Boundary extensions of symmetric spaces in equivariant KK-theory
Ulsnaes, T.

Chapter 1

Preliminaries

This chapter is a gentle introduction the theory we will need for the subsequent part of the thesis. The reader familiar with the basics of C*-algebras and K-theory can safely skip ahead to Section 1.7 where we define the geodesic compactification and introduce a fundamental example.

Unless stated otherwise, all algebras will be assumed to be over the complex numbers.

1.1 C*-algebras

Let A be an algebra over $\mathbb{C}$. By an involution on A we mean a map

$$* : A \to A \qquad a \mapsto *(a) := a^*$$

satisfying the following properties for any $a, b \in A$ and $\lambda \in \mathbb{C}$

- $(a+b)^* = a^* + b^*$
- $(ab)^* = b^* a^*$
- $(\lambda a)^* = \overline{\lambda} a^*$.

Definition 1.1. A (complex) pre-C*-algebra is a (complex) algebra A with an involution $* : A \to A$, and a submultiplicative norm (meaning $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$) satisfying

$$\|a^* a\| = \|a\|^2 \quad \forall\, a \in A \tag{1.1}$$

A (semi-)norm satisfying equation (1.1) will be called a C*(-semi-)-norm. A C*-algebra is a complete pre-C*-algebra. We call a C*-algebra unital if there is an element $1 \in A$ such that

$$a1 = 1a = a$$

for all $a \in A$. A C*-algebra A is called commutative if for all $a_1, a_2 \in A$ we have

$$a_1 a_2 = a_2 a_1.$$

We call the innocuous looking equation (1.1) the C*-identity. It has profound consequences for the theory of C*-algebras, which separates it from the theory of, say, Banach *-algebras (complete normed algebras with an isometric involution).

A *morphism or *-homomorphism* $\phi : A \to B$ between two C*-algebras is an algebra homomorphisms which commutes with the involution, i.e.

$$\phi(a^*) = \phi(a)^*.$$

If A and B are unital C*-algebras, then a morphism

$$\phi : A \to B$$

is called unital if it maps the unit in A to the unit in B.

Example 1.2. A *representation* of a C*-algebra A is a morphism $\pi : A \to B(H)$ for some Hilbert space H. The representation is said to be *faithful* if π is injective and *non-degenerate* if $\pi(A)H$ is dense in H.

The prototypical example of a C*-algebra is the algebra of bounded linear maps on a fixed complex Hilbert space H, denoted $B(H)$, with the supremum norm given on $T \in B(H)$ by

$$\|T\| := \sup_{\|x\| \leq 1} \|T(x)\|.$$

The involution in $B(H)$ is given by sending a map to its adjoint.

It is convenient to have a unit in the C*-algebra. The next example gives the to most common ways to add a unit to a non-unital C*-algebra.

Example 1.3. Let A be a nonunital C*-algebra. The *unitization* of A, denoted A^+, is the universal C*-algebra satisfying the following property:

It is the smallest unital C*-algebra containing A as an ideal in the sense that any morphism $f : A \to B$ from A to a unital C*-algebra B lifts to a unique unital morphism $\hat{f} : A^+ \to B$.

The *multiplier algebra* of A, denoted $M(A)$, is the universal C*-algebra satisfying the following property:

It is the largest C*-algebra containing A as an essential ideal (Definition 1.9). Equivalently, let B be any C*-algebra containing A as an essential ideal. Then there is a unique map $B \to M(A)$ such that the inclusion $\iota_A : A \to M(A)$ factors through

$$A \to B \to M(A).$$

We will see concrete realisations of these two algebras later when we define Hilbert C*-modules in Section 1.4, but let us mention that for a locally compact (but noncompact) Hausdorff space X we have

$$C_0(X)^+ = C(X \cup \{\infty\}),$$

where $X \cup \{\infty\}$ is the one-point compactification or the Alexandroff compactification of X and

$$M(C_0(X)) = C(\beta X)$$

with βX the Stone–Cech compactification of X.

The next two results are some of the many consequences of equation (1.1).

Theorem 1.4 ((Gelfand-Naimark) [41, Theorem 3.4.1])**.** *Any C*-algebra is a C*-subalgebra of $B(H)$ for some Hilbert space H.*

Proposition 1.5 ([41] Cor. 2.1.2)**.** *Let A be an algebra with an involution $* : A \to A$. Then there is at most one norm on A, making it a C*-algebra.*

Example 1.6. Let A be any C*-algebra. Then define a norm on the n'fold direct sum of A (treated as a vector space over $\mathbb{C}$)

$$\underbrace{A \oplus \cdots \oplus A}_{n}$$

by

$$\|(b_1, \ldots, b_n)\|_2 := \|b_1^* b_1 + \cdots + b_n^* b_n^2\|^{1/2}.$$

Now define the algebra of A-valued $n \times n$-matrices $M_n(A)$. This is an algebra over $\mathbb{C}$ with an involution given by the complex conjugate

$$(a_{i,j})^* = (a_{j,i}^*),$$

and product given by the usual matrix multiplication:

$$(a_{i,j})(b_{ij}) = (\sum_{k=1}^{n} a_{ik} b_{kj}).$$

The norm

$$\|(a_{i,j})\| := \sup_{\|(b_1,\ldots,b_n)\|_2 = 1} \left\| (\sum_{j=1}^{n} (a_{1,j} b_j), \ldots, \sum_{j=1}^{n} (a_{n,j} b_j) \right\|_2$$

determines a C*-norm on $M_n(A)$ making it into a C*-algebra, hence it is the unique C*-norm on $M_n(A)$.

Any morphism $\phi : A \to B$ induces a map

$$\phi : M_n(A) \to M_n(B) \qquad (a_{i,j}) \mapsto \phi(a_{i,j}) \tag{1.2}$$

which can be shown to be a *-homomorphism of C*-algebras.

Proposition 1.5 tells us that a C*-norm on A is uniquely determined by the algebraic structure of A. This gives a close connection between the algebraic properties of A and the topological properties of A, which makes it possible to translate theorems of algebra into the language of C*-algebras. For example, here is a C*-analogue of the classical Wedderburn theorem[1]

Example 1.7. Any finite dimensional C*-algebra A (i.e. $A \subset B(H)$ for a finite dimensional Hilbert space H), is isomorphic to a direct sum of full matrix algebras

$$A \simeq \bigoplus_{i=n}^{k} M_{n_i}(\mathbb{C}).$$

Example 1.8. Let us give another important example of C*-algebras, namely commutative C*-algebras $C_0(X)$, of complex-valued functions on a locally compact Hausdorff space X vanishing at infinity. Recall that a function f on a locally compact space X is said to vanish at infinity if for any $\epsilon > 0$ there is a compact subset $K_\epsilon \subset X$ such that

$$|f(x)| < \epsilon \qquad \forall\, x \notin K_\epsilon.$$

We can multiply and add two functions $f, h \in C_0(X)$ pointwise as follows

$$(fh)(x) = f(x)h(x) \qquad (f+h)(x) = f(x) + h(x), \qquad x \in X.$$

Similarly we define scalar multiplication by $(\lambda f)(x) = \lambda f(x) \qquad x \in X, \lambda \in \mathbb{C}$. The norm on $C_0(X)$ making it a C*-algebra, called the supremum norm, is defined by

$$\|f\| := \sup_{x\in X} |f(x)|. \tag{1.3}$$

Note that $C_0(X)$ is unital if and only if X is compact, in which case $C_0(X) = C(X)$, with unit given by the constant function $X \ni x \mapsto 1 \in \mathbb{C}$.

If X is any topological space, then the algebra $C_0(X)$ can be defined just as in Example 1.8 and does produce a C*-algebra with respect to the supremum norm (eq. (1.3)). The reason we restrict ourselves to locally compact Hausdorff spaces is that given a commutative C*-algebra $C_0(X)$ there always exists a (unique up to homeomorphism) locally compact Hausdorff space Y and an isomorphism of C*-algebras

[1] Another example is Kadison's transitivity theorem which generalizes Jacobson's transitivity theorem, but we will not need it here.

$$C_0(Y) \simeq C_0(X).$$

Given a commutative C*-algebra A, we can always write $A = C_0(Y)$ for some locally compact Hausdorff space Y. For this reason, from now on X will always denote a locally compact Hausdorff space. The space Y is called the Gelfand dual (or the spectrum) of A which, as a set, consists of all *-homomorphisms

$$C_0(Y) \to \mathbb{C}.$$

The assignment

$$Y \mapsto C_0(Y)$$

determines a contravariant functor from the category of locally compact Hausdorff spaces with morphisms given by proper continuous maps, to the category of commutative C*-algebras, by sending a proper map $f : X \to X'$ to the map

$$C_0(X') \to C_0(X) \qquad h \mapsto h \circ f \qquad h \in C_0(X').$$

Note that we need properness of f to ensure that $f \circ h$ vanishes at infinity on X. The Gelfand transform determines a contravariant equivalence of categories between the unital commutative C*-algebras with unital morphisms and the category of compact Hausdorff spaces. In the non-unital case we need to take care of what morphisms we allow [2]. If $f : X \to Y$ is a proper map, then the induced map

$$C_0(Y) \to C_0(X) \qquad h \mapsto h \circ f$$

sends approximate units in $C_0(Y)$ to approximate units in $C_0(X)$ [3], so we cannot find a map $X \times Y \to X$ corresponding to the inclusion into the first factor

$$C(X) \to C(X) \oplus C(Y) = C(X \sqcup Y).$$

Definition 1.9. Let A be a C*-algebra. A closed $*$-invariant subalgebra of $I \subset A$ is called a C*-subalgebra of A. If $I \subset A$ is a C*-subalgebra for which

$$aI, Ia \subset I \qquad \forall\, a \in A$$

then I is called an ideal of A.

An ideal $I \subset A$ is called essential, if for any other ideal $J \subset A$, we have

$$I \cap J = \{0\} \Rightarrow J = 0.$$

[2]See for instance [39, p. 9] for a category which is equivalent to the commutative C*-algebra category.

[3]An approximate unit of a C*-algebra A is a net e_α of positive elements such that for all $x \in A$ we have $\|e_i x - x\| \to 0$ as $i \to \infty$.

An example of an essential ideal is $C_0((0,1))$ inside $C([0,1])$. A typical non-example is the following: Let $I \subset A$ be an ideal of A and B any (non-trivial) C*-algebra, then $I \oplus \{0\} \subset A \oplus B$ is a non-essential ideal.

Let us collect some basic properties of C*-algebras.

Proposition 1.10 ([41] Chapter 2)**.** *Let A, B be C*-algebras, and $\phi : A \to B$ a morphism of C*-algebras. Then*

1. *The map ϕ is contractive, meaning $\|\phi(a)\| \leq \|a\|$;*
2. $\mathrm{Ker}(\phi)$ *is an ideal in A;*
3. *The inclusion $\phi(A) \subset B$ is a closed C*-subalgebra of B;*
4. *There is an isomorphism*
$$A/\mathrm{Ker}(\phi) \simeq \phi(A);$$
5. *If A and B are two C*-algebras, then their direct sum $A \oplus B$ is a C*-algebra with respect to the operations*
$$\begin{aligned}(a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2)\\ (a_1, b_1)(a_2, b_2) &= (a_1a_2, b_1b_2)\\ (a_1, b_1)^* &= (a_1^*, b_1^*)\\ \lambda(a_1, b_1) &= (\lambda a_1, \lambda b_1)\end{aligned}$$
for $(a_i, b_i) \in A \oplus B$ and $\lambda \in \mathbb{C}$, and norm given by $\|(a, b)\| = \max(\|a\|, \|b\|)$.

Tensor products of C*-algebras are more subtle as there are several choices of norms on the algebraic tensor product of two C*-algebras. We will not delve into the theory of tensor products for C*-algebras here, but refer the interested reader to the very thorough exposition in [11]. Let us just define one norm, which in some interesting cases turns out to be the only pre-C*-norm on the algebraic tensor product.

We denote by $A_1 \odot A_2$ the algebraic tensor product of two C*-algebras A_1, A_2. This is the linear span of the simple tensors $(a_1 \odot a_2)$ with involution
$$(a_1 \odot a_2)^* = a_1^* \odot a_2^*,$$
and product
$$(a_1 \odot a_2)(a_1' \odot a_2') = a_1a_1' \odot a_2a_2'.$$
The tensor product of two Hilbert spaces $H_1 \otimes H_2$ is a Hilbert space in its own right with respect to the inner product
$$\langle v \otimes w, v' \otimes w'\rangle := \langle v, v'\rangle\langle w, w'\rangle \qquad v, v' \in H_1, w, w' \in H_2.$$

Definition 1.11. Let A_1 and A_2 be two C*-algebras with faithful representations $\pi_i : A \to B(H_i)$. Then we have an injective *-preserving algebra homomorphism

$$\pi_1 \otimes \pi_2 : A_1 \odot A_2 \to B(H_1 \otimes H_2)$$

given by

$$\pi_1 \otimes \pi_2(a \odot a')(h \otimes h') := \pi_1(a)(h) \otimes \pi_2(a')(h').$$

The *minimal* or *spatial* tensor product, denoted $A_1 \otimes A_2$, is the completion of $A_1 \odot A_2$ in $B(H_1 \otimes H_2)$.

Since $A_1 \odot A_2$ is a subalgebra of $B(H \otimes H)$ closed under involution, it is clear that the minimal tensor product norm is a pre-C*-norm on the algebraic tensor product. It remains to be verified that the norm is independent of choice of faithful representations π_i. We refer the reader to [11, Chap. 3] for the proof of this fact. In case one of the algebras is commutative, this is the only norm on the algebraic tensor product making its completion a C*-algebra.

We will also need the following definition:

Definition 1.12. A short exact sequence of C*-algebras $0 \to B \to E \to A \to 0$ is called an extension of B by A. Note that B then must be isomorphic to an ideal of E.

The theory of extensions of C*-algebras will be covered in some detail later in Section 1.6. For now, let us give a simple example: For a commutative C*-algebra (ref. Ex 1.8) if $V \subset X$ is a closed subset, we can define a C*-subalgebra given by

$$I(V) := \{f \in C_0(X) \mid f|_V = 0\}$$

which is easily seen to be an ideal of $C_0(X)$. All ideals of $C_0(X)$ arise in this way for some closed subset of X. Letting $O = X \backslash V$ be the complement of V then it is easy to show we have an isomorphism

$$I(V) = C_0(O) \subset C_0(X)$$

and the quotient algebra is given by

$$C_0(X)/C_0(O) \simeq C_0(X \backslash O) = C_0(V).$$

This gives us an extension of C*-algebras

$$0 \to C_0(O) \to C_0(X) \to C_0(V) \to 0. \tag{1.4}$$

For example if $\overline{M}$ is a manifold with boundary ∂M and interior M, we get an extension

$$0 \to C_0(M) \to C_0(\overline{M}) \to C_0(\partial M) \to 0.$$

An *operator system* is a closed and $*$-invariant subspace $F \subset A$ of a unital C*-algebra A, such that $1_A \in F$.

Definition 1.13 ([11, Definition 3.7.5])**.** Let E be a unital C*-algebra. An extension

$$0 \to B \to E \xrightarrow{\pi} A \to 0$$

is called locally split if for every finite dimensional operator system $F \subset A$ there exists a unital completely positive map $\sigma : F \to E$ such that $\pi \circ \sigma = id_F$.

Definition 1.14. A C*-algebra A is said to be *nuclear* if for any C*-algebra B there is a unique pre-C*-norm on $A \odot B$.

Definition 1.15. A C*-algebra A is called exact if for any exact sequence of C*-algebras

$$0 \to I \xrightarrow{\iota} E \xrightarrow{p} B \to 0$$

the sequence

$$0 \to I \otimes A \xrightarrow{\iota \otimes id} E \otimes A \xrightarrow{p \otimes id} B \otimes A \to 0$$

is exact, where $\iota \otimes id$ and $p \otimes id$ are the maps given on simple tensors in $I \odot A$ and $B \odot A$ by

$$\iota \otimes id(m \odot a) = \iota(a) \odot a \quad \text{and} \quad p \otimes id(b \odot a) = p(b) \odot a$$

respectively, and extended to the C*-tensor product.

We mention without proof that any nuclear C*-algebra is exact, but the converse is in general not true. One family of examples of exact C*-algebras which are not nuclear are the reduced group C*-algebra of discrete non-amenable subgroups of $GL_n(\mathbb{C})$ (see Theorem 1.37). Further, the following example shows that not all nuclear C*-algebras are commutative.

Example 1.16. Let H be a Hilbert space. A finite rank operator $F : H \to H$ is a linear map of the form

$$F(x) = \sum_{n=1}^{k} y_i \langle x_i, x \rangle$$

for some finite set $x_i, y_i \in H$.

Let $B_{\text{fin}}(H)$ denote the collection of all finite rank operators on H and let $\mathbb{K} := \mathbb{K}(H)$ denote the closure of $B_{\text{fin}}(H)$. The algebra $\mathbb{K}$ can be shown to be nuclear C*-algebra sitting in $B(H)$ as an essential ideal.

Definition 1.17. An operator $T \in \mathbb{K}(H)$ defined in Example 1.16 is called a compact operator. A compact operator is a bounded linear map $T : H \to H$ satisfying any of the following equivalent conditions ([41, Chap. 2.4])

1. For any bounded $U \subset H$, $T(U)$ has compact closure;

2. T is the norm limit of finite rank operators

3. T is in the norm limit of finite sums of rank 1 projections

4. $T \in \bigcap_{I \subset B(H)} I$ where I runs over all ideals of $B(H)$ containing $B_{\text{fin}}(H)$.

Let $\mathbb{K} = \mathbb{K}(l^2(\mathbb{N}))$ be the C*-algebra of compact operators on a separable infinite dimensional Hilbert space (i.e. a Hilbert space with a countable infinite orthonormal basis).

Definition 1.18. A C*-algebra A is called stable if $A \otimes \mathbb{K} \simeq A$. For any C*-algebra B, the C*-algebra $B \otimes \mathbb{K}$ is called the stabilization of B.

The stabilization $B \otimes \mathbb{K}$ is stable, since the tensor product is associative and $\mathbb{K} \otimes \mathbb{K} \simeq \mathbb{K}$.

Positive elements If $A = C(X)$ is a commutative C*-algebra, then the subset $A_+ = \{f \in C(X) \mid f \geq 0\}$ of real-valued positive functions on X has the following properties:

1. Each element $f \in A_+$ has a unique square root in A_+;

2. The set A_+ is a cone in A, meaning it is closed under addition and multiplication by $\mathbb{R}_{\geq 0}$;

3. Every element in A_+ is of the form $|f|^2 = \overline{f} f$ for some function $f \in A$;

4. Every $f \in A$ can be written as

$$f = f_1 - f_2 + i(f_3 - f_4)$$

   for some $f_i \in A_+$.

The elements in A_+ are called positive elements of A. For a general C*-algebra we have a similar definition:

Definition 1.19. Let A be any C*-algebra and define

$$A_+ := \{a^* a \mid a \in A\}.$$

The elements in A_+ are called positive, and A_+ is called the positive cone of A.

We have:

Proposition 1.20 ([41] Sec. 2.2)**.** *The positive cone satisfies the following properties*

1. *The set A_+ is a cone in A, meaning it is closed under sums and multiplication by $\mathbb{R}_{\geq 0}$*

2. *Every* $a \in A_+$ *has a unique square root in* A_+*, i.e. there is an element* $b \in A_+$ *such that* $b^2 = a$*;*

3. *every* $a \in A$ *can be written as*

$$a = a_1 - a_2 + i(a_3 - a_4)$$

for $a_i \in A_+$*.*

Now if $\phi : A \to B$ is a morphism of C*-algebras then

$$\phi(a^*a) = \phi(a)^*\phi(a)$$

hence $\phi(A_+) \subset B_+$. Similarly, the induced maps on matrix algebras

$$\phi : M_n(A) \to M_n(B) \qquad \phi[(a_ij)] = [\phi(a_{ij})]$$

also preserves positive elements (being themselves morphisms of C*-algebras).

The following definition gives a weakening of the notion of morphisms of C*-algebras, that is useful in applications (see Example 1.49)

Definition 1.21. A bounded linear map $\phi : A \to B$ between C*-algebras is called positive if

$$\phi(A_+) \subset B_+.$$

It is called contractive if $\|\phi(a)\| \leq \|a\|$ for all $a \in A$.

A positive map is called completely positive if the induced map

$$\phi : M_n(A) \to M_n(B) \qquad [a_{ij}] \mapsto [\phi(a_{ij})],$$

is positive for all $n \in \mathbb{N}$. Similarly it is called a completely positive contractive map if the induced maps

$$\phi : M_n(A) \to M_n(B) \qquad [a_{ij}] \mapsto [\phi(a_{ij})]$$

are positive and contractive for all $n \in \mathbb{N}$.

1.2 Group actions and crossed products

Throughout this thesis, all groups will be assumed to be locally compact and Hausdorff topological groups, and, unless mentioned otherwise, unimodular, meaning the left and right Haar measures agree (Definition 2.21). In this section we will see what happens when a group G acts on a C*-algebra A. A good reference for this material is [44].

Definition 1.22. A group action of a locally compact topological group G on a C*-algebra is a group homomorphism

$$G \to \mathrm{Aut}(A).$$

The action is called continuous if the map

$$G \times A \to A \qquad (g, a) \mapsto \alpha_g(a)$$

is continuous. It is called strongly continuous if for all $a \in A$, the map

$$G \to A \qquad g \mapsto \alpha_g(a)$$

is continuous.

Clearly a continuous action is strongly continuous, but the converse may fail. So it would be better to call it weakly continuous, but we will adhere to the convention of Definition 1.22. Whenever there is a group action on a C*-algebra, it will *always be assumed to be strongly continuous.*

Example 1.23. Assume X is a space with a continuous action of a group G, i.e. the map

$$G \times X \to X \times X \qquad (g, x) \mapsto (x, gx)$$

is continuous. Then the action of G induces a strongly continuous action of G on $C_0(X)$ by

$$(gf)(x) = f(g^{-1}x). \tag{1.5}$$

Definition 1.24. A *G-C*-algebra* or a *C*-dynamical system* (A, α), is a C*-algebra A together with a strongly continuous action $\alpha : G \to \mathrm{Aut}(A)$ of a group G.

A *morphism* of G-C*-algebras

$$\phi : (A, \alpha) \to (B, \beta)$$

is a morphism of C*-algebras

$$\phi : A \to B$$

that commutes with the action of G, that is,

$$\phi(\alpha_g(a)) = \beta_g(\phi(a)).$$

Similar to Definition 1.12, we have the following:

Definition 1.25. A short exact sequence

$$0 \to B \to E \to A \to 0$$

of G-C*-algebras, where each map commutes with the group action, is called an equivariant extension of G-C*-algebras.

We will often write A for a G-C*-algebra (A, α), when the action is clear from the context.

Example 1.26. Let G be a group. A unitary representation of G is a continuous group homomorphism

$$\phi : G \to U(B(H)) \qquad \phi(g) := U_g$$

from G to the group of unitaries on a Hilbert space H. We get a strongly continuous action on $B(H)$ and $\mathbb{K}(H)$ by

$$\alpha_g(T) := U_g T U_g^*.$$

for $T \in B(H)$ or $T \in \mathbb{K}(H)$ respectively.

Definition 1.27. A function $f : G \to A$ is called compactly supported if

$$\mathrm{supp}(f) := \overline{\{g \in G \mid f(g) \neq 0\}} \subset G$$

is compact. The set $C_c(G, A)$ of compactly supported A-valued functions admits the structure of an algebra over $\mathbb{C}$ with respect to the operations given, for any $f, h \in C_c(G, A)$, by

- $(f + h)(g) = f(g) + h(g)$
- $(f \star h)(g) = \int_G f(s)\alpha_s(h(s^{-1}g))ds.$

where the integral is the (Bochner) integral with respect to the Haar measure ds on G. The product $\star : C_c(G, A) \to C_c(G, A)$ is called the convolution product. We can also define an involution on $C_c(G, A)$ by

- $f^*(g) = \alpha_g(f(g^{-1})^*).$

turning $C_c(G, A)$ into a $*$-algebra.

The natural substitute for representations of $C_c(G, A)$ are the integrated forms of covariant representations. Let us go through the definitions.

Definition 1.28. A covariant representation (or covariant pair) for a G-C*-algebra (A, α) is a pair (π, u) where

$$\pi : A \to B(H)$$

is a non-degenerate representation of A, and

$$u : G \to U(H) \qquad g \mapsto u_g$$

is a unitary representation (Example 1.26) of G satisfying the so called *covariance relation*:

$$\pi(\alpha_g(a)) = u_g \pi(a) u_g^*. \tag{1.6}$$

Given a covariant representation of a G-C*-algebra (A, α) there is a natural way to induce a *-preserving algebra homomorphism of $C_c(G, A)$ on $B(H)$ called the integrated form of (ϕ, u).

Definition 1.29. Given a covariant representation (π, u) of (A, α), the integrated form is the representation

$$\pi \rtimes u : C_c(G, A) \to B(H)$$

given for an $f \in C_c(G, A)$ by the Bochner integral

$$(\pi \rtimes u)(f) := \int_{s \in G} \pi(f(s)) u_s ds.$$

The operator $(\pi \rtimes u)(f) := \int_{s \in G} \pi(f(s)) u_s ds$ is the unique operator in $B(H)$ which acts on a vector $v \in H$ by

$$v \mapsto \int_{s \in G} \pi(f(s))(u_s(v)) ds.$$

There may be no C*-norm on $C_c(G, A)$ but we can always find pre-C*-norms whose completion give a C*-algebras (see Definition 1.1). We will now define the two most common norms on $C_c(G, A)$, denoted by $\| \cdot \|_r$ and $\| \cdot \|$.

Recall that the left regular representation of G is the unitary representation

$$\lambda : G \to U(L^2(G)) \qquad g \mapsto \lambda_g$$

where

$$\lambda_g(f)(t) = f(g^{-1}t) \qquad \text{for all } f \in L^2(G).$$

We define $L^2(G, H)$ to be the Hilbert space completion of $C_c(G, H)$ with respect to the inner product

$$\langle f, h \rangle := \int_G \langle f(g), h(g) \rangle dg \qquad (f, h \in C_c(G, H))$$

where dg denotes the Haar measure on G (see [54, Appendix I.4]). Let $\pi : A \to B(H)$ be any faithful non-degenerate representation of our C*-algebra A, then we can extend π to a representation $\hat{\pi} : A \to B(L^2(G, H))$ given by

$$\hat{\pi}(a)(h)(g) := \pi(\alpha_{g^{-1}}(a))h(g) \qquad a \in \mathcal{A}, h \in B(L^2(G, \mathcal{A})), g \in G.$$

A long, but simple computation will show that the pair $(\hat{\pi}, \lambda)$ is a covariant pair for (A, α).

Definition 1.30. The *regular representation* of a G-C*-algebra is the integrated form of the covariant pair $(\hat{\pi}, \lambda)$. That is, it is the representation

$$\hat{\pi} \rtimes \lambda : C_c(G, A) \to B(L^2(G, H))$$

given for any $f \in C_c(G,A)$, $h \in L^2(G,A)$ and $g \in G$ by

$$(\hat{\pi} \rtimes \lambda)(f)(h)(g) = \int_G \pi(\alpha_{g^{-1}}(f(s)))h(g^{-1}s)ds.$$

Definition 1.31. The *reduced crossed product* of a C*-dynamical system (A, α), is the completion of $C_c(G,A)$ with respect to the norm

$$\|f\|_r := \|\hat{\pi} \rtimes \lambda(f)\| \qquad f \in C_c(G,A).$$

The completion of $C_c(G,A)$ is denoted

$$A \rtimes_{r,\alpha} G \qquad \text{or} \qquad A \rtimes_r G.$$

Definition 1.32. The *maximal* or *universal* crossed product is the completion of $C_c(G,A)$ with respect to the norm

$$\|f\| := \sup_{(\pi,u)} \|\pi \rtimes u(f)\|$$

where (π, u) runs over all covariant representations of (A, α). The completion of $C_c(G,A)$ is denoted

$$A \rtimes_\alpha G \qquad \text{or} \qquad A \rtimes G.$$

We refer the reader to the book of Williams [54] for the proof that these are indeed pre-C*-norms on $C_c(G,A)$ satisfying the C*-identity (eq. (1.1)). For crossed products by discrete groups the book of Phillips [46] is also recommended.

Let us state the following property showing the maximality of the universal C*-norm among all "sensible" norms on $C_c(G,A)$:

Lemma 1.33. *Let $\|\cdot\|_t$ be a pre-C*-norm on $C_c(G,A)$ given by a representation of*

$$\pi_t : C_c(G,A) \to B(H)$$

which is norm-decreasing with respect to the L^1 norm on $C_c(G,A)$ i.e. the norm

$$\|f\|_1 := \left\| \int_G f(g)^* f(g) d\mu(g) \right\|.$$

Then for all $f \in C_c(G,A)$ we have $\|f\|_t \leq \|f\|$.

Proof. The assertion follows from [54, Corollary 2.46]. □

The reduced crossed product is in general not the minimal pre-C*-norm on $C_c(G,A)$, however.

Example 1.34 (Group C*-algebras)**.** Let G be a group. Let $id : G \to \mathbb{C}$ be the trivial G action on $\mathbb{C}$ (i.e. the trivial representation of G) $id(g) = 1$ for all $g \in G$. Then the crossed product

$$C_r^*(G) := \mathbb{C} \rtimes_{r,id} G$$

is called the *reduced group C*-algebra* of G (see [54, Example 7.9]), while

$$C^*(G) := \mathbb{C} \rtimes_{id} G$$

is the *(full) group C*-algebra* of G (see [54, Example 2.33]).

Both the reduced and universal crossed product C*-algebras retain some information about the underlying dynamical system. In general, however, one loses information about the underlying dynamics when passing to the associated crossed product C*-algebra, as there are several examples of isomorphic crossed product C*-algebras arising from vastly different topological dynamical systems $(C_0(X), \alpha)$ (see [45] for several examples of this).

1.3 Exactness of the reduced crossed product functor

Given an equivariant morphism $\phi : A \to B$ of two G-C*-algebras, we can define a map

$$\phi : C_c(G, A) \to C_c(G, B) \qquad \phi(f)(g) = \phi(f(g)). \tag{1.7}$$

which induces maps

$$\phi : A \rtimes G \to B \rtimes G \qquad \phi : A \rtimes_r G \to B \rtimes_r G.$$

The equivariance of ϕ makes the induced maps a $*$-homomorphism of the crossed products C*-algebras. A natural question is: Under which conditions an extension of G-C*-algebras

$$0 \to B \to E \to A \to 0$$

gives us an extension of the associated crossed product C*-algebras

$$0 \to B \rtimes_{(r),\beta} G \to E \rtimes_{(r),\gamma} G \to A \rtimes_{(r),\alpha} G?$$

Definition 1.35. A group G is called *exact* (or C*-exact) if for any extension of G-C*-algebras

$$0 \to I \to A \to A/I \to 0$$

the associated sequence of reduced crossed products

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0$$

is also exact, i.e., $- \rtimes_r G$ is an exact functor on the category of G-C*-algebras with equivariant morphisms.

It is known that the full crossed product functor $- \rtimes G$ is always an exact functor for any group G.

Some authors call G exact if $C^*_r(G)$ is an exact C*-algebra, which amounts to the minimal tensor product functor $- \otimes C^*_r(G)$ being exact (Definition 1.15). In general this is weaker than our definition, but we will see the two definitions agree for discrete groups. Here are some of the main theorems regarding exactness of groups:

Theorem 1.36 ([19, Theorem 6])**.** *Let K be a field, $n \geq 1$ a positive integer and $G \subset GL_n(K)$ any discrete subgroup. Then the reduced group C*-algebra $C^*_r(G)$ is exact.*

It turns out that if Γ is any discrete group we have

$$C^*_r(\Gamma) \text{ is an exact C*-algebra} \quad \Leftrightarrow \quad - \rtimes_{r,\alpha} \Gamma \text{ is an exact functor}$$

The implication $\Leftarrow$ is always true for any locally compact group since if Γ acts trivially on a C*-algebra A, then $A \rtimes_r \Gamma \simeq A \otimes_{min} C^*_r(\Gamma)$. The other direction is proved in [31, Theorem 5.2]. Hence we have the following

Theorem 1.37 ([31, Theorem 5.2])**.** *If K is any field and Γ any discrete linear subgroup of $GL_n(K)$ then the functor*

$$- \rtimes_r \Gamma$$

is exact.

The authors of [31] prove this theorem by proving the following, slightly stronger statement: For any discrete group Γ and a Γ-equivariant extension $0 \to B \to E \to A \to 0$ the sequence

$$0 \to B \rtimes_r \Gamma \to E \rtimes_r \Gamma \to A \rtimes_r \Gamma \to 0 \tag{1.8}$$

is exact if and only if the sequence

$$0 \to (B \rtimes \Gamma) \otimes C^*_r(\Gamma) \to (E \rtimes \Gamma) \otimes C^*_r(\Gamma) \to (A \rtimes \Gamma) \otimes C^*_r(\Gamma) \to 0 \tag{1.9}$$

is exact. This gives us a way to determine if a sequence of crossed products is exact even in cases where the functor $- \rtimes_r \Gamma$ is not exact. Before listing some of these cases, we will need the following definition:

Definition 1.38 ([11, Definition 4.3.5])**.** An action of a discrete group Γ on a compact space X is called *amenable* if there exists a net of weak*-continuous maps

$$m_i : X \to \text{Prob}(\Gamma) \qquad x \mapsto m_i^x$$

such that for each $s \in \Gamma$

$$\lim_{i \to \infty} \sup_{x \in X} \|s_\star m_i^x - m_i^{sx}\|_1 = 0$$

where the norm is given for a measure $\mu \in \text{Prob}(\Gamma)$ by

$$\|\mu\|_1 = \sum_{\gamma \in \Gamma} |\mu(\gamma)|$$

Any action of an amenable group on a compact space X is amenable in the sense of Definition 1.38, but there are plenty of amenable actions by non-amenable groups. Amenable actions on a compact space X induce strongly continuous group actions on $C(X)$, and there is a similar notion of amenability on the C*-algebraic level, though they become quite technical. We refer the interested reader to [11] section 4.5 for arbitrary unital C*-algebras and discrete groups. For us the following will suffice as our definition

Proposition 1.39. *Let Γ be a discrete group and A a nuclear Γ-C^*-algebra. Then the action of Γ on A is amenable if and only if*

$$A \rtimes_r \Gamma$$

is nuclear.

We are mostly interested in amenable actions due to the following theorem:

Theorem 1.40 ([11, Theorem 4.3.4])**.** *Let A be any C^*-algebra and Γ a discrete group acting amenably on A. Then*

$$A \rtimes_r \Gamma = A \rtimes \Gamma.$$

So in case the action of Γ is amenable there is a unique pre-C*-norm on $C_c(G, A)$. Let us see an example of amenable actions

Example 1.41. Let G be a unimodular Lie group and $\Gamma \subset G$ a discrete subgroup with $\Gamma \backslash G$ of finite volume with respect to the restricted Haar measure of G. Let $H \subset G$ be a closed subgroup. Then the action of Γ on G/H is amenable if and only if H is amenable. If Γ is an arbitrary discrete subgroup of G then the action of Γ on G/H is amenable if H is amenable (see [55, Corollary 4.3.7]).

As a special case of this, if G is a connected semisimple Lie group with finite center and maximal compact subgroup K, $\Gamma \subset G$ a lattice (meaning $\Gamma \backslash G$ has finite volume) and $P \subset G$ is a parabolic subgroup (Definition 2.47), then the action of Γ on G/P is amenable if and only if P is a minimal parabolic subgroup (as these are the only amenable parabolic subgroups of G).

We will now list a few of the cases where the sequence of equation (1.8) is exact in the next lemma.

Lemma 1.42. *For Γ any discrete group, the sequence (1.8) is exact in the following cases*

1. *The action of Γ is amenable on all C^*-algebras in the sequence.*

2. *A is nuclear and the action of Γ on A is amenable.*

3. *There is a unique C^*-norm on the algebraic tensor product $A \rtimes \Gamma \odot C^*_r(\Gamma)$.*

4. *The sequence $0 \to B \rtimes \Gamma \to E \rtimes \Gamma \to A \rtimes \Gamma \to 0$ is locally split (or semisplit).*

5. *The sequence* $0 \to B \to E \to A \to 0$ *is locally split by* Γ*-equivariant maps (or equivariantly semisplit).*

Proof. **Case 1)** follows from the fact that for any Γ-C*-algebra B with an amenable Γ action we have $B \rtimes_r \Gamma = B \rtimes \Gamma$, and $- \rtimes \Gamma$ is an exact functor.

Case 2): The conditions assure that the crossed product $A \rtimes \Gamma$ is nuclear, hence the claim follows from Case 3 above.

Note that $B \rtimes \Gamma$ is nuclear always implies that B is nuclear. If the action is amenable, the converse also holds. For transformation groupoids given by the action of a discrete group on a locally compact Hausdorff space, Theorem 3.5 of [2] $A \rtimes_r \Gamma$ is nuclear if and only if the action of Γ is amenable.

Case 3) follows from the correspondence of the sequences (1.8) and (1.9) and Corollary 3.7.3 [11], which states that the sequence

$$0 \to B \otimes D \to E \otimes D \to A \otimes D \to 0$$

is exact if the algebraic tensor product $A \odot D$ admits a unique C*-norm, which happens for instance when either A or D are nuclear.

Case 4) If the sequence $0 \to B \to E \to A \to 0$ is locally split (or semisplit), then by Proposition 3.7.6 of [11] the sequence

$$0 \to B \otimes D \to E \otimes D \to A \otimes D \to 0$$

is exact for any C*-algebra D.

Case 5) If the sequence $0 \to B \to E \to A \to 0$ is equivariantly locally split/semisplit, then the sequence $0 \to I \rtimes \Gamma \to A \rtimes \Gamma \to (A/I) \rtimes \Gamma \to 0$ is locally split/semisplit respectively, so we can use Case 4). □

1.4 Hilbert C*-modules

If we think of $\mathbb{C}$ as a C*-algebra, a complex Hilbert space H is nothing but a special module over the C*-algebra $\mathbb{C}$ with an inner product taking values in $\mathbb{C}$. The next definition is a natural generalization of the notion of Hilbert spaces where $\mathbb{C}$ is replaced by an arbitrary C*-algebra -

Definition 1.43 ([34] p. 2)**.** A right pre-Hilbert A-module, is a (complex) linear space H, with a right A-module structure, together with a map $\langle -, - \rangle : H \times H \to A$ satisfying

1. $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in H$, $a \in A$ and $\lambda \in \mathbb{C}$,
2. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all $x, y, z \in H$, and all $\alpha, \beta \in \mathbb{C}$,

3. $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in H$ and all $a \in A$,

4. $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in H$,

5. $\langle x, x\rangle \geq 0$; and $\langle x, x\rangle = 0 \Rightarrow x = 0$ for all $x \in H$,

If H is complete with respect to the norm

$$\|x\| := \|\langle x, x\rangle\|^{1/2}$$

it is called a right Hilbert A-module.

It should be clear that Definition 1.43 is modelled on the definition Hilbert spaces, in fact since $\mathbb{C}$ is a C*-algebra, it is easy to show that

Example 1.44. A right Hilbert $\mathbb{C}$-module is a Hilbert space.

The analogy between Hilbert spaces and right Hilbert C*-modules is not perfect though. Recall that for a Hilbert space H, any bounded linear map $T : H \to H$ has an adjoint, meaning there is a map $T^* : H \to H$ satisfying

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for all $x, y \in H$. The map T^* is uniquely determined and automatically linear and bounded if T is. For Hilbert C*-modules we have a similar notion

Definition 1.45. Let H be a right Hilbert A-module. By an *operator* on H we mean a bounded $\mathbb{C}$-linear map $L : H \to H$ for which

$$L(xa) = L(x)a \qquad \text{for all } x \in H, a \in A.$$

An operator $L : H \to H$ is called adjointable if there exists another operator L^* such that

$$\langle L(x), y\rangle = \langle x, L^*(y)\rangle \qquad \text{for all } x, y \in H.$$

The set of adjointable operators on H are denoted $\mathcal{L}_A(H)$ or $\mathcal{L}(H)$.

As Definition 1.45 would suggest, there can be non-adjointable operators on Hilbert C*-modules. An example of non-adjointable operators can be found in [34, p. 8]. We have the following:

Lemma 1.46 ([34] p.8)**.** *Let H be a Hilbert A-module. Then $\mathcal{L}(H)$ is a C*-algebra with respect to the norm*

$$\|T\| := \sup_{\substack{x\in H\\ \|x\|=1}} \|T(x)\| = \sup_{\substack{x\in H\\ \|x\|=1}} \sup_{\substack{y\in H\\ \|y\|=1}} \|\langle Tx, y\rangle\|. \tag{1.10}$$

We shall now look at a class of operators which are always adjointable

Definition 1.47. Let A be a C*-algebra and H be a Hilbert A-module (Definition 1.43). The *rank one* operators are the operators of the form

$$\theta_{x,y} : H \to H \qquad \theta_{x,y}(z) = x\langle y, z\rangle \qquad \text{for all } z \in H$$

where $x, y \in H$. The compact operators in H, denoted $\mathbb{K}(H)$ is the closed linear span

$$\mathbb{K}(H) := \overline{\mathrm{Span}\{\theta_{x,y} \mid x, y \in H\}}$$

where the closure is taken with respect to the operator norm defined by eq. (1.10).

A little bit of work shows that for any $x, y \in H$ $\theta_{x,y}$ is adjointable with adjoint given by

$$\theta^*_{x,y} = \theta_{y,x}.$$

It follows that $\mathbb{K}(H) \subset \mathcal{L}(H)$.

Example 1.48. Let us look at a useful example of a Hilbert module. As a byproduct we will get a concrete realization of the multiplier algebra defined in Example 1.3. Let A be a C*-algebra. Define a map

$$\langle -, - \rangle : A \times A \to A \qquad \langle a, b\rangle := a^*b, \quad a, b \in A. \tag{1.11}$$

This can be shown to be an A-valued inner product satisfying properties (1)-(4) of Definition 1.43 hence A is a pre-Hilbert module over itself. Completeness follows from completeness of A using the C*-identity (Equation (1.1)):

$$\|\langle a, a\rangle\|^{1/2} = (\|a\|^2)^{1/2} = \|a\|.$$

So A is a Hilbert A-module with respect to the inner product of equation (1.11).

It turns out that if A is nonunital, we have

$$\mathcal{L}(A) = M(A) \quad \text{and} \quad \mathbb{K}(A) = A.$$

Example 1.49 ([34] p. 7)**.** Let A be a unital C*-algebra and, $B \subset A$ any C*-subalgebra containing the unit of A. If H is a Hilbert A-module, let $\phi : A \to B$ be a map satisfying

1. $\phi|_B = \mathrm{Id}_B$;

2. ϕ is a contractive completely positive map (Definition 1.21);

The map ϕ satisfying property (1) and (2) is called a conditional expectation from A to B. By a result of Tomiyama ([11, Theorem 1.5.10]) if ϕ is contractive and satisfies property (1), it is automatically completely positive and B-linear, meaning

$$\phi(ab) = \phi(a)b \qquad \text{for all } a \in A \text{ and } b \in B.$$

As in Example 1.48 we may treat A as a Hilbert A-module with inner product

$$\langle a_1, a_2 \rangle_A := a_1^* a_2$$

The Hilbert module A becomes a Hilbert B-module with respect to the B-valued inner product

$$\langle x, y \rangle_B := \phi(\langle x, y \rangle_A) \qquad x, y \in H.$$

The complete positivity of ϕ is exactly what is needed to ensure that $\langle -, - \rangle_B$ satisfies property (5) of Definition 1.43, since if $a_i \in H$ $(i = 1, \ldots, n)$ is any finite sequence of elements in H, then

$$\phi\left(\langle \sum_{i=1}^n a_i, \sum_{j=1}^n a_j \rangle\right) = \sum_{i=1}^n \sum_{j=1}^n \phi(\langle a_i, a_j \rangle)$$

which is is positive for all finite sums $\sum_{i=1}^n a_i$ in A if and only if ϕ is completely positive (see the proof of [34, Lemma 4.3(i)]).

1.5 K-theory

Operator K-theory is an example of what is called a generalized homology theory on C*-algebras extending the topological K-theory for compact topological spaces through Gelfand duality. We will define both operator K-theory and (compactly supported) topological K-theory in this section and return to this material later when we introduce equivariant KK-theory in the next section. Let us start with topological K-theory.

As previously, X denotes a locally compact Hausdorff space, and all vector bundles will be assumed to be complex and of finite rank. Additionally we will assume any vector bundle $\pi : E \to X$ to be trivial outside a compact set. Note that saying that a bundle $E \to X$ on a locally compact but noncompact Hausdorff space X is trivial outside a compact set, is equivalent to saying that E is the restriction of a bundle $E^\infty :\to X \cup \{\infty\}$ on the one point compactification of X.

We denote by $e^n = X \times \mathbb{C}^n$ the trivial bundle of rank n over X.

Definition 1.50. We define the Whitney sum of two vector bundles $\pi_i : E_i \to X$ $(i = 1, 2)$ to be the vector bundle

$$E_1 \oplus E_2 := \{(v_1, v_2, x) \in E_1 \times E_2 \times X \mid \pi_1(v_1) = \pi_2(v_2) = x\}$$

Definition 1.51 ([22] p. 39)**.** Two vector bundles $E_i \to X$ $(i = 1, 2)$ are called stably isomorphic, denoted $E_1 \simeq_s E_2$, if there is a trivial vector bundle $e^n \to X$ such that

$$E_1 \oplus e^n = E_2 \oplus e^n.$$

Definition 1.52. Given a bundle $\pi : E \to X$ over X and a continuous map $f : Y \to X$, the pullback of E by f is the bundle

$$f^*E \to Y$$

with total space

$$f^*E = \{(v, y) \in E \oplus Y \mid f(y) = \pi(v)\}$$

and bundle map $(f^*\pi) : f^*E \to Y$ given by projection onto the second factor.

It should be clear that the pullback of a bundle preserves the rank of the bundle, and that the pullback of a trivial bundle is trivial. We also have the that for any set $O \subset Y$ and continuous map $\phi : X \to Y$ and bundle $E \to Y$, we have $\phi^*(E|_O) = (\phi^*E)|_{\phi^{-1}(O)}$. Combined, these three properties give us the following:

Lemma 1.53. *Let $\pi : E \to X$ be a bundle, and $K \subset X$ a compact subset such that the restriction$\pi|_{X\backslash K} : E|_{X\backslash K} \to X\backslash K$ is trivial. Let $\phi : Y \to X$ be a proper map. Then ϕ^*E is trivial outside a compact set in Y.*

Denote by $[E]$ and $[E']$ the stable isomorphism class of the bundles E and E' respectively. Then the equivalence class of their sum $[E \oplus E']$ only depends on the stable isomorphism class of E and E', so we may define an operation on the collection of stable isomorphism classes by

$$[E] \oplus [E'] := [E \oplus E'].$$

Denote by $V(X)$ the collection of all stable isomorphism classes of bundles over X. This is an abelian semigroup with respect to Whitney sums, and the class $[e^0]$ (the rank–zero bundle) is the zero element

The next proposition has important consequences for the theory, and is one of the reasons we restrict our attention to vector bundles that are trivial outside a compact set.

Proposition 1.54 ([42, Proposition 1.7.9])**.** *If X is compact, then any vector bundle over X is a subbundle of a trivial bundle e^n for some n. Equivalently, for any bundle $E \to X$, there is a bundle $E' \to X$, and $n \in \mathbb{N}$ such that*

$$E \oplus E' = e^n.$$

One consequence of Proposition 1.54 is that the semigroup $V(X)$ has the cancellation property, meaning

$$[E_1] \oplus [E_2] = [E_1] \oplus [E_3] \Rightarrow [E_2] = [E_3]$$

since we may add a bundle E_1' to each side for which $E_1' \oplus E_1 = e^m$ for some m.

Knowing this, we can form the Grothendieck group $\mathrm{Gr}(V(X))$ as the collection of formal differences

$$\{[E_1] - [E_2] \mid [E_i] \in V(X)\}/\sim$$

with addition defined by $([E_1]-[E_2])+([E_1']-[E_2']) := ([E_1 \oplus E_1']-[E_2 \oplus E_2'])$ and identifying elements

$$[E_1] - [E_2] \sim [E_1'] - [E_2'] \Leftrightarrow [E_1 \oplus E_2'] = [E_1' \oplus E_2].$$

Definition 1.55. The (unreduced compactly supported complex) topological K-theory group of X is defined to be

$$K^0(X) := Gr(V(X)).$$

Now, to steer things towards operator K-theory, assume that X is compact, so that $C_0(X) = C(X)$ and let

$$M_\infty(\mathbb{C}) = \lim_{n\to\infty} M_n(\mathbb{C})$$

be the algebraic direct limit of $n \times n$-matrices with complex coefficients and connecting morphisms

$$M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C}) \qquad x \mapsto \mathrm{diag}(x, 0).$$

The Serre-Swan theorem tells us that there is a 1-1 correspondence between isomorphism classes of complex vector bundles on X and homotopy classes of projection valued continuous maps $X \to M_\infty(\mathbb{C})$ (see [26, Section 4.1]).

The Whitney sum of two projection valued functions $p, p' \in C(X, M_\infty(\mathbb{C}))$ is given by

$$p \oplus p' = \mathrm{diag}(p, p'),$$

i.e. by the block diagonal matrix with entries p and p' along the diagonal. The stable equivalence relations of vector bundles translate to certain equivalence relations the corresponding projection valued functions, which produces a semigroup under addition. Taking the Grothendieck group gives us the operator K-theory groups for $C(X)$, denoted $K_0(C(X))$ which agrees with $K^0(X)$.

With this example in mind, let A be an arbitrary unital C*-algebra and consider the projections in the algebraic direct limit

$$C(A, M_\infty(\mathbb{C})) = M_\infty(A) = \lim_{n\in\mathbb{N}} M_n(A),$$

where the connecting morphisms are simply the maps

$$M_n(A) \to M_{n+1}(A) \qquad M \mapsto M \oplus 0$$

given by padding a matrix in M_n with zeros. Recall that an element in M_∞ is represented by some element in $M_n(A)$ for some $n \in \mathbb{N}$ (see Example 1.6). For any two elements in $p, q \in M_\infty(A)$ we define their sum $p \oplus q$ to be the block diagonal matrix $\mathrm{diag}(p, q)$.

Projections in $M_\infty(A)$ are, as for C*-algebras, the elements $p \in M_\infty(A)$ for which

$$p = p^2 = p^*.$$

We are now ready to define the equivalence relation:

Definition 1.56. Two projections $p, q \in M_\infty(A)$ are called equivalent, denoted $p \sim q$, if there is a rectangular A valued matrix v such that

$$p = v^*v \qquad q = vv^*.$$

Assuming A is unital, we define:

Definition 1.57. Two projections $p, q \in M_\infty(A)$ are called stably equivalent if there is an $n \in \mathbb{N}$ such that

$$I_n \oplus p \sim I_n \oplus q$$

where I_n is the $n \times n$-identity matrix.

Let $V(A)$ be the collection of stable equivalence classes of projections in $M_\infty(A)$. This is a semigroup with the cancellation property, with respect to addition given by diagonal concatenation of block matrices. We define, just as in the topological case:

Definition 1.58. Let A be a unital C*-algebra, then the K-theory group of A is

$$K_0(A) = \mathrm{Gr}(V(A)).$$

If A is nonunital, then we define

$$K_0(A) = \mathrm{Ker}(\iota_* : K_0(A^+) \to K_0(\mathbb{C}) = \mathbb{Z})$$

where $\iota : A^+ \to \mathbb{C}$ is the canonical unital *-homomorphism from the unitization of A.

Bott periodicity and the six–term exact sequence of K-theory

Definition 1.59. Let A be a C*-algebra. The *suspension* of A is the C*-algebra

$$SA := C_0(\mathbb{R}, A)$$

of continuous functions from $\mathbb{R}$ to A vanishing at infinity. Similarly, we denote then n-fold suspension by

$$S^nA := C_0(\mathbb{R}^n, A).$$

For a *-homomorphism $\phi : A \to B$ we define the mapping cone of ϕ to be

$$C_\phi := \{(a, f) \in A \otimes C([0,1], B) \mid f(0) = \phi(a), f(1) = 0\}.$$

We define higher K-groups as follows

Definition 1.60. Let A be any C*-algebra, then for $n \in \mathbb{N}$ define

$$K_n(A) := K_0(S^n A).$$

Let us go through some of the basic properties of the $K_i(A)$-groups, starting with functoriality: If

$$\phi : A \to B$$

is a *-homomorphism, we get a morphism of the associated matrix C*-algebras

$$M_n(A) \to M_n(B) \qquad (a_{i,j}) \mapsto (\phi(a_{i,j})).$$

This gives us a map

$$\phi_* : V(S^n A) \to V(S^n B) \qquad \phi[p] := [\phi(p)]$$

which can be shown to commute with addition, so by the universal properties of the Grothendieck group induces a map

$$\phi_* : K_n(A) \to K_n(B).$$

Hence we have the following

Proposition 1.61. *For any $n \in \mathbb{N}$, the assignment*

$$A \to K_n(A)$$

is a functor from the category of C-algebras to the category of abelian groups.*

Next, let us state the following fundamental theorem

Proposition 1.62 (Bott periodicity [41, Theorem 7.5.1 7])**.** *For any C*-algebra A there is an isomorphism*

$$\delta : K_2(A) \to K_0(A).$$

Let us see what the K-theory functor does to extensions of C*-algebras

Theorem 1.63 ([41, Theorem 7.5.18])**.** *Given an extension*

$$0 \to B \xrightarrow{\iota} E \xrightarrow{p} A \to 0$$

of C-algebra, we have a 6-term exact sequence of K-groups:*

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\iota_*} & K_0(E) & \xrightarrow{p_*} & K_0(A) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ K_1(A) & \xleftarrow[p_*]{} & K_1(E) & \xleftarrow[\iota_*]{} & K_1(B) \end{array}$$

The maps p_* and ι_* are induced by functoriality of K_i. Let us see how the maps ∂, called the connecting morphisms, are constructed in [41]. Let

$$0 \to B \xrightarrow{\iota} E \xrightarrow{p} A \to 0$$

be an extension of C*-algebras. We have natural maps

$$j : B \to C_p \qquad k : SA \to C_p$$

to the mapping cone of p (Definition 1.59) where $j(b) = (\iota(b), 0)$ and $k(f) = (0, f)$ are the inclusion maps. It is shown in [41, Lemma 7.5.12] that $j_* : K_0(B) \to K_0(C_p)$ is an isomorphism. We now define the connecting map to be

$$\partial = (j_*)^{-1}k_* : K_0(SA) = K_1(A) \to K_0(B)$$

Using a similar construction for the extension

$$0 \to SB \to SE \to SA \to 0$$

and Bott periodicity obtain the map

$$\partial : K_0(A) \to K_1(B).$$

1.6 Equivariant extensions and KK-theory

The interplay between extensions of C*-algebras and K-homology dates back to the now classical work of Brown, Douglas and Fillmore ([9], [10]) where the authors, motivated by the study of essentially normal operators, set out to classify extensions of the form

$$0 \to \mathbb{K} \to E \to C(X) \to 0$$

where $\mathbb{K} := \mathbb{K}(l^2(\mathbb{Z}))$ are the compact operators on a separable infinite dimensional Hilbert space. By what seems a coincidence the Ext-group they defined turned out to be isomorphic to the K-homology of the space X, that is the abstract homology theory dual to topological K-theory defined in the previous section. The definition of the Ext-group was later extended to include extensions of the form

$$0 \to \mathbb{K} \to E \to A \to 0$$

for arbitrary stable C*-algebras A (see Definition 1.18) at the expense of making Ext a semigroup. Voiculescu then showed [52] that if A is separable, the semigroup is actually a monoid (i.e. has a zero element) with unit the class of any split extension. The invertible elements of $\mathrm{Ext}(A)$ were later characterized by Arveson in [3] (see also [27, Theorem 3.2.9]) as those extensions for which the associated "Busby invariant" (Def. 1.69) map $\phi : A \to Q(B)$ lifts to a completely positive contractive map $\psi : A \to M(B)$, which is easily proved to be equivalent to having a completely positive contractive splitting of the quotient map in the extension. This is automatic if A is nuclear, by the lifting theorem of Choi and Effros [12].

The theory was greatly generalized by Kasparov in [29] and later in [30] laying the foundations of KK-theory, the bivariant K-theory which bears his name. A new semigroup $\mathrm{Ext}(A, B)$ was defined which extends the definition of $\mathrm{Ext}(A)$ to extensions of the form

$$0 \to B \to E \to A \to 0$$

so $\mathrm{Ext}(A) = \mathrm{Ext}(\mathbb{K}, A)$. Kasparov then proved using his generalized Stinespring dilation theorem, that

$$\mathrm{KK}_1(A, B) = \mathrm{KK}(SA, B) = \mathrm{Ext}(A, B)^0$$

where $\mathrm{Ext}(A, B)^0$ denotes the subgroup of $\mathrm{Ext}(A, B)$ of invertible elements and $SA = C(0, 1)\otimes A$ is the suspension of A. The group KK-groups introduced by Kasparov generalize K-theory and its dual theory K-homology in the sense that (assuming A is separable)

$$\mathrm{KK}(\mathbb{C}, A) = K_0(A) \qquad \mathrm{KK}(A, \mathbb{C}) = K^0(A).$$

The isomorphism uses the characterization of invertible elements in $\mathrm{Ext}(A, B)$ given by the existence of a completely positive splitting $\phi : A \to E$ of the quotient map $p : E \to A$ of the extension. The splitting also gives a concrete realization of the KK^1-cycle representing the extension.

Parallel to this, a theory of equivariant extensions and equivariant KK-theory emerged. The equivariant KK-groups were defined by Kasparov in [30], but it was not linked to an equivariant Ext-group. The equivariant version of the Ext-semigroup was defined in [51] which also gave a characterization of invertibility of elements by means of equivariant completely positive splittings, and proved an isomorphism

$$\mathrm{Ext}_G(A, B)^0 \simeq \mathrm{KK}^1_G(A, B)$$

just as in the non-equivariant case, but this time we need B to be equivariantly $\mathbb{K}_G := \mathbb{K}(\bigoplus_{n\in\mathbb{N}} L^2(G))$-stable rather than just $\mathbb{K}$-stable. We will focus entirely on the equivariant side of the story here, but the non-equivariant case can always be recovered by setting $G = \{0\}$.

1.6.1 Equivariant KK-theory

Let us start by recalling the definition of equivariant KK-theory, and the equivariant extension semigroup Ext_G. For a good reference to KK-theory see for instance [5] where there is a short description on equivariant KK-theory in Chapter VIII.20. The book [27] is also a good reference to KK-theory, which focuses heavily on extensions of C*-algebras, but equivariant KK-theory is not covered.

Throughout this section G will denote a locally compact group and all C*-algebras will be *assumed to be separable*, meaning they have a countable dense subset or equivalently, can be represented on a Hilbert space with a countable basis. Recall that we call a C*-algebra A a G-C*-algebra (Def. 1.24) if it is endowed with a strongly continuous action of G, that is, an action for which the map

$$G \to A \qquad g \mapsto \alpha_g(a)$$

is continuous for all $a \in A$, where $\alpha_g \in \mathrm{Aut}(A)$ denotes the group action of g on A.

If (A, G, α) is a G-C*-algebra, a right Hilbert A-module will be assumed to also have a group action satisfying the following compatibility conditions:

Definition 1.64 ([5, Definition 20.1.1])**.** Let (A, G, α) be a G-C*-algebra. An equivariant right Hilbert A-module is a right Hilbert A-module H (in the sense of Definition 1.43) with an action of G by bounded invertible linear transformations for which

$$G \mapsto \mathbb{R} \quad g \mapsto \|\langle gx, gx\rangle\| \tag{1.12}$$

is continuous and

$$g(xa) = (gx)\alpha_g(a).$$

From now on, any Hilbert module over a G-C*-algebra, will be assumed to be equivariant.

An action of G on H satisfying equation (1.12) is called a *continuous G-action.* As in the case of Hilbert spaces, an action of G on H induces an action of G on the adjointable operators $\mathcal{L}(H)$ by $(gT)(x) = gT(g^{-1}x)$, which is not in general continuous with respect to the operator topology on $\mathcal{L}(H)$, just strictly continuous, i.e. for each $x \in H$ the map $g \mapsto (gT)(x)$ is continuous.

Definition 1.65 ([5] Definition 20.1.2)**.** An operator $T \in \mathcal{L}(H)$ for which $g \mapsto gT$ is operator norm-continuous is called a G-continuous operator.

We are now ready to define equivariant Kasparov modules:

Definition 1.66 ([5, Definition 20.2.1])**.** An odd Kasparov G-module for the G-algebras (A, B) is a triple (H, ϕ, F) where H is a countably generated (equivariant) right Hilbert B-module with a continuous action of G, $\phi : A \to \mathcal{L}(H)$ is an equivariant *-homomorphism and $F \in \mathcal{L}(H)$ is G-continuous operator such that

- $[F, \phi(a)] \in \mathbb{K}(H)$
- $(F^2 - 1)\phi(a) \in \mathbb{K}(H)$
- $(F^* - F)\phi(a) \in \mathbb{K}(H)$
- $(gF - F)\phi(a) \in \mathbb{K}(H)$

for all $a \in A$ and $g \in G$.

An equivariant Kasparov module is call *degenerate* if we have

$$[F, \phi(a)] = (F^2 - 1)\phi(a) = (F^* - F)\phi(a) = (gF - F)\phi(a) = 0.$$

The set of degenerate Kasparov G-modules will be denoted by $\mathbb{D}_G(A, B)$.

Given a Kasparov G-module (H, ϕ, F) for (A, B), let $f_A : C \to A$ and $f^B : B \to D$ be two equivariant *-homomorphisms. Then the pullback of (H, ϕ, F) by f_A is a Kasparov G-module for (C, B) given by

$$f_A^*(H, \phi, F) = (H, \phi \circ f_A, F).$$

Similarly, the pushforward of (H, ϕ, F) by f^B is a Kasparov G-module over (A, D) given by

$$f_*^B(H, \phi, F) = (H \otimes_B D, \phi \otimes 1, F \otimes 1)$$

Two Kasparov G-modules (H_1, ϕ_1, F_1), (H_2, ϕ_2, F_2) for (A, B) are said to be *unitarily equivalent*, denoted by $(H_1, \phi_1, F_1) \simeq_u (H_2, \phi_2, F_2)$, if there is a unitary $u \in L(H_1, H_2)$ intertwining the action of G, ϕ_i and F_i. They are said to be *homotopic* if there is a Kasparov G-module $(\hat{H}, \hat{\phi}, \hat{F})$ for $(A, C([0, 1], B))$ such that

$$(ev_0)_*(\hat{H}, \hat{\phi}, \hat{F}) \simeq_u (H, \phi, F) \qquad \text{and} \qquad (ev_1)_*(\hat{H}, \hat{\phi}, \hat{F}) \simeq_u (H', \phi', F'),$$

where $ev_t : C([0, 1], B) \to B$ is the evaluation at t. We can add two Kasparov modules, just as in the non-equivariant case, by defining

$$(H, \phi, F) \oplus (H', \phi', F') := (H \oplus H', \phi \oplus \phi', F \oplus F') \tag{1.13}$$

The collection of all Kasparov G-modules for (A, B) becomes an abelian semigroup with respect to addition. We denote this semigroup by $E_G(A, B)$. Finally

Definition 1.67. Denote by $\mathrm{KK}_G^1(A, B)$ the quotient of $E_G(A, B)$ by the relation of homotopy equivalence.

Proposition 1.68 ([5, Proposition 20.2.3])**.** $\mathrm{KK}_G^1(A, B)$ *is an abelian group with respect to addition given on Kasparov modules by eq.* (1.13).

Next, let us define the equivariant extension group $\mathrm{Ext}_G(A, B)$ following [51]. An extension of G-C*-algebras

$$0 \to B \xrightarrow{\iota} E \xrightarrow{p} A \to 0$$

is called an equivariant extension of B by A if both ι and p are equivariant, though some authors prefer to call this an equivariant extension of A by B. For ease of notation we will refer to the extensions by its middle algebra and write (E). Two G-extensions (E) and (E') of B by A are said to be isomorphic if there is a $*$-homomorphism $E \to E'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

The homomorphism $E \to E'$ is then necessarily an equivariant $*$-isomorphism ([51] Theorem 2.2). As with non-equivariant extensions, there is a 1 to 1 correspondence between isomorphism classes of G-extensions and elements in $\mathrm{Hom}_G(A, Q(B))$, the set of equivariant $*$-homomorphisms from A to the Corona algebra $Q(B) = M(B)/B$. The construction of the extension associated with a given Busby map $\phi : A \to Q(B)$, is given by the pullback diagram

$$\begin{array}{ccc} E_\phi & \xrightarrow{\quad p \quad} & \mathcal{A} \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \phi} \\ M(B) & \xrightarrow{\ q_B\ } & Q(B) \end{array}$$

where $q_B : M(B) \to Q(B)$ is the quotient map and T and p are the canonical maps of the pullback construction. Explicitly, we have

$$E_\phi = \{(a, m) \in A \oplus M(B) \mid \phi(a) = q_B(m)\},$$

with p and T the projections onto the first and second factor respectively and the associated extension being

$$0 \to B \to E_\phi \to A$$

where the inclusion $B \to E_\phi$ is induced by the inclusion of $B \to M(B)$.

Conversely, given a G-extension we associate to it a map $\phi : A \to Q(B)$. To define ϕ, we need to first define the map $T : E \to M(B)$ which is given implicitly by the equation

$$\iota(T(e)b) = e\iota(b).$$

The injectivity of ι ensures T is uniquely determined.

The map T restricts to the identity on B, since for any $b, b' \in B$ we have

$$\iota(T(\iota(b'))b) = \iota(b')\iota(b) = \iota(b'b).$$

So $T(\iota(b'))$ acts by left multiplication by b' on B in $M(B)$, which is exactly how b' is imbedded into $M(B)$. Given any splitting s (not necessarily linear) of $p : E \to A$, we define:

Definition 1.69. The Busby map for the extension (E) is the map

$$\phi = q_B \circ T \circ s.$$

To see that ϕ does not depend on the choice of splitting, assume $s_1, s_2 : A \to E$ are two splittings of the quotient map $p : E \to A$ of the extension (E), that is $p \circ s_i = id_A$. We have that for any $a \in A$ $p(s_1(a) - s_2(a)) = 0$, hence $s_1(a) - s_2(a) \in \iota(B)$. Thus

$$q_B \circ T \circ s_1(a) - q_B \circ T \circ s_2(a) = q_B(T(s_1(a) - s_2(a))) = 0$$

since $T(\iota(B))$ is contained in $B \subset M(B)$. Two G-extensions are called *unitarily equivalent* if there is a unitary $u \in M(B)$ with $gu - u \in B$ for all $g \in G$ and a $*$-homomorphism $E \to E'$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Ad_u} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

The map $E \to E'$ is necessarily an isomorphism, but *need not be equivariant.* In what follows we use $\simeq_u$ to indicate unitary equivalence of extensions. The next lemma shows what this amounts to at the level of Busby maps

Lemma 1.70. *Given two equivariant $*$-homomorphisms*

$$\phi_i \in \mathrm{Hom}_G(A, Q(B))$$

their extensions (E_{ϕ_i}) $(i = 1, 2)$ are unitarily equivalent if and only if there is a $u \in M(B)$ such that

- $gq_B(u) = q_B(u)$ *for all* $g \in G$;
- $Ad(u) \circ \phi_1 = \phi_2$.

Proof. Assume $u \in M(B)$ is as in the Lemma. We have

$$E_i := E_{\phi_i} = \{(a, m) \in A \oplus M(B) \mid \phi_i(a) = q_B(m)\}$$

but $\phi(a) = q_B(m) \Leftrightarrow \psi(a) = Ad_{q_B(u)}(q_B(m)) = q_B(Ad_u(m))$ hence the map $id \oplus Ad_u : A \oplus M(B) \to A \oplus M(B)$ restricts to an isomorphism $E_1 \to E_2$ which makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Ad_u} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Conversely, assume the above diagram commutes for some $u \in M(B)$ and an arbitrary isomorphism $f : E_1 \to E_2$. Using the implicit definition of the maps $T_i : E_i \to M(B)$:

$$\iota_i(T_i(e)b) = e\iota_i(b)$$

substituting $\iota_1 = f \circ \iota_2 \circ Ad_{u^*}$ the left hand side becomes

$$\iota_1(T_1(e)b) = (f \circ \iota_2)(Ad_{u^*}(T_1(e))Ad_{u^*}(b)),$$

while the right hand side reads

$$e\iota_1(b) = f \circ \iota_2(T_2(f^{-1}(e))Ad_{u^*}(b)).$$

Putting these together gives us

$$T_1 \circ f = Ad_u \circ T_2.$$

Let $s_1 : A \to E_1$ be any splitting of the quotient maps $p_1 : E_1 \to A$. The map $s_2 = f \circ s_1$, is then a splitting for p_2 and the Busby map thus related by

$$\begin{aligned} \phi_1 &= q_B \circ T_1 \circ s_1 \\ &= q_B \circ (Ad_u \circ T_2 \circ f^{-1}) \circ (f \circ s_2) \\ &= q_B \circ (Ad_u \circ T_2 \circ s_2) \\ &= Ad_{q_B(u)} \circ (q_B \circ T_2 \circ s_2) = Ad_{q_B(u)} \circ \phi_2 \end{aligned}$$

Finally, for a unitary $u \in M(B)$ we have $gu - u \in B$ if and only if $gq_B(u) = q_B(u)$, which concludes the proof. □

Remark 1.71. Since $q_B(u)$ is a unitary in $Q(B)$, the reader may wonder why we did not simply pick a unitary in $Q(B)$ when defining unitary equivalence. There is nothing seriously wrong with this approach, but the resulting Ext-groups (to be defined shortly) have been less used in practice.

More precisely, if in the definition of unitary equivalence we pick an invariant unitary $u \in Q(B)$ rather than one in $M(B)$ we get what is called “weak” unitary equivalence in [5], and another Ext_G group that does not agree with the usual Ext_G-group of Definition

1.72 (see for instance [15] section V.6 for an example using the Cuntz algebra in the non-equivariant case). The reason they differ boils down to the fact that not all unitaries in $Q(B)$ can be lifted to unitaries in $M(B)$.

To define addition of two extensions, we will need to make some extra assumptions on the algebra B. We need to assume B is stable (Def. 1.18), i.e. that there is a *-isomorphism

$$B \simeq B \otimes \mathbb{K}$$

where $\mathbb{K}$ has the trivial G-action. Assuming B is stable, let (E_i) $(i = 1, 2)$ be two G-extensions of B by A. Then we define their sum to be the extension

$$0 \to M_2(B) \xrightarrow{\hat{\iota}} \widehat{E} \xrightarrow{\iota} A \to 0$$

where

$$\widehat{E} = \left\{ \begin{bmatrix} e_1 & b_1 \\ b_2 & e_2 \end{bmatrix} \;\middle|\; e_i \in E_i,\ b_i \in \mathcal{B},\ p_1(e_1) = p_2(e_2) \right\}$$

the quotient map is given by

$$\hat{p} = p_1 \oplus p_2 : \begin{bmatrix} e_1 & b_1 \\ b_2 & e_2 \end{bmatrix} \mapsto p_1(e_1)$$

and the inclusion $\hat{\iota}$ is the obvious one. This is an extension of B by A since when B is stable we have $M_2(B) \simeq B$. At the level of Busby invariants this additive structure takes the form

$$\phi_1 \oplus \phi_2 = \phi'$$

where

$$\phi'(a) = Ad_{q_B(V_1)}(\phi_1(a)) + Ad_{q_B(V_2)}(\phi_2(a))$$

for any choice of G-invariant isometries $V_i \in M(B)$ with $V_1V_1^* + V_2V_2^* = 1$. To prove the existence of such isometries, we use the fact that B is stable, since then we have an imbedding $M(\mathbb{K}) \otimes M(B) \subset M(B \otimes \mathbb{K}) \simeq M(B)$ and any two isometries $W_i \in M(\mathbb{K})$ with $W_1W_1^* + W_2W_2^*$ give isometries $V_i = W_i \otimes 1 \in M(\mathbb{K}) \otimes M(B) \subset M(B)$ which are G-invariant and satisfy $V_1V_1^* + V_2V_2^* = 1$.

Similar to the non-equivariant case, an extension (E_ϕ) associated with the map $\phi \in \mathrm{Hom}_G(A, Q(B))$ is called *degenerate* if ϕ lifts to an equivariant *-homomorphism

$$\hat{\phi} : A \to M(B).$$

This is equivalent to the quotient map $p : E \to A$ being split by an equivariant $*$-homomorphism. To see why, let $s : A \to E$ be a splitting of p assumed without loss of generality to be linear. Then

$$T(s(aa') - s(a)s(a')) = \hat{\phi}(aa') - \hat{\phi}(a)\hat{\phi}(a') = 0.$$

Hence $s(aa') - s(a)s(a') \in \mathrm{Ker}T$. Now $p(s(aa') - s(a)s(a')) = aa' - aa' = 0$ hence

$$s(aa') - s(a)s(a') \in \mathrm{Ker}T \cap B,$$

but since T acts as the identity on B, this means $\mathrm{ker}T \cap B = \{0\}$ and so s is multiplicative. The fact that s is $*$-preserving and equivariant can be proved similarly.

We can now define an equivalence relation on $\mathrm{Hom}_G(A, Q(B))$ by saying $\phi \sim \phi'$ if and only if there are degenerate $*$-homomorphisms $\phi_0, \phi_0' \in \mathrm{Hom}_G(A, Q(B))$ such that[4]

$$\phi \oplus \phi_0 \simeq_u \phi' \oplus \phi_0'.$$

Definition 1.72. The equivariant extension semigroup is defined as

$$\mathrm{Ext}_G(A, B) = \mathrm{Hom}_G(A, Q(B))/\sim$$

and $\mathrm{Ext}_G(A, B)^0$ denotes the subgroup of invertible elements in $\mathrm{Ext}_G(A, B)$.

As in the non-equivariant case, there is a way to characterize G-extensions which are invertible using splittings. Let $\mathbb{K}_G := \mathbb{K}(\bigoplus_{n\in\mathbb{N}} L^2(G))$ with G acting diagonally by the regular representation. Then we have

Theorem 1.73 ([51, Theorem 8.1])**.** *An extension*

$$0 \to B \otimes \mathbb{K}_G \to E \to A \otimes \mathbb{K}_G \to 0 \tag{1.14}$$

is invertible if and only if the sequence

$$0 \to B \otimes \mathbb{K}_G \otimes \mathbb{K}_G \to E \otimes \mathbb{K}_G \to A \otimes \mathbb{K}_G \otimes \mathbb{K}_G \to 0 \tag{1.15}$$

obtained by tensoring everything with $\mathbb{K}_G$, is equivariantly semisplit.

Clearly if the extension $0 \to B \otimes \mathbb{K}_G \to E \to A \otimes \mathbb{K}_G \to 0$ is equivariantly semisplit in the first place, by the equivariant completely positive map

$$s : A \otimes \mathbb{K}_G \to E$$

then the sequence of equation (1.15) would also be equivariantly semisplit, by the map

$$s \otimes \mathrm{id} : A \otimes \mathbb{K}_G \otimes \mathbb{K}_G \to E \otimes \mathbb{K}_G$$

so being equivariantly semisplit after tensoring with $\mathbb{K}_G$ is weaker than being equivariantly semisplit.

We will need the following definition:

[4] This should be reminiscent of the way Kasparov's KK-groups are defined as operator homotopy equivalence classes of Kasparov modules modulo degenerate modules.

Definition 1.74 ([51, Definition 9.1]). A G-C*-algebra B is called K-proper if for any countably generated Hilbert B-module E with a continuous G-action, there is an isomorphism

$$E \oplus (B \otimes L^2(G)^\infty) \simeq B \otimes L^2(G)^\infty$$

as Hilbert B-modules, where

$$L^2(G)^\infty = \bigoplus_{n \in \mathbb{N}} L^2(G)$$

is the infinite sum of $L^2(G)$.

So a C*-algebra is K-proper if it satisfies an equivariant version of Kasparov's stabilization theorem. The definition is a generalization of proper actions on topological spaces, in the sense that if $B = C_0(X)$ for some locally compact Hausdorff proper G-space X, then B is K-proper.

This is by no means the only candidate for a generalization of proper actions to G-C*-algebras (see [48] [38], [30]). See also [37, Theorem 8.5] for several equivalent definitions of K-proper actions. We are now ready to state the main theorem of [51] which reads

Theorem 1.75 ([51] Theorem9.2). *Assume A, B are G-C*-algebras with B K-proper. Then*

$$\mathrm{Ext}_G(A, B \otimes \mathbb{K}_G)^0 \simeq \mathrm{KK}^1_G(A, B)$$

We will not repeat the proof here. However, in the next section we will see how to assign a class to an equivariantly semisplit G-extension of commutative C*-algebras (see Example 1.84). Commutative C*-algebras are very far from being stable though, so we will first need make precise what it means for an equivariant KK-cycle to be "associated" with an extension of non-separable and/or non K-proper C*-algebra. If

$$0 \to B \xrightarrow{\iota} E \xrightarrow{p} A \to 0$$

is a G-extension, then by tensoring everything with $\mathbb{K}_G$ we get a G-extension

$$0 \to B \otimes \mathbb{K}_G \xrightarrow{\iota \otimes id} E \otimes \mathbb{K}_G \xrightarrow{p \otimes id} A \otimes \mathbb{K}_G \to 0$$

which in turn gives an element in $\mathrm{Ext}_G(B \otimes \mathbb{K}_G, A \otimes \mathbb{K}_G)^0 = KK^1_G(B, A)$. We refer to this element as the class corresponding to the extension.

1.7 An example from harmonic analysis

We will now introduce an example, studied in [36], where classical harmonic analysis is used to produce an equivariant completely positive splitting of an extension and a class in equivariant KK-theory.

First, we will need to define the geodesic compactification of symmetric spaces of non-positive curvature. Our main reference will be [16, Section 1.7]:

Definition 1.76. A Hadamard manifold is a complete simply connected Riemannian manifold with non-positive sectional curvature.

The important feature of Hadamard manifolds is the following:

Theorem 1.77 (Hadamard)**.** *For any Hadamard manifold X and any point $x_0 \in X$, the exponential map* $\exp : T_{x_0}X \to X$ *is a global diffeomorphism.*

Let X be a Hadamard manifold of dimension n.

Definition 1.78. For each $x \in X$ and $v \in T_xX$, we denote by $\gamma_{x,v} : \mathbb{R} \to X$ the unique geodesic in X with

$$\gamma_{x,v}(0) = x \qquad \text{and} \qquad \gamma'_{x,v}(0) = v$$

The geodesic $\gamma_{x,v}$ is called a directed geodesic (or a geodesic ray) centered at x in direction v.

Now fix a point $x_0 \in X$ and write $\gamma_v := \gamma_{x_0,v}$. We define:

Definition 1.79 ([16, Section 1.27 1.7])**.** The geodesic boundary of X is defined to be

$$X(\infty) := \{\gamma_v \mid \|v\| = 1\}$$

We topologize $X(\infty)$ in such a way that the map

$$X(\infty) \to S^{n-1} \subset T_{x_0}X, \qquad \gamma_v \mapsto v$$

is a homeomorphism, i.e. the pullback or weak topology induced by the map $X(\infty) \to S^{n-1}$.

For any open set $U \subset X(\infty) \simeq S^{n-1} \subset T_{x_0}X$ and $r > 0$, define

$$S_r^U := \left(\bigcup_{v \in U \ t > r} \gamma_v(t) \right) \sqcup U$$

which is a truncated cone of rays centered at x_0 in the direction of vectors in U.

Definition 1.80 ([16] Sec. 1.7)**.** The geodesic compactification of X is (as a set) the disjoint union

$$\overline{X} = X \cup X(\infty).$$

We endow $X \cup X(\infty)$ with the topology generated by the open sets of X together with S_r^U, where U ranges over all open sets in S^{n-1} and r over all positive numbers.

The topology on $X \cup X(\infty)$ given in Definition 1.80 is called the *cone topology*, and makes $X \cup X(\infty)$ homeomorphic to the closed n-ball, with X its interior points and $X(\infty) \simeq S^{n-1}$ its boundary.

It is useful to have the following convergence criterion in mind: Given a sequence of points $(x_i)_{i\in\mathbb{N}}$ in X, then $x_i \to v \in X(\infty)$ if and only if $d(x_i, x_0) \to \infty$ and the vector $v_i \in T_{x_0}X$ pointing in the direction of the geodesic connecting x_0 and x_i converges to v in $S^{n-1} \subset T_{x_0}X$. Thus for any geodesic ray γ_v and any sequence of points $x_i \in \gamma_v(\mathbb{R}^+)$ for which $d(x_0, x_i) \to \infty$, the sequence $(x_i)_{i\in\mathbb{N}}$ converges in the geodesic compactification to $v \in S^{n-1} \simeq X(\infty)$. Not all convergent sequences are of this form though, as there are sequences $x_i \in X$ such that $x_i \to v \in X(\infty)$, but $d(x_i, \gamma_v) := \inf_{t>0} d(x_i, \gamma_v(t)) \to \infty$ as $i \to \infty$.

There is another way to describe the points in $X(\infty)$, sometimes more convenient in practice, which we will now define. Let

$$\mathcal{P} = \{\gamma_{x,v} \mid x \in X, v \in S^{n-1} \subset T_xX\}$$

be the set of all (unit speed) geodesic rays in X (Definition 1.78). Then define an equivalence relation on $\mathcal{P}$ by

$$\gamma \sim \gamma' \Leftrightarrow \sup_{t\in\mathbb{R}^+} d(\gamma(t), \gamma'(t)) < \infty \tag{1.16}$$

We have the following lemma

Lemma 1.81 ([16, Proposition 1.7.3]). *For any $x_0 \in X$, the set $\mathcal{P}/\sim$ is in bijection with the geodesic rays of the form*

$$\gamma_{x_0,v} \qquad \text{for } v \in S^{n-1} \subset T_{x_0}X.$$

Example 1.82. It may be helpful to think of what Lemma 1.81 looks like when X is the euclidean space $\mathbb{R}^n$. In this case the Lemma simply states that if γ is any geodesic ray in $\mathbb{R}^n$, then γ is parallel to a unique geodesic ray centered at 0. So Lemma 1.81 is an analogue of Euclid's parallel postulate[5] for Hadamard manifolds.

However, the case of $\mathbb{R}^n$ may leave the reader wondering why we use geodesic rays at all, and not just geodesics. After all in $\mathbb{R}^n$ for two geodesics $\gamma, \gamma' : \mathbb{R} \to \mathbb{R}^n$ (i.e. straight lines) the conditions

$$\sup_{t\in\mathbb{R}^+} d(\gamma(t), \gamma'(t)) < \infty \tag{1.17}$$

and

$$\sup_{t\in\mathbb{R}^-} d(\gamma(t), \gamma'(t)) < \infty \tag{1.18}$$

[5] which is equivalent to the statement that any straight line is parallel to a unique straight line through a point

and

$$\sup_{t\in\mathbb{R}} d(\gamma(t),\gamma'(t)) < \infty$$

are all equivalent to the condition that $d(\gamma(t),\gamma'(t))$ is constant in t. We mention that in case the space X has negative curvature, there are geodesics which satisfy (1.17) but do not satisfy (1.18), and that are not constant in t in either direction. That is, there are geodesics which are "parallel" in the positive direction, but diverge in the negative direction.

Under the bijection established by Lemma 1.81 the set $\mathcal{P}/\sim$ inherits a topology from $X(\infty)$. It is thus possible to view $X(\infty)$ as a quotient of the space of all geodesics in X. One bonus of this description is that the G-action on $X(\infty)$ is now simple to express. It is just

$$g[\gamma_{x,v}] = \gamma_{gx,(dg)v}, \tag{1.19}$$

which is the image of $\gamma_{x,v}$ under the action of G on X.

With X any Riemannian space, we denote by $\mathrm{Iso}(X)$ its isometry group, and $\mathrm{Iso}(X)^0$ the connected component of the isometry group of X.

Proposition 1.83 ([16] p. 30)**.** *Any isometry $\phi : X \to X$ extends to a homeomorphism*

$$\overline{\phi} : X \cup X(\infty) \to X \cup X(\infty).$$

The extension makes $X \cup X(\infty)$ a $G = \mathrm{Iso}(X)$-space. The action of G on $X(\infty)$ is given by equation (1.19)*.*

The case of the hyperbolic spaces Now assume $X = \mathbb{H}^n$ is the real hyperbolic n-space. Then $\mathbb{H}^n$ is a Hadamard manifold, and thus we can define its geodesic compactification

$$\overline{\mathbb{H}^n} := \mathbb{H}^n \cup \partial\mathbb{H}^n.$$

By Proposition 1.83, the isometry group $G = \mathrm{Iso}(\mathbb{H}^n)^0 = SO^0(1,n)$, acts on $\overline{\mathbb{H}^n}$ by homeomorphisms that extend the isometric action of G on $\mathbb{H}^n$.

Indeed if $n = 2$, then $\mathbb{H}^2$ can be identified with the disk $\mathbb{D} \subset \mathbb{R}^2 \simeq \mathbb{C}$. An orientation preserving isometry acts on $\mathbb{H}^2$ by a Möbius transformation:

$$z \mapsto e^{i\theta}\frac{z-a}{1-\overline{a}z} \qquad a \in \mathbb{D}, \theta \in \mathbb{R}.$$

The Möbius actions act by homeomorphisms on the closed disk $\overline{\mathbb{D}}$ but they are not isometries with respect to the angular metric on S^1 unless $a = 0$. We then get an equivariant extensions of G-C*-algebras

$$0 \to C_0(\mathbb{H}^n) \xrightarrow{\iota} C(\overline{\mathbb{H}^n}) \xrightarrow{q} C(\partial\mathbb{H}^n) \to 0. \tag{1.20}$$

Let now Γ be a discrete torsion-free subgroup $\Gamma \subset G = \mathrm{Iso}(\mathbb{H}^n)^0 = SO(n,1)^0$. Using Theorem 1.37, we get an exact sequence of crossed products

$$0 \to C_0(\mathbb{H}^n) \rtimes_r \Gamma \xrightarrow{\iota \rtimes id} C(\overline{\mathbb{H}^n}) \rtimes_r \Gamma \xrightarrow{q \rtimes id} C(\partial\mathbb{H}^n) \rtimes_r \Gamma \to 0. \tag{1.21}$$

By Theorem 1.63 this induces a six term exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(C_0(\mathbb{H}^n) \rtimes_r \Gamma) & \longrightarrow & K_0(C(\overline{\mathbb{H}^n}) \rtimes_r \Gamma) & \longrightarrow & K_0(C(\partial\mathbb{H}^n) \rtimes_r \Gamma) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C(\partial\mathbb{H}^n) \rtimes_r \Gamma) & \longleftarrow & K_1(C(\overline{\mathbb{H}^n}) \rtimes_r \Gamma) & \longleftarrow & K_1(C_0(\mathbb{H}^n) \rtimes_r \Gamma) \end{array}$$

For any torsion free discrete subgroup Γ of G, Γ acts properly and freely on $\mathbb{H}^n$ and it is well known that this implies

$$K_i(C_0(\mathbb{H}^n) \rtimes_r \Gamma) = K_i(C_0(\mathbb{H}^n/\Gamma)).$$

In case Γ satisfies certain technical conditions, which are known to hold if Γ is cocompact[6], we also have

$$K_i(C(\overline{\mathbb{H}}^n) \rtimes_r \Gamma) = K_i(C_r^*(\Gamma)).$$

Thus the K-theory of two interesting C*-algebras make their appearance in the above 6-term sequence: One is the continuous functions on the space $\mathbb{H}^n/\Gamma$, which is a classifying space for free and proper Γ actions, the other is the reduced group C*-algebra of Γ (see Ex. 1.34) $C_r^*(\Gamma)$.

The extension of equation (1.20) is equipped with a completely positive equivariant splitting

$$C(\partial\mathbb{H}^n) \to C(\overline{\mathbb{H}^n})$$

defined by integration against the Poisson kernel which for $\mathbb{H}^n$ with the ball model takes the form

$$P : \mathbb{H}^n \times \partial\mathbb{H}^n \to \mathbb{R} \qquad P(x,\xi) = \frac{1-|x|^2}{|x-\xi|^n} \tag{1.22}$$

where $|-|$ is the euclidean distance. It is a well-known fact that all bounded harmonic functions f on $\mathbb{H}^n$ which can be extended continuously to $\partial\mathbb{H}^n$ can be written as an integral

$$f(x) = \int_{\partial\mathbb{H}^n} P(x,\xi)\hat{f}(\xi)d\lambda(\xi),$$

for some function $\hat{f} \in C(\partial\mathbb{H}^n)$. The assignment $\hat{f} \mapsto f$ has the following properties:

[6]The condition on Γ we are tacitly requiring is that it should satisfy the Baum-Connes conjecture with coefficients in $\mathbb{C}$ and $C(\overline{H}^n)$. Then $K(C_0(\mathbb{H}^n) \rtimes_r \Gamma) = K_i^{top}(\Gamma, C_0(\mathbb{H}^n)) = K_i^{top}(\Gamma, \mathbb{C}) = K_i(C_r^*(\Gamma))$ where the first and third equality follows from the Baum-Connes conjecture, and the second is the content of Lemma 5 of [17]

- The function f extends to the whole geodesic compactification $\overline{\mathbb{H}}^n$ and restricts to $\hat{f}$ on $\partial\mathbb{H}^n \simeq S^{n-1}$;
- The map $\hat{f} \mapsto f$ is a G-equivariant completely positive splitting of the quotient map $q : C(\overline{\mathbb{H}^n}) \to C(\partial\mathbb{H}^n)$.

We saw in Section 1.6 how these equivariant semisplittings are used to determine if the extension can be represented by a class in $\mathrm{KK}^1_\Gamma(C(\partial\mathbb{H}^n), C_0(\mathbb{H}^n))$. We will see later in Chapter IV how to create a KK^1_G-cycle representing a G-equivariant extension using a G-equivariant semisplitting. In the case of the extension of equation (1.20) however, we can clearly construct the corresponding Kasparov module as follows:

Example 1.84. We repeat here the construction in Section 3.2 of [36]. Let $G = \mathrm{Iso}(\mathbb{H}^n)^0 = SO(1,n)^0$ and write $\mu_x = P(x,-)dv$ for $x \in \mathbb{H}^n$, where dv is the Lebesgue measure on $\partial\mathbb{H}^n \simeq S^{n-1}$. The family μ_x satisfies

$$\mu_{gx} = g_*\mu_x, \qquad g \in G,\ x \in \mathbb{H}^n$$

and μ_x varies continuously with x in the sense that if $x_i \to x$, then

$$\int_{\partial\overline{\mathbb{H}^n}} f(v)d\mu_{x_i}(v) \to \int_{\partial\overline{\mathbb{H}^n}} f(v)d\mu_x(v)$$

for all $f \in C(\partial\overline{\mathbb{H}^n})$. Now define

$$T_1\mathbb{H}^n = \mathbb{H}^n \times \partial\overline{\mathbb{H}}^n.$$

The map

$$\rho : C_c(T_1\mathbb{H}^n) \to C_c(\mathbb{H}), \qquad \rho(\Psi)(x) := \int_{\partial\overline{\mathbb{H}^n}} \Psi(x,v)d\mu_x(v)$$

is a conditional expectation, hence (see Example 1.49) we get a pre-Hilbert $C_0(\mathbb{H}^n)$-module structure on $C_c(T_1\mathbb{H}^n)$ given by the inner product

$$\langle f, f'\rangle(x) := \rho(\overline{f}f')(x), \qquad x \in \mathbb{H}^n,\ f, f' \in C_c(T_1\mathbb{H}^n).$$

Denote by $L^2(T_1\mathbb{H}^n, \mu_x)_{C_0(\mathbb{H}^n)}$ the completion of $C_c(T_1\mathbb{H}^n)$ with respect to this inner product. As the notation suggests, we think of $L^2(T_1\mathbb{H}^n, \mu_x)_{C_0(\mathbb{H}^n)}$ as sections of a continuous field of Hilbert spaces $(L^2(C(\partial\overline{\mathbb{H}^n})), \mu_x)$, fibered over $\mathbb{H}^n$, which vanish at infinity. We also have an action of $C(\partial\overline{\mathbb{H}^n})$ on $L^2(T_1\mathbb{H}^n, \mu_x)_{C_0(\mathbb{H}^n)}$ by adjointable operators, given by multiplication:

$$(f \cdot \Psi)(x,v) = f(v)\Psi(x,v) \qquad f \in C(\partial\overline{\mathbb{H}^n}), \Psi \in C_c(T_1\mathbb{H}^n).$$

Let $p \in \mathcal{L}(L^2(T_1\mathbb{H}^n, \mu_x)_{x \in C_0(\mathbb{H}^n)})$ be the adjointable operator given by

$$p(\Psi)(x,v) = \int_{\partial\overline{\mathbb{H}^n}} \Psi(x,w)d\mu_x(w).$$

It can be shown that p is a projection, i.e. that $p^* = p^2 = p$. Moreover p commutes with the action of G since

$$\begin{aligned} p(g\Psi)(x,v) &= \int_{\partial\overline{\mathbb{H}^n}} \Psi(g^{-1}x, g^{-1}w)d\mu_x(w) = \Psi(g^{-1}x, g^{-1}w)d\mu_x(w) \\ &= \Psi(g^{-1}x, w)(g^{-1})_* d\mu_x(w) = \Psi(g^{-1}x, w)d\mu_{g^{-1}}(w) = g(p\Psi)(x,v). \end{aligned}$$

The triple

$$(L^2(T_1\mathbb{H}^n, \mu_x)_{C_0(\mathbb{H}^n)}, C(\partial\overline{\mathbb{H}^n}), 2p-1)$$

is an Kasparov G-module representing the class of extension (1.20) in $\mathrm{KK}^1_\Gamma(C(\partial\overline{\mathbb{H}^n}), C_0(\mathbb{H}^n))$ (see [36, Theorem 3.4]).

The space $\mathbb{H}^n$ is an example of a symmetric space of noncompact type of rank 1. In the next sections, we will discuss what happens when we try to extend the construction in Example 1.84 to a larger class of spaces, namely the symmetric spaces of noncompact type (of arbitrary rank).