CM-values of p-adic Theta-functions

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CHAPTER 4

Genus theory and quaternion algebras

In this chapter we carry out **Step 1** of our p-adic analytic proof of Theorem B as outlined in Section 2.4. First, we construct two exact sequences using elementary arithmetic in biquadratic fields and basic class field theory. Next, we will explore genus theory, a study pioneered by Carl Friedrich Gauß, which allows us to describe a map connecting these two exact sequences to form a longer one. We proceed to define an F-quadratic form $\det_F$ on B_q that refines the quaternion norm in the sense that $\mathrm{tr} \circ \det_F = \mathrm{Nm}$ and we give various formulae to compute it concretely. Finally, we combine these preliminaries to prove the bijective nature of an explicit and purely algebraic construction that links quaternions in a rational quaternion algebra with class number 1 (or equivalently, with type number 1) to $\mathcal{O}_L$-ideals of a specified norm. This construction is used to rewrite the left hand side of Theorem B into a more useful form in Section 6.1.

Throughout this chapter, we will let $\epsilon_F \in \mathcal{O}_F^\times$ denote a fundamental unit for the field F. By Dirichlet's Unit Theorem, the group $\mathcal{O}_L^\times$ is, up to torsion, also free of rank 1 and as such, we may also specify a fundamental unit $\epsilon_L \in \mathcal{O}_L^\times$. In particular, this means that

$$\mathcal{O}_F^\times = \{\pm 1\} \times \langle \epsilon_F \rangle \quad \text{and} \quad \mathcal{O}_L^\times = \mu_L \times \langle \epsilon_L \rangle,$$

where $\mu_L \subset \mathcal{O}_L^\times$ denotes the finite subgroup of roots of unity in L.

Our efforts are aimed at studying B_q; the definite quaternion algebra over $\mathbb{Q}$ with discriminant $-q$. Recall that $R_q \subset B_q$ denotes a maximal order and that we assume that it is *unique* up to conjugation. In Section 4.4, we will explain how the two embeddings $\alpha_i : \mathcal{O}_i \to R_q$ for $i \in \{1, 2\}$ allow us to view B_q as a 1-dimensional L-vector space. In Section 4.5, we explain how the set of all pairs of embeddings $\mathcal{O}_i \to B_q$ for $i \in \{1, 2\}$ carries a faithful action of $\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ and how one can associate a *reflex ideal* $\mathfrak{q}_1 \subset \mathcal{O}_F$ to any pair of embeddings in the orbit of (α_1, α_2).

For any choice of $[c_1] \in \mathrm{Pic}(K_1)$ and $[c_2] \in \mathrm{Pic}(K_2)$, we will show the existence of an F-quadratic form $\det_F[c_1, c_2] : B_q \to F$ with the property that $\mathrm{tr} \circ \det_F[c_1, c_2] = \mathrm{Nm}$. The following is proved in [HY12] and encompasses the main result of this chapter.

Theorem 4.0.1. *For any $\nu \in F^+$, the number of triples*

$$(b, [c_1], [c_2]) \in (\mathcal{O}_1^\times \setminus R_q \,/\, \mathcal{O}_2^\times) \times \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$$

satisfying the property that $\det_F[c_1, c_2](b) = \nu$, equals $\rho(\nu \mathfrak{q}_1^{-1} \mathcal{D}_F)$.

Proof. The assumption that R_q is the only maximal order of B_q means for $i \in \{1, 2\}$ that any elliptic curve E_i with CM by $\mathcal{O}_i$ with supersingular

reduction at q must satisfy $\mathrm{End}(E_i) \cong R_q$. Then $\mathrm{Hom}(E_1, E_2) \cong R_q$ and we may identify the form $\deg_{\mathrm{CM}}$ in [HY12] by the form $\det_F$ introduced above. Without defining $O_\ell(\nu, E_1, E_2)$ here, we remark that the proof of Proposition 2.18 in [HY12] shows that the quantity we are trying to count is equal to

$$\frac{2}{\#\mathcal{O}_1^\times \#\mathcal{O}_2^\times} \sum_{[c_1],[c_2]} \sum_{\substack{\phi \in B \\ \det_F [c_1,c_2](\phi) = \nu}} \mathbb{1}_{R_q}(\phi) = \prod_\ell O_\ell(\nu, E_1, E_2) = \rho(\nu \mathfrak{q}_1^{-1} \mathcal{D}_F),$$

where the second equality follows from Corollary 2.34 in [HY12]. □

In [HY12], this counting problem is tackled using an adèlic approach, which has the drawback that it is not very explicit. The main purpose of this chapter is to describe an explicit bijection between the two sets whose cardinalities are related in Theorem 4.0.1 in the case that the quaternion algebra B_q has a unique maximal order, which has the added benefit of being global and not adèlic in nature. We sketch this now.

Any pair $([c_1], [c_2]) \in \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ yields an L-vector space structure on B and thus allows us to choose an L-linear isomorphism $\iota[c_1, c_2] : B_q \xrightarrow{\sim} L$. Even though this isomorphism is not unique, one may define an L-ideal associated with $b \in B_q$ by writing $I[c_1, c_2]_b := \iota[c_1, c_2](b)/\iota[c_1, c_2](R_q)$. The following is this chapter's main result.

Theorem 4.0.2. *For any $\nu \in F^+$, the association $(b, [c_1], [c_2]) \mapsto I[c_1, c_2]_b$ establishes a bijection between the set of*

$$(b, [c_1], [c_2]) \in (\mathcal{O}_1^\times \setminus R_q \,/\, \mathcal{O}_2^\times) \times \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$$

with the property that $\det_F[c_1, c_2](b) = \nu$ and the set of integral ideals $I \subset \mathcal{O}_L$ such that $\mathrm{Nm}_{L/F}(I) = \nu \mathfrak{q}_1^{-1} \mathcal{D}_F$.

Before we start, we record here some important properties of the field F and the character χ that we will use numerous times throughout all remaining chapters.

Lemma 4.0.3. *It holds that $\chi(\mathcal{D}_F) = -1$. In addition, $\epsilon_F \gg 0$, and therefore $\#\mathrm{Pic}(F)^+ = 2\#\mathrm{Pic}(F)$.*

Proof. The extension L/F is a CM-extension unramified at all finite places, so χ is totally odd and the first statement is immediate. Therefore, the ideal $(\sqrt{D})$ cannot be trivial in the narrow class group. If ϵ_F were not totally positive, then $(\sqrt{D}) = (\epsilon_F \sqrt{D})$ would have been trivial; therefore $\epsilon_F \gg 0$. The final statement also follows. □

4.1 Two exact sequences

To get started, we need the following simple lemma.

Lemma 4.1.1. *The torsion subgroup $\mu_L \subset \mathcal{O}_L^\times$ is given by $\mathcal{O}_1^\times \mathcal{O}_2^\times$.*

Proof. For $\zeta_n \in L$ it must hold that $\varphi(n) \leq 4$, where φ denotes Euler's totient function. This leaves only a few options: if $n \in \{5, 10\}$, then $L = \mathbb{Q}(\zeta_5)$ is not biquadratic. If $\zeta_8 \in L$, then $L = \mathbb{Q}(\zeta_8)$ and L has the quadratic subfields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$, but none of these have coprime discriminants, so $L = \mathbb{Q}(\zeta_8)$ is not among the fields we consider. If $\zeta_{12} \in L$, then $L = \mathbb{Q}(\zeta_{12}) = \mathbb{Q}(i, \sqrt{3})$, satisfying the claim from the lemma. In all other cases, the torsion of $\mathcal{O}_L^\times$ only defines at most a quadratic extension of $\mathbb{Q}$, and as such will come from one of the imaginary quadratic subfields of L. □

Proposition 4.1.2. *The following sequence is exact:*

$$1 \to \{\pm 1\} \to \mathcal{O}_1^\times \times \mathcal{O}_2^\times \xrightarrow{(x,y)\mapsto xy} \mathcal{O}_L^\times \xrightarrow{\mathrm{Nm}_{L/F}} \mathcal{O}_F^{\times,+}.$$

Proof. We check each entry. Exactness at $\{\pm 1\}$ is trivial. Exactness at $\mathcal{O}_1^\times \times \mathcal{O}_2^\times$ is also clear, because we assume $D_1 \neq D_2$ and because $\mathcal{O}_i^\times = \{\pm 1\}$ unless $D \in \{-3, -4\}$, both unit groups only share $\{\pm 1\}$ in all cases. It remains to prove exactness at $\mathcal{O}_L^\times$. Clearly, for any $u_i \in \mathcal{O}_i^\times$, we have that

$$\mathrm{Nm}_{L/F}(u_i) = \mathrm{Nm}_{K_i/\mathbb{Q}}(u_i) = 1,$$

because the norm from an imaginary quadratic field is positive definite. For the other inclusion, let $u \in \mathcal{O}_L^\times$ be given such that $\mathrm{Nm}_{L/F}(u) = 1$. By Lemma 4.1.1 above, we have

$$\mathcal{O}_L^\times = (\mathcal{O}_1^\times \mathcal{O}_2^\times) \times \langle \epsilon_L \rangle$$

As a result, we may write that $u = u_1 u_2 \epsilon_L^k$ for some $u_1 \in \mathcal{O}_1^\times$ and $u_2 \in \mathcal{O}_2^\times$, so that

$$1 = \mathrm{Nm}_{L/F}(u) = \mathrm{Nm}_{L/F}(u_1)\mathrm{Nm}_{L/F}(u_2)\mathrm{Nm}_{L/F}(\epsilon_L^k) = \mathrm{Nm}_{L/F}(\epsilon_L)^k.$$

So $\mathrm{Nm}_{L/F}(\epsilon_L) \in F$ is a kth root of unity. However, F is totally real, and as such, it follows that if $k \neq 0$, it must hold that $\mathrm{Nm}_{L/F}(\epsilon_L) \in \{\pm 1\}$. However, this would imply that the image of $\mathcal{O}_L^\times$ under the map $\mathrm{Nm}_{L/F}$ is at most $\{\pm 1\}$, which is nonsense, as $\mathcal{O}_F^\times \subset \mathcal{O}_L^\times$ and on this free subgroup of rank 1, the map $\mathrm{Nm}_{L/F}$ is just squaring. □

Remark 4.1.3. Note that if $u \in \mathcal{O}_L^\times$, then

$$u^2 \in \mathcal{O}_1^\times \mathcal{O}_2^\times \mathrm{Nm}_{L/F}(\mathcal{O}_L^\times) \subset \mathcal{O}_1^\times \mathcal{O}_2^\times \mathcal{O}_F^{\times,+}.$$

Indeed, since $\mathrm{Nm}_{K_i/\mathbb{Q}}$ is positive definite, it follows for any $u \in \mathcal{O}_L^\times$ that

$$\mathrm{Nm}_{L/\mathbb{Q}}(u) = \mathrm{Nm}_{K_i/\mathbb{Q}}(\mathrm{Nm}_{L/K_i}(u)) = 1.$$

We may then employ the pretty trick of invoking the equality

$$\begin{aligned} u^2 &= u^2 \mathrm{Nm}_{L/\mathbb{Q}}(u) = u^3 \sigma_1(u)\sigma_2(u)\sigma_F(u) \\ &= \mathrm{Nm}_{L/K_1}(u) \cdot \mathrm{Nm}_{L/K_2}(u) \cdot \mathrm{Nm}_{L/F}(u), \end{aligned}$$

showing the claim. This also shows that

$$[\mathcal{O}_F^{\times,+} : \mathrm{Nm}_{L/F}(\mathcal{O}_L^\times)] \leq 2.$$

There are natural maps $\mathrm{Pic}(K_i) \to \mathrm{Pic}(L)$ by sending $I \subset \mathcal{O}_i$ to the ideal $I\mathcal{O}_L \subset \mathcal{O}_L$. Multiplied together, these combine to form a map

$$\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2) \to \mathrm{Pic}(L).$$

For the second exact sequence, we will again require a small lemma.

Lemma 4.1.4. *For any* $[J] \in \mathrm{Pic}(L)$,

$$[J] = [\sigma_F(J)] \in \mathrm{Pic}(L) \big/ \big(\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)\big).$$

Proof. We use a trick similar to the one in Remark 4.1.3. Indeed, we have that

$$\mathrm{Nm}_{L/\mathbb{Q}} : \mathrm{Pic}(L) \to \mathrm{Pic}(\mathbb{Q}) = \{0\}$$

is the trivial map. As such, we find that

$$\begin{aligned} [J] &= [J] + [\mathrm{Nm}_{L/\mathbb{Q}}(J)] = 2[J] + [\sigma_1(J)] + [\sigma_2(J)] + [\sigma_F(J)] \\ &= [\mathrm{Nm}_{L/K_1}(J)] + [\mathrm{Nm}_{L/K_2}(J)] + [\sigma_F(J)] \\ &\equiv [\sigma_F(J)] \mod \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2), \end{aligned}$$

completing the proof. □

Proposition 4.1.5. *The following sequence is exact:*

$$\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2) \to \mathrm{Pic}(L) \xrightarrow{\mathrm{Nm}_{L/F}} \mathrm{Pic}(F)^+ \xrightarrow{\chi} \{\pm 1\} \to 1.$$

Proof. This time we start checking exactness on the left hand side. At $\{\pm 1\}$ it is clear from the definition and the surjectivity of the restriction map $\mathrm{Gal}(H_F^+/F) \to \mathrm{Gal}(L/F)$. For exactness at $\mathrm{Pic}(F)^+$, we employ a commutative square from class field theory:

$$\begin{array}{ccc} \mathrm{Pic}(L) & \xrightarrow{\sim} & \mathrm{Gal}(H_L/L) \\ {\scriptstyle \mathrm{Nm}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Pic}(F)^+ & \xrightarrow{\sim} & \mathrm{Gal}(H_F^+/F). \end{array}$$

The image of the norm map in the left column coincides with the image of the restriction map on the right, which is equal to $\mathrm{Gal}(H_F^+/L)$. By definition, this equals the kernel of χ. Finally, we show exactness at $\mathrm{Pic}(L)$. First, similar to before, we have for any $[I_i] \in \mathrm{Pic}(K_i)$,

$$\mathrm{Nm}_{L/F}(I_i) = \mathrm{Nm}_{K_i/\mathbb{Q}}(I_i) \in \mathrm{Pic}(\mathbb{Q}) = \{0\},$$

and as such, indeed $[I_i]$ will be in the kernel. We now let $M \subset H_L$ denote the fixed field of the image of $\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ inside of $\mathrm{Pic}(L) \cong \mathrm{Gal}(H_L/L)$. By the above, we may again use class field theory to note that

$$\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2) \subset \ker(\mathrm{Nm}_{L/F})$$

if and only if

$$\mathrm{Gal}(H_L/M) \subset \ker\big(\mathrm{Gal}(H_L/L) \to \mathrm{Gal}(H_F^+/F)\big) = \mathrm{Gal}(H_L/H_F^+).$$

As such, it follows that $H_F^+ \subset M$. It also follows that to show exactness, it suffices to show that the fields H_F^+ and M are in fact equal. To this end, let $\tau \in \mathrm{Gal}(H_L/L)$ be arbitrary. Then by class field theory, it corresponds to some class $[J] \in \mathrm{Pic}(L)$ and the class $[\sigma_F(J)]$ will correspond to $\sigma_F \tau \sigma_F^{-1} \in \mathrm{Gal}(H_L/L)$. We know that $[J]$ and $[\sigma_F(J)]$ agree up to $\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$, and as such, it follows that τ and $\sigma_F \tau \sigma_F^{-1}$ agree in the quotient group $\mathrm{Gal}(M/L)$, i.e. their restrictions to M coincide.

Note that M/L is unramified everywhere because $M \subset H_L$. Hence M/F is only ramified at infinity. If it would be abelian, then by the maximiality of H_F^+ for these properties, it would follow that $M \subset H_F^+$, completing the proof. Indeed, let $\tau \in \mathrm{Gal}(M/L)$. Then the above says that $\tau = \sigma_F \tau \sigma_F^{-1}$ on M. In other words, τ and σ_F commute. Now $\mathrm{Gal}(M/L)$ is abelian by definition of H_L and since $\mathrm{Gal}(M/F)$ is generated by $\mathrm{Gal}(M/L)$ and σ_F, all of which commute, it follows that the whole group $\mathrm{Gal}(M/F)$ is abelian, as desired. □

Our next goal will be to describe a rather subtle map $\mathcal{O}_F^{\times,+} \to \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ that connects the two exact sequences from Propositions 4.1.2 and 4.1.5. To do this, however, we will need some results from genus theory.

4.2 Genus theory

We briefly review Gauß's genus theory in the language of group cohomology. Throughout this section, we let $I(F)$ denote the group of $\mathcal{O}_F$-ideals and $P(F) \subset I(F)$ the subgroup of principal ideals. Also, we let $P(F)^+ \subset P(F)$ denote the subgroup of principal ideals generated by a totally positive element. Then by definition, we have

$$\mathrm{Pic}(F) := I(F)/P(F) \quad \text{and} \quad \mathrm{Pic}(F)^+ := I(F)/P(F)^+.$$

Because $F/\mathbb{Q}$ is of degree 2, its Galois group $G := \mathrm{Gal}(F/\mathbb{Q}) = \langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$ is cyclic. We may use the elements

$$\mathcal{N} := 1 + \sigma \quad \text{and} \quad \Delta := 1 - \sigma$$

in the group ring $\mathbb{Z}[G]$ to define for any G-module the group $H^0(G, A) := \ker(\Delta) = A^\sigma$, the group of σ-invariants, and for any $n \geq 1$:

$$H^{2n-1}(G, A) := \frac{\ker(\mathcal{N})}{\mathrm{im}(\Delta)} \quad \text{and} \quad H^{2n}(G, A) := \frac{A^\sigma}{\mathrm{im}(\mathcal{N})}$$

which makes sense since $\mathcal{N}\Delta = \Delta\mathcal{N} = 1 - \sigma^2 = 0$. We get to where we want to be through a series of quick lemmas.

Lemma 4.2.1. *It holds that $H^1(G, F^{\times,+}) = 0$ and $H^2(G, \mathcal{O}_F^{\times,+}) = 0$.*

Proof. For the first claim, let $x \in F^{\times,+}$ be such that $\mathrm{Nm}(x) = 1$. We must find $y \in F^{\times,+}$ such that $y/\sigma(y) = x$. The existence of such a $y \in F^\times$ is assured by Hilbert 90, so it suffices to verify that it can be chosen totally positively. Indeed, if not, then $y/\sigma(y)$ would be negative under some embedding $F \to \mathbb{R}$, contradicting that $x \gg 0$.

For the second, we take $x \in \mathcal{O}_F^{\times,+}$ such that $\sigma(x) = x$. This means that $x \in \mathbb{Z}^{\times,+} = \{1\}$, and as such, the cohomology group is trivial. □

Lemma 4.2.2. *It holds that $H^1(G, \mathcal{O}_F^{\times,+}) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. We consider those $x \in \mathcal{O}_F^{\times,+}$ such that $\mathrm{Nm}(x) = 1$. Because they are totally positive, this set equals all of $\mathcal{O}_F^{\times,+}$. Since $\mathcal{O}_F^\times = \{\pm 1\} \times$

$\langle \epsilon_F \rangle$, using Lemma 4.0.3, we find that $\mathcal{O}_F^{\times,+} = \langle \epsilon_F \rangle$. Since $\epsilon_F \sigma(\epsilon_F) = \mathrm{Nm}(\epsilon_F) = 1$, it follows that $\epsilon_F^k / \sigma(\epsilon_F^k) = \epsilon_F^{2k}$, so the elements of the form $y/\sigma(y)$ as $y \in \mathcal{O}_F^{\times,+}$ form a subgroup of index 2. □

Lemma 4.2.3. *It holds that* $P(F)^{+,\sigma}/\mathbb{Q}^{\times,+} \cong \mathbb{Z}/2\mathbb{Z}$. *In addition, it also holds that* $H^1(G, P(F)^+) = 0$.

Proof. The short exact sequence defining $P(F)^+$ reads

$$1 \to \mathcal{O}_F^{\times,+} \to F^{\times,+} \to P(F)^+ \to 1.$$

Using that $F^\sigma = \mathbb{Q}$ and Lemma 4.2.1, part of the long exact sequence associated with this will then read

$$0 \to \mathbb{Q}^{\times,+} \to P(F)^{+,\sigma} \to H^1(G, \mathcal{O}_F^{\times,+}) \to 0 \to H^1(G, P(F)^+) \to 0 \to \dots$$

By Lemma 4.2.2, the first claim follows. Since the latter group we are after is now in between two zeroes, it must be zero itself as well. □

Lemma 4.2.4. *It holds that* $\mathrm{Pic}(F)^+[2] = \mathrm{Pic}(F)^{+,\sigma}$.

Proof. Note that for any ideal $I \in I(F)$, it holds that

$$I \cdot \sigma(I) = (\mathrm{Nm}(I)) \in P(F)^+, \quad \text{and thus} \quad [I] + [\sigma(I)] = 0 \in \mathrm{Pic}(F)^+.$$

As such, it follows that

$$[I] \in \mathrm{Pic}(F)^+[2] \iff [I] = [\sigma(I)] \in \mathrm{Pic}(F)^+ \iff [I] \in \mathrm{Pic}(F)^{+,\sigma},$$

as claimed. □

Theorem 4.2.5. *Let* s *denote the number of primes dividing* $D = D_1 D_2 > 0$. *Then* $\mathrm{Pic}(F)^+[2]$ *is a finite abelian 2-group of rank* $s - 1$, *generated by the ramified primes.*

Proof. The short exact sequence defining $\mathrm{Pic}(F)^+$ reads

$$1 \to P(F)^+ \to I(F) \to \mathrm{Pic}(F)^+ \to 1.$$

As such, the start of its long exact sequence will read

$$1 \to P(F)^{+,\sigma} \to I(F)^\sigma \to \mathrm{Pic}(F)^+[2] \to 1,$$

where we used both Lemma 4.2.3 and Lemma 4.2.4 above. It thus suffices to determine the rank of

$$\frac{I_F^\sigma}{P(F)^{+,\sigma}} \cong \frac{I(F)^\sigma/\mathbb{Q}^{\times,+}}{P(F)^{+,\sigma}/\mathbb{Q}^{\times,+}} \cong \frac{I(F)^\sigma/\mathbb{Q}^{\times,+}}{\mathbb{Z}/2\mathbb{Z}},$$

where we again used Lemma 4.2.3. We complete the proof by establishing an isomorphism

$$I(F)^\sigma/\mathbb{Q}^{\times,+} \to \prod_{r|D} \mathbb{Z}/2\mathbb{Z},$$

in which we send an ideal $I \in I(F)^\sigma$ to the s-tuple $\{v_r(\mathrm{Nm}(I))\}_{r|D}$ mod 2. Since all ramified prime ideals $\mathfrak{r}$ lying over $r \mid D$ are fixed by the Galois action, it is clear that this map is surjective. To show injectivity, let $I = \sigma(I)$ be any ideal in the kernel. By dividing out $\mathfrak{r}^2 = (r) \in \mathbb{Q}^+$ enough times, we may assume without loss of generality that $\mathrm{Nm}(I)$ is coprime with D. Similarly, we may divide I by any inert prime to assume without loss of generality that I is only divisible by primes that split in $F/\mathbb{Q}$. But if $(t) = \mathfrak{t}\sigma(\mathfrak{t})$ for some prime t and $\mathfrak{t} \mid I$, then $\sigma(\mathfrak{t}) \mid \sigma(I) = I$ and as such, it follows that even $t \mid I$. Thus we may divide out such primes too to find that there are no primes left to consider; $I = (1)$ and the claim is proved. □

In other words, the s ramified primes generate the 2-torsion in the narrow class group but since its rank is $s-1$, there must be some relation between them. We wish to identify this relation explicity. It turns out that the origin of this relation depends on the fundamental unit of F.

Theorem 4.2.6. *There exists some $y_F \in \mathcal{O}_F$ with the properties that $\epsilon_F = y_F/\sigma(y_F)$ and $\mathrm{Nm}(y_F) \mid D$. Further,*

$$\sum_{r|\mathrm{Nm}(y_F)} [\mathfrak{r}] = 0 \in \mathrm{Pic}(F)^+.$$

Proof. Since $\epsilon_F \gg 0$ by Lemma 4.0.3, it follows that $\mathrm{Nm}(\epsilon_F) = 1$. The existence of some $y_F \in F^\times$ such that $y_F/\sigma(y_F) = \epsilon_F$ is then assured by Hilbert 90. As in Lemma 4.2.1, it follows that even $y_F \in F^{\times,+}$. In particular it follows that $(y_F) = (\sigma(y_F))$ and as such $(y_F) \in I(F)^\sigma$. Using the same argument as in the proof of Theorem 4.2.5, it follows that we may adjust (y_F) by a rational number $c \in \mathbb{Q}$ to obtain an ideal (cy_F) of norm dividing D. Since $cy_F/\sigma(cy_F) = y_F/\sigma(y_F) = \epsilon_F$, we may rename cy_F to y_F to obtain $\mathrm{Nm}(y_F) \mid D$. Finally, it follows from its norm that the ideal $[(y_F)] = 0 \in \mathrm{Pic}(F)^+$ must factor as the product of the prime ideals lying over the ramified primes dividing it.

It now remains to show that this relation is non-trivial. In other words, we must exclude that $\mathrm{Nm}(y_F) = 1$, or equivalently, that $y \in \mathcal{O}_F^{\times,+}$. However, this is immediate from the proof of Lemma 4.2.2, which shows that the generator ϵ_F of $\mathcal{O}_F^{\times,+}$ cannot be of the form $y_F/\sigma(y_F)$ if y_F itself is taken from $\mathcal{O}_F^{\times,+}$. □

4.3 The key exact sequence

We can now describe the map

$$\varphi : \mathcal{O}_F^{\times,+} \to \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$$

whose existence was claimed before.

- If the map $\mathrm{Nm}_F^L : \mathcal{O}_L^\times \to \mathcal{O}_F^{\times,+}$ is surjective, then $\varphi := 0$.
- Otherwise, we let $\varphi(\epsilon_F)$ be the pair $([I_1],[I_2]) \in \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ where I_1 and I_2 are such that $\mathrm{Nm}(I_1) \cdot \mathrm{Nm}(I_2) = \mathrm{Nm}(y_F) \mid D$, where y_F is as in the second part of Theorem 4.2.6.

Note that the I_i for $i \in \{1,2\}$ are uniquely determined as the product over all primes in $\mathcal{O}_i$ above the primes dividing $\gcd(D_i, \mathrm{Nm}(y_F))$. We now verify that these choices correctly combine the exact sequences from Propositions 4.1.2 and 4.1.5. This connection requires a relation between the sizes of the class groups involved in terms of the fundamental units of the fields F and L. This relation is given by the analytic class number formula. Recall that the Dedekind zeta-function associated with a number field M is defined by

$$\zeta_M(s) = \prod_{\mathfrak{r} \subset \mathcal{O}_M} \left(1 - \mathrm{Nm}_{M/\mathbb{Q}}(\mathfrak{r})^{-s}\right)^{-1}.$$

One conventionally writes $\zeta_{\mathbb{Q}} = \zeta$, the Riemann-zeta function.

Proposition 4.3.1. *The following equality holds true:*

$$\zeta_L(s) \cdot \zeta(s)^2 = \zeta_{K_1}(s) \cdot \zeta_{K_2}(s) \cdot \zeta_F(s).$$

Proof. We use the Euler product expansion to reduce to showing an equality between the factors contributed by a single rational prime r. There are three cases: either r splits completely in L, or it is inert in precisely two of the fields K_1, K_2 and F and split in the third, or it is ramified in precisely two subfields. For primes of the first kind, we must verify that

$$\left(1 - r^{-s}\right)^4 \cdot \left(1 - r^{-s}\right)^2 = \left(1 - r^{-s}\right)^2 \cdot \left(1 - r^{-s}\right)^2 \cdot \left(1 - r^{-s}\right)^2,$$

which is obviously true. For primes of the second kind, we check that

$$\left(1 - r^{-2s}\right)^2 \cdot \left(1 - r^{-s}\right)^2 = \left(1 - r^{-2s}\right) \cdot \left(1 - r^{-2s}\right) \cdot \left(1 - r^{-s}\right)^2,$$

which is again clearly true. In the final case, depending on the splitting of the prime in the unramified field extension, we must check that

$$\left(1-r^{-s}\right)^2\cdot\left(1-r^{-s}\right)^2=\left(1-r^{-s}\right)\cdot\left(1-r^{-s}\right)\cdot\left(1-r^{-s}\right)^2,$$

and that

$$\left(1-r^{-2s}\right)\cdot\left(1-r^{-s}\right)^2=\left(1-r^{-s}\right)\cdot\left(1-r^{-s}\right)\cdot\left(1-r^{-2s}\right).$$

This completes the proof. □

Recall the *analytic class number formula* for any number field M,

$$\lim_{s\to 1}(s-1)\zeta_M(s)=\frac{2^r(2\pi)^s\mathrm{Reg}_M h_M}{w_M\sqrt{D_M}},$$

where r denotes the number of real embeddings of M and s the number of pairs of complex embeddings; Reg_M denotes the regulator, h_M the class number, w_M the number of roots of unity and D_M the discriminant. In combination with Proposition 4.3.1, this yields the following.

Proposition 4.3.2. *The following equality holds true;*

$$2|\log|\epsilon_L||h_L=|\log|\epsilon_F||h_1h_2h_F.$$

Proof. We obtain the following residues from the class number formula, using Lemma 4.1.1 to write $w_L=w_1w_2$,

$$\begin{aligned}
\lim_{s\to 1}(s-1)\zeta(s)&=1;\\
\lim_{s\to 1}(s-1)\zeta_{K_1}(s)&=\frac{2\pi h_1}{w_1\sqrt{D_1}};\\
\lim_{s\to 1}(s-1)\zeta_{K_2}(s)&=\frac{2\pi h_2}{w_2\sqrt{D_2}};\\
\lim_{s\to 1}(s-1)\zeta_F(s)&=\frac{|\log|\epsilon_F||h_F}{\sqrt{D_1D_2}};\\
\lim_{s\to 1}(s-1)\zeta_L(s)&=\frac{(2\pi)^2|\log|\epsilon_L||h_L}{w_1w_2\sqrt{D_L}/2}.
\end{aligned}$$

Hence Proposition 4.3.1 gives us that

$$\frac{(2\pi)^2|\log|\epsilon_L||h_L}{w_1w_2\sqrt{D_L}/2}=\frac{2\pi h_1}{w_1\sqrt{D_1}}\cdot\frac{2\pi h_2}{w_2\sqrt{D_2}}\cdot\frac{|\log|\epsilon_F||h_F}{\sqrt{D_1D_2}}.$$

Cancelling some terms on both sides, we obtain

$$\frac{2|\log|\epsilon_L||h_L}{\sqrt{D_L}} = \frac{|\log|\epsilon_F||h_1h_2h_F}{D_1D_2}.$$

According to Theorem 3 in [Wil70], we have $D_L = D_1^2D_2^2$ because we assumed D_1 and D_2 to be coprime, and as such, the equality reduces to the one from the proposition. □

Theorem 4.3.3. *With $\varphi : \mathcal{O}_F^{\times,+} \to \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ as defined at the start of this section, the following sequence is exact:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathcal{O}_1^\times \times \mathcal{O}_2^\times & \longrightarrow & \mathcal{O}_L^\times & \longrightarrow & \mathcal{O}_F^{\times,+} \\ \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2) & \longrightarrow & \mathrm{Pic}(L) & \longrightarrow & \mathrm{Pic}(F)^+ & \longrightarrow & \{\pm 1\} & \longrightarrow & 1. \end{array}$$

Proof. By Proposition 4.1.2 and 4.1.5, it suffices to show exactness at $\mathcal{O}_F^{\times,+}$ and $\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$. Recall from Lemma 4.0.3, we know that $\epsilon_F \gg 0$ and $2h_F = h_F^+$. We must distinguish two cases.

First suppose that the map $\mathcal{O}_L^\times \to \mathcal{O}_F^{\times,+} = \langle \epsilon_F \rangle$ is surjective, so that $\varphi = 0$ by definition. The sequence is then trivially a complex, so we reduce to showing that the map $\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2) \to \mathrm{Pic}(L)$ is injective. To this end, we note that $\mathrm{Nm}_F^L(\epsilon_L) = \epsilon_F$ and from Remark 4.1.3 we conclude that $\epsilon_L^2 \in \mathcal{O}_1^\times \mathcal{O}_2^\times \langle \epsilon_F \rangle$. As such, $\epsilon_L^2 = \zeta \epsilon_F^k$ for some root of unity ζ, and by applying σ_F to this expression, also $\sigma_F(\epsilon_L)^2 = \zeta^{-1}\epsilon_F^k$. Multiplying these two expressions yields $\mathrm{Nm}_F^L(\epsilon_L)^2 = \epsilon_F^{2k}$ and so it follows that $k = 1$. Taking absolute values, we find that $|\epsilon_L|^2 = |\epsilon_F|$ and as such, $2\log|\epsilon_L| = \log|\epsilon_F|$. Appealing to Proposition 4.3.2, which now yields that $h_L = h_1h_2h_F$, we obtain that $2h_L = h_1h_2h_F^+$ and so Proposition 4.1.5 implies the desired injectivity.

It remains to examine the case in which $\epsilon_F \gg 0$ and the map $\mathcal{O}_L^\times \to \mathcal{O}_F^{\times,+} = \langle \epsilon_F \rangle$ is *not* surjective. We first show that our choice of φ makes the whole into a complex. Indeed, the composition

$$\mathcal{O}_L^\times \to \mathcal{O}_F^{\times,+} \xrightarrow{\varphi} \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$$

is trivial because $\mathrm{Nm}_F^L(\epsilon_L) = \epsilon_F^2$ maps to the pair $([I_1^2], [I_2^2]) \in \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$; these classes are trivial because I_1 and I_2 are supported only at ramified primes, which are all 2-torsion. Next, we note that

$$\langle \epsilon_F \rangle = \mathcal{O}_F^{\times,+} \xrightarrow{\varphi} \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2) \to \mathrm{Pic}(L)$$

is always the zero map. Indeed, the ideal $I_1 I_2 \subset \mathcal{O}_L$ by construction contains the same primes as the ideal $y_F \mathcal{O}_L$, so they must be equal.

Exactness at $\mathcal{O}_F^{\times,+}$ is equivalent to the claim that $\varphi(\epsilon_F)$ is nontrivial. Suppose the contrary and write $I_1 = (y_1)$ and $I_2 = (y_2)$ for some $y_1 \in \mathcal{O}_1$ and $y_2 \in \mathcal{O}_2$. Now set $u = y_F/(y_1 y_2) \in L$. Then it is immediate that $\mathrm{Nm}(u) = 1$. We even claim that $u \in \mathcal{O}_L^\times$. Indeed, its possible non-zero valuations are supported on the ramified primes and by the norm equality, one will divide y_F if and only if it divides one of the y_i. The ramified primes in two quadratic subfields extend to the same primes in L, hence they will cancel out. We may now compute that

$$\mathrm{Nm}_F^L(u) = u\sigma_F(u) = \frac{y_F^2}{\mathrm{Nm}(y_1)\mathrm{Nm}(y_2)} = \frac{y_F^2}{\mathrm{Nm}(y_F)} = \frac{y_F^2}{y_F \sigma(y_F)} = \epsilon_F,$$

by construction of the element y_F. This contradicts the norm map $\mathcal{O}_L^\times \to \mathcal{O}_F^{\times,+} = \langle \epsilon_F \rangle$ not being surjective.

Finally, as $\mathrm{Nm}_F^L(\epsilon_L) = \epsilon_F^2$ now, using an argument analogous to the one used in the previous case, we deduce that $\log|\epsilon_L| = \log|\epsilon_F|$ in this case. One final application of Proposition 4.3.2, combined with $2h_F = h_F^+$, now yields that $4h_L = h_1 h_2 h_F^+$. Proposition 4.1.5 shows that the kernel of the map $\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2) \to \mathrm{Pic}(L)$ contains exactly 1 non-trivial element. As the image $\varphi(\epsilon_F)$ hits such an element, it surjects onto the kernel and exactness of the whole sequence has been established. □

4.4 An F-quadratic form

We now pivot to a seemingly unrelated algebraic construction, which mimics the construction of the form $\deg_{\mathrm{CM}}$ in [HY12] and which will turn out to be intimately linked with the results proved hitherto. The setting for the remainder of this chapter will be as follows.

Recall that we fixed two embeddings $\alpha_i : K_i \to B_q$ for $i \in \{1, 2\}$. This turns B_q into a 1-dimensional L-vector space as follows. Let $x \in K_1$ and $y \in K_2$. Then the action of the element $xy \in L$ on some $\gamma \in B_q$ is defined by $(xy) * \gamma := \alpha_1(x)\gamma\alpha_2(y)$. We may extend this definition to the whole of L by $\mathbb{Q}$-linearity and the observation that $L = K_1 \otimes K_2$. Since $[L : \mathbb{Q}] = [B_q : \mathbb{Q}] = 4$, indeed B_q must be 1-dimensional over L.

For notational convenience, we define

$$A := \alpha_1(\sqrt{D_1}) \quad \text{and} \quad B := \alpha_2(\sqrt{D_2}).$$

Note that $A^2 = \alpha_1(\sqrt{D_1})^2 = \alpha_1(D_1) = D_1$ and similarly $B^2 = D_2$. Furthermore, as they are traceless, we have $\overline{A} = -A$ and $\overline{B} = -B$.

The goal of this section is to refine the usual $\mathbb{Q}$-quadratic norm form $\mathrm{Nm} : B_q \to \mathbb{Q}$ to an F-quadratic form $B_q \to F$ whose trace to $\mathbb{Q}$ coincides with the reduced norm pairing. To this end, let us commence with a brief intermezzo on linear algebra over various base fields. In the forthcoming, both K and M number fields such that M/K is a finite extension. Let V be a finite dimensional M-vector space.

Lemma 4.4.1. *Every K-linear map $f : V \to K$ is uniquely of the form $\mathrm{tr}_{M/K} \circ g$ for some M-linear map $g : V \to M$.*

Proof. Let $d = \dim_M(V)$ and $n = [M : K]$, so that $\dim_K(V) = nd$. It is then obvious that the set $\{f : V \to K \mid f \text{ is } K\text{-linear}\}$ is an nd-dimensional K-vector space. Similarly, the set $\{g : V \to M \mid g \text{ is } M\text{-linear}\}$ is a d dimensional M-vector space, and hence also an nd-dimensional K-vector space. Composing with the trace gives a K-linear map between these two vector spaces, so to establish the bijection it suffices to show injectivity. If $\mathrm{tr} \circ g = 0$, then restricted to any 1-dimensional M-subspace V' of V, the map $\mathrm{tr} \circ g|_{V'} : V' \cong M \to M$ is the zero map. Every M-linear map $M \to M$ is given by mutliplication by some element $a \in M$. However, since the trace form is non-degenerate, the trace of this map is only identically zero when $a = 0$. It follows that $g|'_V = 0$ for all 1-dimensional $V' \subset V$, and thus $g = 0$, proving the injectivity and hence the claim. □

Recall that a K-bilinear form $f : V \times V \to K$ is called *M-equivariant* if $f(v, \lambda w) = f(\lambda v, w)$ for all $v, w \in V$ and $\lambda \in M$.

Lemma 4.4.2. *Every M-equivariant K-bilinear form $f : V \times V \to K$ is the trace of a unique M-bilinear form $g : V \times V \to M$.*

Proof. Let $w \in V$ be arbitrary. Then the map $f(-, w) : V \to K$ is K-linear and thus by the above uniquely of the form $\mathrm{tr}(g_w(-))$ for some M-linear map $g_w : V \to M$. We claim that the formula $g(v, w) = g_w(v)$ constitutes the desired M-bilinear map. This is clearly M-linear in the first component. For the second, for $\lambda \in M$, note that since

$$\begin{aligned}\mathrm{tr}(g_{w_1+\lambda w_2}(v)) &= f(v, w_1 + \lambda w_2) = f(v, w_1) + f(v, \lambda w_2) \\ &= \mathrm{tr}(g_{w_1}(v)) + f(\lambda v, w_2) = \mathrm{tr}(g_{w_1}(v) + \lambda g_{w_2}(v)),\end{aligned}$$

the non-degeneracy of the trace allows us to conclude that $g_{w_1+\lambda w_2} = g_{w_1} + \lambda g_{w_2}$. This establishes M-linearity in the second argument. The uniqueness is clear from the construction. □

Proposition 4.4.3. *There exists a unique F-quadratic form $\det_F : B_q \to F$ with the property that $\mathrm{tr}_{F/\mathbb{Q}}(\det_F(\gamma)) = \mathrm{Nm}(\gamma)$ for all $\gamma \in B_q$. In addition, $\det_F(\gamma)$ is totally positive for any $\gamma \in B_q^\times$.*

Proof. Consider the $\mathbb{Q}$-bilinear form

$$f : B_q \times B_q \to \mathbb{Q} : (\gamma_1, \gamma_2) \mapsto \mathrm{tr}\left(\gamma_1 \overline{\gamma_2}\right)/2.$$

We claim that it is F-equivariant. Indeed, by $\mathbb{Q}$-linearity it suffices to compute, using cyclicity of the trace, that

$$f\left(\sqrt{D} * \gamma_1, \gamma_2\right) = \mathrm{tr}\left(A\gamma_1 B\overline{\gamma_2}\right)/2 = \mathrm{tr}\left(\gamma_1 \overline{A\gamma_2 B}\right)/2 = f\left(\gamma_1, \sqrt{D} * \gamma_2\right),$$

where we used the anticommutivity of conjugation. We may then apply Proposition 4.4.2 to the F-equivariant quadratic form $f(\gamma, \gamma) = \mathrm{Nm}(\gamma)$ to obtain a unique F-bilinear form $g : B_q \times B_q \to F$ whose trace equals f. It follows that $\det_F(x) = g(x, x)$ satisfies the required property.

It remains to verify that its values are totally positive. Let $\gamma \in B_q^\times$ be arbitrary and write $\det_F(\gamma) = a$ for convenience. Since $\det_F$ is F-quadratic, we have that $\det_F(x\gamma) = x^2 a$ for all $x \in F$. Since B_q is positive definite, it follows that $\mathrm{tr}(x^2 a) > 0$ for all $x \in F^\times$.

Identify a with its image under one of the real embeddings, so that $\sigma(a)$ is the image under the other. We see that $a + (\sigma(x)/x)^2 \sigma(a) > 0$ for all possible $x \in F^\times$. Choosing any $x \in F^\times$ with $|\sigma(x)| < |x|$ ensures that $\lim_{n\to\infty}(\sigma(x)/x)^n = 0$; whence $a > 0$ and similarly $\sigma(a) > 0$. □

Even though it is convenient to know that a form

$$\det_F : B_q \to F$$

satisfying the properties from Proposition 4.4.3 exists, it might also be desirable to be able to compute it. It is natural to start with identifying the F-bilinear form $g : B_q \times B_q \to F$ whose trace is supposed to equal the $\mathbb{Q}$-bilinear form $f : B_q \times B_q \to \mathbb{Q} : (\gamma_1, \gamma_2) \mapsto \mathrm{tr}\left(\gamma_1 \overline{\gamma_2}\right)/2$. This is established in the next lemma.

Lemma 4.4.4. *The form*

$$g : B_q \times B_q \to F : (\gamma_1, \gamma_2) \mapsto \frac{\mathrm{tr}(\gamma_1 \overline{\gamma_2})}{4} + \frac{\mathrm{tr}(A\gamma_1 B\overline{\gamma_2})}{4\sqrt{D}}$$

is F-bilinear and satisfies $\mathrm{tr} \circ g = f$.

Proof. The second property is clear, and so by $\mathbb{Q}$-linearity it suffices to show that

$$g(\sqrt{D} * \gamma_1, \gamma_2) = \sqrt{D} g(\gamma_1, \gamma_2) = g(\gamma_1, \sqrt{D} * \gamma_2)$$

for any $\gamma_1, \gamma_2 \in B_q$. To this end, we compute that

$$\begin{aligned} g(\sqrt{D} * \gamma_1, \gamma_2) &= \frac{\mathrm{tr}(A\gamma_1 B\overline{\gamma_2})}{4} + \frac{\mathrm{tr}(A^2\gamma_1 B^2\overline{\gamma_2})}{4\sqrt{D}} \\ &= \frac{\mathrm{tr}(A\gamma_1 B\overline{\gamma_2})}{4} + \sqrt{D}\frac{\mathrm{tr}(\gamma_1\overline{\gamma_2})}{4} \\ &= \sqrt{D} g(\gamma_1, \gamma_2). \end{aligned}$$

Similarly, we may use cyclicity of the trace to compute that

$$\begin{aligned} g(\gamma_1, \sqrt{D} * \gamma_2) &= \frac{\mathrm{tr}(\gamma_1\overline{A\gamma_2 B})}{4} + \frac{\mathrm{tr}(A\gamma_1 B\overline{A\gamma_2 B})}{4\sqrt{D}} \\ &= \frac{\mathrm{tr}(\gamma_1 B\overline{\gamma_2} A)}{4} + \frac{\mathrm{tr}(A\gamma_1 B^2\overline{\gamma_2} A)}{4\sqrt{D}} \\ &= \frac{\mathrm{tr}(A\gamma_1 B\overline{\gamma_2})}{4} + D_2\frac{\mathrm{tr}(A^2\gamma_1\overline{\gamma_2})}{4\sqrt{D}} \\ &= \sqrt{D} g(\gamma_1, \gamma_2); \end{aligned}$$

this proves the lemma. □

Corollary 4.4.5. *It holds that*

$$\det_F : B_q \to F : \gamma \mapsto \frac{\mathrm{Nm}(\gamma)}{2} + \frac{\mathrm{tr}(A\gamma B\overline{\gamma})}{4\sqrt{D}}.$$

Proof. This is immediate from the above by restricting to the diagonal where $\gamma_1 = \gamma_2$, as this is how $\det_F$ is constructed. □

We next want to relate this expression to the fixed points τ_i, τ_i' for the actions of K_i for $\mathbb{C}_p$ for $i \in \{1, 2\}$. The following lemma connects the trace above to these fixed points.

Lemma 4.4.6. *Let $a, b, c \in \mathbb{C}_p$ be such that τ_1 is a root of the polynomial $aT^2 + bT + c$, normalised such that a equals the bottom left entry of A. Let $\gamma \in B_q$ be arbitrary and let $x, y, z \in \mathbb{C}_p$ be such that $\gamma\tau_2$ is a root of the polynomial $xT^2 + yT + z$, normalised such that x equals the bottom left entry of $\gamma B\gamma^{-1}$. Then*

$$-2\mathrm{tr}(A\gamma B\overline{\gamma}) = (2cx + 2az - by)\mathrm{Nm}(\gamma).$$

Proof. We prove this first for $\gamma = 1$, and we write

$$A = \begin{pmatrix} s_1 & s_2 \\ s_3 & -s_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} t_1 & t_2 \\ t_3 & -t_1 \end{pmatrix}.$$

Then

$$\mathrm{tr}(AB) = \mathrm{tr}\begin{pmatrix} s_1t_1 + s_2t_3 & s_1t_2 - s_2t_1 \\ s_3t_1 - s_1t_3 & s_3t_2 + s_1t_1 \end{pmatrix} = 2s_1t_1 + s_2t_3 + s_3t_2.$$

Since the matrices A and B fix τ_1 and τ_2 respectively, these numbers must satisfy the equations

$$s_3\tau_1^2 - 2s_1\tau_1 - s_2 = 0 \quad \text{and} \quad t_3\tau_2^2 - 2t_1\tau_2 - t_2 = 0.$$

Now note that the discriminants of these equations are given by

$$4(s_1^2 + s_2s_3) = 4D_1 \quad \text{and} \quad 4(t_1^2 + t_2t_3) = 4D_2$$

respectively, where these equalities are a result of the equations $A^2 = D_1$ and $B^2 = D_2$ respectively. It follows that

$$(a, b, c) = (s_3, -2s_1, -s_2) \quad \text{and} \quad (x, y, z) = (t_3, -2t_1, -t_2).$$

We may then finally compute that

$$2cx + 2az - by = -2s_2t_3 - 2s_3t_2 - 4s_1t_2 = 2\,\mathrm{tr}(AB);$$

this proves the claim for $\gamma = 1$. Now for arbitrary $\gamma \in B_q$, consider the embedding $\alpha_2' : K_2 \to B_q$ defined as $\gamma\alpha_2(-)\gamma^{-1}$. It is easy to see that now $\gamma\tau_2$ is a fixed point for the image of α_2'. For these two embeddings, the special case above implies that

$$-2\mathrm{tr}(A\gamma B\gamma^{-1}) = 2cx + 2az - by.$$

Multiplying both sides by $\mathrm{Nm}(\gamma)$ and using that $\gamma^{-1} = \overline{\gamma}/\mathrm{Nm}(\gamma)$ then yields the desired formula. □

Corollary 4.4.7. *Let $a, b, c, x, y, z \in \mathbb{C}_p$ be as in Lemma 4.4.6. Then*

$$\det_F : B_q \to F : \gamma \mapsto \mathrm{Nm}(\gamma)\left(\frac{1}{2} - \frac{2cx + 2az - by}{8\sqrt{D}}\right).$$

Proof. This is a direct consequence of combining Corollary 4.4.5 and Lemma 4.4.6 above. □

The following is crucial, though it will require a slightly laborious computation to verify. However, it will allow us to deduce the expression for $\det_F$ that will be most useful for our purposes. We invite the reader to compare these expressions to Equation 2.2.

Proposition 4.4.8. *Let $a, b, c, x, y, z \in \mathbb{C}_p$ be as in Lemma 4.4.6. Then*

$$\frac{(\tau_1 - \gamma\tau_2)(\tau_1' - \gamma\tau_2')}{(\tau_1 - \tau_1')(\gamma\tau_2 - \gamma\tau_2')} = \frac{1}{2} \pm \frac{2cx + 2az - by}{8\sqrt{D}};$$

the change of sign being caused by exchanging τ_i, τ_i' for some $i \in \{1, 2\}$.

Proof. We may label the two roots τ_1 and τ_1' of $aT^2 + bT + c$ in such way that $\tau_1 - \tau_1' = 2\sqrt{D_1}/a$, which follows from its equation and discriminant. Similarly, we choose τ_2 and τ_2' in such a way that $\gamma\tau_2 - \gamma\tau_2' = 2\sqrt{D_2}/x$. Then one computes that

$$\frac{(\tau_1 - \gamma\tau_2)(\tau_1' - \gamma\tau_2')}{(\tau_1 - \tau_1')(\gamma\tau_2 - \gamma\tau_2')} = ax\frac{(\tau_1 - \gamma\tau_2)(\tau_1' - \gamma\tau_2')}{4\sqrt{D}}.$$

One may now expand $(\tau_1 - \gamma\tau_2)(\tau_1' - \gamma\tau_2')$ to find it equals

$$\begin{aligned}
&\left(\frac{-b + 2\sqrt{D_1}}{2a} - \frac{-y + 2\sqrt{D_2}}{2x}\right)\left(\frac{-b - 2\sqrt{D_1}}{2a} - \frac{-y - 2\sqrt{D_2}}{2x}\right) \\
&= \frac{(-bx + 2x\sqrt{D_1} + ay - 2a\sqrt{D_2})(-bx - 2x\sqrt{D_1} + ay + 2a\sqrt{D_2})}{(2ax)^2} \\
&= \frac{(b^2 - 4D_1)x^2 + a^2(y^2 - 4D_2) - 2axby + 8ax\sqrt{D_1 D_2}}{(2ax)^2} \\
&= \frac{4acx^2 + 4a^2xz - 2axby + 8ax\sqrt{D}}{(2ax)^2} \\
&= \frac{2cx + 2az - by + 4\sqrt{D}}{2ax}.
\end{aligned}$$

Combining this with the above proves the proposition. □

Theorem 4.4.9. *The form $\det_F : B_q \to F$ is given by*

$$\det_F(\gamma) = \mathrm{Nm}(\gamma)\frac{(\tau_1 - \gamma\tau_2)(\tau_1' - \gamma\tau_2')}{(\tau_1 - \tau_1')(\gamma\tau_2 - \gamma\tau_2')},$$

for all $\gamma \in B_q$ and where τ_i and τ_i' for $i \in \{1, 2\}$ are the fixed points for the action of the image of $\alpha_i : \mathcal{O}_i \to B_q \to M_2(\mathbb{Q}_p)$ on $\mathbb{C}_p$.

Proof. This follows from combining the results of Corollary 4.4.7 and Proposition 4.4.8 above. □

4.5 From quaternions to ideals

The goal of this section is to prove Theorem 4.0.1 from [HY12] using an explicit, global bijection. Recall that the embeddings α_1 and α_2 turn B_q into a 1-dimensional L-vector space, and as such, we may choose some isomorphism

$$\iota : B_q \xrightarrow{\sim} L$$

of L-vector spaces. We use this to define for any $b \in B_q$ the ideal

$$I_b := \iota(b)/\iota(R_q) \subset L.$$

Lemma 4.5.1. *The set $\iota(R_q)$ defines a fractional ideal in L and for any $b \in R_q$, the ideal I_b is both integral and independent of the choice of isomorphism $\iota : B_q \xrightarrow{\sim} L$ of L-vector spaces.*

Proof. Recall that a fractional ideal in L is a finitely generated sub-$\mathcal{O}_L$-module of L, so since it is clearly finitely generated, for the first claim it suffices to verify that $\iota(R_q)$ is an $\mathcal{O}_L$-module. To this end, we observe that if $x \in \mathcal{O}_1$ and $y \in \mathcal{O}_2$, it follows that for any $b \in R_q$,

$$(xy)\iota(b) = \iota((xy) * b) = \iota(\alpha_1(x)b\alpha_2(y)) \in \iota(R_q),$$

because for $i \in \{1, 2\}$, the embeddings α_i map $\mathcal{O}_i$ into R_q, so that indeed $\alpha_1(x)b\alpha_2(y) \in R_q$. Now to complete the proof of the first claim we need merely observe that, by the coprimality of D_1 and D_2, we have

$$\mathcal{O}_L = \mathcal{O}_1 \otimes_{\mathbb{Z}} \mathcal{O}_2.$$

To see that I_b is an integral ideal for any $b \in R_q$, by definition

$$\iota(R)^{-1} = \{z \in L \mid z\iota(R_q) \subset \mathcal{O}_L\},$$

so that indeed for any $z \in \iota(R_q)^{-1}$, it holds that in $z\iota(b) \in \mathcal{O}_L$, whence $I_b \subset \mathcal{O}_L$. Finally, to see the independence of I_b from ι, we note that any two isomorphisms of 1-dimensional L-vector spaces agree up to a scalar, so any other ι' can be written as $\lambda\iota$ for some $\lambda \in L^\times$. Then indeed

$$\begin{aligned}
\iota'(b)\iota'(R_q)^{-1} &= \lambda\iota(b) \cdot \{z \in L \mid z\iota'(R_q) \subset \mathcal{O}_L\} \\
&= \{\lambda\iota(b)z \in L \mid \lambda z\iota(R_q) \subset \mathcal{O}_L\} \\
&= \{\iota(b)z' \in L \mid z'\iota(R_q) \subset \mathcal{O}_L\} \\
&= \iota(b)\iota(R_q)^{-1},
\end{aligned}$$

using the bijective substitution $z' = \lambda z$. □

It is through the norm of the ideal I_b from L down to F that the importance of the form $\det_F : B_q \to F$ from the previous section becomes apparent. We will use some of the results proved in [HY12]. Their strategy to compute this norm is to do this locally at every prime. There are two cases to consider: those primes above q, and all the others. This gives rise to the following two results from [HY12], the proofs of which we do not include here for to avoid needless repetition.

Proposition 4.5.2. *Let $\ell \neq q$ be any prime number. Then there is some $\delta_\ell \in F_\ell^\times$ that is a generator of the $\mathcal{O}_{F,\ell}$-ideal $\mathcal{D}_{F,\ell}$ such that we may choose the isomorphism $\iota : B_q \to L$ in such a way that*

$$\det_F(-) = \delta_\ell^{-1}\mathrm{Nm}_{L_\ell/F_\ell}(\iota(-))$$

and such that ι takes $R_q \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ to $\mathcal{O}_{L,\ell}$.

Proof. This is Lemma 2.16 in [HY12]. □

To state the second result, we must define the *reflex ideal* of $\mathcal{O}_F$ above the rational prime q associated with the pair of embeddings α_1, α_2. Let $\Pi \in R_q$ be an element of norm q. Then the *reflex ideal* is defined as

$$F \cap \ker\left(\mathcal{O}_L \to R_q \to R_q/\Pi \cong \mathbb{F}_{q^2}\right),$$

where the first map, given by $z \mapsto z * 1$, is not a ring morphism, but by the commutativity of the rightmost ring, it is not difficult to see that this composite is actually a ring morphism and as such, defines an ideal in $\mathcal{O}_F$. We make our choice of $\mathfrak{q}_1, \mathfrak{q}_2 \subset \mathcal{O}_F$ in such a way that $\mathfrak{q}_1$ is the relevant reflex ideal for the fixed pair of embeddings α_1, α_2.

Proposition 4.5.3. *For some $\beta \in F_q^\times = F_{\mathfrak{q}_1}^\times \times F_{\mathfrak{q}_2}^\times$ such that $v_{\mathfrak{q}_1}(\beta) = 1$ and $v_{\mathfrak{q}_2}(\beta) = 0$ we can choose ι in such a way that*

$$\det_F(-) = \beta\ \mathrm{Nm}_{L_q/F_q}(\iota(-))$$

and such that ι takes $R_q \otimes_{\mathbb{Z}} \mathbb{Z}_q$ to $\mathcal{O}_{L,q}$.

Proof. This is Proposition 2.22 in [HY12]. □

These two results combine to prove the following key result.

Proposition 4.5.4. *For any $b \in B_q$, the ideal I_b satisfies*

$$\mathrm{Nm}_{L/F}(I_b) = \det_F(b)\mathfrak{q}_1^{-1}\mathcal{D}_F.$$

Proof. By unique ideal factorisation into primes, it suffices to check the claimed equality everywhere locally. First consider any prime $\ell \neq q$. Then on the right hand side, we have the ideal $\det_F(b)\mathcal{D}_{F,\ell}$. On the left hand side, we use the independence of I_b from our choice of ι to make the choice from Proposition 4.5.2. This maps $\iota(R)^{-1}$ simply to the unit ideal, whereas from $\det_F(b) = \delta_\ell^{-1}\mathrm{Nm}_{L_\ell/F_\ell}(\iota(b))$ it follows that the ideal generated by the norm of $\iota(b)$ agrees with $\delta_\ell \det_F(b)$. By definition of δ_ℓ, these two ideals agree. For the primes above q, the argument is similar, now noting that the right hand side of the theorem localises to $\det_F(b)\mathfrak{q}_1^{-1}$ because we assumed q to be coprime to D_1 and D_2. As β from Proposition 4.5.3 generates the ideal $\mathfrak{q}_1$, the claim follows. □

Lemma 4.5.5. *Let $u_i \in \mathcal{O}_i^\times$ for $i \in \{1,2\}$. Then for any $b \in B_q$,*

$$\det_F(u_1u_2 * b) = \det_F(b).$$

Proof. We use Corollary 4.4.5 to reduce the lemma to observing that $\mathrm{Nm}(u_1u_2 * b) = \mathrm{Nm}(u_1bu_2) = \mathrm{Nm}(b)$ by multiplicativity, and

$$\begin{aligned}\mathrm{tr}(A(u_1u_2 * b)B(\overline{u_1u_2 * b})) &= \mathrm{tr}(Au_1bu_2B\overline{u_2}\overline{b}\overline{u_1}) \\ &= \mathrm{tr}(\overline{u_1}u_1AbBu_2\overline{u_2}\overline{b}) = \mathrm{tr}(AbB\overline{b}),\end{aligned}$$

using both cyclicity of the trace and the fact that $u_1 \in K_1$ and $u_2 \in K_2$ commute with A and B respectively. □

Any attempt at constructing a bijection between quaternions and ideals using only one choice of embeddings is doomed to fail for the simple fact that $\iota(R_q)$, and as such I_b, will always be in the same ideal class. We must take into account the action of the Picard groups on the embeddings, the definition of which we shall now recall. Recall that we assume that B_q has class number 1. Under this assumption, Corollary 30.4.23 in [Voi21] gives us group actions of $\mathrm{Pic}(K_i)$ for $i \in \{1,2\}$ on the set of $R_q^\times$-conjugacy classes of embeddings $\mathcal{O}_i \to R_q$. For $[J] \in \mathrm{Pic}(K_1)$, this action is given by

$$[J] \cdot \alpha_1(-) = \xi^{-1}\alpha_1(-)\xi \quad \text{where} \quad \alpha_1(J)R_q = \xi R_q.$$

The action of $\mathrm{Pic}(K_2)$ is similar, but now $\alpha_2(J)$ acts from the right. From now on, we will write $\iota[c_1, c_2]$ for an isomorphism of 1-dimensional L-vector spaces $B \to L$ where B is equipped with the L-vector space structure induced by the embeddings $[c_1] \cdot \alpha_1$ and $[c_2] \cdot \alpha_2$, where $[c_1] \in \mathrm{Pic}(K_1)$ and $[c_2] \in \mathrm{Pic}(K_2)$ are any representatives. This leaves some ambiguity coming from conjugation by $R_q^\times$, but this does not affect the

counting problem we aim to solve, so we may ignore it. Further, let $I[c_1, c_2]_b$ denote the ideal associated with b using the embedding $\iota[c_1, c_2]$ and let $\det_F[c_1, c_2]$ be the resulting F-bilinear quadratic form.

Proposition 4.5.6. *Let* $[c_1] \in \mathrm{Pic}(K_1)$ *and* $[c_2] \in \mathrm{Pic}(K_2)$ *be given. Then the class of* $I[c_1, c_2]_b$ *inside of* $\mathrm{Pic}(L)$ *is given by* $[c_1] + [c_2] + [I_b]$.

Proof. It suffices to show this for $[c_2] = 0 \in \mathrm{Pic}(K_2)$, for two applications of this special case are enough to deduce the general case. Let now $J \subset \mathcal{O}_1$ be any ideal and let ι' be any L-vector space isomorphism $\iota' : B \to L$ using the modified action $(xy) \star b = ([J] \cdot \alpha_1)(x) b \alpha_2(y)$. We must show that

$$[\iota'(R_q)] = [\iota(R_q)] + [J] \in \mathrm{Pic}(L).$$

Without loss of generality we set $\iota(1) = \iota'(1) = 1$, so that $\iota(x * 1) = x\iota(1) = x$ for all $x \in L$ and similarly for $\star$. If $x = yz$ for $y \in K_1$ and $z \in K_2$, we may then write that

$$\begin{aligned}
x \in \iota'(R_q) &\iff x \star 1 \in R_q \iff (yz) \star 1 \in R_q \\
&\iff (J \cdot \alpha_1)(y)\alpha_2(z) \in R_q \iff \xi^{-1}\alpha_1(y)\xi\alpha_2(z) \in R_q \\
&\iff \alpha_1(y)\xi\alpha_2(z) \in \xi R_q = \alpha_1(J)R_q \\
&\iff (yz) * \xi \in \alpha_1(J)R_q \iff x * \xi \in \alpha_1(J)R_q \\
&\iff x\iota(\xi) \in \iota(J)\iota(R_q) = J\iota(R_q).
\end{aligned}$$

This holds for all $x \in L$ by linearity. Since $\iota(\xi)$ is a fixed scalar, it follows that $[\iota'(R_q)] = [\iota(R_q)] + [J]$, as desired. □

The following result connects this construction with the most subtle part of the exact sequence from Theorem 2.3.3. Indeed, now the explicit map $\varphi : \mathcal{O}_F^{\times,+} \to \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ comes into the picture. The following proposition relates the values of the $\det_F$-function for two pairs of ideals in $\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ whose image in $\mathrm{Pic}(L)$ are the same.

Proposition 4.5.7. *Let* $([c_1], [c_2]), ([d_1], [d_2]) \in \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ *be two* distinct pairs *satisfying that* $[c_1] + [c_2] = [d_1] + [d_2] \in \mathrm{Pic}(L)$. *Then*

$$\det_F[c_1, c_2](R_q) \cap \epsilon_F \cdot \det_F[d_1, d_2](R_q) \neq \varnothing,$$

where $\det_F[-,-](R_q)$ *denotes the set of values of* $\det_F[-,-]$ *on the elements of* R_q.

Proof. The existence of such distinct pairs is by Theorem 4.3.3 ensured when $\mathrm{Nm}_F^L(\mathcal{O}_L^\times)$ does not surject onto $\mathcal{O}_F^{\times,+}$, so we find ourselves in this case. For notational convenience, we will also assume that $([d_1],[d_2]) = ([0],[0])$, so that $([c_1],[c_2]) = \varphi(\epsilon_F)$. The reader will find it easy to deduce the general case from this special one.

Denote the form induced by $([c_1],[c_2])$ by $\det'_F$. We observe that it suffices to establish an element $b \in R_q$ such that $\det'_F(1) = \epsilon_F \det_F(b)$. Still under the simplifying assumption that R_q has class number 1, we may choose $\xi_1, \xi_2 \in R_q$ such that $\alpha_1(c_1)R_q = \xi_1 R_q$ and $R_q\alpha_2(c_2) = R_q\xi_2$. We then claim that

$$f : B_q \to F : b \mapsto \frac{\mathrm{Nm}(\xi_1)}{\mathrm{Nm}(\xi_2)}\det_F(\xi_1^{-1}b\xi_2).$$

satisfies the defining property for $\det'_F(b)$. Indeed,

$$\mathrm{Tr}(f(b)) = \frac{\mathrm{Nm}(\xi_1)}{\mathrm{Nm}(\xi_2)}\mathrm{Tr}(\det_F(\xi_1^{-1}b\xi_2)) = \frac{\mathrm{Nm}(\xi_1)}{\mathrm{Nm}(\xi_2)}\mathrm{Nm}(\xi_1^{-1}b\xi_2) = \mathrm{Nm}(b).$$

Also, it is F-quadratic for the action $\star$ induced by the embeddings $\alpha'_1 = \xi_1\alpha_1\xi_1^{-1}$ and $\alpha'_2 = \xi_2\alpha_2\xi_2^{-1}$, because

$$\begin{aligned} f(\sqrt{D}\star b) &= \frac{\mathrm{Nm}(\xi_1)}{\mathrm{Nm}(\xi_2)}\det_F\big(\xi_1^{-1}\xi_1\alpha_1(\sqrt{D_1})\xi_1^{-1}b\xi_2\alpha_2(\sqrt{D_2})\xi_2^{-1}\xi_2\big) \\ &= \frac{\mathrm{Nm}(\xi_1)}{\mathrm{Nm}(\xi_2)}\det_F(\sqrt{D} * \xi_1^{-1}b\xi_2) = D\cdot f(b); \end{aligned}$$

proving our claim. By considering the index of both ideals, it must follow that $\mathrm{Nm}(\xi_i) = \mathrm{Nm}(c_i)$ for $i \in \{1,2\}$. By our definition of the connecting map φ in Theorem 4.3.3, we thus have $\mathrm{Nm}(\xi_1\xi_2) = \mathrm{Nm}(y_F)$ where $y_F/\sigma(y_F) = \epsilon_F$. We now set

$$b = \frac{\sigma(y_F)}{\mathrm{Nm}(\xi_2)} * (\xi_1^{-1}\xi_2) \in R_q.$$

Indeed, using the F-linearity of $\det_F$, we compute that

$$\begin{aligned} \epsilon_F\det_F(b) &= \frac{y_F}{\sigma(y_F)}\det_F\left(\frac{\sigma(y_F)}{\mathrm{Nm}(\xi_2)} * (\xi_1^{-1}\xi_2)\right) = \frac{y_F\sigma(y_F)}{\mathrm{Nm}(\xi_2)^2}\det_F(\xi_1^{-1}\xi_2) \\ &= \frac{\mathrm{Nm}(y_F)}{\mathrm{Nm}(\xi_2)^2}\frac{\mathrm{Nm}(\xi_2)}{\mathrm{Nm}(\xi_1)}\det'_F(1) = \det'_F(1), \end{aligned}$$

as claimed. This completes the proof. □

We are now ready to prove the main result of this chapter.

Theorem 4.0.2. *For any $\nu \in F^+$, the association $(b, [c_1], [c_2]) \mapsto I[c_1, c_2]_b$ establishes a bijection between the set of*

$$(b, [c_1], [c_2]) \in (\mathcal{O}_1^\times \setminus R_q \,/\, \mathcal{O}_2^\times) \times \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$$

with the property that $\det_F[c_1, c_2](b) = \nu$ and the set of integral ideals $I \subset \mathcal{O}_L$ such that $\mathrm{Nm}_{L/F}(I) = \nu \mathfrak{q}_1^{-1} \mathcal{D}_F$.

Proof. Well-definedness of the association has already been established. Our strategy is to take some ideal $I \subset \mathcal{O}_L$ with norm as described and to reason that there is a unique triple $(b, [c_1], [c_2])$ mapping to it.

We start by observing that I maps under the norm map to the ideal class of $[\mathfrak{q}^{-1}\mathcal{D}_F]$ inside $\mathrm{Pic}(F)^+$. Similarly, for any given $(b, [c_1], [c_2])$, we have by Proposition 4.5.4 that the ideal class of $\mathrm{Nm}_F^L(I[c_1, c_2]_b)$ equals the class of $[\mathfrak{q}^{-1}\mathcal{D}_F]$ as well. By Theorem 4.3.3, these two ideal classes in $\mathrm{Pic}(L)$ must therefore differ up to an element from $\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$. Using Proposition 4.5.6, this means that there exist $[c_1] \in \mathrm{Pic}(K_1)$ and $[c_2] \in \mathrm{Pic}(K_2)$ such that the class $[I]$ agrees with the class $[I[c_1, c_2]_b]$ for any choice $b \in B_q$.

The ambiguity in the choice of $([c_1], [c_2]) \in \mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2)$ here is, by Theorem 4.3.3, measured by $\mathcal{O}_F^{\times,+} / \mathrm{Nm}_{L/F}(\mathcal{O}_L^\times)$. On the other hand, for any such choice of $([c_1], [c_2])$, the ideal $I \cdot \iota[c_1, c_2](R_q)$ will be principal. Because $\iota[c_1, c_2]$ is bijective, we find some $b \in B_q$ with $I \cdot \iota[c_1, c_2](R_q) = (\iota[c_1, c_2](b))$; in other words, we obtain some $b \in B_q$ with $I = I[c_1, c_2]_b$. Using Proposition 4.5.4, comparing norms to F yields that $\det_F(b) \in F$ and $\nu \in F$ generate the same $\mathcal{O}_F$-ideal, and as such, they must agree up to a unit from $\mathcal{O}_F^{\times,+}$, where we used that $\det_F(b)$ and ν are both totally positive. We may only modify b by an element of $\mathcal{O}_L^\times$ without changing $I[c_1, c_2]_b$, which will modify $\det_F(b)$ by an element of $\mathrm{Nm}_{L/F}(\mathcal{O}_L^\times)$ by a computation similar to that in the proof of Lemma 4.5.5. The failure to equate $\det_F(b)$ and ν is thus again measured by $\mathcal{O}_F^{\times,+} / \mathrm{Nm}_{L/F}(\mathcal{O}_L^\times)$. By Proposition 4.5.7, there is a unique choice of $([c_1], [c_2])$ such that this obstruction can be lifted.

We make this choice and find some $b \in B_q$ such that $I = I[c_1, c_2]_b$ and $\det_F(b) = \nu$. Because the first condition determines b uniquely up to multiplication by an element $u \in \mathcal{O}_L^\times$ and $\mathrm{Nm}_F^L(u) = 1$ if and only if $u \in \mathcal{O}_1^\times \mathcal{O}_2^\times$, this shows that b is determined uniquely up to an element from $\mathcal{O}_1^\times \mathcal{O}_2^\times$. By Lemma 4.5.5, this ambiguity remains. To complete the proof, we must still show that $b \in R_q$. One can check this locally using Proposition 4.5.2 and Proposition 4.5.3; in this instance, we opt to leave the details to the reader. □