Sweeping vacuum gravitational waves under the rug

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CHAPTER 6

Hadamard regularization of the graviton stress tensor

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6.1 Introductory remarks

In this chapter we reexamine what was done in Chapter 4 in a fully covariant way. Having reviewed in the previous chapter the necessary tools to study GWs as a massless spin-2 particle on a curved spacetime, we revisit the regularization and renormalization of vacuum tensor perturbations. This allows us to highlight that the FRLW foliation formulation studied in the previous chapters does not feature all the subtleties that one can encounter in renormalizing the stress energy tensor of a quantum field on a curved spacetime. Thus, even if in fixing the renormalization conditions we refer to the study of vacuum GWs on a RD universe in order to compare our findings with the previous results, this chapter is intended as a generalization of Chapter 4.

In the following we first derive the effective action for GWs in Section 6.2. To do so we expand the Einstein Hilbert action up to second order in the perturbation $h_{\mu\nu}$ and we fix the de Donder gauge using the Faddeev-Popov method.
Once we have obtained the action that, if variated with respect to the background metric, gives the stress energy tensor of a massless spin-2 particle on a generic background, we proceed to regularize in Section 6.3 using Hadamard regularization. We first introduce Hadamard regularization by reviewing the definition of the regulator σ^{μ}, the Hadamard form of the Feynman propagator and the link with the Hadamard representation of Green's function. We then rewrite both the action of GWs and the action of the Faddeev-Popov ghosts in terms of the Hadamard representation of Green's function and we compute the counterterms needed to reabsorb the divergencies appearing in computing the stress energy tensor.
In Section 6.4 we then proceed in renormalizing the divergences by fixing the renormalization conditions. Before doing so we comment on the finite contributions

of the regularized result, stressing that the finite leftovers of renormalization are not uniquely defined and need to be fixed by experiments. The renormalization conditions are then imposed by specifying the background metric, left unspecified throughout the previous sections. In this way, by using the $P(X)$ theory to derive the action that reproduces the stress tensor for the background radiation fluid, we can reconnect Chapter 4 with and study the effects of a massless spin-2 particle on a RD universe.
We conclude by commenting the results of the fully covariant description of this chapter in Section 6.5 and we leave the details of the Hadamard regularization of the GWs action in the Appendix A.

6.2 Vacuum stress energy tensor from the effective action

Our treatment proceeds from the standard effective action via the background field method (see Chapter 5 for more details). As in Chapter 2, we define $h_{\mu\nu}$ as the perturbations around some background, defined as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{6.1}$$

where $g_{\mu\nu}$ is the background metric, which we leave unspecified for the time being, and the field $h_{\mu\nu}$ represents a massless spin-2 particle. We then consider the effective action

$$S_{\rm cl} := S_{\rm EH} + S_{\rm M} + S_{\rm ct}, \tag{6.2}$$

where $S_{\rm EH}$, which represents the Einstein Hilbert action, and $S_{\rm M}$, representing the matter content that sources the background expansion, constitute the tree level action for the classical background and $S_{\rm ct}$ represents the counterterms needed to subtract the UV divergences that arise at any given loop order, with finite remainders that are to be fixed through renormalization conditions.
We expand the classical Einstein-Hilbert action to quadratic order in perturbations $h_{\mu\nu}$ and obtain [18, 34]

$$\begin{aligned} S^{(2)} = \frac{\kappa^2}{2}\int d^4x\sqrt{-g}\Bigg[&\frac{1}{2}h_{\rho\sigma}\Box h^{\rho\sigma} + h\nabla^\rho\nabla^\sigma h_{\sigma\rho} + \nabla^\alpha h_\alpha{}^\rho\nabla^\sigma h_{\sigma\rho} - \frac{1}{2}h\Box h \\ &+R^\beta{}_{\rho\alpha\sigma}h_\beta{}^\alpha h^{\rho\sigma} + h_\alpha{}^\rho h^{\alpha\sigma}R_{\rho\sigma} - hh^{\rho\sigma}R_{\rho\sigma} - \frac{1}{2}h^{\rho\sigma}h_{\rho\sigma}R + \frac{1}{4}hhR\Bigg], \end{aligned} \tag{6.3}$$

where we have defined $\kappa^2 \equiv \frac{1}{8\pi G_N}$. The stress tensor for GWs is obtained by variation with respect to the background metric – a process that is equivalent to perturbing the Einstein equations to second order and bringing the quadratic terms over to the other side to act as a source for the background.

Before proceeding, we recall that if one is interested in studying GWs for which there is a prior scale separation, a number of approximations and simplifications are possible. A detailed overview of such simplifications can be found in Section 2.2, where we study the derivation of the stress energy tensor, that this is valid for spectra of bounded support sourced by some physical production mechanism (and hence sub-horizon). However, taking the result of such derivation in the context of cosmological stochastic backgrounds ought to be treated with caution. This caution should be amplified when one encounters divergences that need to be regularized, as is the case under analysis. Consequently, we define the stress energy tensor for GWs as the variation of the graviton action with respect to the background metric:

$$T^{\rm gw}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \left\langle \frac{\delta\left(-S_{\rm gw}\right)}{\delta g^{\mu\nu}} \right\rangle_{\rm in,in} \tag{6.4}$$

where the in-in expectation value implicitly traces over some initial density matrix. When this state is taken as the adiabatic vacuum, one obtains the stress tensor for vacuum tensor perturbations.

In order to define the action of GWs $S_{\rm gw}$, we must first consider the process of gauge-fixing, for which de Donder gauge presents a particularly efficient choice. We proceed via the Faddeev-Popov method [97] (see Section 5.2.1 for more details) and add a gauge breaking term which fixes the chosen gauge condition to Eq. 6.3, defined as $\nabla_\mu h^{\mu\nu} = \frac{1}{2}\nabla_\nu h$, along with a ghost term that accounts for the measure factor induced by gauge-fixing:

$$S_{\rm gb} = -\frac{\kappa^2}{2}\int d^4x\sqrt{-g}\left(\nabla^\mu h_{\mu\nu} - \frac{1}{2}\nabla_\nu h\right)\left(\nabla^\alpha h_\alpha{}^\nu - \frac{1}{2}\nabla^\nu h\right), \tag{6.5}$$

$$S_{\rm gh} = \frac{\kappa^2}{2}\int d^4x\sqrt{-g}\left[\bar\eta^\mu\left(g_{\mu\nu}\Box - R_{\mu\nu}\right)\eta^\nu\right]. \tag{6.6}$$

In the above, η^ρ represents the ghost field that accounts for the residual gauge freedom by subtracting the spurious DoFs from the action $S^{(2)}$. Consequently, the gauge-fixed action for the gravitational sector is given by:

$$\begin{aligned} S_{\rm gw} =& S^{(2)} + S_{\rm gb} + S_{\rm gh} \\ =& \frac{\kappa^2}{2}\int d^4x\sqrt{-g}\left[\frac{1}{2}h_{\rho\sigma}\Box h^{\rho\sigma} - \frac{1}{4}h\Box h + R^\beta{}_{\rho\alpha\sigma}h_\beta{}^\alpha h^{\rho\sigma} + h_\alpha{}^\rho h^{\alpha\sigma}R_{\rho\sigma} - hh^{\rho\sigma}R_{\rho\sigma}\right. \\ & \left. -\frac{1}{2}h^{\rho\sigma}h_{\rho\sigma}R + \frac{1}{4}hhR + \bar\eta^\mu\left(g_{\mu\nu}\Box - R_{\mu\nu}\right)\eta^\nu\right]. \end{aligned} \tag{6.7}$$

In this way, starting with the action Eq. 6.3 for a rank-2 symmetric tensor field nominally consisting of ten DoFs, we obtain the action of a massless spin-2 particle $S_{\rm gw}$ with only two propagating DoFs. From Eq. 6.4, the stress energy tensor of vacuum tensor perturbations is given by the sum of contributions

$$T^{\rm gw}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left\langle \frac{\delta}{\delta g^{\mu\nu}}\left(-S_{\rm gr} - S_{\rm gh}\right)\right\rangle_{\rm in,in} = T^{\rm gr}_{\mu\nu} + T^{\rm gh}_{\mu\nu}, \tag{6.8}$$

where we have defined $S_{\rm gr} \equiv S^{(2)} + S_{\rm gb}$. The angled brackets above denote the time ordered in-in correlation function $\langle ...\rangle := \langle {\rm in, vac}|T[...]|{\rm in, vac}\rangle$, which inevitably exhibits divergences for field bilinears in the coincident limit, the regularization of which will be studied in the next section.

Before proceeding with the regularization and renormalization of the effective action, a final and no less consequential comment is necessitated by the question of whether we are obliged to work with the on-, or off-shell formulation of the effective action in our computations. To 1-loop, the former can be obtained by expanding the action to quadratic order in fluctuations and evaluating the resulting functional determinant as a function of the background:

$$\Gamma_1 = \frac{i}{2} \ln\det\left\{\frac{\delta^2 S_{\rm cl}[\bar{\phi}^a]}{\delta\varphi^b\delta\varphi^c}\right\}, \tag{6.9}$$

where, referring to the notation of Section 5.2, $\bar{\phi}^a$ is the relevant background field, and φ^a denote fluctuations around it. The result of differentiating the above with respect to the background yields the radiation currents Eqs. 6.18 and 6.19, whose contractions make explicit that we are differentiating the 1-loop 1PI vacuum graph [1]. Unlike the classical action, however, Eq. 6.9 is not a scalar and is dependent on how one parametrizes field space in addition to also depending on the background field gauge in the presence of gauge symmetries.
Instead, an effective action that does not suffer from these drawbacks was arrived at by Vilkovisky and DeWitt ([36, 37]) by working covariantly in field space and writing down the equivalent of the functional determinant of the field covariant second variational derivative of the action:

$$\Gamma_1^{\rm VdW} = \frac{i}{2} \ln\det\left\{\frac{\delta^2 S_{\rm cl}[\varphi^a]}{\delta\varphi^b\delta\varphi^c} - \Gamma^d_{bc}[\bar{\phi}^a]\frac{\delta S_{\rm cl}[\bar{\phi}^a]}{\delta\varphi^d}\right\}, \tag{6.10}$$

where Γ^d_{bc} is the connection on field space. When the background field φ^a minimizes $S_{\rm cl}$ (i.e. one is working on shell) the two forms are equivalent. However, the two forms will in general differ for any quantity obtained from differentiating the effective action when the field space connection is non-vanishing even when evaluated on shell. Therefore, the renormalized stress tensor obtained from the Vilkovisky-DeWitt effective action will have additional contributions relative to the stress tensor obtained from the 'standard theory'. However, the difference is only in terms of additional finite contributions, which moreover vanish for vacuum contributions on maximally symmetric spacetimes and for one-particle states on Minkowski space [18]. On a general FRLW background, with a homogeneous and isotropic fluid sourcing the background expansion, these are non-vanishing and provide additional finite contributions that depend on the quantum state.
Nevertheless, the procedure we follow allows us to work with the standard form of the effective action, as the divergences that need to be regularized are unaffected by the Vilkovisky-DeWitt correction term, and any finite contributions are absorbed by the process of fixing renormalization conditions. What is crucial for the subtraction process, however, is the identification of the scale factor dependence of the various finite and state dependent terms, for which we determine recursion relations with initial coefficients that can be fixed with a sufficient number of measurements at the renormalization scale.

6.3 Regularization

The regularization of the stress tensor for any propagating DoFs on a general background must proceed with care, all the more so when gauge redundancies are present. Differently from what we did in Capter 4, where we worked directly at the level of the stress tensor obtained by variation of Eq. 6.3, gauge-fixing by hand, and then imposing the SVT decomposition to extract the stress tensor for the propagating spin-2 polarizations when evaluated as an expectation value, in this section we proceed to do so in a covariant manner. This procedure can be related to the covariant method detailed below by a series of Ward identities that we shall return to further on. In what follows, we proceed with the first step of the renormalization procedure reviewed in Section 1.2.2 and we regularize the divergences encountered in the evaluation of Eq. 6.8 by adopting Hadamard regularization techniques [18, 120, 193], which are an extension of the covariant point-splitting method[1].

6.3.1 Hadamard point splitting

The point-split version of a tensor $U^{\mu\nu}(x)$ is defined as the coincidence limit of the bitensor $U^{\mu\nu'}(x,x')$ defined in a neighbourhood of x^μ

$$U^{\mu\nu}(x) = \lim_{\sigma^\mu\to 0} U^{\mu\nu'}(x,x'), \tag{6.11}$$

where primed indices refer to the point $x^{\mu'}$ and σ^μ is the geodesic distance between x^μ, and $x^{\mu'}$. Doing so allows us to isolate the divergent from finite contributions to Eq. 6.8 with σ^μ as the UV regulator. Hadamard regularization of the effective action proceeds through the intermediary of the Feynman propagators (in the notation of [18]):

$$\begin{aligned}
G^{\mu\nu\alpha'\beta'}(x,x') &= \frac{i}{32\pi G_N}\frac{\langle\psi|T\left(h^{\mu\nu}(x)h^{\alpha'\beta'}(x')\right)|\psi\rangle}{\langle\psi\mid\psi\rangle}\\
&= \frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma+i\varepsilon}\left(g^{\alpha'(\mu}g^{\nu)\beta'}\right)+V^{\mu\nu\alpha'\beta'}\ln(\Lambda^2(\sigma+i\varepsilon))+W^{\mu\nu\alpha'\beta'}\right]\\
\tilde{G}^{\mu\alpha'}(x,x') &= \frac{i}{32\pi G_N}\frac{\langle\psi|T\left(\bar{\eta}^{\mu}(x)\eta^{\alpha'}(x')\right)|\psi\rangle}{\langle\psi\mid\psi\rangle}\\
&= \frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma+i\varepsilon}g^{\mu\alpha'}+\tilde{V}^{\mu\alpha'}\ln(\Lambda^2(\sigma+i\varepsilon))+\tilde{W}^{\mu\alpha'}\right],
\end{aligned} \tag{6.12}$$

where Λ is some arbitrary mass scale so that the argument of the logarithms are dimensionless, and primed indices are geodesically transported from $x^{\mu'}$ to x^μ by using the bivector of parallel displacement $g^{\alpha'}{}_{\alpha}$, defined by the differential equation

[1] cf. [196, 172] for applications of point splitting to gauge theories on flat space, and [72, 55, 30, 83, 42] for applications of Hadamard techniques to scalar, vector and fermionic DoFs on curved backgrounds.

[81, 87]

$$\nabla_\rho g_{\alpha'\beta}\nabla^\rho\sigma = 0, \tag{6.13}$$

with the boundary condition

$$\lim_{x\to x'} g_{\alpha'\beta}(x,x') = g_{\alpha\beta}(x). \tag{6.14}$$

The $i\varepsilon$ appearing above is characteristic of the Feynman propagator. We note that one should in general be careful in distinguishing and extracting quantities relevant to the Cauchy problem, namely, in-in currents and expectation values as opposed to the corresponding in-out quantities, both of which can be extracted from the Euclidean effective action through different choices of boundary conditions [36, 37, 38]. Using the notation of the former references, we first consider the following definitions for the effective background:[2]

$$\varphi^a_{\rm F} = \frac{\langle \text{out},\text{vac}|\varphi^a|\text{in},\text{vac}\rangle}{\langle \text{out},\text{vac}|\text{in},\text{vac}\rangle}, \tag{6.15}$$

which is of interest in applications when in and out states can be defined (e.g., scattering problems), and

$$\varphi^a_{\rm IN} = \langle \text{in},\text{vac}|\varphi^a|\text{in},\text{vac}\rangle, \tag{6.16}$$

which is of primary relevance to problems where only the initial state is specified. Both fields can be obtained from the effective equations of motion, which can be brought into the form:

$$\frac{\delta S_{\rm cl}}{\delta\varphi^a_{\rm F}} + J^{\rm F}_a = 0, \qquad \frac{\delta S_{\rm cl}}{\delta\varphi^a_{\rm IN}} + J^{\rm IN}_a = 0, \tag{6.17}$$

where $S_{\rm cl}$ is the 'classical action' and the J_a are the so-called radiation currents. Different diagrammatic rules apply when attempting to determine $J^{\rm F}_a$ or $J^{\rm IN}_a$. The radiation current $J^{\rm F}_a$ can be obtained via techniques relevant to the computation of transition amplitudes, and is given to 1-loop given by [36]:

$$J^{\rm F}_a = -\frac{i}{2}\frac{\delta^3 S_{\rm cl}}{\delta\varphi^a_{\rm F}\delta\varphi^b_{\rm F}\delta\varphi^c_{\rm F}} G^{cb}_{\rm F}, \qquad G^{cb}_{\rm F} = i\frac{\langle \text{out},\text{vac}|T[\varphi^c\varphi^b]|\text{in},\text{vac}\rangle}{\langle \text{out},\text{vac}|\text{in},\text{vac}\rangle}, \tag{6.18}$$

where $G^{cb}_{\rm F}$ is the Feynman propagator. Similarly, the current $J^{\rm IN}_a$ is given to 1-loop by

$$J^{\rm IN}_a = -\frac{i}{2}\frac{\delta^3 S_{\rm cl}}{\delta\varphi^a_{\rm IN}\delta\varphi^b_{\rm IN}\delta\varphi^c_{\rm IN}} G^{cb}_{\rm IN}, \qquad G^{cb}_{\rm IN} = i\langle \text{in},\text{vac}|T[\varphi^c\varphi^b]|\text{in},\text{vac}\rangle, \tag{6.19}$$

where the latter can be evaluated as it appears, or with the full regalia of the Schwinger-Keldysh formalism. $G^{cb}_{\rm F}$ and $G^{cb}_{\rm IN}$ differ in terms of their boundary conditions: although in the specific case of future and past asymptotic flatness one has

[2] Where we also adopt DeWitt's condensed notation, and to avoid a proliferation of indices, the composite index a can also be taken to denote a pair of spacetime indices $a := \{\mu,\nu\}$.

$|\text{out}, \text{vac}\rangle = |\text{in}, \text{vac}\rangle$ so that $G_{\text{F}}^{cb} \equiv G_{\text{IN}}^{cb}$, in general $G_{\text{F}}^{cb} \neq G_{\text{IN}}^{cb}$. Nevertheless, for the purposes of regularization, only the short distance divergence structure of G_{IN}^{cb} is relevant, which is identical to that of G_{F}^{cb}. The reason for this can be inferred from the fact that if the two Green's functions differ only in their boundary conditions, completeness dictates that the short distance modes of the two vacua must be related to each other by a Bogoliubov rotation that tends to zero for short wavelengths, otherwise one would represent an infinite energy excitation relative to the other (see also [48] for an expanded discussion on this point). The long wavelength behavior of G_{IN}^{cb} will certainly differ from that of G_{F}^{cb}; however, the difference will manifest as finite and non-local terms that will be absorbed in the process of fixing renormalization conditions.
We stress this point as it offers the possibility to adapt computations that make use of Feynman Green's functions for the purposes of identifying the local counterterms necessitated by the subtraction procedure (as done in [18]). However, since within the present context it is more convenient to work with the Hadamard Green's functions, we follow [42] in deriving their representations from the Hadamard form of the Feynman propagator. By using the identities

$$\frac{1}{\sigma + i\varepsilon} = \mathcal{P}\frac{1}{\sigma} - i\pi\delta(\sigma), \quad \ln(\sigma + i\varepsilon) = \ln|\sigma| + i\pi\Theta(-\sigma), \tag{6.20}$$

where $\mathcal{P}$ and Θ denote the Cauchy principal value and the Heaviside theta function respectively, we can rewrite Eq. 6.12 as:

$$G_F^{ab}(x, x') = G_A^{ab}(x, x') + \frac{i}{2}G^{ab}(x, x'), \tag{6.21}$$

where[3]

$$\begin{aligned} G_A^{\mu\nu\alpha'\beta'}(x, x') &= \frac{1}{8\pi}\left[\Delta^{1/2}\left(g^{\alpha'(\mu}g^{\nu)\beta'}\right)\delta(\sigma) - V^{\mu\nu\alpha'\beta'}\Theta(-\sigma)\right] \\ \tilde{G}_A^{\mu\alpha'}(x, x') &= \frac{1}{8\pi}\left[\Delta^{1/2}g^{\mu\alpha'}\delta(\sigma) - \tilde{V}^{\mu\alpha'}\Theta(-\sigma)\right], \end{aligned} \tag{6.22}$$

are the average of the advanced and retarded Green's functions, and

$$\begin{aligned} G^{\mu\nu\alpha'\beta'}(x, x') &= \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}\left(g^{\alpha'(\mu}g^{\nu)\beta'}\right) + V^{\mu\nu\alpha'\beta'}\ln(\Lambda^2\sigma) + W^{\mu\nu\alpha'\beta'}\right] \\ \tilde{G}^{\mu\alpha'}(x, x') &= \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}g^{\mu\alpha'} + \tilde{V}^{\mu\alpha'}\ln(\Lambda^2\sigma) + \tilde{W}^{\mu\alpha'}\right], \end{aligned} \tag{6.23}$$

are the Hadamard Green's functions.

The state $|\psi\rangle$ appearing in Eq. 6.23 is somewhat circularly defined as any quantum state – the Hadamard state – such that the short distance divergence structure is of the forms indicated in the square brackets, where $\sigma = \frac{1}{2}\sigma_\mu\sigma^\mu$ denotes the square

[3] Round/square parenthesis denote symmetrization/anti-symmetrization of indices.

of the geodesic distance between x^μ and $x^{\mu'}$, Δ is the Van Vleck-Morette determinant and the bitensors $V^{\mu\nu\alpha'\beta'}$, $W^{\mu\nu\alpha'\beta'}$, $\tilde{V}^{\mu\alpha'}$ and $\tilde{W}^{\mu\alpha'}$ are smooth functions in the limit $\sigma \to 0$ of the form:

$$V^{\mu\nu\alpha'\beta'} = \textstyle\sum_{n=0}^{\infty} V_n^{\mu\nu\alpha'\beta'} \sigma^n \qquad W^{\mu\nu\alpha'\beta'} = \displaystyle\sum_{n=0}^{\infty} W_n^{\mu\nu\alpha'\beta'} \sigma^n$$

$$\tilde{V}^{\mu\alpha'} = \textstyle\sum_{n=0}^{\infty} \tilde{V}_n^{\mu\alpha'} \sigma^n \qquad \tilde{W}^{\mu\alpha'} = \displaystyle\sum_{n=0}^{\infty} \tilde{W}_n^{\mu\alpha'} \sigma^n. \tag{6.24}$$

It is to be stressed that the bitensors $V_n^{\mu\nu\alpha'\beta'}$ and $\tilde{V}_n^{\mu\alpha'}$ depend only on the local geometry, whereas the bitensors $W_n^{\mu\nu\alpha'\beta'}$ and $\tilde{W}_n^{\mu\alpha'}$ depend on the boundary conditions and the precise choice of the state $|\psi\rangle$. The finite contributions are what is misinterpreted in the literature and treated as the theoretical estimate of observables. On the contrary, as we will show in in Section 6.4, those are the part that is absorbed by renormalizing the couplings in the effective action through the process of imposing renormalization conditions.

Each of the $V_n^{\mu\nu\alpha'\beta'}$, $W_n^{\mu\nu\alpha'\beta'}$, $\tilde{V}_n^{\mu\alpha'}$ and $\tilde{W}_n^{\mu\alpha'}$ bitensors can be rewritten in the form of a covariant Taylor expansion for x^μ in the neighbourhood of $x^{\mu'}$:

$$\begin{aligned} V_n^{\mu\nu\alpha'\beta'} &= g^{\alpha'}{}_{\alpha} g^{\beta'}{}_{\beta} \left[v_n^{\mu\nu\alpha\beta} + v_n^{\mu\nu\alpha\beta}{}_{\gamma}\sigma^\gamma + \frac{1}{2} v_n^{\mu\nu\alpha\beta}{}_{\gamma\tau}\sigma^\gamma\sigma^\tau + ... \right], \\ \tilde{V}_n^{\mu\alpha'} &= g^{\alpha'}{}_{\alpha} \left[\tilde{v}_n^{\mu\alpha} + \tilde{v}_n^{\mu\alpha}{}_{\gamma}\sigma^\gamma + \frac{1}{2}\tilde{v}_n^{\mu\alpha}{}_{\gamma\tau}\sigma^\gamma\sigma^\tau + ... \right], \end{aligned} \tag{6.25}$$

and similarly for $W_n^{\mu\nu\alpha'\beta'}$ and $\tilde{W}_n^{\mu\alpha'}$. By expanding the Hadamard Green's function of Eq. 6.23 using the Taylor expansions in Eq. 6.25, we obtain

$$\begin{aligned} G^{\rho\sigma\alpha'\beta'}(x,x') =& \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}\left(g^{\alpha'(\rho}g^{\sigma)\beta'}\right) + V^{\rho\sigma\alpha'\beta'}\ln(\mu^2\sigma) + W^{\rho\sigma\alpha'\beta'}\right] \\ =& \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{2\sigma}\left(g^{\alpha'\rho}g^{\sigma\beta'} + g^{\alpha'\sigma}g^{\rho\beta'}\right) + g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}v_0^{\rho\sigma\alpha\beta}\ln(\mu^2\sigma)\right. \\ & + g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}v_0^{\rho\sigma\alpha\beta}{}_{\gamma}\sigma^\gamma\ln(\mu^2\sigma) + \frac{1}{2}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}v_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}\sigma^\gamma\sigma^\varepsilon\ln(\mu^2\sigma) \\ & + \frac{1}{2}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}v_1^{\rho\sigma\alpha\beta}\sigma_\gamma\sigma^\gamma\ln(\mu^2\sigma) + g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}w_0^{\rho\sigma\alpha\beta} \\ & + g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}w_0^{\rho\sigma\alpha\beta}{}_{\gamma}\sigma^\gamma + \frac{1}{2}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}w_0^{\rho\sigma\alpha\beta}{}_{\gamma\tau}\sigma^\gamma\sigma^\tau \\ & \left. + \frac{1}{2}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}w_1^{\rho\sigma\alpha\beta}\sigma^\gamma\sigma_\gamma\right], \end{aligned} \tag{6.26}$$

$$\begin{aligned}\tilde{G}^{\mu\alpha'}(x,x') =& \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}g^{\mu\alpha'} + \tilde{V}^{\mu\alpha'}\ln(\mu^2\sigma) + \tilde{W}^{\mu\alpha'}\right] \\ =& \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}g^{\mu\alpha'} + g^{\alpha'}{}_{\alpha}\tilde{v}_0^{\mu\alpha}\ln(\mu^2\sigma) + g^{\alpha'}{}_{\alpha}\tilde{v}_0^{\mu\alpha}{}_{\gamma}\sigma^{\gamma}\ln(\mu^2\sigma)\right. \\ &+\frac{1}{2}g^{\alpha'}{}_{\alpha}\tilde{v}_0^{\mu\alpha}{}_{\gamma\varepsilon}\sigma^{\gamma}\sigma^{\varepsilon}\ln(\mu^2\sigma) + \frac{1}{2}g^{\alpha'}{}_{\alpha}\sigma_{\gamma}\sigma^{\gamma}\tilde{v}_1^{\mu\alpha}\ln(\mu^2\sigma) + g^{\alpha'}{}_{\alpha}\tilde{w}_0^{\mu\alpha} \\ &\left.+g^{\alpha'}{}_{\alpha}\tilde{w}_0^{\mu\alpha}{}_{\gamma}\sigma^{\gamma} + \frac{1}{2}g^{\alpha'}{}_{\alpha}\tilde{w}_0^{\mu\alpha}{}_{\gamma\tau}\sigma^{\gamma}\sigma^{\tau} + \frac{1}{2}g^{\alpha'}{}_{\alpha}\tilde{w}_1^{\mu\alpha}\sigma^{\gamma}\sigma_{\gamma}\right],\end{aligned} \tag{6.27}$$

where higher orders in powers of σ vanish in the limit $\sigma^\mu \to 0$, and the tensors contributing to the divergent part ($v_0^{\rho\sigma\alpha\beta}$, $v_0^{\rho\sigma\alpha\beta}{}_{\gamma}$, $v_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}$, $v_1^{\rho\sigma\alpha\beta}$, $\tilde{v}_0^{\rho\alpha}$, $\tilde{v}_0^{\rho\alpha}{}_{\gamma}$, $\tilde{v}_0^{\rho\alpha}{}_{\gamma\varepsilon}$ and $\tilde{v}_1^{\rho\alpha}$) can be found in the appendix 6.6.

The Hadamard Green's functions as expressed in Eqs. 6.26 and 6.27 facilitate the regularization of Eq. 6.7 in that they can be viewed as the point split expression:

$$\begin{aligned}\langle S_{\mathrm{gr}}\rangle =& \lim_{\sigma^\mu\to 0}\left\{\int d^4x\sqrt{-g}\left[\left(-\frac{1}{2}g_{\rho\alpha'}g_{\sigma\beta'} + \frac{1}{4}g_{\rho\sigma}g_{\alpha'\beta'}\right)\nabla_\tau\nabla^{\tau'}G^{\rho\sigma\alpha'\beta'}\right.\right. \\ &\left.\left.+\left(R_{\alpha'\rho\beta'\sigma} + g_{\beta'\sigma}R_{\rho\alpha'} - g_{\alpha'\beta'}R_{\rho\sigma} - \frac{1}{2}R\,g_{\rho\alpha'}g_{\sigma\beta'} + \frac{1}{4}R\,g_{\rho\sigma}g_{\alpha'\beta'}\right)G^{\rho\sigma\alpha'\beta'}\right]\right\}\end{aligned} \tag{6.28}$$

$$\langle S_{\mathrm{gh}}\rangle = \lim_{\sigma^\mu\to 0}\left\{\int d^4x\sqrt{-g}\left[-g_{\mu\alpha'}\nabla_\tau\nabla^{\tau'}\tilde{G}^{\mu\alpha'} - R_{\mu\alpha'}\tilde{G}^{\mu\alpha'}\right]\right\}. \tag{6.29}$$

Recalling that the divergence structure of the Hadamard Green's functions is completely captured by the terms containing the Van Vleck-Morette determinant Δ, and the bitensors $V^{\mu\nu\alpha'\beta'}$ and $\tilde{V}^{\mu\alpha'}$, the divergent part of the gravitational sector of the effective action Eq. 6.7 must be of the form

$$\langle S_{\mathrm{gw}}\rangle_{\mathrm{div}} \sim \lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[\gamma_1(\sigma)R + \gamma_2(\sigma)R^2 + \gamma_3(\sigma)R^{\mu\nu}R_{\mu\nu} + \gamma_4(\sigma)\Box R\right]. \tag{6.30}$$

We compute the coefficients $\gamma_1(\sigma)$, $\gamma_2(\sigma)$, $\gamma_3(\sigma)$, and $\gamma_4(\sigma)$ in the next subsection, from which one can immediately identify the counterterms required to subtract them.

6.3.2 Counterterms

The process of determining the counterterms needed to regularize the effective action begins with rewriting the coincidence limits as:

$$\lim_{\sigma^\mu\to 0}R_{\mu\alpha'}\tilde{G}^{\mu\alpha'} = R_{\mu\alpha}\lim_{\sigma^\mu\to 0}g_{\alpha'}{}^{\alpha}\tilde{G}^{\mu\alpha'} \tag{6.31}$$

so that the tensors contracted with the Hadamard Green's functions are tensors in x^μ and scalars in $x^{\mu'}$. Furthermore, as the coincidence limit depends on the path by which σ^μ approaches zero, it is necessary to specify a path-averaging procedure. Following [14], we use the so-called *elementary averaging procedure*, whereby one makes the replacements:

$$\begin{aligned}
\sigma_\lambda\sigma_\mu &\to \frac{1}{4}g_{\lambda\mu}\sigma_\rho\sigma^\rho = \tfrac{1}{2}g_{\lambda\mu}\sigma, \\
\sigma_\lambda\sigma_\mu\sigma_\gamma\sigma_\delta &\to \frac{\sigma^2}{6}\left(g_{\gamma\delta}g_{\lambda\mu} + g_{\gamma\lambda}g_{\delta\mu} + g_{\gamma\mu}g_{\delta\lambda}\right), \\
\sigma_\alpha\sigma_\beta\sigma_\lambda\sigma_\mu\sigma_\gamma\sigma_\delta &\to \frac{\sigma^3}{24}\left[g_{\alpha\beta}\left(g_{\lambda\mu}g_{\nu\delta} + g_{\lambda\gamma}g_{\mu\delta} + g_{\lambda\delta}g_{\mu\nu}\right) + g_{\alpha\lambda}\left(g_{\beta\mu}g_{\gamma\delta} + g_{\beta\gamma}g_{\mu\delta}\right.\right. \\
&\quad \left. +g_{\beta\delta}g_{\mu\gamma}\right) + g_{\alpha\mu}\left(g_{\beta\lambda}g_{\gamma\delta} + g_{\beta\nu}g_{\lambda\delta} + g_{\beta\delta}g_{\lambda\gamma}\right) + g_{\alpha\gamma}\left(g_{\beta\lambda}g_{\mu\delta}\right. \\
&\quad \left.\left. +g_{\beta\mu}g_{\lambda\delta} + g_{\beta\delta}g_{\lambda\mu}\right) + g_{\alpha\delta}\left(g_{\beta\lambda}g_{\mu\nu} + g_{\beta\mu}g_{\lambda\gamma} + g_{\beta\gamma}g_{\lambda\mu}\right)\right] \quad (6.32)
\end{aligned}$$

The singularity structure is then extracted by expanding at the endpoints and iteratively solving the equation of motion of the propagator to find the Taylor coefficients of $V^{\mu\nu\alpha'\beta'}$ and $\tilde{V}^{\mu\alpha'}$ (see appendix 6.6 for more details). From this, the divergent contributions to $S_{\rm gh}$ are found to be

$$\langle S_{\rm gh}\rangle_{\rm div} = \frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[-\frac{3}{\sigma}R + \ln(\Lambda^2\sigma)\left(R_{\mu\nu}R^{\mu\nu} + \frac{1}{6}R^2 - \frac{1}{24}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{5}{12}\Box R\right)\right], \tag{6.33}$$

whereas the divergent contributions to S_{gw} are given by:

$$\langle S_{\rm gr}\rangle_{\rm div} = \frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[-\frac{7}{6\sigma}R + \ln(\Lambda^2\sigma)\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{4}R^2 - \frac{13}{24}\Box R\right)\right]. \tag{6.34}$$

By use of the Bianchi identity and the Gauss-Bonnet theorem, the divergent contribution of the action of vacuum tensor fluctuations results from the difference of Eq. 6.34 and Eq. 6.33 (accounting for the statistics of the ghost contributions), and is given by:

$$\langle S_{\rm gw}\rangle_{\rm div} = \frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[\frac{11}{6\sigma}R + \ln(\Lambda^2\sigma)\left(\frac{1}{6}R_{\mu\nu}R^{\mu\nu} - \frac{11}{24}R^2 - \frac{23}{24}\Box R\right)\right], \tag{6.35}$$

which is of the form of Eq. 6.30, with the $\gamma_i(\sigma)$ coefficients straightforwardly read off from the above.

6.4 Renormalization

Having regularized the divergent contributions of the effective action, that once variated with respect to the background metric leads to the divergent contribution to the stress energy tensor, allows us to proceed with the remaining steps of the renormalization procedure (see Section 1.2.2 for more details). In what follows, we first subtract the divergent contributions with the appropriate counterterms and, as widely discussed in the previous chapters, we then have to fix the finite contribution of the regularized action by imposing renormalization conditions before being able to infer constraints from experiments. This leads to a qualitatively different result in the interpretations of existing constraints, such as the $N_{\rm eff}$ bounds. In this section, we first comment on the finite contribution of the renormalized action and then we proceed to specify our otherwise general results in a RD universe to subtract the divergent contributions, impose renormalization conditions and find results that can be compared with Section 4.4.

We begin by considering all contributions to the action (cf. Eqs. 6.2 and 6.7):

$$\langle S\rangle = S_{\rm EH} + S_{\rm RD} + S_{\rm ct} + \langle S_{\rm gw}\rangle\,, \tag{6.36}$$

where $S_{\rm M}$ is the matter content that sources the background expansion and where we have defined

$$S_{\rm ct} = \frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\Big[\alpha_1(\sigma,\mu)R+\alpha_2(\sigma,\mu)R_{\mu\nu}R^{\mu\nu}+\alpha_3(\sigma,\mu)R^2+\alpha_4(\sigma,\mu)\Box R\Big], \tag{6.37}$$

where μ is an arbitrary mass scale whose meaning will become clear shortly. The divergent contributions in Eq. 6.35 can be subtracted with the following choices for the α_i:

$$\begin{aligned}
\alpha_1(\mu,\sigma) &= -\frac{11}{6}\frac{1}{\sigma} + \alpha_1^{\rm F}(\mu),\\
\alpha_2(\mu,\sigma) &= -\frac{1}{6}\log(\mu^2\sigma) + \alpha_2^{\rm F}(\mu),\\
\alpha_3(\mu,\sigma) &= \frac{11}{24}\log(\mu^2\sigma) + \alpha_3^{\rm F}(\mu),\\
\alpha_4(\mu,\sigma) &= \frac{23}{24}\log(\mu^2\sigma) + \alpha_4^{\rm F}(\mu),
\end{aligned} \tag{6.38}$$

where the $\alpha_i^{\rm F}$ are finite contributions that we leave unspecified for now. With this the action can be expressed as:

$$\begin{aligned}
\langle S\rangle = &\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G(\mu)}R + \bar\alpha_2(\mu)R_{\mu\nu}R^{\mu\nu} + \bar\alpha_3(\mu)R^2 + \bar\alpha_4(\mu)\Box R\right]\\
&+ S_{\rm RD} + \langle S_{\rm gw}\rangle_{\rm fin}\,,
\end{aligned} \tag{6.39}$$

where $\langle S_{\rm gw}\rangle_{\rm fin} := \langle S_{\rm gw}\rangle - \langle S_{\rm gw}\rangle_{\rm div}$ and where

$$\begin{aligned}
\frac{1}{16\pi G(\mu)} &= \frac{1}{16\pi G_B} + \frac{\alpha_1^{\rm F}(\mu)}{4\pi^2},\\
\bar\alpha_2(\mu) &= \frac{1}{4\pi^2}\left[\frac{1}{6}\log\frac{\Lambda^2}{\mu^2} + \alpha_2^{\rm F}(\mu)\right],\\
\bar\alpha_3(\mu) &= \frac{1}{4\pi^2}\left[-\frac{11}{24}\log\frac{\Lambda^2}{\mu^2} + \alpha_3^{\rm F}(\mu)\right],\\
\bar\alpha_4(\mu) &= \frac{1}{4\pi^2}\left[-\frac{23}{24}\log\frac{\Lambda^2}{\mu^2} + \alpha_4^{\rm F}(\mu)\right].
\end{aligned} \tag{6.40}$$

The subscripts B appearing above are to denote bare quantities.

6.4.1 Finite contributions

The finite remainder given by $\langle S_{\rm gw}\rangle_{\rm fin}$ can be separated into parts that are uniquely determined by the background geometry – i.e. the terms determined by the bitensors $V^{\mu\nu\alpha'\beta'}$ and $\tilde{V}^{\mu\alpha'}$ which we denote as $\langle S_{\rm gw}\rangle^{\rm bg}_{\rm fin}$ – and those that depend on the state, determined by the bitensors $W^{\mu\nu\alpha'\beta'}$ and $\tilde{W}^{\mu\alpha'}$, which we denote as $\langle S_{\rm gw}\rangle^{\rm sd}_{\rm fin}$.

Computing $\langle S_{\rm gw}\rangle^{\rm sd}_{\rm fin}$, can be reduced to express the following four terms

$$
\begin{aligned}
(\mathrm{I}):&\qquad R_{\mu\alpha}\lim_{\sigma^\mu\to0} g_{\alpha'}{}^{\alpha}\tilde{G}^{\mu\alpha'} &(6.41)\\
(\mathrm{II}):&\qquad g_{\rho}{}^{\gamma}\lim_{\sigma^\mu\to0} g_{\tau'}{}^{\tau}g_{\alpha'\gamma}\nabla_\tau\nabla^{\tau'}\tilde{G}^{\rho\alpha'} &(6.42)\\
(\mathrm{III}):&\qquad \mathcal{P}_{\mu\nu\alpha\beta}\lim_{\sigma^\mu\to0} g_{\alpha'}{}^{\alpha}g_{\beta'}{}^{\beta}G^{\mu\nu\alpha'\beta'} &(6.43)\\
(\mathrm{IV}):&\qquad \mathcal{Q}_{\mu\nu}{}^{\gamma\delta}\lim_{\sigma^\mu\to0} g_{\tau'}{}^{\tau}g_{\alpha'\gamma}g_{\beta'\delta}\nabla_\tau\nabla^{\tau'}G^{\mu\nu\alpha'\beta'}, &(6.44)
\end{aligned}
$$

where we defined

$$
\begin{aligned}
\mathcal{P}_{\mu\nu\gamma\delta} &\equiv R_{\gamma\mu\delta\nu} + g_{\delta\nu}R_{\mu\gamma} - g_{\gamma\delta}R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\gamma}g_{\nu\delta} + \frac{1}{4}R\, g_{\mu\nu}g_{\gamma\delta},\\
\mathcal{Q}_{\mu\nu\gamma\delta} &\equiv -\frac{1}{2}g_{\mu\gamma}g_{\nu\delta} + \frac{1}{4}g_{\mu\nu}g_{\gamma\delta}, \qquad (6.45)
\end{aligned}
$$

as a function of the state-dependent terms of the Taylor expansions of Eqs. 6.26 and 6.27 – $w_0^{\rho\sigma\alpha\beta}$, $w_0^{\rho\sigma\alpha\beta}{}_{\gamma}$, $w_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}$, $w_1^{\rho\sigma\alpha\beta}$, $\tilde{w}_0^{\rho\alpha}$, $\tilde{w}_0^{\rho\alpha}{}_{\gamma}$, $\tilde{w}_0^{\rho\alpha}{}_{\gamma\varepsilon}$ and $\tilde{w}_1^{\rho\alpha}$. Considering only the state-dependent part of Eqs. 6.26 and 6.27, one can straightforwardly isolate the state-dependent part of Eqs. 6.41 and 6.43 which results

$$
\begin{aligned}
(\mathrm{I})^{\rm sd}_{\rm fin}:&\qquad \frac{i}{8\pi^2}R_{\mu\alpha}\tilde{w}_0^{\mu\alpha}\\
(\mathrm{III})^{\rm sd}_{\rm fin}:&\qquad \frac{i}{8\pi^2}\mathcal{P}_{\mu\nu\alpha\beta}w_0^{\mu\nu\alpha\beta}. &(6.46)
\end{aligned}
$$

Extracting the state-dependent part of Eqs. 6.42 and 6.44 is less straightforward: following the procedure of the appendix 6.6 we obtain

$$
\begin{aligned}
(\mathrm{II})^{\rm sd}_{\rm fin}:&\qquad \frac{i}{8\pi^2}\left[-\nabla_\tau\tilde{w}_{0\,\mu}^{\mu}{}^{\tau} - \tilde{w}_{0\,\mu}^{\mu}{}^{\tau}{}_{\tau} - 4\tilde{w}_{1\,\mu}^{\mu}\right]\\
(\mathrm{IV})^{\rm sd}_{\rm fin}:&\qquad \frac{i}{8\pi^2}\left[\mathcal{Q}_{\mu\nu}{}^{\gamma\delta}\left(-\nabla_\tau w_0^{\mu\nu}{}_{\gamma\delta}{}^{\tau} - w_0^{\mu\nu}{}_{\gamma\delta}{}^{\tau}{}_{\tau} - 4w_1^{\mu\nu}{}_{\gamma\delta}\right)\right]. &(6.47)
\end{aligned}
$$

Finally, $\langle S_{\rm gw}\rangle^{\rm sd}_{\rm fin}$ can be expressed as

$$\begin{aligned}
\langle S\rangle^{\rm sd}_{\rm fin} &= \int d^4x\sqrt{-g}\left[\Big((\mathrm{III})^{\rm sd}_{\rm fin}+(\mathrm{IV})^{\rm sd}_{\rm fin}\Big)-\Big(-(\mathrm{I})^{\rm sd}_{\rm fin}-(\mathrm{II})^{\rm sd}_{\rm fin}\Big)\right]\\
&= \frac{1}{4\pi^2}\int d^4x\sqrt{-g}\left[\mathcal{Q}_{\mu\nu}{}^{\gamma\delta}\left(-\nabla_\tau w_0^{\mu\nu}{}_{\gamma\delta}{}^{\tau}-w_0^{\mu\nu}{}_{\gamma\delta}{}^{\tau}{}_{\tau}-4w_1^{\mu\nu}{}_{\gamma\delta}\right)\right.\\
&\quad\left.+\mathcal{P}_{\mu\nu\alpha\beta}w_0^{\mu\nu\alpha\beta}+R_{\mu\alpha}\tilde{w}_0^{\mu\alpha}-\nabla_\tau\tilde{w}_0{}^{\mu}{}_{\mu}{}^{\tau}-\tilde{w}_0{}^{\mu}{}_{\mu}{}^{\tau}{}_{\tau}-4\tilde{w}_1{}^{\mu}{}_{\mu}\right].
\end{aligned}\tag{6.48}$$

There is one more step that can be done to simplify the form of $\langle S_{\rm gw}\rangle^{\rm sd}_{\rm fin}$. We note that we can obtain the Taylor coefficients of $W_1^{\mu\nu\alpha'\beta'}$ in terms of the Taylor coefficients of $W_0^{\mu\nu\alpha'\beta'}$ (we focus on the graviton contribution, but a similar procedure will give us the analogous Taylor coefficients for the ghost contributions). By iteratively solving order by order in σ^μ in the equation of motion for the propagator, we find[4] (cf. [18] for more details):

$$\begin{aligned}
&n(n+1)W_n^{\mu\nu}{}_{\alpha'\beta'}+nW_n^{\mu\nu}{}_{\alpha'\beta';\rho}\sigma^\rho-nW_n^{\mu\nu}{}_{\alpha'\beta'}\Delta^{-1/2}\Delta^{1/2}_{;\rho}\sigma^\rho+(2n+1)V_n^{\mu\nu}{}_{\alpha'\beta'}\\
&+V_n^{\mu\nu}{}_{\alpha'\beta';\rho}\sigma^\rho-V_n^{\mu\nu}{}_{\alpha'\beta'}\Delta^{-1/2}\Delta^{1/2}_{;\rho}\sigma^\rho+\frac{1}{2}D_{\rho\sigma}{}^{\mu\nu}\,W^{\rho\sigma}_{n-1\,\alpha'\beta'}=0,
\end{aligned}\tag{6.49}$$

where

$$\begin{aligned}
D_{\mu\nu}{}^{\alpha\beta}&=\Box g_{\mu}^{(\alpha}\,g_{\nu}^{\beta)}-P_{\mu\nu}{}^{\alpha\beta},\\
P_{\mu\nu}{}^{\alpha\beta}&=-2R_{(\mu}{}^{\alpha}{}_{\nu)}{}^{\beta}+\frac{1}{2}g_{\mu\nu}R^{\alpha\beta}+\frac{1}{2}g^{\alpha\beta}R_{\mu\nu}-\frac{1}{4}Rg_{\mu\nu}g^{\alpha\beta}+\frac{1}{2}Rg_{(\mu}{}^{\alpha}g_{\nu)}{}^{\beta}.
\end{aligned}\tag{6.50}$$

By specifying the recursion relation Eq. 6.49 for $n=1$ and expanding at the 0^{th} order in σ we obtain $w_1^{\mu\nu}{}_{\alpha\beta}$ as a function of the Taylor coefficients of $W_0^{\mu\nu\alpha'\beta'}$

$$2w_1^{\mu\nu}{}_{\alpha\beta}=-3v_1^{\mu\nu}{}_{\alpha\beta}-\frac{1}{2}g_\rho^{(\mu}g_\sigma^{\nu)}\left[\Box w_0^{\rho\sigma}{}_{\alpha\beta}+\nabla_\tau w_0^{\rho\sigma}{}_{\alpha\beta}{}^{\tau}+\frac{1}{2}w_0^{\rho\sigma}{}_{\alpha\beta}{}^{\tau}{}_{\tau}\right]+\frac{1}{2}P^{\mu\nu}{}_{\rho\sigma}w_0^{\rho\sigma}{}_{\alpha\beta}.\tag{6.51}$$

In the above, $w_0^{\rho\sigma\alpha\beta}$, $w_0^{\rho\sigma\alpha\beta}{}_{\gamma}$ and $w_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}$ are the 'initial' inputs for the recursion relations corresponding to the specifics of the state.

6.4.2 Renormalization conditions

In this section we fix the finite contributions and impose the renormalization conditions. In doing so we specify the background expansion as, although the treatment that follows generalizes to any background, we aim to compare our results with the FLRW foliation studied in Chapter 4.

To connect with the findings of Chapter 4, $S_{\rm M}$ is now specified by $S_{\rm RD}$ to denote radiation domination. For concreteness, we work with the action formulation for a

[4] Semi-colons denote covariant derivatives with respect to the background metric.

barotropic fluid expressed in terms of a derivatively coupled scalar [50, 28], so that

$$S_{\rm RD} = \int d^4x\sqrt{-g}P^{\rm bg}(X), \tag{6.52}$$

where $P^{\rm bg}(X) = X^2$, with $X := -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$, reproduces the stress tensor for the background radiation fluid[5] (See section 5.3 for more details). In order to proceed and fix the finite contributions, we are obliged to make use of leading order field equations to eliminate redundant higher order correction terms containing second derivatives and time derivatives of what were auxiliary fields in the tree level action [216, 61]. That is, one can substitute the tree level equations of motion $R = -8\pi G(\mu)T^{\rm bg}$ and $R_{\mu\nu} = 8\pi G(\mu)[{T^{\rm bg}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\rm bg}]$ into the above, where

$${T^{\rm bg}}_\nu{}^\mu(X_B) = \delta^\mu{}_\nu P^{\rm bg} - {P^{\rm bg}}_{,X}\partial^\mu\psi_B\partial_\nu\psi_B \tag{6.53}$$

is obtained from Eq. 6.52 through variation with the background metric. The functional form $P^{\rm bg}(X_B) = X_B^2$ ensures that $T^{\rm bg} \equiv 0$, so that the regularized action takes the form:

$$\langle S\rangle = \int d^4x\sqrt{-g}\left[\frac{1}{2}M^2(\mu)R + \widetilde{P}^{\rm bg}(X_B,\mu)\right] + \langle S_{\rm gw}\rangle_{\rm fin}\,, \tag{6.54}$$

where we have dropped total derivatives and defined $M^{-2}(\mu) := 8\pi G(\mu)$, which we use interchangeably in what follows, and where

$$\widetilde{P}^{\rm bg}(X_B,\mu) := X_B^2 + 12X_B^4\frac{\bar\alpha_2(\mu)}{M^4(\mu)}, \tag{6.55}$$

with $\bar\alpha_2$ defined as in Eq. 6.40. We immediately notice that the stress tensor associated with the shifted matter sector $\widetilde{P}(X_B,\mu)$ is no longer traceless:

$${\widetilde{T}^{\rm bg}}{}_\mu{}^\mu = -48X_B^4\frac{\bar\alpha_2(\mu)}{M^4(\mu)}. \tag{6.56}$$

This is because Einstein gravity is not conformally invariant, and therefore neither are the field equations governing GWs, even if the background is conformally flat [177]. Hence, the stress tensor for GWs will not be exactly traceless unless $\bar\alpha_2 \equiv 0$, and this feature gets imported into the matter sector via operator redundancy at 1-loop, a point which we will return to shortly.

Before proceeding with the renormalization conditions, we comment on how one could compare the finite remainder $\langle S_{\rm gw}\rangle_{\rm fin}$ with the TT-gauge-fixed, foliation specific treatment of 4.4.

We recall that in fixing de Donder gauge with the Faddeev Popov method, we considered the action for a rank-2 symmetric tensor field (describing 10 DoFs) and we fixed the gauge by adding the gauge breaking term in Eq. 6.5 (which fixes 4 DoFs)

[5] A feature that remains true to all orders in perturbation theory [50].

and the ghost term (which subtracts the remaining 4 spurious DoFs). The ghost term and gauge breaking term have the same properties as the spurious eight DoFs present in the initial action, but with fermionic statistics that subtracts them from all on shell quantities. In order to obtain a fully gauge-fixed final result of the action in terms of the graviton only (equivalently for stress energy tensor, once one variates with respect to the background metric) we have to determine the ghost propagator in terms of the graviton propagator. This is done by using the generalization of the Ward identities discussed in [18], but now evaluated on a background that does not correspond to a vacuum spacetime. While addressing this matter falls outside the current investigation's scope, fully completing it, considering the remaining symmetries of FRLW spacetimes, would represent a valuable and practical computation worth pursuing further.

We now proceed to fix the finite parts of the relevant couplings via renormalization conditions. We recall the shifted tadpole condition, which is defined by the requirement that the background effective field $g_{\mu\nu}$ must be put on shell in all final expressions. That is, we demand that

$$\frac{1}{8\pi G(\mu)}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=\widetilde{T}^{\mathrm{bg}}_{\mu\nu}+\langle T^{\mathrm{gw,fin}}_{\mu\nu}\rangle. \tag{6.57}$$

We first note that Newton's constant can only be fixed via a Cavendish-type experiment, where we have knowledge of the masses whose strength of gravitational interactions we are attempting to fix. Assuming that this is done at mm scales, we impose the renormalization condition $[8\pi G(\mu)]^{-1}\equiv M^2(\mu_*)\equiv M^2_{\mathrm{pl}}$ where the latter is given by the reduced Planck mass $M^2_{\mathrm{pl}}=2.435\times 10^{18}$ GeV. From Eq. 6.40 it therefore folows that

$$8\pi G_N(\mu)=\frac{1}{M^2_{\mathrm{pl}}}\left[1+\frac{\alpha_1^{\mathrm{F}}(\mu)-\alpha_1^{\mathrm{F}}(\mu_*)}{2\pi^2M^2_{\mathrm{pl}}}\right]^{-1}, \tag{6.58}$$

which can be used to express Eq. 6.57 in covariant form as

$$G_{\mu\nu}=\frac{1}{M^2_{\mathrm{pl}}}\left[\widetilde{T}^{\mathrm{bg}}_{\mu\nu}+\langle T^{\mathrm{gw,fin}}_{\mu\nu}\rangle\right]\left[1+\frac{\alpha_1^{\mathrm{F}}(\mu)-\alpha_1^{\mathrm{F}}(\mu_*)}{2\pi^2M^2_{\mathrm{pl}}}\right]^{-1}. \tag{6.59}$$

We see from the above how additive renormalization of Newton's constant is equivalent to the multiplicative renormalization of the matter sector via the tadpole condition.

In order to fix the remaining finite contribution appearing in $\bar{\alpha}_2$ in combination with those coming from $\langle T^{\mathrm{gw,fin}}_{\mu\nu}\rangle$, we have to appeal to additional observations. Before we do so, we note that $\langle T^{\mathrm{gw,fin}}_{\mu\nu}\rangle$ has a series of contributions that can be recursively obtained. We recall that if on one hand the background dependent finite contributions can be uniquely determined by the recursion relations of the EOM, this is not the case for the state-dependent part (see Section 6.4.1). However, the adiabatic vacuum by definition is invariant under the symmetries of the background geometry, and so all the initial state-dependent inputs must themselves be constructed out of geometric invariants. This implies that both the background and the state-dependent contributions to the finite part will result in the generation of a handful of terms that redshift as radiation, along with a series of additional slow quenching terms that decay much faster, as we will se in the following. We also note that the example studied in Chapter 4 implied that the contribution Eq. 6.56 is canceled by a compensating term from the state-dependent part of $\langle T^{\mathrm{gw,fin}}_{\mu\nu}\rangle$ in the adiabatic vacuum. Regardless of this fact, in addition to admitting the possibility of other operators in the effective action from the presence of additional DoFs, one simply extracts that part of the renormalized expression that scales as radiation and proceeds accordingly.

Consider for example the measurement of the equation of state parameter w during what we presume to be radiation domination. In principle, any other measurement of a dimensionless ratio will do (e.g. H^2/M_{pl}^2), as what follows transcribes straightforwardly. Making such a measurement at the scale μ_{R} results in the renormalization condition:

$$3H_{\mathrm{R}}^2(3\omega_{\mathrm{R}}-1) = -48X_B^4\frac{\bar{\alpha}_2(\mu_{\mathrm{R}})}{M_{\mathrm{pl}}^6}+\frac{\beta^{\mathrm{F}}(\mu_{\mathrm{R}})}{M_{\mathrm{pl}}^2}, \tag{6.60}$$

where $\beta^{\mathrm{F}}(\mu_{\mathrm{R}}) := \langle T^{\mathrm{gw,fin}}\rangle$. Regardless of how suppressed the right hand side may appear, it in principle fixes the remaining finite remainder in $\bar{\alpha}_2$ defined in Eq. 6.40, up to the state dependence of the terms that appear in $\beta^{\mathrm{F}}(\mu_{\mathrm{R}})$. If the right hand side cancels exactly, the effective background corresponds to RD expansion. If the right hand side does not cancel, it would still correspond to RD expansion up to suppressed slow quenching terms[6], which, moreover, dilute much faster than radiation. Hence, the most one can conclude from the shifted tadpole condition Eq. 6.59 is:

$$3H^2 = \frac{1}{M_{\mathrm{pl}}^2}\left(\widetilde{\rho}_{\mathrm{bg}}+\rho^{\mathrm{rd}}_{\mathrm{gw,fin}}\right)\left[1+\frac{\alpha_1^{\mathrm{F}}(\mu)-\alpha_1^{\mathrm{F}}(\mu_*)}{2\pi^2M_{\mathrm{pl}}^2}\right]^{-1} \approx \frac{\widetilde{\rho}_{\mathrm{bg}}}{M_{\mathrm{pl}}^2}\left(1+\delta_{\mathrm{Z}}\right), \tag{6.61}$$

whith $\widetilde{\rho}_{\mathrm{bg}}$ corresponding to the time-time component of $\widetilde{T}^{\mathrm{bg}}_{\mu\nu}$, $\rho^{\mathrm{rd}}_{\mathrm{gw,fin}}$ denoting any (possibly vanishing) state-dependent contributions from $\langle T^{\mathrm{gw,fin}}_{\mu\nu}\rangle$ that scales as radiation, and δ_{Z} is some constant by this definition. The latter defines the wavefunction renormalization of the otherwise unobservable bare thermal potential ψ_B:

$$\psi := \left(1+\delta_{\mathrm{Z}}\right)^{1/4}\psi_B, \tag{6.62}$$

where now ψ denotes a dressed quantity. Therefore one concludes that the net effect of the renormalization procedure is to simply mimic shifts in the definition of otherwise unaccessible quantities[7] a conclusion that is in agreement with the results of Section 4.4.

[6]In practice, the best accuracy with which we can ever hope to constrain the left hand side of Eq. 6.60 means that the right hand side can only be taken in practice to be consistent with zero.

[7]Which is consistent with the results reviewed in Section 1.2.

6.5 Conclusions

Motivated by the results of Chapter 4, in which we concluded that one has to follow through the renormalization process to completion in order to make contact with cosmological observations, in this chapter we generalized the previously studied FLRW foliation derivations to a generic background metric in a fully covariant formulation. In doing so we confirmed the findings of Chapter 4, namely that any attempts to extract N_{eff} bounds from vacuum tensor perturbations is inextricable from the process of background renormalization, but also we highlighted subtleties that were not evident in studying a specific foliation.

Differently from the previous results, we find that higher order corrections to the Einstein Hilbert action are required in order to define the counterterms that subtract the divergences arising in computing the stress energy tensor of GWs. Furthermore, we obtain that the regularized stress energy tensor including the contribution of both radiation and GWs is no longer traceless, due to the fact that a massless spin-2 particle is not conformally invariant. This leads to additional slow quenching terms that decay much faster than radiation. We then showed that $\sim R^2$ corrections, together with the anomalous trace, vanish once one imposes a radiation-like solution, which is in agreement with the result of Chapter 4.

6.6 Appendix A: Details of Hadamard regularization

In this appendix, we present further details as to how one can obtain the divergent contributions in Eq. 6.35.

Regularizing the contributions of Eqs. 6.28 and 6.29 in order to obtain Eqs. 6.33 and 6.34, comes down to to regulating the four terms defined in Section 6.4.1

$$\text{(I)}: \qquad R_{\mu\alpha} \lim_{\sigma^\mu \to 0} g_{\alpha'}{}^{\alpha} \tilde{G}^{\mu\alpha'} \tag{6.63}$$

$$\text{(II)}: \qquad g_\rho{}^\gamma \lim_{\sigma^\mu \to 0} g_{\tau'}{}^{\tau} g_{\alpha'\gamma} \nabla_\tau \nabla^{\tau'} \tilde{G}^{\rho\alpha'} \tag{6.64}$$

$$\text{(III)}: \qquad \mathcal{P}_{\mu\nu\alpha\beta} \lim_{\sigma^\mu \to 0} g_{\alpha'}{}^{\alpha} g_{\beta'}{}^{\beta} G^{\mu\nu\alpha'\beta'} \tag{6.65}$$

$$\text{(IV)}: \qquad \mathcal{Q}_{\mu\nu}{}^{\gamma\delta} \lim_{\sigma^\mu \to 0} g_{\tau'}{}^{\tau} g_{\alpha'\gamma} g_{\beta'\delta} \nabla_\tau \nabla^{\tau'} G^{\mu\nu\alpha'\beta'}. \tag{6.66}$$

By iteratively solving at orders σ^0, $\sigma^{\frac{1}{2}}$ and σ the equations of motion for the graviton propagator[8] we find $v_0^{\rho\sigma\alpha\beta}$, $v_0^{\rho\sigma\alpha\beta}{}_\gamma$, $v_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}$ and $v_1^{\rho\sigma\alpha\beta}$([18])

$$\begin{aligned}
v_{0\mu\nu}{}^{\rho\gamma} =& -\frac{1}{12}R\left(g_{(\mu}{}^{\rho}g_{\nu)}{}^{\gamma} - \frac{1}{2}g_{\mu\nu}g^{\rho\gamma}\right) + \frac{1}{2}P_{\mu\nu}{}^{\rho\gamma} - \frac{1}{4}g^{\rho\gamma}P_{\mu\nu e}{}^{e} \\
v_{0\mu\nu}{}^{\rho\gamma}{}_{\alpha} =& -\frac{1}{2}v_{0a\nu}{}^{\nu\gamma}{}_{;\alpha} - \frac{1}{6}g_{(\mu}{}^{(\nu}\left(R_{|\alpha|\nu)}{}^{;\gamma)} - R_{|\alpha|}{}^{\gamma)}{}_{;\nu)}\right) \\
v_{0\mu\nu}{}^{\nu\gamma}{}_{\alpha\beta} =& \frac{1}{2}\left(v_0{}^{\nu\gamma}{}_{\mu\nu(\alpha;\beta)} - v_{0\mu\nu}{}^{\nu\gamma}{}_{(\alpha;\beta)}\right) + \frac{1}{6}P_{\mu\nu}{}^{\rho\gamma}{}_{;;(\alpha\beta)} + \frac{1}{12}P_{\mu\nu}{}^{\rho\gamma}R_{\alpha\beta} - \frac{1}{12}g^{\rho\gamma}P_{\mu\nu\sigma}{}^{\sigma}{}_{;(\alpha\beta)} \\
& - \frac{1}{24}g^{\rho\gamma}P_{\mu\nu\sigma}{}^{\sigma}R_{\alpha\beta} + \frac{1}{6}g_{(\mu}{}^{(\rho}R_{\nu)\sigma\epsilon(\alpha}R^{\gamma)\sigma\epsilon}{}_{\beta)} - \frac{1}{6}g_{\mu}{}^{(\sigma}g_{\nu}{}^{\epsilon)}R_{\alpha}{}^{\delta(\rho}R_{\beta\delta\epsilon}{}^{\gamma)} \\
& - \frac{1}{6}\left(g_{(\mu}{}^{\rho}g_{\nu)}{}^{\gamma} - \frac{1}{2}g_{\mu\nu}g^{\rho\gamma}\right)\left(\frac{1}{30}R_{\sigma\epsilon\delta\alpha}R^{\sigma\epsilon\delta}{}_{\beta} + \frac{1}{30}R_{\alpha\sigma\beta\epsilon}R^{\sigma\epsilon} - \frac{1}{15}R_{\alpha\sigma}R_{\beta}{}^{\sigma}\right. \\
& \left. + \frac{1}{12}RR_{\alpha\beta} + \frac{3}{20}R_{;\alpha\beta} + \frac{1}{20}\Box R_{\alpha\beta}\right) \\
v_{1\mu\nu}{}^{\rho\gamma} =& \frac{1}{48}\left(g_{\mu}{}^{\rho}g_{\nu}{}^{\gamma} + g_{\mu}{}^{\gamma}g_{\nu}{}^{\rho} - g_{\mu\nu}g^{\rho\gamma}\right)\left(\frac{1}{30}R_{\sigma\epsilon\delta\varsigma}R^{\sigma\epsilon\delta\varsigma} - \frac{1}{30}R_{\sigma\epsilon}R^{\sigma\epsilon} + \frac{1}{12}R^2 + \frac{1}{5}\Box R\right) \\
& - \frac{1}{24}(\Box + R)P_{\mu\nu}{}^{\rho\gamma} + \frac{1}{8}P_{\mu\nu}{}^{\sigma\epsilon}P_{\sigma\epsilon}{}^{\rho\gamma} + \frac{1}{48}g^{\rho\gamma}\left[(\Box + R)P_{\mu\nu\sigma}{}^{\sigma} - 3P_{\mu\nu}{}^{\sigma\epsilon}P_{\sigma\epsilon\delta}{}^{\delta}\right] \\
& + \frac{1}{24}\left(R_{\sigma\epsilon(\mu}{}^{(\rho}R_{\nu)}{}^{\gamma)\sigma\epsilon} - g_{(\mu}{}^{(\rho}R_{\nu)\sigma\epsilon\delta}R^{\gamma)\sigma\epsilon\delta}\right).
\end{aligned} \tag{6.67}$$

where

$$P_{\mu\nu}{}^{\rho\gamma} = -2R^{\gamma}_{(\mu\nu)}{}^{\rho} + \frac{1}{2}\left(g_{\mu\nu}R^{\rho\gamma} + g^{\rho\gamma}R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\rho\gamma}\right) + \frac{1}{2}Rg_{(\mu}{}^{\gamma}g_{\nu)}{}^{\rho}. \tag{6.68}$$

In the same way, we can find $\tilde{v}_0^{\rho\alpha}$, $\tilde{v}_0^{\rho\alpha}{}_\gamma$, $\tilde{v}_0^{\rho\alpha}{}_{\gamma\varepsilon}$ and $\tilde{v}_1^{\rho\alpha}$ by solving order by order in

[8]Higher orders in σ do not contribute to the divergent part.

σ^μ for the ghost propagator and obtain

$$
\begin{aligned}
\tilde{v}_0^{\mu\nu} &= -\frac{1}{12}g^{\mu\nu}R - \frac{1}{2}R^{\mu\nu} \\
\tilde{v}_0^{\mu\nu\alpha} &= - -\frac{1}{2}\tilde{v}_0^{\mu\nu;\alpha} - \frac{1}{6}R^{\alpha[\mu;\nu]} \\
\tilde{v}_0^{\mu\nu\alpha\beta} &= -\tilde{v}_0^{[\mu\nu](\alpha;\beta)} + \frac{1}{12}R^{\mu\rho\gamma(\alpha}R^{\beta)}{}_{\gamma\rho}{}^{\nu} - \frac{1}{6}R^{\mu\nu;(\alpha\beta)} - \frac{1}{12}R^{\mu\nu}R^{\alpha\beta} \\
&\quad + g^{\mu\nu}\left(-\frac{1}{180}R^{\rho\gamma\sigma\alpha}R_{\rho\gamma\sigma}{}^{\beta} - \frac{1}{180}R_{\rho\gamma}R^{\alpha\rho\beta\gamma} + \frac{1}{90}R^{\alpha\rho}R^{\beta}{}_{\rho} - \frac{1}{72}RR^{\alpha\beta}\right. \\
&\quad \left. -\frac{1}{40}R^{;\alpha\beta} - \frac{1}{120}\Box R^{\alpha\beta}\right) \\
\tilde{v}_1^{\mu\nu} &= -\frac{1}{48}R^{\mu\rho\gamma\sigma}R_{\rho\gamma\sigma}{}^{\nu} + \frac{1}{24}\Box R^{\mu\nu} + \frac{1}{24}RR^{\mu\nu} + \frac{1}{8}R^{\mu\rho}R_{\rho}{}^{\nu} \\
&\quad + g^{\mu\nu}\left(\frac{1}{720}R^{\rho\gamma\sigma\epsilon}R_{\rho\gamma\sigma\epsilon} - \frac{1}{720}R^{\rho\gamma}R_{\rho\gamma} + \frac{1}{288}R^2 + \frac{1}{120}\Box R\right).
\end{aligned}
\tag{6.69}
$$

Eqs. 6.63 and 6.65 are then regularized by subtracting the divergent terms of the expansions in Eqs. 6.26 and 6.27 with the appropriate counterterms. These divergences, using Eqs. 6.67 and 6.69, are straightforwardly given by

$$
\begin{aligned}
(\mathrm{I})_{\mathrm{div}}: &\qquad -\frac{1}{16\pi^2}\lim_{\sigma^\mu\to 0}\left[\frac{1}{\sigma}R + \ln(\mu^2\sigma)\left(-\frac{1}{12}R^2 - \frac{1}{2}R_{\mu\nu}R^{\mu\nu}\right)\right] \\
(\mathrm{III})_{\mathrm{div}}: &\qquad -\frac{1}{16\pi^2}\lim_{\sigma^\mu\to 0}\left[-3\frac{1}{\sigma}R + \ln(\mu^2\sigma)\left(\frac{3}{2}R_{\mu\nu}R^{\mu\nu} - R^2 - \frac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right.\right. \\
&\qquad \left.\left. -\frac{1}{2}R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma}\right)\right],
\end{aligned}
\tag{6.70}
$$

and the requisite counterterms are readily identified. Regularizing Eq. 6.64 and Eq. 6.66 is less straightforward, as in order to regularize $\nabla_\tau\nabla^{\tau'}\tilde{G}^{\rho\alpha'}$ and $\nabla_\tau\nabla^{\tau'}G^{\mu\nu\alpha'\beta'}$ we need to sequentially:

1. Compute the derivative of the Hadamard Green's functions using the expansions in 6.26 and 6.27 and keeping the terms that are divergent in the limit $\sigma^\mu \to 0$.

2. Expand the result in powers of σ^μ using the endpoint expansions in [14].

3. Use the averages in Eq. 6.32 to obtain a direction independent result.

Following these steps, one obtains

$$
\begin{aligned}
(\mathrm{II})_{\mathrm{div}} :& -\frac{1}{16\pi^2}\lim_{\sigma^\mu\to 0}\left[\frac{2}{\sigma}R+\ln(\mu^2\sigma))\left(-\frac{1}{2}R_{\mu\nu}R^{\mu\nu}-\frac{1}{12}R^2+\frac{1}{12}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right.\right.\\
&\left.\left.-\frac{1}{12}R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma}-\frac{5}{12}\Box R\right)\right]\\
(\mathrm{IV})_{\mathrm{div}} :& -\frac{1}{16\pi^2}\lim_{\sigma^\mu\to 0}\left[\frac{11}{6}\frac{1}{\sigma}R+\ln(\mu^2\sigma)\left(-\frac{1}{2}R_{\mu\nu}R^{\mu\nu}+\frac{3}{4}R^2+\frac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right.\right.\\
&\left.\left.+\frac{1}{2}R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma}-\frac{13}{24}\Box R\right)\right].
\end{aligned}
\tag{6.71}
$$

In summary, considering the extra minus in front of the ghost terms accounting for the different statistics, the divergent contributions in Eq. 6.35 are given by

$$
\begin{aligned}
\langle S\rangle_{\mathrm{div}} =& \int d^4x\sqrt{-g}\left[\Big((\mathrm{III})_{\mathrm{div}}+(\mathrm{IV})_{\mathrm{div}}\Big)-\Big(-(\mathrm{I})_{\mathrm{div}}-(\mathrm{II})_{\mathrm{div}}\Big)\right]\\
=&\frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[\frac{11}{6}\frac{1}{\sigma}R+\ln(\mu^2\sigma)\left(\frac{1}{6}R_{\mu\nu}R^{\mu\nu}-\frac{11}{24}R^2-\frac{23}{24}\Box R\right)\right]
\end{aligned}
\tag{6.72}
$$

where we have used the Bianchi identity $R^{\mu\nu\rho\sigma}+R^{\mu\rho\sigma\nu}+R^{\mu\sigma\nu\rho=0}$ to obtain $2R_{\mu\nu\rho\sigma}R^{\mu\rho\nu\sigma}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and the Gauss-Bonnet theorem[9] to rewrite the Riemann squared terms in terms of the Ricci tensor and scalar.

[9]The Gauss-Bonnet theorem implies that we can add to the action the Gauss-Bonnet action $S_{GB}=\int d^4x\sqrt{-g}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-4R^{\mu\nu}R_{\mu\nu}+R^2\right)$ without consequences on the theory.