}_{\mathbb{T}}(C/\mathfrak{m}_f C, \mathbb{T}/\mathfrak{m}_f) \to H^1(G_F, \mathbb{Q}_p(\chi)) : \psi \mapsto \psi \circ c.$$

Proof. To show that the maps are well-defined, we write out the equations for the off-diagonal entries we obtain when we require π to be a homomorphism. These yield respectively that

$$b(\sigma\tau) = d(\tau)b(\sigma) + a(\sigma)b(\tau) \quad \text{and} \quad c(\sigma\tau) = a(\tau)c(\sigma) + d(\sigma)c(\tau).$$

We may now write out, using that a and d take values in $\mathbb{T}$ and as such commute with the application of ϕ by its assumed $\mathbb{T}$-linearity,

$$\begin{aligned}(\chi \cdot (\phi \circ b))(\sigma\tau) &= \chi(\sigma\tau)\phi(b(\sigma\tau)) \\ &= \chi(\sigma\tau)\big(d(\tau)\phi(b(\sigma)) + a(\sigma)\phi(b(\tau))\big) \\ &\equiv \chi(\sigma\tau)\big(\chi(\tau)\phi(b(\sigma)) + \phi(b(\tau))\big) \mod \mathfrak{m}_f \\ &= (\chi \cdot (\phi \circ b))(\sigma) + \chi(\sigma)(\chi \cdot (\phi \circ b))(\tau),\end{aligned}$$

showing well-definedness. A very similar calculation shows that $\psi \circ c \in Z^1(G_F, \mathbb{Q}_p(\chi))$. It remains to show that, if the result is a coboundary, then ϕ must have been identically zero, again because the proof for j_C will be very similar. So let us suppose that for some $\lambda \in \mathbb{T}/\mathfrak{m}_f$, it holds that $\chi \cdot (\phi \circ b) = \lambda(1 - \chi)$. In particular, this means that $G_L \subset \ker(\chi \cdot (\phi \circ b))$, or equivalently, $G_L \subset \ker(\phi \circ b)$. Any other element of G_F can be written as $\gamma\tau$ for some $\tau \in G_L$. Using that $b(\gamma) = 0$, as can be read off from the matrix $\pi(\gamma)$, we compute that

$$(\phi \circ b)(\gamma\tau) = d(\tau)(\phi \circ b)(\gamma) + a(\gamma)(\phi \circ b)(\tau) \equiv d(\tau) \cdot \phi(0) + 1 \cdot 0 = 0 \mod \mathfrak{m}_f,$$

showing that $\phi \circ b$ is trivial everywhere. Since ϕ is defined on the module generated by all the images of b, it must be trivial itself. □

We continue to exploit the knowledge that π is nearly ordinary to obtain certain local information about the entries b and c, which we will later use to deduce further global properties of the modules B and C. It is quite remarkable how a subtle dance between global properties and information about local restrictions turns out to provide us with all the conclusions that we need.

Lemma 5.4.7. *It holds that* $\epsilon_1 \equiv \mathbb{1} \mod \mathfrak{m}_f$ *and* $\delta_1 \equiv \chi \mod \mathfrak{m}_f$. *Similarly, it holds that* $\epsilon_2 \equiv \chi \mod \mathfrak{m}_f$ *and* $\delta_2 \equiv \mathbb{1} \mod \mathfrak{m}_f$.

Proof. We have seen before that $\mathrm{Tr}(\pi(\sigma)) \equiv 1 + \chi(\sigma) \mod \mathfrak{m}_f$ for all $\sigma \in G_F$, so if we have shown the above claims for δ_1 and δ_2, the results for ϵ_1 and ϵ_2 follow. To determine δ_1, we recall from Theorem 5.4.1 that $\delta_1(x) = U_x$ for all $x \in G_{\mathfrak{p}_i}^{\mathrm{ab}} \cong F_{\mathfrak{p}_i}^\times$. By construction, the p-stabilisation $f = E_{1,\chi}^{(p)} = (1 - V_{\mathfrak{p}_1})(1 + V_{\mathfrak{p}_2})E_{1,\chi}$ is a U_{π_1}-eigenvector with eigenvalue -1 for all uniformisers π_1 inside $F_{\mathfrak{p}_i}^\times$. Therefore, $U_{\pi_1} \equiv -1 = \chi(\mathfrak{p}_1) \mod \mathfrak{m}_f$ and so $\delta_1 \equiv \chi \mod \mathfrak{m}_f$; the proof for δ_2 is similar. □

Proposition 5.4.8. *The map* $c \mod \mathfrak{m}_f C$ *is a coboundary when restricted to* $G_{\mathfrak{p}_1}$. *Similarly, the map* $\chi \cdot b \mod \mathfrak{m}_f B$ *is a coboundary when restricted to* $G_{\mathfrak{p}_2}$.

Proof. We only show the claim about $c \mod \mathfrak{m}_f C$, for the result for $\chi \cdot b \mod \mathfrak{m}_f B$ can be proved analogously. Let

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

be the change of basis matrix that changes π into the upper triangular form from Theorem 5.4.1 on $G_{\mathfrak{p}_1}$. Then it must satisfy for all $\sigma \in G_{\mathfrak{p}_1}$,

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} \epsilon_1(\sigma) & * \\ 0 & \delta_1(\sigma) \end{pmatrix}.$$

By Lemma 4.6 in [DKV18], we may assume that $x, z \in \mathbb{K}^\times$. Comparing the bottom left entries, we obtain that

$$xc(\sigma) + zd(\sigma) = z\epsilon_1(\sigma), \quad \text{or equivalently,} \quad c(\sigma) = \frac{z}{x}\big(\epsilon_1(\sigma) - d(\sigma)\big).$$

Choose any $\tau \in G_{\mathfrak{p}_1}$ such that $\chi(\tau) = -1$. Using Lemma 5.4.7 above in combination with Lemma 5.4.3, we find that $\epsilon_1(\tau) \equiv 1 \not\equiv -1 \equiv d(\tau) \mod \mathfrak{m}_f$, and as a result, it follows that $\epsilon_1(\tau) - d(\tau) \notin \mathfrak{m}_f$. In other words, it belongs to $\mathbb{T}^\times$. This shows that

$$\frac{z}{x} = c(\tau) \cdot \big(\epsilon_1(\tau) - d(\tau)\big)^{-1} \in C,$$

since C was the $\mathbb{T}$-module *generated* by the images of c. Therefore

$$c(\sigma) \equiv \frac{z}{x}\big(\epsilon_1(\sigma) - d(\sigma)\big) \equiv \frac{z}{x}\big(1 - \chi(\sigma)\big) \mod \mathfrak{m}_f C,$$

which shows that c is a coboundary $\mod \mathfrak{m}_f C$ on $G_{\mathfrak{p}_1}$. □

Corollary 5.4.9. *The map $c \mod \mathfrak{m}_f C$ is* not *a coboundary when restricted to $G_{\mathfrak{p}_2}$. Similarly, the map $\chi \cdot b \mod \mathfrak{m}_f B$ is* not *a coboundary when restricted to $G_{\mathfrak{p}_1}$.*

Proof. Again, we only show the claim about $c \mod \mathfrak{m}_f C$, for the result for $\chi \cdot b \mod \mathfrak{m}_f B$ can be proved analogously. Let us therefore suppose that c is a coboundary when restricted to both $G_{\mathfrak{p}_1}$ and $G_{\mathfrak{p}_2}$. We claim that c is then a coboundary globally.

To see this, first note that $c \in H^1(G_F, C/\mathfrak{m}_f C)$ if the action of G_F on $C/\mathfrak{m}_f C$ is through the character χ. Indeed, by Lemma 5.4.3,

$$c(\sigma\tau) = a(\tau)c(\sigma) + d(\sigma)c(\tau) \equiv c(\sigma) + \chi(\sigma)c(\tau) \mod \mathfrak{m}_f C.$$

Through a proof similar to that of Corollary 5.1.6, we may now conclude that the natural restriction maps

$$H^1(G_F, C/\mathfrak{m}_f C) \xrightarrow{\sim} H^1(G_{\mathfrak{p}_1}, C/\mathfrak{m}_f C) \oplus H^1(G_{\mathfrak{p}_2}, C/\mathfrak{m}_f C)$$

induce an isomorphism; here we have implicitly used that $C/\mathfrak{m}_f C$ is a finitely generated module over $\mathbb{T}/\mathfrak{m}_f\mathbb{T} \cong \mathbb{Q}_p$, as shown in Lemme 4 in [BC06]. Any cocycle that is a coboundary at the two places above p of F, must thus have been globally trivial to begin with. It follows that there exists some $\lambda \in C/\mathfrak{m}_f C$ such that $c(\sigma) \equiv \lambda \cdot (1 - \chi(\sigma)) \mod \mathfrak{m}_f C$. In particular, it follows that $0 = c(\gamma) \equiv 2\lambda \mod \mathfrak{m}_f C$, and as such, it follows that $\lambda \equiv 0 \mod \mathfrak{m}_f C$ and hence $C/\mathfrak{m}_f C = 0$. By Nakayama's Lemma, it follows from this that $C = 0$ globally, and as such, c must be the zero-cocycle. However, π is irreducible; this is a contradiction and this completes the proof. □

Proposition 5.4.10. *Both B and C are free $\mathbb{T}$-modules of rank 1.*

Proof. Using the same argument as Lemme 4 in [BC06], it follows that both B and C are $\mathbb{T}$-modules of finite type. We claim that it suffices to show that the images of the maps j_B and j_C are 1-dimensional inside $H^1(G_F, \mathbb{Q}_p(\chi))$. Indeed, if this is true, by the injectivity established by Lemma 5.4.6, using that $\mathbb{T}/\mathfrak{m}_f\mathbb{T} \cong \mathbb{Q}_p$, it would follow that the space $\mathrm{Hom}_{\mathbb{T}}(B/\mathfrak{m}_f B, \mathbb{Q}_p)$ is 1-dimensional. In other words, $B/\mathfrak{m}_f B$ is generated by a single element. Using Nakayama's Lemma, the same must then hold for B itself, completing the proof. Of course, the proof for C is analogous. Now, Proposition and Corollary 5.4.9 combine to imply that both b and c will be locally trivial at precisely one of the two places above p. These constraints cut out a 1-dimensional subspace $H^1(G_{\mathfrak{p}_i}, \mathbb{Q}_p(\chi)) \subset H^1(G_F, \mathbb{Q}_p(\chi))$; this proves the proposition. □

Corollary 5.4.11. *There exists a basis of $\mathbb{K}^2$ such that the image of π takes values in $\mathbb{T}^2$ and is upper triangular $\mod \mathfrak{m}_f$. The $\mathbb{T}$-module spanned by these basis vectors is G_F-stable.*

Proof. By Proposition 5.4.10 above, we can find an element $b_0 \in B$ generating the $\mathbb{T}$-module B. Now consider the basis $\{b_0e_1, e_2\}$, in which π looks like

$$\pi(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma)b_0^{-1} \\ c(\sigma)b_0 & d(\sigma) \end{pmatrix}.$$

Since $B = \mathbb{T} \cdot b_0$, it follows that $b(\sigma)b_0^{-1} \in \mathbb{T}$ for all $\sigma \in G_F$. By Lemma 5.4.4, it further follows that $c(\sigma)b_0 \in \mathfrak{m}_f$ for all $\sigma \in G_F$. This means that π takes values in $\mathbb{T}^2$, and as such, it stabilises the lattice $M = \langle b_0e_1, e_2\rangle$. □

We now rescale our original choice of basis vectors as in the corollary above, to avoid having to continue to take b_0 with us in all our future calculations.

Proposition 5.4.12. *The representation π in the basis above is, up to a scalar unit multiple of one of the basis vectors, a lift of ρ_η.*

Proof. From Lemma 5.4.3 and the corollary above, we know that

$$a \equiv \mathbb{1} \mod \mathfrak{m}_f, \quad d \equiv \chi \mod \mathfrak{m}_f \quad \text{and} \quad c \equiv 0 \mod \mathfrak{m}_f,$$

so it suffices to show that $b(\sigma) \equiv \lambda \cdot \eta \mod \mathfrak{m}_f$ for some $\lambda \in \mathbb{Q}_p^\times$. Recall that η is characterised up to such a scalar λ by being a generator for the 1-dimensional subspace of $H^1(G_F, \mathbb{Q}_p(\chi))$ of cocycles that vanish on the decomposition group $G_{\mathfrak{p}_2} \subset G_F$. This means that it suffices to show that $b(\sigma) \in \mathfrak{m}_f$ for all $\sigma \in G_{\mathfrak{p}_2}$ to complete the proof. However, we already assume that $b \mod \mathfrak{m}_f$ is a coboundary when restricted to $G_{\mathfrak{p}_2}$, and similarly to before, evaluating at γ then yields that $b \mod \mathfrak{m}_f$ must vanish on all of $G_{\mathfrak{p}_2}$. As a result, it must be a scalar multiple of η. Since $b \mod \mathfrak{m}_f$ cannot be trivial when restricted to $G_{\mathfrak{p}_1}$, as a result of Corollary 5.4.9, it follows that $b \mod \mathfrak{m}_f$ is even a non-zero scalar multiple of η. As such, we may scale the basis by a further unit to ensure equality, completing the proof. □

Finally, to conclude that π in any basis that satisfies the conclusion from Proposition 5.4.12 is actually a *deformation* of ρ_η when it comes to the specific nearly ordinary deformation problem at hand, it remains to identify free direct summands of rank 1 inside $\mathbb{T}^2$ on which suitable restrictions of π act scalar. We are given the existence of such lines in $\mathbb{K}^2$, and to obtain the necessary lines in $\mathbb{T}^2$, we follow the general reasoning in the proof of Proposition 2.3.1 in [Poz19].

Theorem 5.4.13. *Consider π from Theorem 5.4.1 in any basis satisfying the conclusion from Proposition 5.4.12. Then π defines a nearly ordinary deformation of ρ_η.*

Proof. It suffices to exhibit for $i \in \{1,2\}$ a free direct summand L_i of rank 1 inside $\mathbb{T}^2$ which is stable under the restriction $\pi|_{G_{\mathfrak{p}_i}}$ and which lifts e_i. We demonstrate how to obtain L_1; the construction of L_2 is analogous. Recall that we can find matrices satisfying

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} \epsilon_1(\sigma) & * \\ 0 & \delta_1(\sigma) \end{pmatrix},$$

with $x, z \in \mathbb{K}^\times$, yielding the equalities

$$b(\sigma) = \frac{x}{z}\big(\epsilon_1(\sigma) - a(\sigma)\big) \quad \text{and} \quad c(\sigma) = \frac{z}{x}\big(\epsilon_1(\sigma) - d(\sigma)\big).$$

After this change of basis, the stable line for the image of π will be generated by

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} e_1 = \begin{pmatrix} x \\ z \end{pmatrix}.$$

If $\epsilon_1 \equiv \chi \mod \mathfrak{m}_f$, we would find using identical reasoning as in the proof of Lemma 5.4 that $b \mod \mathfrak{m}_f$ must be a coboundary on $G_{\mathfrak{p}_1}$, contradicting that c is already a coboundary there in view of Corollary 5.4.9. As such, it follows that $\epsilon_1 \equiv 1 \mod \mathfrak{m}_f$ and again choosing some $\tau \in G_{\mathfrak{p}_1} \setminus G_L$, we find that

$$\frac{z_1}{x_1} = c(\tau)\big(\epsilon_1(\tau) - d(\tau)\big)^{-1} \in \mathfrak{m}_f.$$

where we used that c takes values in $\mathfrak{m}_f$. We thus set

$$L_1 = \left\langle \begin{pmatrix} 1 \\ z/x \end{pmatrix} \right\rangle.$$

Indeed, this line is fixed by π and as $z/x \in \mathfrak{m}_f$, the line L_1 reduces to $\langle e_1 \rangle$. This completes the proof. $\square$

5.5 A modularity theorem

The goal of this section will be to prove an isomorphism $R^{\text{no}}_{\rho_\eta} \cong \mathbb{T}$. We have already constructed the map; indeed, Theorem 5.4.13 claims that there exists a nearly ordinary deformation π of ρ_η to $\mathbb{T}$ satisfying various properties. By the universal property of $R^{\text{no}}_{\rho_\eta}$, this induces a map

$$\mathcal{T} : R^{\text{no}}_{\rho_\eta} \to \mathbb{T}$$

that induces this deformation from ρ^{univ}.

Lemma 5.5.1. *The map $\mathcal{T} : R^{\text{no}}_{\rho_\eta} \to \mathbb{T}$ is surjective.*

Proof. Let $\Lambda = \mathbb{Q}_p[\![X, Y, Z]\!]$ be as defined in Section 2.2 and 3.1 in [BDS20]. Both $R^{\text{no}}_{\rho_\eta}$ and $\mathbb{T}$ carry a natural Λ-algebra structure and the map $\mathcal{T}$ defined above is generally Λ-linear. Since $\mathbb{T}$ is generated over Λ by the operators $T_{\mathfrak{l}}$, $\langle \mathfrak{l} \rangle$ and U_x for $x \in \mathcal{O}_F \otimes \mathbb{Z}_p$, it suffices to show that these are contained in the image of $\mathcal{T}$. We use the defining relation that $\pi = \mathcal{T} \circ \rho^{\text{univ}}$ to show for $\mathfrak{l} \nmid p$ that

$$\mathcal{T}\big(\text{Tr}(\rho^{\text{univ}}(\text{Frob}_{\mathfrak{l}}))\big) = \text{Tr}\big(\mathcal{T}(\rho^{\text{univ}}(\text{Frob}_{\mathfrak{l}}))\big) = \text{Tr}(\pi(\text{Frob}_{\mathfrak{l}})) = T_{\mathfrak{l}}.$$

Similarly,

$$\mathcal{T}\big(\det(\rho^{\text{univ}}(\text{Frob}_{\mathfrak{l}}))\big) = \det(\pi(\text{Frob}_{\mathfrak{l}})) = \langle \mathfrak{l} \rangle \text{Nm}(\mathfrak{l}).$$

It now suffices to consider the operators U_x for $x \in F \otimes \mathbb{Z}_p \cong F_{\mathfrak{p}_1} \times F_{\mathfrak{p}_2}$. Use the stable lines for ρ^{univ} and π to construct bases. If we then let Δ denote the bottom right entry of ρ^{univ}, we then obtain for $x \in F^\times_{\mathfrak{p}_1}$ that $\mathcal{T}\big(\Delta(x)\big) = \delta_1(x) = U_x$. The top-left entry yields the same result for $x \in F^\times_{\mathfrak{p}_2}$, completing the proof. $\square$

To show that the map $\mathcal{T} : R^{\text{no}}_{\rho_\eta} \to \mathbb{T}$ must in fact be an isomorphism, we use some general results from commutative algebra. Namely, it turns out that we have now gathered enough data to proceed by commutative algebraic considerations.

Lemma 5.5.2. *Let k be a field and $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be local k-algebras. Suppose that $\dim_k(\mathfrak{m}_A/\mathfrak{m}_A^2) = \dim(B) < \infty$ and that there is a surjective map of k-algebras $A \to B$. Then A and B are both regular local rings with the same finite Krull dimension.*

Proof. Write $\dim_k(\mathfrak{m}_A/\mathfrak{m}_A^2) = \dim(B) = d$. Using the surjective map $A \to B$, it follows that $\dim(A) \geq \dim(B) = d$. By Krull's Principal Ideal

Theorem, see the Stacks Project [00KD], it also holds that $\dim(A) \leq \dim_k(\mathfrak{m}_A/\mathfrak{m}_A^2) = d$. It follows that $\dim(A) = d$ and as such, A is regular. Similarly, the existence of the surjection implies that $\dim_k(\mathfrak{m}_B/\mathfrak{m}_B^2) \leq \dim_k(\mathfrak{m}_A/\mathfrak{m}_A^2) = d$. However, again by Krull's Principal Ideal Theorem, we know that also $\dim_k(\mathfrak{m}_B/\mathfrak{m}_B^2) \geq \dim(B) = d$. We conclude that also $\dim_k(\mathfrak{m}_B/\mathfrak{m}_B^2) = d$ and so, B must also be regular and of the same dimension as A. □

Proposition 5.5.3. *Let k be a field and let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be Noetherian regular local k-algebras with the same finite Krull dimension. Then every surjective map $f : A \to B$ must be an isomorphism.*

Proof. Because f is a map of local rings, it follows that $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$ and as such, $f(\mathfrak{m}_A^n) \subset \mathfrak{m}_B^n$ for any integer $n \geq 1$. This means that for any integer $n \geq 1$, the morphism f descends to a map

$$f_n : A/\mathfrak{m}_A^n \to B/\mathfrak{m}_B^n.$$

We will show that each f_n is an isomorphism. Assuming this, it would follow that f itself must be an isomorphism. Indeed, if $f(a) = 0$ for some $a \in A$, then also $f_n(a) = 0$ for all $n \geq 1$ and as such, $a \in \mathfrak{m}_A^n$ for all $n \geq 1$. But by Krull's Intersection Theorem, see the Stacks Project [00IP], we may then conclude that $a = 0$, completing the proof.

It remains to show that each f_n is an isomorphism. We prove this with induction, starting with the observation that the map $A \to B/\mathfrak{m}_B \cong k$ is a surjective map to a field, so its kernel must be a maximal ideal of A. Since A is a local ring, it follows that this maximal ideal must be $\mathfrak{m}_A$ and as such, $f_1 : A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ must be an isomorphism.

Now suppose that for some $n \geq 1$, we have shown that f_n is an isomorphism. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1} & \longrightarrow & A/\mathfrak{m}_A^{n+1} & \longrightarrow & A/\mathfrak{m}_A^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \\ 0 & \longrightarrow & \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} & \longrightarrow & B/\mathfrak{m}_B^{n+1} & \longrightarrow & B/\mathfrak{m}_B^n & \longrightarrow & 0 \end{array}$$

If we can show that the map $\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1} \to \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}$ is always an isomorphism of k-vector spaces, then by the five lemma we would conclude our induction step.

To this end, we let $b_1, \ldots, b_d \in \mathfrak{m}_B$ be a k-basis of the tangent space $\mathfrak{m}_B/\mathfrak{m}_B^2$. Because f is surjective, we can find $a_1, \ldots, a_d \in A$ such that $f(a_i) = b_i$ for all $1 \leq i \leq d$. Because f_1 was an isomorphism, we know that even $a_1, \dots, a_d \in \mathfrak{m}_A$. In fact, we claim that these elements form a basis for the tangent space $\mathfrak{m}_A/\mathfrak{m}_A^2$. Indeed, they are linearly independent over k, for if they were not, we would find non-zero $\lambda_i \in k$ such that

$$\sum_{i=1}^{d} \lambda_i a_i \in \mathfrak{m}_A^2 \overset{f(\cdot)}{\Longrightarrow} \sum_{i=1}^{d} \lambda_i b_i \in f(\mathfrak{m}_A^2) \subset \mathfrak{m}_B^2.$$

However, this contradicts the linear independence of the b_i. By the regularity assumption and the fact that A and B have the same dimension, their tangent spaces must also have the same dimension. This shows that the map $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ induced by f is an isomorphism.

To finish the proof, we appeal to Lemma 10.106.1 in the Stacks Project [00NO], which implies that k-bases for $\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ and $\mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}$ are given by

$$\left\{ \prod_{i=1}^{d} a_i^{n_i} \;\middle|\; n_i \geq 0,\ \sum_{i=1}^{d} n_i = n \right\} \quad \text{and} \quad \left\{ \prod_{i=1}^{d} b_i^{m_i} \;\middle|\; m_i \geq 0,\ \sum_{i=1}^{d} m_i = n \right\}$$

respectively. Now we remark that

$$f\left(\prod_{i=1}^{d} a_i^{n_i} \right) = \prod_{i=1}^{d} b_i^{n_i}$$

to conclude that f takes one basis to another, thus inducing an isomorphism of k-vector spaces. This concludes the proof. ☐

Theorem 5.0.1. *The map $\mathcal{T} : R_{\rho_\eta}^{\mathrm{no}} \to \mathbb{T}$ is an isomorphism.*

Proof. Corollary 5.3.11 shows that $\dim(t_{\rho_\eta}^{\mathrm{no}}) = 3$ and from Proposition A.5.5, it follows that $\mathbb{T}$ is equidimensional of dimension 3. By Lemma 5.5.1, the map $\mathcal{T}$ is surjective. Now Lemma 5.5.2 implies that both $R_{\rho_\eta}^{\mathrm{no}}$ and $\mathbb{T}$ are regular of the same Krull dimension; namely 3. Because they are also Noetherian, Proposition 5.5.3 implies that the surjective map $\mathcal{T}$ must in fact be an isomorphism, completing the proof. ☐