Sweeping vacuum gravitational waves under the rug
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CHAPTER 4

(Im)possibility of constraining primordial GWs with N_{eff} bounds

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4.1 Introductory remarks

In this chapter we question the validity of the results reviewed in Chapter 2 and show the implications of both re-deriving the stress energy tensor of GWs and renormalizing the energy density of GWs following the renormalization procedure reviewed in Section 1.2.2. We apply what we learned in the previous chapter to an example of great interest in cosmology: the contribution of primordial vacuum tensor fluctuation to N_{eff}, being the latter tightly constrained by BBN bounds.

In Section 4.2 we improve the definition of the stress energy tensor of GWs being mindful of avoiding the scale separation on which the result derived in Section 2.2 relies. We obtain an original result that is valid independently from the frequency of the signal and takes into account the effects of curvature. Before proceeding with the renormalization of the energy density, we comment on the physical meaning of the improved definition.
In Section 4.3 we follow trough the procedure studied in the previous chapter in order to regularize the energy density of vacuum GWs. We compare the result obtained by regularizing using physical hard cutoffs and dimensional regularization and, as we learned in the scalar case, we show that even if the coefficients of the log-divergences agree in the two regularization schemes, dimensional regularization is to be preferred to obtain meaningful results.
We proceed in Section 4.4 with the second step of renormalization that consists in imposing renormalization conditions. We stress that this is a fundamental step that, even if it might appear of secondary importance, as it is often overlooked in the literature of renormalization of divergences, it is essential to understand the role of BBN bounds on constraining primordial GWs. By carefully imposing the renormalization conditions to fix the finite, scheme-dependent leftover of the renormalized result, we demonstrate that the BBN constraints fail to constrain the

production of primordial vacuum tensor perturbations.
We conclude in Section 4.5 and comment on the possibility of constraining vacuum GWs with $N_{\rm eff}$ bounds.

4.2 Stress energy tensor (re)definition

As reviewed in Section 2.2, the formula for the stress energy tensor of GWs[1]

$$\rho_{\rm gw}^{\rm Isc} = \frac{1}{32\pi a^2 G_N} \left\langle \hat{h}'_{ij}(\tau,k)\hat{h}'^{ij}(\tau,k)\right\rangle \tag{4.1}$$

was derived in the context of astrophysically sourced GWs. It is nevertheless extensively referenced in computations of the stress energy tensor of cosmological gravitational wave backgrounds[2]. However, retracing the steps used in its derivation, it becomes immediately clear that one needs to proceed with extra care for applications in a cosmological context, in particular when considering questions of renormalized stress tensors in prescriptions that involve the integral over all momenta[3]. This is because, as reviewed in detail in Section 2.2, the Isaacson form presumes a prior scale separation between fast and slow frequencies defined relative to the curvature scale of the background (a criterion that is time dependent), and repeatedly relies on this scale separation along with the implicit averaging prescription to facilitate various approximations resulting in the simplified final form Eq. 4.1. In the context of an expanding spacetime, with modes of observational relevance that might have crossed any a-priori defined scale more than once over cosmic evolution, undoing the steps that relied on these approximations is warranted. Therefore, it behooves us to re-examine the derivation of the stress tensor for GWs without resorting to approximations that may be at odds with the need to incorporate wavelengths beyond this approximation in any intermediate steps in following through the process of renormalization on cosmological spacetimes. In the following we re-derive Eq. 4.1 without the assumptions of having a high-frequency signal propagating on a flat background.

We compute the stress energy tensor of GWs $\hat{T}^{\rm gw}_{\mu\nu}$ following the derivation of Section 2.2, where the stress energy tensor of GWs is defined as the averaged second

[1] Implicit in the expectation value of Eq. 4.1 is an additional (Brill-Hartle) spatial and temporal averaging prescription discussed in Section 2.2. We presume this as implicit when discussing the Isaacson form of the stress tensor in what follows.
[2] e.g. leading to Eq. 2.26. See also [68] and references therein.
[3] A feature that can be bypassed entirely in the context of Hadamard regularization [18, 171].

order perturbed Einstein equations[4]

$$\begin{aligned}\langle \hat{T}^{\mathrm{gw}}{}_{\mu}{}^{\nu}\rangle &:= -\frac{1}{8\pi G_N}\langle \delta^2 G^{\nu}_{\mu}\rangle \\ &= -\frac{1}{8\pi G_N}\Big\langle \delta^2 g^{\nu\alpha}R_{\mu\alpha} + \delta^1 g^{\nu\alpha}\delta^1 R_{\mu\alpha} + g^{\nu\alpha}\delta^2 R_{\mu\alpha} \\ &\quad -\frac{1}{2}\delta^{\nu}_{\mu}\left(\delta^2 g^{\alpha\beta}R_{\alpha\beta} + \delta^1 g^{\alpha\beta}\delta^1 R_{\alpha\beta} + g^{\alpha\beta}\delta^2 R_{\alpha\beta}\right)\Big\rangle . \end{aligned} \tag{4.2}$$

Using the expansions in Eqs. 2.2 and fixing the gauge ($D_\rho \hat{h}^{\rho\nu} = 0$, $\hat{h}_\mu{}^\mu = \hat{h} = 0$) the stress energy tensor in (4.2) results

$$\begin{aligned}\langle \hat{T}^{\mathrm{gw}}{}_{\mu}{}^{\nu}\rangle = -\frac{1}{8\pi G_N}\Big\langle & -\frac{1}{2}\hat{h}^{\nu}{}_{\alpha}\hat{h}_{\mu}{}^{\sigma}R^{\alpha}{}_{\sigma} + \frac{1}{2}\hat{h}^{\rho}{}_{\alpha}\hat{h}_{\rho}{}^{\nu}R^{\alpha}{}_{\mu} + \hat{h}_{\alpha}{}^{\nu}\hat{h}_{\sigma}{}^{\rho}R^{\sigma\alpha}{}_{\rho\mu} + \frac{1}{2}\hat{h}^{\nu}{}_{\alpha}\Box\hat{h}^{\alpha}{}_{\mu} \\ & +\frac{1}{4}D_\mu\hat{h}^{\beta}{}_{\alpha}D^{\nu}\hat{h}_{\beta}{}^{\alpha} + \frac{1}{2}D_\sigma\hat{h}^{\nu}{}_{\alpha}D^{\sigma}\hat{h}_{\mu}{}^{\alpha} - \frac{1}{2}D_\sigma\hat{h}^{\nu}{}_{\alpha}D^{\alpha}\hat{h}_{\mu}{}^{\sigma} + \frac{1}{2}\hat{h}^{\sigma}{}_{\alpha}D_\mu D^{\nu}\hat{h}_{\sigma}{}^{\alpha} \\ & +\frac{1}{2}\hat{h}^{\sigma}{}_{\alpha}D^{\alpha}D_{\sigma}\hat{h}_{\mu}{}^{\nu} - \frac{1}{2}\hat{h}^{\sigma}{}_{\alpha}D^{\alpha}D_{\mu}\hat{h}_{\sigma}{}^{\nu} - \frac{1}{2}\hat{h}^{\alpha}{}_{\sigma}D_{\alpha}D^{\nu}\hat{h}_{\mu}{}^{\sigma} \\ & -\frac{\delta^{\nu}{}_{\mu}}{2}\left(\hat{h}_{\beta}{}^{\alpha}\hat{h}_{\sigma}{}^{\rho}R^{\sigma\beta}{}_{\rho\alpha} + \hat{h}^{\beta}{}_{\alpha}\Box\hat{h}^{\alpha}{}_{\beta} + \frac{3}{4}D_\rho\hat{h}^{\alpha}{}_{\beta}D^{\rho}\hat{h}_{\alpha}{}^{\beta} - \frac{1}{2}D_\sigma\hat{h}^{\beta}{}_{\alpha}D^{\alpha}\hat{h}_{\beta}{}^{\sigma}\right)\Big\rangle . \end{aligned} \tag{4.3}$$

At this stage, Brill-Hartle averaging would result in the Maccallum-Taub averaged stress tensor [149], which upon further integrations by parts within the spatial and temporal averaged integrals in addition to commuting covariant derivatives would bring the latter into the Isaacson form Eq. 2.5. Instead, we persist with Eq. 4.3 as it is. To find the energy density of GWs ρ_{gw}, we have to specify the covariant derivatives in Eq 4.3. Using that in conformal time, the only non-vanishing Christoffel symbols on a FLRW universe are $\Gamma^0_{00} = \frac{a'}{a}$, $\Gamma^i_{0j} = \frac{a'}{a}\delta_j{}^i$ and $\Gamma^0_{ij} = \frac{a'}{a}\delta_{ij}$, and considering only the transverse traceless part of the metric as the propagating DoFs (see Section 2.2.1 for more details), $\hat{T}_0{}^0$ then results

$$\langle \hat{T}_0{}^0\rangle = \frac{-1}{64\pi a^2 G_N}\Big\langle \hat{h}'_j{}^i\hat{h}'^j_i - 3\partial_k\hat{h}_j{}^i\partial^k\hat{h}_i{}^j - 4\hat{h}_j{}^i\partial_k\partial^k\hat{h}_i{}^j + 8\mathcal{H}\hat{h}_j{}^i\hat{h}'^j_i + 2\partial_k\hat{h}_j{}^i\partial^j\hat{h}_i{}^k\Big\rangle . \tag{4.4}$$

Consequently, in the coincidence limit ($\rho_{\mathrm{gw}} := \lim_{y\to x}\rho_{\mathrm{gw}}(\tau;x,y)$) the energy density of GWs can be expressed as:

$$\rho^{\mathrm{Imp}}_{\mathrm{gw}} = \frac{1}{64\pi a^2 G_N}\Big\langle \hat{h}'_{ij}\hat{h}'^{ij} - 3\partial_k\hat{h}_{ij}\partial^k\hat{h}^{ij} - 4\hat{h}_{ij}\partial_k\partial^k\hat{h}^{ij} + 2\partial_k\hat{h}_{ij}\partial^j\hat{h}^{ik} + 8\mathcal{H}\hat{h}_j{}^i\hat{h}'^j_i\Big\rangle \tag{4.5}$$

where $\mathcal{H} := a'/a$. In spite of not invoking any additional averaging prescriptions, the expectation value featuring in the improved stress tensor above is still doing a lot of heavy lifting. We stress that this is purely a quantum expectation value (strictly,

[4] One can also independently derive what follows by expanding the action to second order in perturbations and varying with respect to the background metric, taking care to properly address gauge-fixing and ghost terms that are necessary in the context of quantum expectation values, the net result of which will be Eq 4.3 [171].

an in-in correlation function) at some fixed time with no extra spatial or temporal averaging invoked. Furthermore, the expectation value presumes a density matrix, which we take to be Bunch Davies vacuum. A far more interesting story beyond the scope of the present work commences once one incorporates the effects of non-trivial density matrices, and interactions induced by gravitational non-linearities, as well as those induced by matter couplings.

It is worth stressing that Eq. 4.5 is the temporal component of a covariantly conserved tensor, and it is under no obligation to be conserved in isolation. This is a corollary of the fact that on FRLW backgrounds, a locally conserved energy cannot be defined. Moreover, it is also under no obligation to be positive definite. In fact, were one to insist of constructing the spectral density associated with the vacuum expectation value of Eq. 4.5, one can show that it crosses zero at a comoving scale corresponding to the horizon scale at any given time. This is consistent with the operational ambiguity of associating a background/fluctuation split for wavelengths commensurate with the background curvature. However, none of this is of any operational concern for our purposes, as we stress that Eq. 4.5 is to be viewed as the components of a covariantly conserved tensor corresponding to a massless spin-2 excitation whose role in renormalization of background quantities is well understood in the covariant context, but obscured and widely conflated for a physical contribution to the number of relativistic species in foliation-specific computations as we elaborate upon next. The specific computation we are interested in performing is the renormalization of the graviton stress tensor on a background corresponding to radiation domination preceded by a finite period of inflation. In doing so, we will compare and contrast the results reviewed in Chapter 2, obtained from both the Isaacson and improved forms of the stress tensor, before revisiting the question of $N_{\rm eff}$ bounds from vacuum tensor modes and concluding.

Before proceeding, an important aside is due: although one might find statements in the literature that questions the utility of even defining GWs with wavelengths grater than the background curvature scale[5], this would nominally be at odds with the premise of many computations. It is also add odds with observations: GWs from mergers of binary black hole systems have been observed [4, 2, 3]. Moreover, the search for primordial GWs is premised on the fact that they induce local quadrupolar anisotropies in the density field of the primordial plasma at all scales, resulting in a signature B-mode polarization pattern [128, 198]. Both situations feature GWs with wavelengths comparable to or greater than the background curvature radius at some point – the peak frequency emitted from a merger corresponding to wavelengths commensurate to the Schwarzschild radius, and primordial tensor fluctuations having crossed the Hubble radius before sourcing local anisotropies.
Clearly, nature tells us that the notion of tensor perturbations with wavelengths

[5]In section 35.7 of [160] for example, one finds the statement that "One must always have $\mathcal{A} \ll 1$ as well as $\lambda \ll 2\pi\mathcal{R}$ if the concept of gravitational wave is to make any sense", where $\mathcal{A}$ is the dimensionless amplitude of the gravitational wave and $\mathcal{R}^{-2}$ is as defined in footnote 1 (see also [152, 149, 204, 68] for more detailed discussions of this point).

longer than than the background curvature radius has to make sense. By general covariance and the Bianchi identities that follow as a corollary, one must also be able to identify a conserved rank two tensor that plays the role of a stress tensor from direct perturbation of the equations of motion. That one can do so with minimal fuss by simply undoing the averaging prescription as done in this section. It is also the premise of the computation in [18] and the covariant formulation of Chapter 6 where we parametrize vaccum GWs a massless spin-2 DoFs on a curved background, whose regularization and renormalization proceeds via established prescriptions.

Isaacson limit

As a consistency check, we show how, by reintroducing Isaacson's assumptions, we can reduce the improved energy density in Eq. 4.5 to Eq. 4.1.
In order to obtain Eq. 4.1, we have to assume that the curvature is negligible ($k \gg \mathcal{H}$) and that the average scheme allows to integrate by parts both in time and space. Applying these assumptions to Eq. 4.5 we obtain that

- The second and third term can be rewritten as $-3\partial_k \hat{h}_{ij}\partial^k \hat{h}^{ij} - 4\hat{h}_{ij}\partial_k\partial^k \hat{h}^{ij} = 3\hat{h}_{ij}\partial_k\partial^k \hat{h}^{ij} - 4\hat{h}_{ij}\partial_k\partial^k \hat{h}^{ij} = -\hat{h}_{ij}\partial_k\partial^k \hat{h}^{ij}$.
- The EOM results: $\Box \hat{h}_{ij} = 0 \quad \Rightarrow \quad \partial_k \partial^k \hat{h}_{ij} = \hat{h}''_{ij}$.
- The last therm is negligible $\mathcal{H} h_{ij} \dot{h}^{ij} \sim 0$.

In conclusion[6], (4.5) results

$$\rho_{\rm gw}^{\rm Imp} = \frac{1}{64\pi G_N a^2}\left\langle \hat{h}'_{ij}\hat{h}'^{ij} - \hat{h}_{ij}\hat{h}''_{ij}\right\rangle = \frac{1}{32\pi G a^2}\left\langle \hat{h}'_{ij}\hat{h}'^{ij}\right\rangle = \rho_{\rm gw}^{\rm Isc} \tag{4.6}$$

where in the second equality we integrate by parts in time.

4.3 Regularization – Finite inflation

Before delving into the substance of this section, it is informative to contrast the results typically found in the literature. As reviewed in Section 2.3.2, imposing hard cutoffs the the energy density of GWs results in expressions of the form:

$$\rho_{\rm gw} \simeq \frac{A_t}{32\pi G_N}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\frac{1}{2n_t}\frac{1}{a^4} \propto \frac{1}{a^4}\left[\frac{1}{n_t} + \log\frac{k_{\rm UV}}{k_*}\right], \tag{4.7}$$

where k_* is some reference IR scale, and the approximation is only valid when $n_t \to 0$ [158]. We deconstruct and rederive the energy density for vacuum tensor perturbations in what follows, but before doing so, it is useful to compare what one would have obtained in retracing the steps leading to Eq. 4.7 for a massless test scalar field. In doing so, one would obtain the expression

$$\rho \simeq \lim_{n_s\to 1}\frac{A_s}{32\pi G_N}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_s-1}\frac{1}{2(n_s-1)}\frac{1}{a^4} \propto \frac{1}{a^4}\left[\frac{1}{n_s-1} + \log\frac{k_{\rm UV}}{k_*}\right], \tag{4.8}$$

[6] The second last term of Eq. 4.5 does not contribute as once we compute the expectation value in our gauge choice results in $k^i\epsilon_{ij}(k) = 0$ (see Section 4.3 for more details).

which can directly be compared to the divergences computed in Eqs. 3.75 - 3.77 from hard cutoffs in physical momenta and Eq. 3.80 from dimensional regularization, all of which are understood to be intermediate expressions that are to be subtracted and renormalized.
An immediate reservation one might express for expressions such as Eq. 4.7 is the appearance of cutoffs in what should be a physical result. Although one might argue that the effects of UV and IR modes are negligible relative to what can be extracted from Eq. 4.7, its form nevertheless suggests that the expression above is an intermediate result on the way to computing a physical energy density. Moreover, one might be concerned in applying the formula Eq. 4.7 to the case of blue tilted spectra (i.e. for positive $n_t \sim \mathcal{O}(1)$, so that $\rho_{\rm GW} \sim (k_{\rm UV}/k_*)^{n_t}$ as derived in e.g. [158]), that some part of this expression is nothing other than the Fourier transform of a UV divergence that gets subtracted in the usual way.
In the following we continue this process of regularizing and renormalizing the energy density of GWs on a background that transitions from a pre-inflationary RD era to a pure dS inflation and finally back to RD era. This is an example of how one should proceed to ensure that well defined physical observables are been computed.

Going back to the study of finite inflation and tensor vacuum perturbations, we work in TT-gauge and we expand the tensor perturbations as

$$\hat{h}_{ij}(\tau,x) = \sum_{r=+,x} \int \frac{d^3k}{M_{\rm pl}\,(2\pi)^3} e^{ix\cdot k}\left[\epsilon^r_{ij}(k)\hat{a}_k\gamma_k(\tau) + \epsilon^{r*}_{ij}(-k)\hat{a}^\dagger_{-k}\gamma^*_k(\tau)\right]. \tag{4.9}$$

The polarization tensors are normalized as $\epsilon^s_{ij}\epsilon^{r*}_{ij} = 4\delta^{rs}$, $k^i\epsilon^r_{ij} = 0$. The normalizations are chosen so that the relevant mode functions during inflation and the pre and post-inflationary phases of radiation domination are given for each polarization just as for massless minimally coupled scalars[7]. These are:

$$\begin{aligned}
\gamma^{\rm PI}_k(\tau) &= \frac{1}{a}\frac{1}{\sqrt{2k}} e^{-ik\tau_{\rm I}\left(2-\frac{a}{a_{\rm R}}e^{\mathcal{N}_{\rm tot}}\right)} \\
\gamma'^{\rm PI}_k(\tau) &= -\frac{1}{a^2\sqrt{2k}}\frac{a_{\rm I}}{\tau_{\rm I}} e^{-ik\tau_{\rm I}\left(2-\frac{a}{a_{\rm R}}e^{\mathcal{N}_{\rm tot}}\right)}\left(-1+ik\tau_{\rm I}\frac{a}{a_{\rm I}}\right),
\end{aligned} \tag{4.10}$$

during the pre-inflationary phase,

$$\begin{aligned}
\gamma^{\rm I}_k(\tau) &= \frac{H}{\sqrt{2k^3}}\left[\alpha^{\rm I}_k e^{i\frac{k}{aH}}\left(1-\frac{ik}{aH}\right) + \beta^{\rm I}_k e^{-i\frac{k}{aH}}\left(1+\frac{ik}{aH}\right)\right] \\
\gamma'^{\rm I}_k(\tau) &= -\frac{H}{\sqrt{2k^3}}\left[\alpha^{\rm I}_k e^{i\frac{k}{aH}}\frac{k^2}{aH} + \beta^{\rm I}_k e^{-i\frac{k}{aH}}\frac{k^2}{aH}\right],
\end{aligned} \tag{4.11}$$

[7]The EOM for h^j_i in TT-gauge can be derived from the first order expansion of Einstein equation: $\hat{h}''^j_i + 2\mathcal{H}\hat{h}'^j_i - \partial^2_k\hat{h}^j_i = 0$.

during inflation, and

$$\gamma_k^{\rm RD}(\tau) = \frac{1}{a}\frac{1}{\sqrt{2k}}\left[\alpha_k^{\rm R} e^{-ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)} + \beta_k^{\rm R} e^{ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\right]$$
$$\gamma_k'^{\rm RD}(\tau) = \frac{a_{\rm R}}{a^2\tau_{\rm R}\sqrt{2k}}\left[\alpha_k^{\rm R} e^{-ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1-\frac{iak\tau_R}{a_{\rm R}}\right) + \beta_k^{\rm R} e^{ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1+\frac{ik\tau_{\rm R}a}{a_{\rm R}}\right)\right] \tag{4.12}$$

during the terminal RD phase. The corresponding Bogoliubov coefficients are the same as those given in Eq. 3.64 and Eq. 3.65.
An immediate corollary of the above is that the two point correlation function of each graviton polarization is identical to that of a massless, minimally coupled scalar. Therefore, as illustrated in Figs. 3.1, one finds that the IR divergences exhibited on a past infinite dS background are also cured for the graviton two point function on a background corresponding to finite duration inflation.
Using Eq. 4.9, the energy density of GWs in Eq. 4.1 results (with the reduced Planck mass defined as $M_{\rm pl}^2 = \frac{1}{8\pi G_N}$) in:

$$\rho_{\rm gw}^{\rm Isc} = \lim_{y\to x}\rho_{\rm gw}(\tau;x,y) = \frac{1}{\pi^2 a^2}\int_0^\infty dk\; k^2\left[\gamma_k'^{\rm RD}\gamma_k'^{\rm RD*}\right] \tag{4.13}$$

as the tensor mode counterpart of Eq. 3.70, after having summed the contributions from the two independent polarizations. Using the mode functions specified by Eqs. 4.12, the relations 3.65, and tracing through the steps of the previous section, we find

$$\rho_{\rm gw}^{\rm Isc} = \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty}\frac{d^4k}{k}\left[k+\frac{a_{\rm R}^4H^2}{a^2k}\right]\left\{1+\frac{a_{\rm R}^4H^4}{2k^4}+2\left|\beta_k^{\rm I}\right|^2\left(1+\frac{a_{\rm R}^4H^4}{2k^4}\right)\right.$$
$$-i\alpha_k^{\rm I}\beta_k^{\rm I*}e^{2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2}\left(-i+\frac{a_{\rm R}H}{k}+i\frac{a_{\rm R}^2H^2}{2k^2}\right)+i\alpha_k^{\rm I*}\beta_k^{\rm I}e^{-2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2}$$
$$\left.\left(i+\frac{a_{\rm R}H}{k}-i\frac{a_{\rm R}^2H^2}{2k^2}\right)\right\}+\frac{1}{4\pi^4a^4}\int_{-\infty}^{\infty}\frac{d^4k}{k}\left(\frac{a_{\rm R}^4H^2}{a^2k}\right)\left[\alpha_k^{\rm R}\beta_k^{\rm R*}e^{-2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\right.$$
$$\left.\left(1+i\frac{ka}{a_{\rm R}^2H}\right)^2+\alpha_k^{\rm R*}\beta^{\rm R}e^{2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1-i\frac{ka}{a_{\rm R}^2H}\right)^2\right], \tag{4.14}$$

which again can be split into a power law contribution and oscillating but finite contributions in the UV-limit that become relevant when the oscillations freeze as $k\to 0$, softening the IR behavior. As we did for the scalar case, we similarly separate the spectral density into oscillatory and power law contributions

$$\rho_{\rm gw}^{\rm Isc} = \frac{1}{2\pi^2a^4}\int_0^\infty\frac{dk}{k}\,k^4\left[\left(1+\frac{a_{\rm R}^4H^2}{a^2k^2}\right)\left(1+2|\beta_k^{\rm R}|^2_{\rm power}\right)+\{\rm osc\}\right]$$
$$:= \int_0^\infty\frac{dk}{k}\left[\Omega_{\rm power}^{\rm gw}(k,\tau)+\Omega_{\rm osc}^{\rm gw}(k,\tau)\right] \tag{4.15}$$

and neglect $\Omega_{\rm osc}^{\rm gw}(k,\tau)$ in regularizing the UV divergences. Even though the Isaacson form of the stress tensor is not strictly valid when incorporating wavelengths greater than the Hubble scale at any given time, naïvely persisting with it for all wavelengths would show an IR regular spectral density $\Omega^{\rm gw}_{\rm total} \propto k^2$. Consequently, one only has the resulting UV divergent contributions to regulate:

$$\rho^{\rm Isc}_{\rm gw,div} \;=\; \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k\,\left(1+\frac{a_{\rm R}^4H^4+a_{\rm I}^4H^4}{2k^4}+\frac{a_{\rm R}^4H^2}{a^2k^2}\right), \tag{4.16}$$

which up to overall factors, is formally identical to the contributions Eq. 3.74, and whose subtraction proceeds along the same lines.

Were we to consider the improved stress tensor Eq. 4.5 (which does not invoke a scale separation or time averaging prescription), we would obtain

$$\rho^{\rm Imp}_{\rm gw} = \frac{1}{2\pi^2a^2}\int_0^\infty dk\,k^2\left[\gamma_k^{\prime\rm RD}\gamma_k^{\prime\rm RD*}+k^2\gamma_k^{\rm RD}\gamma_k^{\rm RD*}+4\mathcal{H}\left(\gamma_k^{\prime\rm RD}\gamma_k^{\rm RD*}+\gamma_k^{\rm RD}\gamma_k^{\prime\rm RD*}\right)\right] \tag{4.17}$$

as the counterpart of Eq. 4.13 that is now valid for for all wavelengths. Using the mode functions and normalizations specified above, we find

$$\begin{aligned}\rho^{\rm Imp}_{\rm gw} =&\frac{1}{4\pi^4a^4}\int_{-\infty}^{\infty}\frac{d^4k}{k}\left[k-\frac{7a_{\rm R}^4H^2}{a^22k}\right]\Bigg\{1+\frac{a_{\rm R}^4H^4}{2k^4}+2\left|\beta_k^{\rm I}\right|^2\left(1+\frac{a_{\rm R}^4H^4}{2k^4}\right)\\ &-i\alpha_k^{\rm I}\beta_k^{\rm I*}e^{2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2}\left(-i+\frac{a_{\rm R}H}{k}+i\frac{a_{\rm R}^2H^2}{2k^2}\right)+i\alpha_k^{\rm I*}\beta_k^{\rm I}e^{-2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2}\\ &\left(i+\frac{a_{\rm R}H}{k}-i\frac{a_{\rm R}^2H^2}{2k^2}\right)\Bigg\}+\frac{1}{4\pi^4a^4}\int_{-\infty}^{\infty}\frac{d^4k}{k}\Bigg[\alpha_k^{\rm R}\beta_k^{\rm R*}e^{-2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\\ &\left(\frac{k}{2}+\frac{a_{\rm R}^4H^2}{a^22k}\left(1+i\frac{ka}{a_{\rm R}^2H}\right)^2-4\frac{a_{\rm R}^4H^2}{a^2k}\left(1+i\frac{ka}{a_{\rm R}^2H}\right)\right)+\alpha_k^{\rm R*}\beta^{\rm R}e^{2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\\ &\left(\frac{k}{2}+\frac{a_{\rm R}^4H^2}{a^22k}\left(1-i\frac{ka}{a_{\rm R}^2H}\right)^2-4\frac{a_{\rm R}^4H^2}{a^2k}\left(1-i\frac{ka}{a_{\rm R}^2H}\right)\right)\Bigg]\end{aligned} \tag{4.18}$$

and we can again separate the spectral density[8] into oscillatory and power law contributions

$$\begin{aligned}\rho^{\rm Imp}_{\rm gw} \;&=\; \frac{1}{2\pi^2a^4}\int_0^\infty\frac{dk}{k}\,k^4\left[\left(1-\frac{7a_{\rm R}^4H^2}{2a^2k^2}\right)\left(1+2|\beta_k^{\rm R}|^2_{\rm power}\right)+\{\rm osc\}\right],\\ &:=\; \int_0^\infty\frac{dk}{k}\left[\Omega^{\rm gw}_{\rm power}(k,\tau)+\Omega^{\rm gw}_{\rm osc}(k,\tau)\right].\end{aligned} \tag{4.19}$$

We again find that the spectral density summing all contributions in the IR goes as $\Omega^{\rm gw}_{\rm total}\propto k^2$, albeit with a negative overall coefficient (cf. the discussion below Eq. 4.5) with the following divergent contributions in the UV that necessitate

[8]We stress that the spectral density as defined in Eq. 4.19 is to be viewed as only a calculational definition for the purposes of comparison to the literature, and not be to viewed as the power spectral density of vacuum fluctuations without additional caveats.

subtraction

$$\rho_{\rm gw,div}^{\rm Imp} \quad = \quad \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k\,\left(1+\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4} - \frac{7\,a_{\rm R}^4 H^2}{2a^2k^2}\right). \qquad (4.20)$$

In proceeding to regularizing divergences, we can identify the various UV divergences is different schemes, finding identical coefficients for the log divergences in all of them. By imposing cutoffs in physical momenta ($k = a\Lambda_{\rm UV}$) on the identifiably UV-divergent terms above, we obtain

$$\frac{1}{2\pi^2 a^4}\int_0^{a\Lambda_{\rm UV}} dk\,k^3 - \frac{7}{2}\frac{a_{\rm R}^4 H^2}{2\pi^2 a^6}\int_0^{a\Lambda_{\rm UV}} dk\,k = \frac{\Lambda_{\rm UV}^4}{8\pi^2} - \frac{7H^2\Lambda_{\rm UV}^2}{8\pi^2(a/a_{\rm R})^4} \qquad (4.21)$$

and

$$\frac{1}{4\pi^2 a^4}\int_{a\Lambda_{\rm IR}}^{a\Lambda_{\rm UV}} \frac{dk}{k}\left(a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4\right) = \frac{H^4\left(1+e^{-4\mathcal{N}_{\rm tot}}\right)}{4\pi^2(a/a_R)^4}\log\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}. \qquad (4.22)$$

If instead we were to now isolate the power law UV divergent parts and dimensionally regularize, we would obtain

$$\begin{aligned}\rho_{\rm gw,div}^{\rm Imp} \supset & \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k - \frac{7a_{\rm R}^2}{4\pi^4 a^6}\int_{-\infty}^{\infty} d^4k\,\left(\frac{a_{\rm R}^2 H^2}{2k^2}\right)\\ = & \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k\left[\frac{k^2+m^2}{(k^2+m^2)} - \frac{7}{2}\frac{a_{\rm R}^4 H^2}{2a^2}\left(\frac{1}{(k^2+m^2)} + \frac{m^2}{k^2(k^2+m^2)}\right)\right]\end{aligned} \qquad (4.23)$$

which, as for the dimensionally regularized scalar case Eq. 3.78, has the divergent contributions canceling among themselves and not necessitating any counterterms. The logarithmic divergence can be expressed as

$$\begin{aligned}\rho_{\rm gw,div}^{\rm Imp} \quad \supset \quad & \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k\,\left(\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4}\right)\\ = \quad & \frac{H^4\left(1+e^{-4\mathcal{N}_{\rm tot}}\right)}{4\pi^2(a/a_{\rm R})^4}\left[\frac{1}{\delta_{\rm UV}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right].\end{aligned} \qquad (4.24)$$

The results of the regularized improved energy density can be collected and summarized as

$$\rho_{\rm gw,div}^{\rm Imp} = \lim_{\Lambda_{\rm UV}\to\infty}\left\{\frac{1}{2\pi^2}\frac{\Lambda_{\rm UV}^4}{4} - \frac{7}{2}\frac{a_{\rm R}^4 H^2}{2\pi^2 a^4}\frac{\Lambda_{\rm UV}^2}{2} + \frac{H^4\left(1+e^{-4\mathcal{N}_{\rm tot}}\right)}{4\pi^2(a/a_{\rm R})^4}\log\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}\right\}, \text{(cutoff)} \qquad (4.25)$$

and also compared to the result obtained via dimensional regularization

$$\rho_{\rm gw,div}^{\rm Imp} = \lim_{\delta_{\rm UV}\to 0}\left\{\frac{H^4\left(1+e^{-4\mathcal{N}_{\rm tot}}\right)}{4\pi^2(a/a_R)^4}\left[\frac{1}{\delta_{\rm UV}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right]\right\}. \qquad (\text{dim}-\text{reg}) \qquad (4.26)$$

Before commenting on the regularized results in Eqs. 4.25 and 4.26, for illustrative purposes, we plot the power spectral densities of the (time unaveraged) Isaacson and improved forms of the stress tensors in Fig. 4.1. For sub-horizon modes, where a positive spectral density results, the improved stress tensor evaluated at any given time does not exhibit oscillations[9].

Going back to the rengularized results in Eqs. 4.25 and 4.26, as cautioned in the previous chapter, one may not be able to consistently subtract such divergences unless one uses regularization schemes that preserve the symmetries of the background. In section 3.3.2 we showed how hard cutoffs fail to give a counterterm for the leading divergences that is proportional to the metric nor the Ricci tensor. Similarly, we follow through the same logic to study the divergences appearing in computing the energy density and pressure of GWs. Indeed, we show that if we were to regularize the energy density and pressure of GWs using hard cutoffs, it appears that the equation of state would not be satisfied. Following the procedure described in section 4.2, we derive the energy density and pressure of GWs during RD era that result respectively

$$\begin{aligned}
\rho_{\rm gw} &= \frac{1}{64\pi a^2 G_N}\left\langle h'^{\,i}_{j} h'^{\,j}_{i} - 3\partial_k h_{ij}\partial^k h^{ij} - 4h_{ij}\partial_k\partial^k h^{ij} + 8\mathcal{H} h_j{}^i h'^{\,j}_{i} + 2\partial_k h_{ij}\partial^j h^{ik}\right\rangle \\
P_{\rm gw} &= -\frac{1}{3}\frac{1}{64\pi a^2 G_N}\left\langle 5h'^{\,i}_{j} h'^{\,j}_{i} - 3\partial_k h_{ij}\partial^k h^{ij} - 4h_{ij}\partial_k\partial^k h^{ij} + 16\mathcal{H} h_j{}^i h'^{\,j}_{i} + 8h_j{}^i h''^{\,j}_{i}\right\rangle \\
&= \frac{1}{3}\frac{1}{8\pi a^2 G_N}\left\langle -5h'^{\,i}_{j} h'^{\,j}_{i} + 3\partial_k h_{ij}\partial^k h^{ij} - 4h_{ij}\partial_k\partial^k h^{ij}\right\rangle
\end{aligned} \tag{4.27}$$

where in the second equality of the pressure we use the EOM. From the result above it appears that the equation of state is not satisfied. Specifying the result in the case of finite inflation and proceeding as before, we find that by using physical cutoff and dimensional regularization respectively, the regularized pressure results

$$P_{\rm gw,div} = \frac{1}{3}\lim_{\Lambda_{\rm UV}\to\infty}\left\{\frac{\Lambda_{\rm UV}^4}{8\pi^2} - \frac{5}{2}\frac{a_{\rm R}^4 H^2}{2\pi^2 a^4}\frac{\Lambda_{\rm UV}^2}{2} + \frac{H^4\left(1+e^{-4\mathcal{N}_{\rm tot}}\right)}{4\pi^2 (a/a_{\rm R})^4}\log\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}\right\}, \quad \text{(cutoff)} \tag{4.28}$$

$$P_{\rm gw,div} = \frac{1}{3}\lim_{\delta_{\rm UV}\to 0}\left\{\frac{H^4\left(1+e^{-4\mathcal{N}_{\rm tot}}\right)}{4\pi^2 (a/a_R)^4}\left[\frac{1}{\delta_{\rm UV}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right]\right\}. \quad (\text{dim}-\text{reg}) \tag{4.29}$$

By recalling the result for the regularized energy density which are given in Eqs 4.25 and 4.26, we can see that only the result regularized using dimensional regularization gives a traceless stress energy tensor and satisfies the equation of state for radiation-like species. We then find that even if at the operator level this is not evident, once one regularize using dimensional regualrization the equation of state ($P = \frac{1}{3}\rho$ in the case under analysis) is satisfied. Moreover, using dimensional regualarization the

[9] The oscillatory behavior of the time unaveraged Isaacson stress tensor can be understood from the fact that the magnitude of the time derivative squared of a linearized gravitational wave is by itself not a constant of the linearized equations of motion (which requires restoring the spatial derivative contributions), unlike for the improved form.

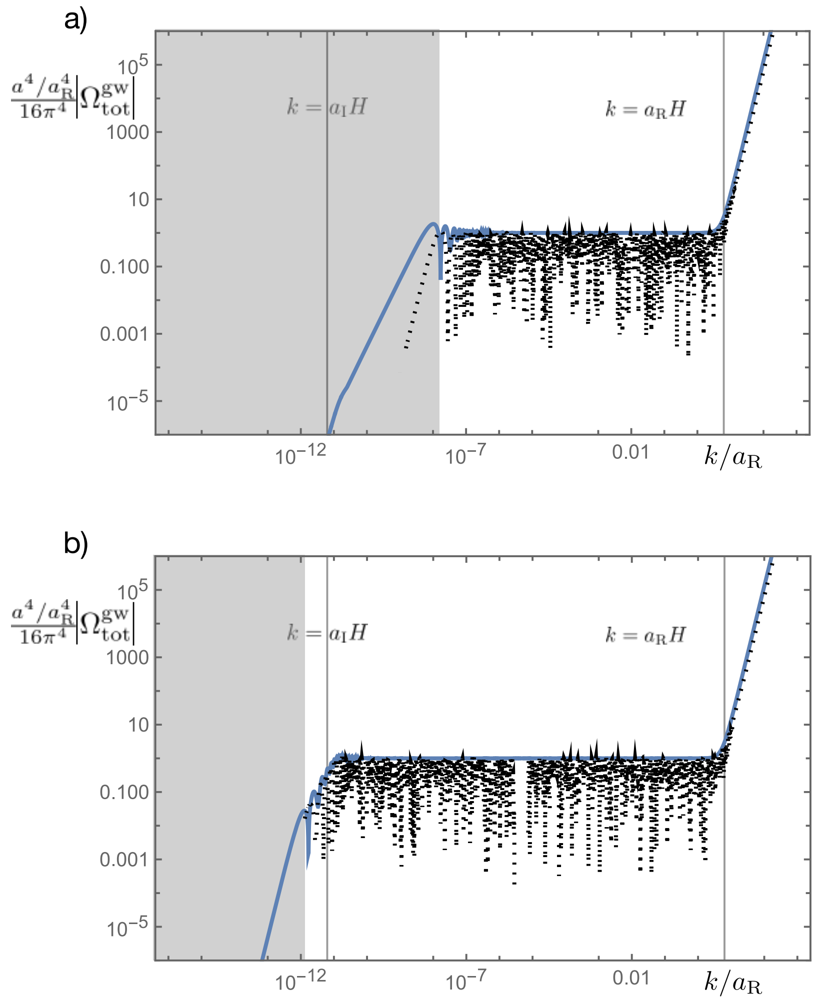

Figure 4.1: Comparison Power spectral densities vacuum GWs

The graph shows the comparison between the power spectral density of the Isaacson stress tensor (dashed lines) and the spectral density of the improved stress tensor (blue line). The gray shaded regions correspond to super-horizon scales, which are outside the domain of validity of the Isaacson stress tensor (where we note that the oscillations would not appear in its time averaged form), and also where the spectral density for the improved stress tensor nominally becomes negative (cf. discussion below Eq. 4.5).
Spectral density evaluated at $a = 10^9 a_{\mathrm{R}}$ (in **a)**) and at $a = 10^{13} a_{\mathrm{R}}$ (in **b)**), with $a_{\mathrm{I}} = 10^{-12} a_{\mathrm{R}}$ in units where H is set to 2π.

trace of the energy tensor is proportional to the Ricci scalar ($T_\mu{}^\mu = R = 0$ in the case under analysis) and the divergences can be subtracted with a counterterm that comes from varying the background curvature with respect to the metric (and thus corresponds to a renormalization of G_N). For this reason, we stick to dimensional regularization in what follows, although covariant point splitting methods also offer a practical alternative[48, 14, 72, 80, 71, 57, 58].

Before proceeding in the next section in reabsorbing the divergences appearing in Eq. 4.26, we comment on the counterterms we need to add to have a predictive theory with which we can calculate the results for any subsequent observations. To understand which counterterms are required to reabsorb the divergences in dimensional regularization, we need only to look at the scale factor dependence of the divergences in Eq. 4.26. We immediately notice that the divergences we have computed do not necessitate higher derivative counterterms beyond those already in the Einstein Hilbert action.
In order to understand why this is, we trace through a treatment that uses adiabatic regularization to renormalize the stress tensor of a non-minimally coupled test scalar field [56], where mode functions are adiabatically expanded in order to regularize divergences and identify counterterms. Higher orders in the adiabatic expansion necessitate successively higher order counterterms. The adiabatic solution of order zero is regularized by a cosmological constant counterterm, the second order solution by a curvature counterterm, with curvature squared counterterms needed to renormalize divergences that appear only at fourth order in the adiabatic expansion. In our case the adiabatic solution of order four is null and non-vanishing curvature squared counterterms would only be necessitated if we were to consider couplings to other matter fields, or by incorporating the effects of loops and higher order gravitational and matter coupling non-linearities (see [138, 109, 98, 156, 24] for corresponding studies of adiabatic expansions for fields of different spins and masses which necessitate higher order counterterms).

4.4 Renormalization and N_{eff} bounds

We turn our attention in this section to the second and most consequential step in the process of renormalization[10] – that of extracting physical observables after imposing renormalization conditions. Before studying the GWs result, we begin the discussion with the relatively academic exercise of renormalizing the stress tensor of a test scalar field on a background that transitions in and out of inflation (considering the results obtained in section 3.3.3). By virtue of not having a classically evolving background and energy density, the only effects of a test scalar will be in renormalizing background couplings in the context of the effective theory of gravity [90, 91, 92, 60]. We return to the more interesting and physically relevant case of primordial GWs next.

Our starting point is the bare matter and gravitational actions, along with the

[10]See Section 1.2.2 for more details.

requisite counterterms:

$$S = S_{\rm EH} + S_{\rm bg} + S_\phi + S_{\rm ct}. \tag{4.30}$$

In order to impose renormalization conditions after having regularized divergences, we first consider the background equations of motion

$$\frac{1}{8\pi G_B}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = T^{\rm bg}_{\mu\nu} + \langle \hat{T}^\phi_{\mu\nu}\rangle + \langle \hat{T}^{\rm ct}_{\mu\nu}\rangle, \tag{4.31}$$

where we stress that the couplings that appear in Eq. 4.30 are to be understood as bare couplings, and the presence of the counterterm shifts the tadpole condition in a manner that we will shortly make precise. Given the consistency of dimensional regularization with general covariance, we can proceed by considering the above for any given component. For the 00 component, we have

$$-\frac{{R_0}^0}{8\pi G_B} = \rho^{\rm cl}_{\rm bg} + \rho_\phi + \rho_{\rm ct}, \tag{4.32}$$

where $\rho^{\rm cl}_{\rm bg}$ is the to be renormalized classical background energy density that sources the expanding geometry around which we have computed the stress tensor for the test scalar field. The corresponding energy density ρ_ϕ is defined as

$$\rho_\phi := \rho^{\rm cl}_\phi - \langle {\hat{T}^0}_0\rangle \equiv -\langle {\hat{T}^0}_0\rangle, \tag{4.33}$$

where $\rho^{\rm cl}_\phi \equiv 0$ for a test scalar by assumption. The background satisfies

$$-{R_0}^0 = -\frac{3}{a^2}\left(\frac{a'}{a}\right)' = \frac{3H^2}{(a/a_{\rm R})^4} \tag{4.34}$$

via Eq. 3.58 during the terminal RD phase, where it is to be stressed again that H is the Hubble constant during the intermediate phase of inflation that enters the definition of the scale factor during radiation domination via Eq. 3.58. We need to identify the counterterm that absorbs the divergence exhibited in Eq. 3.80:

$$\frac{1}{8\pi G_B}\frac{3H^2}{(a/a_R)^4} = \rho^{\rm cl}_{\rm bg} + \frac{H^4(1+e^{-4\mathcal{N}_{\rm tot}})}{8\pi^2(a/a_{\rm R})^4}\left[\frac{1}{\delta_{\rm UV}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right] + \rho_{\rm ct} + \rho_{\phi,{\rm finite}}, \tag{4.35}$$

where $\rho_{\rm finite} \equiv \rho_\phi - \rho_{\phi,{\rm div}}$. We are immediately presented with a choice here – after subtracting the pole, do we proceed to (multiplicatively) renormalize Newton's constant, or (additively) renormalize the background whose unshifted value is determined by $\rho^{\rm cl}_{\rm bg}$? It turns out that this choice is rendered moot by the tadpole condition that determines the background equations of motion at any given order in $\hbar$, in that whichever choice we make will lead us to the same shifted tadpole condition. We thus proceed by reabsorbing the pole by adding a counterterm that multiplies the Ricci scalar in the Einstein Hilbert action with coefficient B defined as

$$\rho_{\rm ct} = \frac{3H^2}{(a/a_R)^4}\left(\frac{B_{-1}}{\delta_{\rm UV}} + B_0\right). \tag{4.36}$$

By assigning

$$B_{-1} = -\frac{H^2(1+e^{-4\mathcal{N}_{\text{tot}}})}{24\pi^2}, \tag{4.37}$$

one subtracts the pole contribution. By defining a scale dependent gravitational coupling as

$$\frac{1}{8\pi G_N(\mu)} = \frac{1}{8\pi G_B} - B_0 - \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\text{tot}}})\left\{1-\gamma_E+\log\left(\frac{\mu}{H}\right)\right\}, \tag{4.38}$$

we can rewrite Eq. 4.35 as

$$\frac{1}{8\pi G_N(\mu)}\frac{3H^2}{(a/a_R)^4} = \rho_{\text{bg}}^{\text{cl}} + \rho_{\phi,\text{finite}}, \tag{4.39}$$

where the finite remainder from the counterterm and $\rho_{\phi,\text{div}}$ are absorbed by the scale dependent gravitational coupling, allowing us to start imposing renormalization conditions to fix the finite parts. We do so by determining the Newtonian constant via a measurement at some energy scale μ_*, with which we can eliminate all reference to G_B and B_0 via

$$\frac{1}{8\pi G_B} = \frac{1}{8\pi G_N(\mu_*)} + B_0 + \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\text{tot}}})\left\{1-\gamma_E+\log\left(\frac{\mu_*}{H}\right)\right\}, \tag{4.40}$$

and substituting the result into Eq. 4.38 to obtain

$$\frac{1}{8\pi G_N(\mu)} = \frac{1}{8\pi G_N(\mu_*)} - \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\text{tot}}})\log\left(\frac{\mu}{\mu_*}\right). \tag{4.41}$$

Picking μ_* to be some scale where we have determined Newton's constant[11] to be $8\pi G_N(\mu_*) = M_{\text{pl}}^{-2}$ where $M_{\text{pl}} = 2.435\times 10^{18}$ GeV, we can finally express $G_N(\mu)$ as

$$8\pi G_N(\mu) = \frac{1}{M_{\text{pl}}^2}\left[1-\frac{H^2(1+e^{-4\mathcal{N}_{\text{tot}}})}{24\pi^2 M_{\text{pl}}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}, \tag{4.42}$$

which can be used to express Eq. 4.39 in its fully covariant form as

$$G_{\mu\nu} = \frac{1}{M_{\text{pl}}^2}T_{\mu\nu}^{\text{bg,shift}}\left[1-\frac{H^2(1+e^{-4\mathcal{N}_{\text{tot}}})}{24\pi^2 M_{\text{pl}}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}, \tag{4.43}$$

where the shifted background stress tensor is defined as the sum of tree level and finite contributions on the left hand side of Eq. 4.39. Several things are to be immediately noted here – foremost is the minuscule nature of the scale dependence of the

[11] Note that measuring the strength of the gravitational coupling can only be done via a Cavendish type experiment, typically done at laboratory scales where we have independent knowledge of the masses whose mutual gravitational force we can determine. This yet another manner in which gravity is distinguished among forces as the only force whose coupling strength we measure in the UV (i.e. mm scale) and run into the IR, rather than the other way around.

gravitational coupling, should we phrase it that way. We could also simply view it as a multiplicative renormalization of the background matter content that sources the expansion history[12]. Given that virtual effects from test scalar fields serve only to renormalize background quantities[13] and impart scale dependence in observables associated to other propagating DoFs (and that too, in a highly suppressed manner [84, 40, 59]), this is the furthest we can take this exercise.

The situation for GWs is more interesting. Retracing the steps above with the dimensionally regularized result for GWs on a finite duration background in Eq. 4.26, we end up with the renormalized background

$$G_{\mu\nu} = \frac{1}{M_{\rm pl}^2} T_{\mu\nu}^{\rm bg,shift}\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{12\pi^2 M_{\rm pl}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}, \tag{4.44}$$

which should be the starting point for determining any constraints on vacuum sourced primordial GWs from $N_{\rm eff}$ bounds. As reviewed in Section 2.3.2, this is in essence the question of how vacuum tensor perturbations renormalize the background expansion through $1/a^4$ contributions that mimic additional relativistic species. Unlike the case for virtual test scalars, however, GWs have a classically evolving background upon which they represent perturbations – the background geometry itself. Therefore, we have additional means to potentially measure the contributions from vacuum tensor modes.
We first reconsider the spectral density for GWs $\Omega^{\rm gw}(k,\tau)$, whose amplitude on the scale invariant plateau for sub-horizon modes is well defined, and given by

$$\Omega^{\rm gw}(k,\tau) = \frac{H^4}{16\pi^4(a/a_{\rm R})^4} \qquad ({\rm sub-horizon}) \tag{4.45}$$

as plotted in Fig. 4.1. Let us presume that we have the means to determine the ratio $H^2/M_{\rm pl}^2$ during inflation via measurement of the tensor to scalar ratio at some pivot scale via B-mode anisotropy observations, or via the measurement of the spectral density of the stochastic gravitational wave background at some fixed scale via interferometeric means, or both. In the context of Eq. 4.44, which can now be re-expressed as

$$\begin{aligned}\frac{3H^2}{(a/a_R)^4} &= \frac{1}{M_{\rm pl}^2}\left(\rho_{\rm bg}^{\rm cl} + \rho_{\rm gw,finite}\right)\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{12\pi^2 M_{\rm pl}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1} \\ &= \frac{\rho_{\rm bg}^{\rm cl}}{M_{\rm pl}^2}\left(1+\delta_{\rm r}\right)\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{12\pi^2 M_{\rm pl}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1},\end{aligned} \tag{4.46}$$

[12] This interpretation is to be preferred if one would like keep the graviton to be canonically normalized throughout all of cosmic history, something that is implicitly taken for granted for most quoted observational results.

[13] Noting that given the vanishing background energy density of the test scalar field, it contributes vanishingly to the curvature perturbation.

where $\delta_{\rm r}$ is a constant since the putative tree level background and the stochastic background of vacuum tensor modes both scale as $\propto (a/a_{\rm R})^4$. Therefore, recalling that the quantities that parametrized the background on the left hand side of Eq. 4.32 were also implicitly bare quantities (see the second step of the renormalization procedure reviewed in Section 1.2.2 for more details), one finds that the shifted tadpole condition which one obtains upon renormalization is indistinguishable from a rescaling of the scale factor normalization at reheating, or to simple shift the temperature redshift relation in an otherwise unobservable manner[14]. This result might not come with surprise after the review on the effects of including quantum corrections in Section 1.2 but leads to substantial consequences on the meaning of the BBN bounds on the amount of GWs produced by vacuum tensor perturbations. It is now evident that the bound on $N_{\rm eff}$ from CMB as presented in Section 2.3.2, cannot be used as a constraint as it is first necessary to fix the renormalization conditions trough independent observations.

4.5 Conclusions

In this chapter we analyzed the physical example of vacuum tensor perturbations and showed that divergences in primordial observables are not something cosmologists have the luxury of ignoring: every cosmological tracer corresponding to a density fluctuation samples and convolves the coincident limit of a bilinear field, necessitating subtraction.

Merely regularizing, however, is not enough. As reviewed in Section 1.2.2, the process of arriving at a physical observable is incomplete unless one follows through by constructing the requisite counterterms and fixing any finite contributions that could accompany any subtraction via the imposition of renormalization conditions. Failure to do so runs the risk of drawing unphysical conclusions that include scheme (e.g. cutoff) dependence in observables where there should not be any, or over-interpreting contributions to physical observables that are absorbed or otherwise accounted for in the process of renormalization. To carry out the renormalization of the energy density of GWs, we needed to improve the definition of the stress tensor for GWs to obtain a result that does not presume a prior scale separation in the context of cosmology. We were then allowed to apply what we learned from the study of the scalar case, perform the integral over all wavelengths and regularize the divergences appearing in computing the energy density. After reabsorbing the divergences by adding the required counterterms, we are left with a finite amount of renormalization conditions to be imposed after measuring physical observables.

Therefore, in agreement with the review on the effects of including quantum corrections in Section 1.2, we conclude that vacuum tensor fluctuations by their very nature only serve to renormalize background quantities. As a consequence, vacuum tensor fluctuations do not enter as an additional effective light species as registered

[14] We should also stress that although we kept the scale dependent factor in Eq. 4.46 for completeness, it can effectively be set to unity given the current upper bounds on the tensor to scalar ratio r_* or about $r_* \lesssim 3\times 10^{-2}$ [67, 10, 9], so that $H^2/(8\pi^2 M_{\rm pl}^2) = \frac{r_*}{16}\Delta^2_{\mathcal{R}} \lesssim 10^{-12}$, in combination with the fact that the log of the ratio between laboratory and Hubble scales is no more than order 10^2.

by N_{eff} bounds. This should be immediately apparent from the physical nature of N_{eff} bounds as measuring the ratio of propagating light species that have undergone freeze-out relative to the entropy density of the universe, which does not apply to vacuum fluctuations. This, is of course, not true for gravitons that are physically produced by some mechanism in the early universe; however, the latter will also feature a bounded integrated spectral density.