Part II

Covariant formulation

CHAPTER 5

QFT in curved spacetime: $T_{\mu\nu}$ from the action

5.1 Introductory remarks

In this chapter, we give the context upon which our work described in Chapter 6 is structured as, by studying simple applications such as quantum scalar fields or QED, we introduce the tools used in the next chapter. Indeed, as we aim to repeat the derivations of chapter 4 in a fully covariant formulation, we introduce the basics of QFT in curved spacetime to derive the semi-classical Einstein equations, the Faddeev-Popov method as gauge-fixing method, and the $P(X)$ theory as the Lagrangian description of a perfect fluid.

In Section 5.2 we start by reviewing the semi-classical approach of QFT in curved spacetime, where the fields appearing as the source of Einstein equations are treated quantum-mechanically and the background metric is parametrized by a classical field. In doing so, we introduce the effective action of a quantum field and the link between the latter with the stress energy tensor of the field. We introduce the background field method as a way to compute the 1-loop corrected effective action and we apply the background field method to the case of a scalar field. We explicitly show that the effective action of a given background field can be computed as the sum of all one-particle-irreducible vacuum graphs where all internal lines correspond to the fluctuations around the background. We then compare the divergences appearing in computing the 1-loop effective action and the stress energy tensor and we show that in general they are not identical. We review with a specific example the differences between the regularized stress energy tensor at the regularized effective action, even if it is possible to connect both results with the divergences arising in the propagator. We then recall the Faddeev-Popov method as a gauge-fixing method which allows to covariantly fix the gauge at the level of the action. By analyzing the example of QED, we review the main features of this gauge-fixing method, such as the appearance of ghosts as unphysical DoFs that fix the remaining spurious DoFs.
Finally, in Section 5.3 we introduce the $P(X)$ theory and we show how radiation can be represented by a scalar field. We derive the stress energy tensor and, by comparing the result with the stress energy tensor of a perfect fluid, we accordingly define energy density and pressure as a function of the scalar field. We then specify

the results in the case of radiation and derive the Lagrangian that reproduces the stress energy tensor for the radiation fluid.
In Section 5.4 we conclude and motivate our work on the covariant formulation of GWs on curved spacetime presented in the following chapter.

5.2 Effective action and background field method

An alternative strategy with respect to how we proceeded in the previous chapter is to derive the stress energy tensor of GWs from the action. We follow chapter 6 of [48] to review how we can treat the computation of the stress energy tensor of a quantum field in a semi-classical approach, where the background metric is parametrized by a classical field while matter fields are treated quantum mechanically[1]. We then seek the action defined as

$$S = S_{\mathrm{g}} + S_{\mathrm{m}} \tag{5.1}$$

that, once variated with respect to the background metric $g_{\mu\nu}$, gives Einstein's field equations

$$R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = -8\pi G_N \langle \hat{T}^{\mathrm{m}}_{\mu\nu} \rangle \tag{5.2}$$

where $\langle T^{\mathrm{m}}_{\mu\nu} \rangle := \langle \mathrm{in}, 0 | \hat{T}^{\mathrm{m}}_{\mu\nu} | 0, \mathrm{in} \rangle$ is the quantum expectation value of the stress energy tensor of the matter content[2] defined as

$$\hat{T}^{\mathrm{m}}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} \tag{5.3}$$

and g is the determinant of the background metric. The first term of the action in Eq. 5.1 is the Einstein Hilbert action

$$S_{\mathrm{g}} = S_{\mathrm{EH}} = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G_N} \right) \tag{5.4}$$

while the second term, that in the classical case would be the classical matter action, is now the effective action $W_{\mathrm{m}}[0]$. The effective action is defined from the generating functional

$$Z[J] = \int \mathcal{D}\phi \exp \left\{ \mathrm{i} S_{\mathrm{m}}[\phi] + \mathrm{i} \int d^4x J(x) \phi(x) \right\} \tag{5.5}$$

where from now on the notation implies a treatment for the scalar field, but the formal manipulations are identical for fields of higher spins. The external current J in Eq. 5.5 causes the initial vacuum state to be unstable and brings the production

[1] This is a generalization of semiclassical theory of electrodynamics ([111]).

[2] Note that with "matter content", we are referring to a generic quantum field that sources the background expansion. Later on, we will be interested in specifying these results for GWs; however, the derived results are valid for fields with generic spin. Furthermore in this section we drop the hat to denote the difference between operators and mode functions as matter fields which are always intended to be operators.

of particles. Taking the limit $J \to 0$ we find the vacuum partition function that in Minkowski is normalized to 1

$$Z[0] := \langle \text{out}, 0|0, \text{in}\rangle = 1. \tag{5.6}$$

Even if in curved spacetime in general $|0, \text{out}\rangle \neq |0, \text{in}\rangle$ in the limit $J \to 0$, path-integral quantization still works and J is interpreted as the current density. We then define the effective action in the limit $J \to 0$ as

$$Z[0] := e^{iW_\phi[0]} \tag{5.7}$$

which, using Eq. 5.6 gives

$$W_\phi[0] = \lim_{J\to 0}(-i \ln Z[J]) = -i \ln\langle \text{out}, 0|0, \text{in}\rangle \tag{5.8}$$

and that, using

$$\delta Z[0] = i\langle \text{out}, 0|\delta S_\text{m}|0, \text{in}\rangle, \tag{5.9}$$

can be used to rewrite Eq. 5.3 as

$$\frac{\langle \text{out}, 0|\hat{T}^\text{m}_{\mu\nu}|0, \text{in}\rangle}{\langle \text{out}, 0|0, \text{in}\rangle} = \frac{2}{\sqrt{-g}}\frac{\delta W_\phi[0]}{\delta g^{\mu\nu}}. \tag{5.10}$$

The appearance of $\langle \text{out}, 0|\hat{T}^\text{m}_{\mu\nu}|0, \text{in}\rangle$ in Eq. 5.10 instead of $\langle \text{in}, 0|\hat{T}^\text{m}_{\mu\nu}|0, \text{in}\rangle$, is related to different boundary conditions. However, as we will discuss in detail in what follows, boundary conditions do not effect the divergences arising in computing the stress energy tensor and enter only in the finite leftover of the renormalized result[3]. The latter, being scheme dependent, needs to be fixed imposing renormalization conditions. As a consequence, in the following we do not worry about the arrangement of vacuum states and proceed in describing the background field method as a procedure to calculate the 1-loop effective action.

The effective action can be determined up to 1-loop corrections in different ways. Functional techniques such as the heat kernel method (through which the DeWitt-Schwinger formalism can be implemented [196, 81, 82, 48, 178]) have the advantage of being fully covariant and therefore particularly suited for computation on arbitrary backgrounds, albeit in Euclidean signature [210].

In the next chapter we work in the context of the background field method, which we now review following [194], as it is particularly useful in computing loop corrections in gauge field theories. The background field method, introduced by DeWitt in [81] in a formalism applicable to one-loop processes ([94, 112, 27, 191, 131]), has the benefit of quantizing gauge field theories without losing explicit gauge invariance. The background field method has been then extended to multi– loop calculations ([207, 86, 51, 1] and it is extensively used in gravity and supergravity theories (i.e., [85, 208, 206]).

We previously reviewed the definition of $W_\text{m}[0]$ in order to derive the stress energy

[3]See Section 1.2.2 for more details.

tensor of a quantum field. We now reintroduce the source J to briefly review the background field method in the simple example of a scalar field (see [1] for more details and applications to gauge theories and [90] for a more recent review) to derive the effective action given by the Legendre transform[4]

$$\Gamma[\phi] = W_\phi[J_\phi] - \int d^4x J_\phi \phi \tag{5.11}$$

where J_ϕ satisfies

$$\left.\frac{\partial W_\phi[J]}{\partial J}\right|_{J_\phi} = \phi. \tag{5.12}$$

Starting from the definition of the generating function in Eq. 5.5, we split the field ϕ into a non-dynamical, but arbitrary background-field $\bar{\phi}$ and a quantum correction φ:

$$\phi = \bar{\phi} + \varphi \tag{5.13}$$

and rewrite the action as $S_{\mathrm{m}}[\bar{\phi} + \varphi]$ so that the generating functional results

$$\tilde{Z}[J] := e^{iW[\bar{\phi},J]} = \int \mathcal{D}\varphi \exp\left\{ \mathrm{i} S_{\mathrm{m}}[\bar{\phi} + \varphi] + \mathrm{i} \int d^4x \; J(x)\varphi(x) \right\}. \tag{5.14}$$

After shifting the integrated field as $\varphi \to \varphi - \bar{\phi}$ we obtain

$$W[\bar{\phi}, J] = W_\varphi[J] - \int d^4x J\bar{\phi} \tag{5.15}$$

and $\Gamma[\bar{\phi}, \varphi]$, computed trough the Legendre transform as in 5.11, results

$$\Gamma[\bar{\phi}, \varphi] = W[\bar{\phi}, J_{\bar{\phi}}] - \int d^4x J_{\bar{\phi}} \varphi \tag{5.16}$$

which has the additional dependence on the background field. Using Eq. 5.15 we obtain

$$\Gamma[\bar{\phi}, \varphi] = W_\varphi[J_{\bar{\phi}}] - \int d^4x J_{\bar{\phi}}(\bar{\phi} + \varphi). \tag{5.17}$$

We then define $\bar{\phi}_J$ as an implicit functional of J satisfying

$$\frac{\partial \Gamma[\bar{\phi}]}{\partial \bar{\phi}} = J, \tag{5.18}$$

which gives the inverse Legendre transform

$$W[\bar{\phi}, J] = \Gamma[\bar{\phi}_J] + \int d^4x J\bar{\phi}_J \tag{5.19}$$

that once variated with respect to J results in

$$\frac{\partial W[\bar{\phi}, J]}{\partial J} = \bar{\phi}_J. \tag{5.20}$$

[4]Note that Eq. 5.11 reduces to Eq. 5.8 in the limit $J \to 0$.

Repeating the same exercise starting from Eq. 5.11 we obtain $\frac{\partial W_\phi[J]}{\partial J} = \phi_J$. In this way, the variation of Eq. 5.15 with respect to J can be rewritten as

$$\bar{\phi}_J = \varphi_J - \bar{\phi}. \tag{5.21}$$

Then, taking $\varphi = \bar{\phi}_J$, $J_{\bar{\phi}_J} = J$ and the result in Eq. 5.21, Eq. 5.17 becomes

$$\Gamma[\bar{\phi}, \bar{\phi}_J] = W_\varphi[J] - \int d^4x J \varphi_J = \Gamma[\varphi_J] = \Gamma[\bar{\phi}_J + \bar{\phi}]. \tag{5.22}$$

Since this is true for any current J, we obtain the final result $\Gamma[\bar{\phi}, \varphi] = \Gamma[\bar{\phi} + \varphi]$, which implies $\Gamma[\bar{\phi}, 0] = \Gamma[\bar{\phi}]$. This means that one can compute the effective action as the sum of all one-particle-irreducible vacuum graphs in the presence of a given background, with all internal lines corresponding to fluctuations around this background.

Going back to the derivation of the stress energy tensor of a quantum field in curved spacetime, there is one last caveat we want to review, before studying the graviton example in the next chapter. Generalizing the derivation in flat spacetime (see [48] for more details) it is possible to connect the effective action $W_\phi[0]$ to the Feynman propagator $G_{\rm F}(x, x')$

$$W_\phi[0] = -\frac{i}{2} \lim_{x \to x'} \mathrm{Tr} \ln(-G_{\rm F}(x, x')) \tag{5.23}$$

where $G_{\rm F}(x, x')$ is to be interpreted in position space normalized by

$$\langle x | x' \rangle = \frac{1}{\sqrt{-g}} \delta^4(x - x') \tag{5.24}$$

so that

$$G_{\rm F}(x, x') = \langle x | G_{\rm F} | x' \rangle \tag{5.25}$$

and where trace is defined as

$$\mathrm{Tr} M = \int d^4x \sqrt{-g} \langle x | M | x \rangle. \tag{5.26}$$

In computing the coincidence limit in Eq 5.23, divergences arise and, considering Eq. 5.10, such divergences affect the stress energy tensor as well[5]. As the divergences arise because of the UV behaviour of the field modes, they are independent on the quantum state and can be expressed in terms of the background geometry only. By defining the effective Lagrangian $\mathcal{L}^{\rm m}_{\rm eff}$ as

$$W_{\rm m}[0] = \int d^4x \sqrt{-g} \mathcal{L}^{\rm m}_{\rm eff}, \tag{5.27}$$

[5]Many examples of the study of such divergences can be found in the literature (see [56, 79, 104, 93, 22, 13, 57, 71, 12, 42, 18] for some examples and [48] for an overview and comparison of different regularization methods), where both fields of different spins and different renormalization methods are investigated.

this implies that the divergences of $\mathcal{L}^{\rm m}_{\rm eff}$ are purely geometrical and can be expressed solely as a function the background metric, even if they arise from the action of the quantum field[6]. The divergencies arising in $\mathcal{L}^{\rm m}_{\rm eff}$ are then reabsorbed by redefining the coupling constant of the background action: after regularizing, to rewrite the divergences in the form

$$\mathcal{L}^{\rm m}_{\rm eff,div} = c_0 + c_1 R + c_2 R^2 + c_3 R_{\mu\nu}R^{\mu\nu} + c_4 \Box R, \tag{5.28}$$

where c_0, c_1, c_2, c_3 and c_4 are divergent in the coincidence limit. Then, by adding higher order corrections to the background action in Eq. 5.1, we can absorb the divergences arising in the effective action of the quantum content by redefining the coupling constants of $S_{\rm g}$ (cosmological constant, Newtonian constant and prefactors of higher order corrections of the Einstein-Hilbert action) and obtain

$$S = S_{\rm g,ren} + W_{\rm m,ren} \tag{5.29}$$

where $W_{\rm m,ren}$ is now finite. By variating with respect to the background metric, we obtain the renormalized semi-classical Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + aH^{(1)}_{\mu\nu} + bH^{(2)}_{\mu\nu} = -8\pi G\frac{\langle {\rm out},0|\hat{T}^{\rm m}_{\mu\nu}|0,{\rm in}\rangle_{\rm ren}}{\langle {\rm out},0|0,{\rm in}\rangle} \tag{5.30}$$

where $H^{(1)}_{\mu\nu}$ and$H^{(2)}_{\mu\nu}$ are the variation of $\int d^4x\sqrt{-g}R^2$ and $\int d^4x\sqrt{-g}R_{\mu\nu}R^{\mu\nu}$ respectively and Λ, G, a and b must be fixed by renormalization conditions.
However, even if the divergences arising in computing the stress energy tensor of a quantum field are connected with those arising in computing the 1-loop effective action, they are not always identical. Indeed, as we show in the following, both the divergences of the tress energy tensor and the effective action can be derived from the regularized propagator $G_{\rm F}(x,x')$. Regardless of this, subtracting the divergences arising in computing $G_{\rm F}(x,x')$ is in general not enough to obtain a finite result for the stress energy tensor derived from Eq. 5.10. Considering the DeWitt-Schwinger representation of the 1-loop effective action ([196, 81])

$$W_{\rm m}[0] = \frac{i}{2}\int {\rm d}^n x[-g(x)]^{\frac{1}{2}} \lim_{x'\to x}\int_0^\infty {\rm d}m^2 G_F\left(x,x'\right), \tag{5.31}$$

we obtain that the regularized form of the effective Lagrangian in the DeWitt-Schwinger expansion results

$$\begin{aligned}\mathcal{L}^{\rm m}_{\rm eff,div} = -\lim_{x'\to x}\frac{\Delta^{\frac{1}{2}}\left(x,x'\right)}{32\pi^2}\int_0^\infty \frac{{\rm d}s}{s^3}{\rm e}^{-i\left(m^2 s-\sigma/2s\right)}\left[a_0\left(x,x'\right) + a_1\left(x,x'\right){\rm i}s\right.\\ \left.+a_2\left(x,x'\right)({\rm i}s)^2\right]\end{aligned} \tag{5.32}$$

where Δ is the Van Vleck-Morette determinant and $a_0\left(x,x'\right)$, $a_1\left(x,x'\right)$ and $a_2\left(x,x'\right)$ depend on the spin of the quantum field we are considering and are functions of

[6]Note that this is not the case for the finite contribution to $\mathcal{L}^{\rm m}_{\rm eff}$, which depends on the quantum state, it corresponds to the long wavelength part and can probe the large scale structure of the manifold.

the background metric only (see [210] for examples of effective Lagrangian in the DeWitt-Schwinger expansion for fields of different spins). On the other hand, the result for the regularized stress energy tensor is given not only by the DeWitt-Schwinger representation of $G_{\mathrm{F}}(x,x')$ as for the effective action, but also by the divergences obtained by differentiating the DeWitt-Schwinger representation of $G_{\mathrm{F}}(x,x')$. Only in the case of conformally trivial systems, in which both the background metric and the quantum field are conformally invariant, the divergences arising in the stress energy tensor are identical to those of the 1-loop renormalized action[7]. However, in general such "short cut" is not available, and it is necessary to separately compute the divergent contribution to the stress energy tensor. As an example, we review the results of Bunch et al. in [57] to show the difference between the regularized effective action and stress energy tensor. Considering a massive scalar field in curved spacetime, the stress energy tensor results

$$\begin{aligned} T^{\phi}_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{2}g_{\mu\nu}m^2\phi^2 \\ &= \lim_{x\to x'}\left(\nabla_{\mu'}\nabla_\nu G^{\phi}_{\mathrm{F}}(x,x') - \frac{1}{2}g_{\mu\nu}\nabla_{\alpha'}\nabla^\alpha G^{\phi}_{\mathrm{F}}(x,x') + \frac{1}{2}g_{\mu\nu}m^2 G^{\phi}_{\mathrm{F}}(x,x')\right) \end{aligned} \tag{5.33}$$

where we use the notation in which primed indices refer to the derivative with respect to x'. As shown in [57], the divergent contribution to the stress energy tensor is given by the sum of the following terms

$$\begin{aligned} \lim_{x\to x'}\left(\nabla_{\mu'}\nabla_\nu G^{\phi}(x,x') - \frac{1}{2}g_{\mu\nu}\nabla_{\alpha'}\nabla^\alpha G^{\phi}(x,x')\right) &= \lim_{\sigma\to 0}(2\pi)^{-1}\left[\left(\frac{1}{\sigma} - \frac{1}{4}m^2 + \frac{1}{24}R\right)\right. \\ \left(g_{\mu\nu} - 2\frac{\sigma_\mu\sigma_\nu}{\sigma_{\alpha^2}\sigma^\alpha}\right) + \frac{1}{60}m^{-2}\left(R_{;\mu\nu} - \frac{1}{2}\Box R g_{\mu\nu}\right) &\left. + O\left(m^{-4}\right) + O\left(\sigma^{1/2}\right)\right] \end{aligned} \tag{5.34}$$

$$\begin{aligned} \lim_{x\to x'}\frac{1}{2}m^2 g_{\mu\nu}G^{\phi}(x,x') = -\lim_{\sigma\to 0}\frac{g_{\mu\nu}}{2\pi}&\left[\frac{m^2}{4}\ln\left|\frac{m^2\sigma}{2}\right| - \frac{1}{24}R - \frac{1}{240m^2}\left(R^2 + 2\Box R\right)\right. \\ &\left. + O\left(m^{-4}\right) + O\left(\sigma^{1/2}\right)\right]. \end{aligned} \tag{5.35}$$

As a consequence, regularizing only $G^{\phi}_{\mathrm{F}}(x,x')$ is not sufficient to obtain the regularized result for the stress energy tensor. Indeed, looking at Eq. (5.34), we notice that there is an extra pole arising in the derivative of $G^{\phi}(x,x')$ that would not appear if we were regularizing the effective action.

5.2.1 Gauge-fixing: Faddeev-Popov method

As mentioned in section 2.2.1, in order to reduce the 10 DoFs of the symmetric rank two tensor $h_{\mu\nu}$ to the two physical DoFs representing the graviton, it is necessary to fix the gauge to get rid of the spurious DoFs. In the following chapter we

[7] See [48] for more details or [73] for some examples. Note that this is not the case for the graviton, see Chapter 6 for more details.

will do so using the so-called Faddeev-Popov method [97], which is reviewed in this section (see [194, 125, 184] for more details).

The Faddeev-Popov method is based on the fact that, in defining the partition function as the integral over all the field configurations, in gauge theories one is including configurations that are equivalent up to a gauge transformation. As a consequence, one is integrating over an infinite set of copies of just one configuration. The Faddeev-Popov method provides a procedure for isolating and computing the integral over the unphysical redundant configurations. We review how this is done in the context of QED, even if, as we will see in the following, QED is a trivial example that does not require Faddeev-Popov ghosts to subtract the spurious DoFs. A more instructive example will be studied in the next chapter, where we use the Faddeev-Popov method to fix the gauge for the graviton action.
We consider the path integral of the electromagnetic field A_μ

$$\int \mathcal{D}A_\mu \, e^{-i\int d^4x \, \mathcal{L}_{\rm EM}[A_\mu]} \tag{5.36}$$

which is invariant under the gauge transformation

$$A_\mu \to A_\mu^\pi = A_\mu + \partial_\mu \pi := \mathcal{G}[A_\mu^\pi]. \tag{5.37}$$

Defining the Faddeev-Popov determinant $\Delta_{\mathcal{G}}[A_\mu] := \det\left(\frac{\partial \mathcal{G}}{\partial \pi}\right)$ such that

$$1 = \Delta_{\mathcal{G}}[A_\mu] \int \mathcal{D}\pi \, \delta(\mathcal{G}[A_\mu^\pi]), \tag{5.38}$$

we can equivalently rewrite Eq. 5.36 as

$$\int \mathcal{D}A_\mu \, \Delta_{\mathcal{G}}[A_\mu] \int \mathcal{D}\pi \, \delta(\mathcal{G}[A_\mu^\pi]) e^{-i\int d^4x \, \mathcal{L}_{\rm EM}[A_\mu]}. \tag{5.39}$$

We perform in the integrand the gauge transformation from A_μ^π to A_μ and obtain

$$\int \mathcal{D}\pi \int \mathcal{D}A_\mu \, \Delta_{\mathcal{G}}[A_\mu] \delta(\mathcal{G}[A_\mu]) e^{-i\int d^4x \, \mathcal{L}_{\rm EM}[A_\mu]} \tag{5.40}$$

where we took advantage of the fact that $\mathcal{L}[A_\mu]$, the Faddeev-Popov determinant and the measure are invariant under gauge transformations. As a result, since in Eq. 5.40 nothing depends on π anymore, the integral over π results in multiplying the partition function of an overall (infinite) constant and we obtain

$$\int \mathcal{D}A_\mu \, \Delta_{\mathcal{G}}[A_\mu] \delta(\mathcal{G}[A_\mu]) e^{-i\int d^4x \, \mathcal{L}_{\rm EM}[A_\mu]}. \tag{5.41}$$

We now note that we can equivalently rewrite the result in Eq. 5.41 by considering the average over a Gaussian-weighted selection of shifts[8] of the gauge condition

[8] We consider that nothing would change if we shift $\mathcal{G}[A_\mu]$ by a constant and that we can multiply by the unity obtained from the Gaussian integral: $1 = N(\xi) \int \mathcal{D}\chi e^{-\frac{i}{2\xi}\int d^4x \chi^2}$, where $N(\xi)$ is a normalization constant.

$\mathcal{G}[A_\mu]$

$$\int \mathcal{D}\chi\, e^{-i\int d^4x \frac{\chi^2}{2\xi}}\, \delta\left(\mathcal{G}[A_\mu]-\chi\right) = e^{-i\int d^4x \frac{1}{2\xi}\mathcal{G}[A_\mu]^2} \tag{5.42}$$

and obtain

$$\int \mathcal{D}A_\mu\, \Delta_{\mathcal{G}}[A_\mu] e^{-i\int d^4x \left(\mathcal{L}_{\rm EM}[A_\mu]+\frac{1}{2\xi}\mathcal{G}[A_\mu]^2\right)}. \tag{5.43}$$

Lastly, we consider that the Faddeev–Popov determinant can be expressed as the integral over an artificial fermion field η

$$\Delta_{\mathcal{G}}[A_\mu] = \det\left(\frac{\partial \mathcal{G}}{\partial \pi}\right) = \int \mathcal{D}\eta\mathcal{D}\bar{\eta}\, e^{i\int d^4x \bar{\eta}\frac{\partial\mathcal{G}}{\partial\pi}\eta} \tag{5.44}$$

to obtain the final result for the gauge-fixed partition function

$$\int \mathcal{D}A_\mu \mathcal{D}\eta\mathcal{D}\bar{\eta}\, e^{-i\int d^4x \left(\mathcal{L}_{\rm EM}[A_\mu]+\frac{1}{2\xi}\mathcal{G}[A_\mu]^2-\bar{\eta}\frac{\partial\mathcal{G}}{\partial\pi}\eta\right)}. \tag{5.45}$$

We now comment on the terms appearing in the gauge-fixed action of Eq 5.45

$$S_{\rm EM} = \int d^4x\, \mathcal{L}_{\rm EM}[A_\mu] + S_{\rm gb} + S_{\rm gh} \tag{5.46}$$

where we defined the so-called gauge breaking action and the Faddeev–Popov ghost action respectively as

$$S_{\rm gb} = \frac{1}{2\xi}\int d^4x\, \mathcal{G}[A_\mu]^2 \tag{5.47}$$

and

$$S_{\rm gh} = -\int d^4x\, \bar{\eta}\frac{\partial\mathcal{G}}{\partial\pi}\eta. \tag{5.48}$$

We notice that the gauge breaking term fixes the gauge condition $\mathcal{G}[A_\mu]$ and, as a consequence, the resulting action is gauge dependent. The fermion field η, called Faddeev–Popov ghost, is an anticommuting Lorentz scalar and, even if the ghost term does not contribute in the case of QED[9], this new unphysical[10] field is the responsible of subtracting the remaining spurious DoFs in non-abelian gauge theories.

5.3 $P(X)$ theory and radiation-like species

In this section, we review the Lagrangian formulation of a perfect fluid ([157, 28]). We follow the notation of Boubekeur et al. in [50] to review the $P(X)$ theory, where the physics of a perfect fluid is derived from a unique scalar field and radiation can be represented by the scalar field ψ.

[9]Since $\frac{\partial\mathcal{G}}{\partial\pi}$ does not depend on the electromagnectic field, the Faddeev–Popov determinant is just a constant and can be dropped.

[10]Faddeev–Popov ghosts violate the spin statisitc theorem, however, since they appear only as internal lines, this is not a problem in computing physical quantities.

We consider the Lagrangian density

$$\mathcal{L} = P(X) = X^2 \quad \text{where} \quad X := -g_{\mu\nu}\partial_\mu\psi\partial^\nu\psi, \tag{5.49}$$

so that the stress energy tensor results[11]

$$T_{\mu\nu} = \mathcal{L}g_{\mu\nu} - \frac{\partial\mathcal{L}}{\partial\partial^\mu\psi}\partial_\nu\psi = 2P'(X)\partial_\mu\psi\partial_\nu\psi + P(X)g_{\mu\nu} \tag{5.50}$$

and the EOM results

$$\partial_\mu\left[P'(X)\partial^\mu\psi\right] = 0. \tag{5.51}$$

Given that the stress energy tensor of a perfect and barotropic fluid is of the form

$$T^{\text{pf}}_{\mu\nu} = (\rho + p(\rho))u_\mu u_\nu + p(\rho)g_{\mu\nu}, \tag{5.52}$$

where u^μ is the 4-velocity of the fluid, we compare with Eq. 5.50 to identify the energy density, the pressure and the 4-velocity as

$$\rho = 2P'(X)X - P(X), \quad p = P(X), \quad u_\mu = \frac{\partial_\mu\psi}{\sqrt{X}} \tag{5.53}$$

and impose that $\partial_\mu\psi$ is everywhere timelike and future directed. In this way, we obtain a consistent Lagrangian formulation of a perfect fluid. Note that in projecting the conservation equation $\partial_\mu T^{\mu\nu}$ along and orthogonal to the fluid flux we obtain the conservation of energy, which results in the EOM in Eq. 5.51, and Euler equation respectively. Furthermore, one can derive the speed of sound c_s that results

$$c_s = \frac{dp}{d\rho} = \frac{P'(X)}{P'(X) + 2P''(X)X}. \tag{5.54}$$

As in the next chapter we use this formalism to describe radiation, we consider the equation of state $p = w\rho$ with w constant. Using the definitions in Eq. 5.53, we then find

$$P(X) = X^{\frac{1+w}{2w}} \tag{5.55}$$

up to a proportionality constant which is irrelevant for the classical theory. In order to find the Lagrangian density for the radiation fluid we consider $w = \frac{1}{3}$, so that the Lagrangian density reduces to

$$\mathcal{L} = X^2 = \left(-\nabla_\mu\psi\nabla^\mu\psi\right)^2. \tag{5.56}$$

Furthermore, it is straightforward to verify that the speed of sound in Eq. 5.54 results $c_s = w = \frac{1}{3}$, the stress energy tensor in Eq. 5.50 is traceless and the EOM results

$$\partial_\mu\left[-2\partial_\nu\psi\partial^\nu\psi\partial^\mu\psi\right] = 0. \tag{5.57}$$

[11] we define $P'(X) = \frac{\partial P(X)}{\partial X}$.

5.4 Comments and motivations for our work

In this chapter, which is to be intended as a continuation of Chapter 2, we introduced the formalism that we will use in the next chapter to study vacuum GWs in a covariant formulation. As we aim to study the caveats introduced in Chapter 2, we introduced the definition of the stress energy tensor of a quantum field in curved spacetime as the variation of the effective action with respect to the background metric and we reviewed the tools needed to parametrize GWs as a massless spin-2 particle on a RD spacetime. This includes the background field method, the Faddeev-Popov method and $P(X)$ theory.

In Chapter 6 we present our work and re-derive in a fully covariant way the results in Chapter 4. In studying the case of the graviton, we derive the regularized Lagrangian that has the form of Eq. 5.28, we reabsorb the divergences by redefining the coupling constants of the background action and we comment on the finite part and on the renormalization conditions to be imposed.
As we will show in the next chapter, the covariant formulation is not only confirming the conclusion of the previous chapter, but also bringing to light features that were hidden in the foliation dependent treatment.