**Sweeping vacuum gravitational waves under the rug**
Negro, A.

CHAPTER 3

# Handbook for divergences in cosmology – scalar case

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*An Étude on the Regularization and Renormalization of Divergences in Primordial Observables*
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## 3.1 Introductory remarks

This chapter is intended as a practical tour for the regularization of nominally divergent quantities on backgrounds of cosmological interest. Before studying vacuum tensor perturbations and the possibility to infer constrains from BBN bounds, we focus on the simplest possible applications one could think of, given the richness of the problems already encountered in cosmology, that of minimally coupled non-interacting test scalar fields. Moreover, we work with a formalism that should be familiar to most practicing cosmologists: two-point functions, power spectra and spectral densities, all constructed through the intermediary of mode functions derived on a FRLW slicing.

In the following, we start with a recap on how UV and IR divergences arise in the computations of standard cosmological observables. In Section 3.2, we show that UV and IR divergences naturally appear in both the stress energy tensor and the two-point function. Taking a cue from non-relativistic QED and QCD, we demonstrate how to split scaleless integrals into UV and IR contributions that would otherwise cancel in dimensional regularization. We derive the regularized power spectrum of a massless scalar field on dS regularizing both using dimensional regularization and physics momenta cutoff and we verify that the logarithmic running of physical quantities does not depend on the regularization method.
In Section 3.3, we focus on non-interacting test scalar fields and the UV log-divergent parts to make sense of results in the literature that impose hard cutoffs in physical momenta. We compare the coefficients for the UV divergent logs, regularizing both using dimensional regularization and physical cutoffs. We find identical coefficients for the power spectrum of a massless scalar field on dS background and massless and light scalar field on a quasi dS background. We then study the divergences

arising in the stress energy tensor and we point out that it is not a straightforward extension of the two-point function. We explicitly show that by using hard cutoffs, one fails to construct a counterterm that subtracts the UV divergences from background geometric invariants. Lastly, we consider the process of regularization on backgrounds that transition in and out of inflation to radiation domination. Although scales corresponding to the beginning and end of inflation can be shown to merely parametrize rather than regulate UV divergences, nominally IR divergent contributions on a dS background are cured in the examples we consider.
We conclude in Section 3.4 by sumarizing the implications of our results.

## 3.2 Regularization of stress tensors and correlation functions

All physical observations entail the exchange of energy or momentum among interacting components and detectors or tracers, whether through direct or indirect means. Correlation functions serve as useful computational tools bridging the gap between any given (effective) description valid during the primordial epoch and observables at late times, yet they are not necessarily directly observable by themselves. They can, however, be related to observations at later times by acting upon them with the appropriate derivatives to extract energy and momentum, and subsequently convolving them with transfer functions that encapsulate how they are processed by the intervening cosmological evolution (see e.g., [70] for a review).
Correlation functions are typically computed in Fourier space, whereas the observations they relate to are made at some fixed temporal and spatial location. Furthermore, correlation functions in real space involve the coincident limits of bilinear or higher point functions, necessitating subtraction of UV divergences associated with this limit. Well defined coincident limits also implicitly depend on the IR behavior of correlation functions, with the difference that any IR divergence appearing in computing physical quantities must cancel between all contributions in convolution with the transfer function (see Section 1.2.1 for more details). This must be the case for any well defined physical observables in any self-consistent calculational setup.
In this chapter we elaborate on the process of extracting these divergences, regularizing quantities of interest on different backgrounds by following the procedure reviewed in Section 1.2.2. In order to do this, in this section we establish some preliminary facts and conventions.

The energy momentum tensor for a non-interacting, minimally coupled test scalar field $\phi$ on an FRLW background is given by

$$T^{\mu}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}\delta^{\mu}_{\nu}\left(g^{\lambda\beta}\partial_{\lambda}\phi\partial_{\beta}\phi + m^2\phi^2\right), \tag{3.1}$$

from which we can extract

$$-T^{0}{}_{0} := \rho = \frac{\phi'^2}{2a^2} + \frac{(\nabla\phi)^2}{2a^2} + \frac{m^2}{2}\phi^2, \tag{3.2}$$

where we work in conformal time, and primes denote derivatives with respect to conformal time. The above is a local density that we can rewrite as the coincident limit of a bilinear form as

$$\rho := \lim_{y\to x} \rho(\tau; x, y), \tag{3.3}$$

where

$$\rho(\tau; x, y) := \frac{1}{2a^2}\left[\phi'(\tau,x)\phi'(\tau,y) + \nabla_x\phi(\tau,x)\cdot\nabla_y\phi(\tau,y) + m^2a^2\phi(\tau,x)\phi(\tau,y)\right]. \tag{3.4}$$

Given the Fourier transform convention for the corresponding field operator $\hat{\phi}$

$$\hat{\phi}(\tau,x) = \int \frac{d^3k}{(2\pi)^3}\hat{\phi}(\tau,k)e^{ik\cdot x}, \tag{3.5}$$

where the argument distinguishes the field operator in position space from its Fourier component, and the definition[1]

$$\frac{k^3}{2\pi^2}\langle\hat{\phi}(\tau,k)\hat{\phi}(\tau,k')\rangle := (2\pi)^3\delta^3(k+k')\mathcal{P}_\phi(\tau,k), \tag{3.6}$$

we can express the two point correlation function as

$$\begin{aligned}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,y)\rangle &= \int \frac{d^3k}{4\pi}\frac{\mathcal{P}_\phi(\tau,k)}{k^3}e^{ik\cdot(x-y)},\\ &= \int\frac{dk}{k}\mathcal{P}_\phi(\tau,k)\frac{\sin(kr)}{kr},\end{aligned} \tag{3.7}$$

where we have assumed statistical isotropy to perform the angular integrals, and $r := |x-y|$, and will work with the adiabatic vacuum associated with a given background in what follows. For inflationary spacetimes, this will be the usual Bunch Davies vacuum state. The coincident limit of Eq. 3.7 can be expressed as

$$\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \int_0^\infty \frac{dk}{k}\mathcal{P}_\phi(\tau,k), \tag{3.8}$$

and the coincident limit of Eq. 3.4 can similarly be expressed as

$$\lim_{y\to x}\rho(\tau;x,y) = \frac{1}{2a^2}\int_0^\infty\frac{dk}{k}\left[\left(k^2+a^2m^2\right)\mathcal{P}_\phi(\tau,k) + \mathcal{P}_{\phi'}(\tau,k)\right], \tag{3.9}$$

where we have used the shorthand

$$\frac{k^3}{2\pi^2}\langle\hat{\phi}'(\tau,k)\hat{\phi}'(\tau,k')\rangle := (2\pi)^3\delta^3(k+k')\mathcal{P}_{\phi'}(\tau,k). \tag{3.10}$$

In terms of canonically normalized mode functions $\phi_k(\tau)$ of the appropriate adiabatic vacuum, the free field operator admits the expansion

$$\hat{\phi}(\tau,k) = \hat{a}_k\phi_k(\tau) + \hat{a}^\dagger_{-k}\phi^*_k(\tau), \tag{3.11}$$

[1] Below and in the rest of what follows, the equal time expectation values are short hand for in-in correlation functions. For non-interacting test scalar fields evaluated in the adiabatic vacuum, this reduces to Eq. 3.6 and the following.

where $[\hat{a}_k, \hat{a}^\dagger_{-k'}] = (2\pi)^3\delta^3(k+k')$, so that

$$\mathcal{P}_\phi(\tau,k) = \frac{k^3}{2\pi^2}|\phi_k(\tau)|^2, \tag{3.12}$$

and so that Eq. 3.9 can be expressed as

$$\lim_{y\to x}\rho(\tau;x,y) = \frac{1}{4\pi^2 a^2}\int_0^\infty k^2\,dk\left[(k^2+a^2m^2)|\phi_k(\tau)|^2+|\phi_k'(\tau)|^2\right]. \tag{3.13}$$

On a purely dS background, the mode functions for a massless field are given by

$$|\phi_k(\tau)|^2 = \frac{H^2}{2k^3}(1+k^2\tau^2), \qquad |\phi_k'(\tau)|^2 = \frac{H^2}{2k^3}k^4\tau^2, \tag{3.14}$$

so that the late time power spectrum for a massless test scalar is exactly scale invariant with amplitude determined from the Bunch Davies vacuum as $\mathcal{P}_\phi = \left(\frac{H}{2\pi}\right)^2$, where $H$ is the Hubble factor that defines the dS background, so that Eq. 3.8 reduces to

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H}{2\pi}\right)^2\int_0^\infty\frac{dk}{k}. \tag{3.15}$$

This above is evidently divergent in both the UV and IR. Similar qualitative divergences arise when constructing the corresponding energy-momentum tensor, as we explore in the subsequent sections. Before delving into that, we take a detour on the different approaches one might adopt to regularize expressions like the above where important caveats immediately arise.

### 3.2.1 Regularization scheme (in)dependence

Given the absence of any scale in the integrand in Eq. 3.15, the integral nominally vanishes in mass independent regularization schemes such as dimensional or zeta function regularization (see e.g. the relevant chapters of [130]). Nevertheless, in what follows we show that it is still possible to cleanly separate UV and IR divergent contributions (using techniques borrowed from matching calculations in non-relativistic QCD and QED [61, 154]), and compare these to the contributions one would have calculated in mass dependent regularization schemes, such as through imposing hard cutoffs in physical momenta. This is useful, as UV and IR divergences have distinct physical interpretations – as discussed in Section 1.2.1, whereas UV divergences are to be subtracted with local counterterms, IR divergences cancel among themselves once physical observables are computed[2].

If instead one attempts to regularize the divergences by imposing hard cutoffs in both the UV and the IR, one is immediately presented with a choice as to precisely how. For instance, Burgess et al. [64] adopts cutoffs in terms of physical momenta on the basis that since IR divergences must cancel in all physical processes, the IR

[2] With the caveat that well defined observables can be identified, which presume that some of the IR divergences in question are not signaling the invalidity of the particular background quantization.

scales associated with these must be expressed in terms of physical scales. On the other hand, Baumgart and Sundrum [41] argue that it is more practical to adopt cutoffs in terms of comoving momenta for IR divergences given that the existence of a pre-inflationary phase ought to serve as a natural regulator where the distinction becomes less relevant (in the sense that it leads to only sub-leading corrections)[3]. In what follows we aim to show that using hard cutoffs in physical momenta works particularly straightforwardly for matching with the logarithmic UV divergences that one can extract from scaleless integrals. The expected scheme dependence drops out of physical quantities once renormalization conditions are imposed consistently[4], and both hard cutoffs in physical momenta and dimensional regularization lead to straightforward identification of the requisite counterterms. Simply put, even if one finds scheme dependence for the coefficients of nominal power law divergences, we show that one finds identical coefficients for UV divergent logarithms whether one dimensionally regularizes or uses hard cutoffs in physical momenta.

### 3.2.2 UV and IR divergences in mass (in)dependent schemes

Reconsider the scaleless integral Eq. 3.15, which can formally be rewritten as

$$\int_0^\infty \frac{dk}{k} = \int_0^\infty \frac{k^3 dk}{k^4} = \frac{1}{2\pi^2}\int_{-\infty}^\infty \frac{d^4k}{k^4}, \tag{3.16}$$

where the formal manipulations that result in the above render this integral to be Euclidean by default. We see that Eq.3.16 is a specific case of the more general form [61]

$$I_D(m^2) = \int \frac{d^Dk}{(2\pi)^D}\frac{k^{2A}}{(k^2+m^2)^B}. \tag{3.17}$$

Formally, Eq. 3.17 evaluates to[5]

$$I_D(m^2) = \frac{\Gamma\left(A+\frac{D}{2}\right)\Gamma\left(B-A-\frac{D}{2}\right)}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}\right)\Gamma(B)}\left(m^2\right)^{A-B+D/2}, \tag{3.18}$$

[3]Logarithmic divergences are special in that they involve the ratios of scales for some observable at some time, where the distinction between comoving and physical cutoffs becomes irrelevant.

[4]That is to say, a suitable counterterm can be subtracted that does not invalidate the background field quantization prescription one has presumed.

[5]The integral evaluates to $\left|\frac{x^{A+D/2}}{2A+D}\,{}_2F_1[B,\,A+D/2,\,1+A+D/2,\,-x]\right|_0^\infty$ with Eq. 3.18 coming from the upper limit along with additional divergent contributions from the upper and lower limits if $A-B+D/2>0$ or if $A+D/2<0$, which are subleading to the divergences in Eq. 3.18 for integer values of $A$ and $B$ as $D\to 4$.

which permits analytic continuation to non-integer values of $D$. Re-expressing the two point function Eq. 3.15 via Eq. 3.16 as[6]

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \frac{H^2}{8\pi^4}\int_{-\infty}^{\infty}\frac{d^4k}{k^4} = \frac{H^2}{8\pi^4}\int_{-\infty}^{\infty}d^4k\left[\frac{1}{k^2(k^2+m^2)}+\frac{m^2}{k^4(k^2+m^2)}\right], \tag{3.19}$$

we see via Eq. 3.18 that this evaluates in $D = 4-\delta$ dimensions to two equal and opposite contributions of the form

$$\pm\frac{H^2}{4\pi^2}\left[\frac{1}{\delta}-\frac{1}{2}\left(\log\frac{m^2}{4\pi\mu^2}+\gamma_E-1\right)\right], \tag{3.20}$$

where $\gamma_E$ is the Euler-Mascheroni constant, and $\mu$ is some arbitrary mass scale necessitated by dimensional deformation. The sum of the two contributions can be written as

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H}{2\pi}\right)^2\left[\frac{1}{\delta_{\rm UV}}-\frac{1}{\delta_{\rm IR}}+\log\frac{\mu_{\rm UV}}{\mu_{\rm IR}}\right], \tag{3.21}$$

where we have artificially given a separate label to the dimensional deformation parameters $\delta$ and $\mu$ from the two terms to distinguish the UV and IR divergent contributions in Eq. 3.19. We do so to illustrate how dimensional regularization works in canceling two separately divergent contributions by default.

It is informative to compare this result with the outcome of imposing hard cut-offs. Whether we chose to do so in Eq. 3.15 in terms of physical momenta (where $\Lambda_{\rm UV/IR} = a\,k_{\rm UV/IR}$) or in terms of comoving momenta (where $\Lambda_{\rm UV/IR} = k_{\rm UV/IR}$) is immaterial in the context of logarithmic divergences which only sees ratios of these scales:

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H}{2\pi}\right)^2\log\left(\frac{k_{\rm UV}}{k_{\rm IR}}\right). \tag{3.22}$$

From this, we see that in spite of the fact that scaleless integrals vanish under dimensional regularization, factorizing them into a sum of scaleful integrals allows one to match the coefficients of the UV divergent logarithms in Eq. 3.21 with those in Eq. 3.22 obtained by imposing hard cutoffs in physical momenta. As we show in subsequent sections, this will be true no matter the background.

## 3.3 Non-interacting test scalar fields

In this section, we examine divergences in the two point correlation function and energy momentum tensor for non-interacting, minimally coupled massless test

[6] Dimensional regularization requires us to deform the dimension of every quantity that depends on spacetime dimension, including the mode functions on the background in question. This is especially crucial in the context of loop corrections in the presence of interactions [199, 84, 40, 59], where spuriously large loop corrections might otherwise be inferred. However, because we only consider bubble diagrams of non-interacting fields in what follows (with no external momenta), the result of doing so contributes only additional finite contributions and will be elided in what follows.

scalar fields on dS and quasi dS backgrounds, as well as that for sufficiently light massive scalar fields on a dS background. All divergent integrals turn out to be scaleless, a feature that persists even when one considers a background that transitions from a pre-inflationary epoch through to inflation and exiting to a terminal phase of radiation domination. Nevertheless, one can extract UV divergences via the techniques elaborated in section 3.2.2 and compare with what one would have obtained with a hard cutoff in physical momenta, recovering identical coefficients for the logarithmic divergences.

### 3.3.1 Regularization – (quasi) dS backgrounds

On backgrounds that deviate from dS, one has to be careful to factor in the difference in the scale factor from the pure dS form of $a(\tau) = -1/(H\tau)$. For constant but non-zero $\epsilon := -\dot{H}/H^2$, as is the case during power law inflation for instance, one has $a(\tau) = 1/[-\tau H_0]^{\nu - 1/2}$, where $\nu = \frac{3-\epsilon}{2(1-\epsilon)}$ (and where the integration constant $H_0$ corresponds to the value of $H$ at $t = 0$ in cosmological time – cf. [70]). The net result for the coincident limit of the two point function for a massless scalar field on a background corresponding to constant $\epsilon$ is given by

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H_0}{2\pi}\right)^2 \frac{\Gamma^2(\nu)}{\pi}2^{2\nu-1}\int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{3-2\nu}\left[1+\frac{k^2\tau^2}{2(\nu-1)}+...\right], \tag{3.23}$$

or more generally,

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle \sim \left(\frac{H_0}{2\pi}\right)^2 \int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{n_s-1}\left[1+\left(\frac{k}{aH_0}\right)^2\frac{a^{n_s-1}}{2-n_s}+...\right] \tag{3.24}$$

where the ellipses denote terms that vanish in the $\tau \to 0$ limit and the $\sim$ indicates a numerical pre-factor that is close to unity for $n_s \approx 1$ (and tends to it in the dS limit).

In the dS limit, we find the relatively simple expression

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H_0}{2\pi}\right)^2 \int_0^\infty \frac{dk}{k}\left[1+\left(\frac{k}{aH_0}\right)^2\right], \tag{3.25}$$

which supplements Eq. 3.15 with a sub-leading correction in the late time limit, notable relative to the analogous expression on quasi dS Eq. 3.24 in that is Eq. 3.25 exact for all times. The scaleless nature of Eq. 3.24 implies that it vanishes when regularized in any mass independent scheme. Instead, one might contemplate imposing a hard UV or IR cutoff in physical momenta by setting $k_{\mathrm{UV/IR}} = a\Lambda_{\mathrm{UV/IR}}$, and subtracting the UV divergence with the necessary counterterm. Doing so for the expression Eq. 3.25, one finds (see e.g. [64])

$$\begin{aligned}
\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle &= \left(\frac{H_0}{2\pi}\right)^2\left[\int_{a\Lambda_{\rm IR}}^{a\Lambda_{\rm UV}}\frac{dk}{k}\left(1+\left(\frac{k}{aH_0}\right)^2\right)\right.\\
&\qquad\left.-\int_{a\mu}^{a\Lambda_{\rm UV}}\frac{dk}{k}\left(1+\left(\frac{k}{aH_0}\right)^2\right)\right]\\
&= \left(\frac{H_0}{2\pi}\right)^2\left\{\log\left(\frac{\mu}{\Lambda_{\rm IR}}\right)+\frac{1}{2H_0^2}\left(\mu^2-\Lambda_{\rm IR}^2\right)\right\}, \qquad (n_s=1)
\end{aligned} \tag{3.26}$$

where $\mu$ is some arbitrary renormalization scale. We note that the counterterm that subtracts the UV divergence in Eq. 3.26 is given by

$$\text{c.t.} = \left(\frac{H_0}{2\pi}\right)^2\left[\log\left(\frac{\mu}{\Lambda_{\rm UV}}\right)+\frac{1}{2H_0^2}\left(\mu^2-\Lambda_{\rm UV}^2\right)\right], \tag{3.27}$$

corresponding to a scheme dependent renormalization of the cosmological constant[7]. That these expressions are independent of time is a consequence of having imposed a physical cutoff on a dS invariant background. The remaining IR divergence is indicative of the fact that we have not yet arrived at something that can be processed into an observable quantity. It could end up being canceled by other contributions when we compute a well defined observable, or it could herald the need for resummation in which case it will also eventually get canceled. It could also indicate that the background we are attempting quantization around is not what it seems [76], and additional physical inputs are required in order to proceed. As we will show explicitly in what follows, were we to abandon the assumption that inflation was past dS eternal, then all IR divergences cancel among themselves and are in effect regulated by the scale corresponding to the beginning of inflation.

It is informative to compare what we would have obtained if we had dimensionally regularized Eq. 3.25 instead. We first note that the scaleless nature of the integral permits us to change variables to $q := k/(aH_0)$ for any fixed time. Hence the integral we need to evaluate can be factorized as in Eq. 3.19 as:

$$\frac{H_0^2}{8\pi^4}\int_{-\infty}^{\infty}\frac{d^4q}{q^4}(1+q^2) = \frac{H_0^2}{8\pi^4}\int_{-\infty}^{\infty}d^4q\left[\frac{1}{(q^2+\widetilde{m}^2)}+\frac{1+\widetilde{m}^2}{q^2(q^2+\widetilde{m}^2)}+\frac{\widetilde{m}^2}{q^4(q^2+\widetilde{m}^2)}\right] \tag{3.28}$$

where $\widetilde{m} := \mu/H_0$ is some auxiliary dimensionless mass scale. The first and second terms above are power law and log divergent in the UV, respectively, whereas the third term is IR divergent. The UV divergences must of course, be subtracted by an appropriate counterterm. One finds, after cancellations between the contributions

[7] We note that imposing a comoving cutoff in the limits of Eq. 3.26 instead of a physical cutoff would have resulted in a time dependence in the required counterterms (at odds with the dS invariance of the vacuum state) and would have necessitated further gymnastics to arrive at any physically meaningful quantity.

from the first and second terms of Eq. 3.28, a remaining UV divergence

$$\text{c.t.} = \left(\frac{H_0}{2\pi}\right)^2 \left\{\log\left(\frac{\mu}{H_0}\right) - \frac{1}{\delta_{\rm UV}} + \frac{1}{2}\left(\gamma_E - 1 - \log 4\pi\right)\right\}, \tag{3.29}$$

which is to be compared to Eq. 3.27.

The situation for quasi dS is more complicated if one were to regularize it in a mass dependent scheme, as is evident from Eq. 3.24. Nevertheless, it is still worth elaborating upon, provided we simplify matters by further specifying that the background corresponds to that of eternal power-law inflation, defined by a constant but non-zero $\epsilon$, so that $n_s - 1 = -2\epsilon - 2\epsilon^2 + \dots$ . We will address the more realistic case of finite duration inflation next. One finds from inspection of the integrand of Eq. 3.24:

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{n_s-1}\left[1 + \left(\frac{k}{aH_0}\right)^2 \frac{a^{n_s-1}}{2-n_s}\right] \tag{3.30}$$

that the first term results

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{n_s-1} = \frac{1}{|n_s-1|}\left(\frac{a\Lambda}{H_0}\right)^{n_s-1}, \tag{3.31}$$

and the second term results

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{aH_0}\right)^{n_s+1}\frac{a^{2(n_s-1)}}{2-n_s} = \frac{a^{2(n_s-1)}}{(2-n_s)(n_s+1)}\left(\frac{\Lambda}{H_0}\right)^{n_s+1} \tag{3.32}$$

where whether $\Lambda$ is an IR or UV cutoff depends on the value of $n_s$. By rewriting Eq. 3.24 as a function of the slow roll parameter $\epsilon$

$$\lim_{\tau\to 0}\langle\hat\phi(\tau,x)\hat\phi(\tau,x)\rangle \sim \left(\frac{H_0}{2\pi}\right)^2 \int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{-2\epsilon}\left[1 + \left(\frac{k}{aH_0}\right)^2\frac{a^{-2\epsilon}}{1+2\epsilon}\right] \tag{3.33}$$

we find that in the quasi dS limit ($\epsilon \ll 1$) the first term gives a logarithmic divergent contribution

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{-2\epsilon} = \frac{1}{2\epsilon} + \log\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}} + \mathcal{O}(\epsilon) \tag{3.34}$$

and the second term a power law UV divergence

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{aH_0}\right)^{2(1-\epsilon)}\frac{a^{-4\epsilon}}{1+2\epsilon} = \frac{a^{-4\epsilon}}{2}\left(\frac{\Lambda_{\rm UV}}{H_0}\right)^2. \tag{3.35}$$

The interpretation of the IR divergences is as discussed above, while the UV divergence must be subtracted with a local counterterm if we are to obtain physically meaningful quantities. For small $\epsilon$, the counterterm is given by

$$\begin{aligned} \text{c.t.} &= \lim_{\tau\to 0}\left(\frac{H_0}{2\pi}\right)^2\left[\frac{1}{2\epsilon}+\log\frac{\mu}{\Lambda_{\rm UV}}+\frac{a^{-4\epsilon}}{2}\left(\left(\frac{\mu}{H_0}\right)^2-\left(\frac{\Lambda_{\rm UV}}{H_0}\right)^2\right)+\mathcal{O}(\epsilon)\right] \\ &= \lim_{\tau\to 0}\left(\frac{H_0}{2\pi}\right)^2\left[\frac{1}{2\epsilon}+\log\frac{\mu}{\Lambda_{\rm UV}}-\frac{a^{-4\epsilon}}{2}\frac{\Lambda_{\rm UV}^2}{H_0^2}+\mathcal{O}(\epsilon)\right] \end{aligned} \tag{3.36}$$

where again $\mu$ is some renormalization scale and given that $\Lambda_{\rm UV} \gg \mu$, we have neglected the sub-leading dependence on $\mu$ in the second equality.
We first note that the required counterterm is time dependent, which should not come as a surprise, given that the background is no longer maximally symmetric. It behooves us to elaborate on the specific form of the local counterterm that could possibly have this particular time dependence. Noting that on the specified power law inflating background, $\dot{H}, H^2 \sim a^{-2\epsilon}$, so that curvature squared invariants evaluated on this background have the required secular dependence. That is, Eq. 3.36 derives from

$$\text{c.t.} \subset \int d^4x\sqrt{-g}\left[c_1(\mu)R^2+c_2(\mu)R_{\mu\nu}R^{\mu\nu}\right], \tag{3.37}$$

which are the usual leading curvature squared counterterms encountered in the effective field theory treatment of gravity [90, 91, 92, 60].

As before, we can retrace the same computation using dimensional regularization. We first consider the separate contributions to the integrand Eq. 3.33. The logarithmic divergent contribution can be re-expressed as

$$\begin{aligned} \int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{-2\epsilon} &= \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{d^4q}{q^4}q^{-2\epsilon} \\ &= \frac{1}{2\pi^2}\int_{-\infty}^{\infty}d^4q\left[\frac{q^{-2\epsilon}}{q^2(q^2+\widetilde{m}^2)}+\frac{\widetilde{m}^2q^{-2\epsilon}}{q^4(q^2+\widetilde{m}^2)}\right], \end{aligned} \tag{3.38}$$

where we have switched to the dimensionless variable $q := k/H_0$, and similarly for $\widetilde{m} := \mu/H_0$. By isolating the UV pole, we find that the presence of the non-integer exponents eliminates the $\delta$ poles and we obtain that in the quasi dS limit the first term results [8]

$$\left(\frac{H_0}{2\pi}\right)^2\int_0^\infty\frac{dk}{k}\left(\frac{k}{H_0}\right)^{-2\epsilon} = \left(\frac{H_0}{2\pi}\right)^2\left[\frac{1}{2\epsilon}-\log\frac{\mu}{H_0}+\frac{\epsilon}{12}\left(\pi^2+24\log\frac{\mu}{H_0}\right)\right]. \tag{3.39}$$

[8]Note that we are required to take the limits in the order $\delta \to 0$ and then $\epsilon \to 0$. To do the opposite would mean to have regularized on a dS background first and then deformed the background in the hopes that no new divergences will have appeared through this process. One can show that this is not justified on general grounds [48], and is seen directly from the proliferation of higher order $\delta$ poles encountered when taking the limits in the opposite order.

The second term of Eq. 3.33 can be rewritten as

$$\begin{aligned}\frac{a^{-4\epsilon}}{1+2\epsilon}\int_0^\infty \frac{dk}{k}\left(\frac{k}{aH_0}\right)^{2(1-\epsilon)} &= \frac{a^{-4\epsilon}}{1+2\epsilon}\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{d^4q}{q^4}q^{2(1-\epsilon)} \\ &= \frac{a^{-4\epsilon}}{1+2\epsilon}\frac{1}{2\pi^2}\int_{-\infty}^{\infty}d^4q\left[\frac{q^{2(1-\epsilon)}}{q^2(q^2+\widetilde{m}^2)}+\frac{\widetilde{m}^2q^{2(1-\epsilon)}}{q^4(q^2+\widetilde{m}^2)}\right],\end{aligned} \tag{3.40}$$

where just as we did on dS space, we work with dimensionless variables $q := k/aH_0$ and $\widetilde{m} := \mu/H_0$. Using Eq. 3.18 we see that power law UV divergences cancel among themselves, and so do not necessitate any counterterms. In conclusion, we find that the counterterm in the quasi dS limit results

$$\text{c.t.} = \lim_{\tau\to 0}\left(\frac{H_0}{2\pi}\right)^2\left[\frac{1}{2\epsilon}+\log\frac{\mu}{H_0}+\mathcal{O}(\epsilon)\right]. \tag{3.41}$$

The generalization to massive fields is straightforward on dS backgrounds, where analytic expressions for the mode functions can be obtained:

$$|\phi_k(\tau)|^2 = \frac{\pi}{4}H_0^2(-\tau)^3|H^{(1)}_{\nu_m}(-k\tau)|^2, \tag{3.42}$$

where

$$\nu_m^2 := \frac{9}{4}-\frac{m^2}{H_0^2}, \tag{3.43}$$

and where $H^{(1)}_{\nu_m}$ is the corresponding Hankel function with degree $\nu_m$. Presuming $0 < m^2 \leq 9H_0^2/4$ so that $\nu_m$ is still real, the coincident limit of the two point function at late times can then be computed as

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \frac{H_0^2 2^{2\nu_m}\Gamma^2(\nu_m)}{8\pi^3}\int_0^\infty\frac{dk}{k}\left[\left(\frac{k}{aH_0}\right)^{3-2\nu_m}+\frac{(k/aH_0)^{5-2\nu_m}}{2(\nu_m-1)}\right]. \tag{3.44}$$

It is notable that in spite of the presence of the additional mass scale $m$, the integral remains scaleless[9]. Here also, we can make the change of variable to $q = k/(aH)$, so that the above can be recast as

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \frac{H_0^2}{16\pi^5}2^{2\nu_m}\Gamma^2(\nu_m)\int_{-\infty}^{\infty}\frac{d^4q}{q^4}q^{3-2\nu_m}\left[1+\frac{q^2}{2(\nu_m-1)}\right], \tag{3.45}$$

where the first thing to note is that the IR divergence encountered for a non-interacting massless scalar on dS is eliminated given that $\nu_m < 3/2$[10]. Proceeding exactly as before, we can compare the UV divergence obtained from imposing a hard

[9]This can be understood from the fact that on dS, the quantity nominally labeled mass in Eq. 3.43 is constructed from eigenvalues of the Casimir operators corresponding to conformal dimension and spin expressed in units of the dS radius, which remains the only scale in the problem.

[10]A short-lived conclusion on dS backgrounds the minute any interactions are incorporated.

cutoff in physical momenta to that obtained by factorizing the scaleless integral in dimensional regularization. In the limit $m^2/H_0^2 \ll 1$, one finds that the required counterterms are given by

$$\text{c.t.} = \frac{H_0^2}{8\pi^2}\frac{2^{2\nu_m}\Gamma^2(\nu_m)}{\pi}\left[\frac{3H_0^2}{m^2} + \log\frac{\mu}{\Lambda_{\rm UV}} - \frac{\Lambda_{\rm UV}^2}{4H_0^2} + ...\right] \qquad \text{(physical cutoff)} \tag{3.46}$$

and

$$\text{c.t.} = \frac{H_0^2}{8\pi^2}\frac{2^{2\nu_m}\Gamma^2(\nu_m)}{\pi}\left[\frac{3H_0^2}{m^2} + \log\frac{\mu}{H_0} + ...\right] \qquad \text{(dim reg)} \tag{3.47}$$

where again, one takes the limit $\delta \to 0$ to regularize the integrals before taking the limit $m^2/H_0^2 \to 0$. In both cases, the required counterterm corresponds to a renormalization of the cosmological constant.

### 3.3.2 Regularization – stress energy tensor

Having labored over the details of regularizing and renormalizing divergences for the two point function of test scalar fields, repeating the exercise for the associated stress tensor may seem a straightforward extension[11]. However, we immediately encounter a difficulty: it is not always possible to construct a counterterm that subtracts the encountered UV divergences from background geometric invariants. This appear to be the case if the regularization scheme itself does not respect this symmetry. This is not to a concern to be casually dismissed[12], as persisting with this regularization scheme could lead us to the erroneous conclusion that the background we attempted to quantize around is transmuting into something else under renormalization if not worked through to the end. It is here that dimensional and related zeta function regularization techniques distinguish themselves in theories with general covariance, as they effortlessly preserve the symmetries of the background even if the precise mechanism by which this occurs can seem non-trivial. For this reason, it is worth explicitly tracing through this process in detail[13].

Consider a minimally coupled non-interacting test scalar field on dS space in the Bunch Davies vacuum. The rotational invariance of the vacuum state allow us to write the non-vanishing components as

$$\rho = \frac{1}{4\pi^2 a^2}\int_0^\infty k^2\, dk\left[\left(k^2 + a^2m^2\right)|\phi_k(\tau)|^2 + |\phi_k'(\tau)|^2\right], \tag{3.48}$$

and

$$p = \frac{1}{4\pi^2 a^2}\int_0^\infty k^2\, dk\left[-\left(\frac{k^2}{3} + a^2m^2\right)|\phi_k(\tau)|^2 + |\phi_k'(\tau)|^2\right], \tag{3.49}$$

[11] See Section 1.2.3 for an introduction on divergences arising in computing the stress energy tensor.
[12] An issue that did not arise for the two-point function given that its nature as a scalar bilinear.
[13] Nevertheless, it remains true that the coefficients of the UV divergent logarithms would match regardless of whether one dimensionally regularizes or one imposes hard cutoffs in physical momenta.

with all other components of the energy momentum tensor vanishing. From the asymptotic forms of Eq. 3.42, one can infer that both integrals are UV divergent. The energy density in the massless limit is straightforwardly computed via Eq. 3.14, and found to be

$$\rho = \frac{H^4}{8\pi^2}\int_0^\infty \frac{dk}{k}\left[\left(\frac{k}{aH}\right)^2 + 2\left(\frac{k}{aH}\right)^4\right], \tag{3.50}$$

and the corresponding pressure is found to be

$$p = \frac{H^4}{8\pi^2}\int_0^\infty \frac{dk}{k}\left[-\frac{1}{3}\left(\frac{k}{aH}\right)^2 + 2\left(\frac{k}{aH}\right)^4\right]. \tag{3.51}$$

Despite one can regularize both components through imposing physical cutoffs, one is immediately confronted by the fact that the required counterterm for the leading divergences will not be proportional to the metric nor the Ricci tensor on the presumed dS background (which would correspond either to renormalizations of the cosmological and Newton's constants). This is not the case when one implements dimensional regularization. Although the fully covariant formalism to regularize stress tensors using mass independent schemes is largely studied in the literature (as elaborated upon in [48, 178]), it is nevertheless informative to trace through this process.

We consider the example of a massive scalar field on an FRLW background, and consider the analogs of Eqs. 3.48 and 3.49 in cosmic time where we can neglect the effects of background expansion. This is in order to be facilitate working with transparent analytic expressions that, moreover, are non-vanishing (as would be the case for a massless scalar field), with the further justification that as we are only interested in computing the relevant counterterms to subtract UV divergences and can thus be forgiven for this approximation for illustrative purposes. Details of how one dimensional regularizes energy momentum tensors for more realistic examples in a fully covariant approach can be found in e.g. [48, 178]. What follows below closely tracks the treatment of [74, 133].
Working in $D = 4 - \delta$ dimensions, where the background metric is of the FRLW form in cosmic time, one finds the following expressions for the vacuum expectation values of the stress tensor:

$$\begin{aligned}\langle \hat{T}_{00}\rangle &= \frac{\mu^{4-D}}{2a^{D-1}(2\pi)^{D-1}}\int \mathrm{d}^{D-1}k\sqrt{m^2+\frac{k^2}{a^2}}\\ \langle \hat{T}_{ii}\rangle &= \frac{\mu^{4-D}}{2a^{D-1}(2\pi)^{D-1}}\frac{1}{D-1}\int \mathrm{d}^{D-1}k\frac{k^2}{\sqrt{m^2+\frac{k^2}{a^2}}}.\end{aligned} \tag{3.52}$$

Both components have the same degree of divergence in the UV, but with different coefficients. Imposing hard cutoffs in physical momenta would necessitate a counterterm that cannot be constructed from background geometric invariants. Instead, we proceed via Eq. 3.18, with $A = 0$, $B = -\frac{1}{2}$ in the first integral above, and $A = 1$,

$B = \frac{1}{2}$ for the second integral, to find

$$\int \mathrm{d}^{D-1}k\sqrt{m^2+\frac{k^2}{a^2}} = \frac{(ma)^D}{a}\frac{(2\pi)^{D-1}}{(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}$$
$$\int \mathrm{d}^{D-1}k\frac{k^2}{\sqrt{m^2+\frac{k^2}{a^2}}} = a\,(ma)^D\frac{(2\pi)^{D-1}}{(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{1}{2}+\frac{D}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}, \quad (3.53)$$

so that the dimensionally regularized components of the stress energy tensor become

$$\begin{aligned}\langle\hat{T}_{00}\rangle &= \frac{\mu^{4-D}}{2a^{D-1}(2\pi)^{D-1}}\frac{(ma)^D}{a}\frac{(2\pi)^{D-1}}{(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}\\ &= \frac{\mu^{4-D}}{2(4\pi)^{\frac{D-1}{2}}}\frac{(ma)^D}{a^D}\frac{\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)}\\ &= \frac{\mu^4}{2(4\pi)^{\frac{D-1}{2}}}\left(\frac{m}{\mu}\right)^D\frac{\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)}, \end{aligned} \quad (3.54)$$

for the energy density, and

$$\begin{aligned}\langle\hat{T}_{ii}\rangle &= \frac{\mu^{4-D}}{2a^{D-1}(2\pi)^{D-1}}\frac{1}{D-1}a\,(ma)^D\frac{(2\pi)^{D-1}}{(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{1}{2}+\frac{D}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}\\ &= \frac{\mu^{4-D}}{2(4\pi)^{\frac{D-1}{2}}}\frac{1}{D-1}\frac{a\,(ma)^D}{a^{D-1}}\frac{\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}\\ &= -\frac{\mu^4a^2}{2(4\pi)^{\frac{D-1}{2}}}\left(\frac{m}{\mu}\right)^D\frac{\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)}, \end{aligned} \quad (3.55)$$

for the pressure components, where in the last equality we have used $\Gamma(x+1) = x\Gamma(x)$ to re-express

$$\frac{\Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)} = -\frac{D-1}{\Gamma\left(-\frac{1}{2}\right)}. \quad (3.56)$$

From this, we see that the stress tensor has a divergence of the form

$$\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{div}} = -g_{\mu\nu}\frac{m^4}{64\pi^2}\left\{\frac{2}{\delta} - \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma_E - \frac{3}{2}\right\}, \quad (3.57)$$

which can straightforwardly be subtracted by a cosmological constant-like countert-erm. For this reason, regularization schemes that preserve general covariance such as dimensional or zeta function regularization are preferred in the context of renor-malizing the stress energy tensor. In spite of this caveat, however, one can still be forgiven for using hard cutoffs in physical momenta when restricted to calculating logarithmic divergences, as one can show the result to be identical to what would have been obtained in a mass independent scheme. We will return to this point in regards the regularization of the stress tensor for GWs in the next chapter.

### 3.3.3 Regularization – finite duration inflation

Having acquainted ourselves with the basics of regularizing and renormalizing divergences for familiar quantities, one would like to generalize this to backgrounds that realistically model the universe we inhabit. In particular, one might wonder if the fact that we only obtained scaleless integrals in the previous subsections is an artifact of considering the unrealistic scenario of infinite duration inflation. We show that this situation persists even when we consider a cosmology that transitions between different epochs.

In order for inflation to have finite duration by definition, we must be considering an epoch before and after inflation, which we choose to be radiation domination in both cases. We consider the scale factor to evolve as

$$\begin{aligned} a(\tau) &= a_{\rm R}\left(2-\frac{\tau}{\tau_{\rm I}}\right)e^{-\mathcal{N}_{\rm tot}} \qquad \tau<\tau_{\rm I} \\ &= a_{\rm R}\left(\frac{\tau_{\rm I}}{\tau}\right)e^{-\mathcal{N}_{\rm tot}} \qquad \tau_{\rm I}<\tau<\tau_{\rm R} \\ &= a_{\rm R}\left(2-\frac{\tau}{\tau_{\rm R}}\right) \qquad \tau_{\rm R}<\tau \end{aligned} \tag{3.58}$$

where $a_{\rm R}$ is the scale factor at reheating, and $\tau_{\rm I}$ and $\tau_{\rm R}$ correspond to the (negative) conformal time at the start and end of inflation[14] respectively, and

$$\mathcal{N}_{\rm tot} = \log(a_{\rm R}/a_{\rm I}) = \log(\tau_{\rm I}/\tau_{\rm R}) \tag{3.59}$$

is the total amount e-folds of inflation. The domain of $\tau$ is $(-\infty,\infty)$ with inflation occurring during negative conformal time. With these definitions, the Hubble rate during inflation is given by

$$H = -\frac{1}{a_{\rm R}\tau_{\rm R}}. \tag{3.60}$$

In what follows, we keep the normalization $a_{\rm R}$ arbitrary, although one can readily set $a_{\rm R}\equiv 1$ for convenience in what follows. Here again, we focus on a massless, minimally coupled non-interacting scalar field for illustrative purposes. The mode functions during the terminal stage of radiation domination can be rewritten for the purposes of a matching to the end of inflation as

$$\begin{aligned} \phi_k^{\rm RD} &= \frac{1}{a}\frac{1}{\sqrt{2k}}\left[\alpha_k^{\rm R}e^{-ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)} + \beta_k^{\rm R}e^{ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\right] \\ \phi_k'^{\mathcal{RD}} &= \frac{a_{\rm R}}{a^2\tau_{\rm R}\sqrt{2k}}\left[\alpha_k^{\rm R}e^{-ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1-i\frac{ak\tau_{\rm R}}{a_{\rm R}}\right) + \beta_k^{\rm R}e^{ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1+i\frac{ak\tau_{\rm R}}{a_{\rm R}}\right)\right]. \end{aligned} \tag{3.61}$$

[14] Note that this puts the initial singularity at $\tau = 2\tau_{\rm I}$, however, the only manner in which the pre-inflationary phase bears on late time observables is through the choice to begin with the adiabatic vacuum evolved to the start of the inflationary epoch at $\tau_{\rm I}$.

The mode functions during inflation are given by

$$\begin{aligned} \phi_k^{\mathrm{I}} &= \frac{H}{\sqrt{2k^3}}\left[\alpha_k^{\mathrm{I}} e^{i\frac{k}{aH}}\left(1-\frac{ik}{aH}\right)+\beta_k^{\mathrm{I}} e^{-i\frac{k}{aH}}\left(1+\frac{ik}{aH}\right)\right] \\ \phi_k^{\prime\mathcal{I}} &= -\frac{H}{\sqrt{2k^3}}\left[\alpha_k^{\mathrm{I}} e^{i\frac{k}{aH}}\frac{k^2}{aH}+\beta_k^{\mathrm{I}} e^{-i\frac{k}{aH}}\frac{k^2}{aH}\right], \end{aligned} \tag{3.62}$$

where the presence of non-trivial Bogoliubov coefficients come from having matched to a radiation dominated pre-inflationary phase initiated in the adiabatic vacuum, where the mode functions are given by

$$\begin{aligned} \phi_k^{\mathrm{PI}} &= \frac{1}{a}\frac{1}{\sqrt{2k}}e^{-ik\tau_{\mathrm{I}}\left(2-\frac{a}{a_{\mathrm{R}}}e^{\mathcal{N}_{\mathrm{tot}}}\right)} \\ \phi_k^{\prime\mathcal{PI}} &= -\frac{1}{a^2\sqrt{2k}}\frac{a_{\mathrm{I}}}{\tau_{\mathrm{I}}}e^{-ik\tau_{\mathrm{I}}\left(2-\frac{a}{a_{\mathrm{R}}}e^{\mathcal{N}_{\mathrm{tot}}}\right)}\left(-1+ik\tau_{\mathrm{I}}\frac{a}{a_{\mathrm{I}}}\right). \end{aligned} \tag{3.63}$$

One first obtains $\alpha_k^{\mathrm{I}}$ and $\beta_k^{\mathrm{I}}$ by matching to the pre-inflationary RD epoch, so that

$$\begin{aligned} \alpha_k^{\mathrm{I}} &= \left(i-i\frac{a_{\mathrm{I}}^2H^2}{2k^2}+\frac{a_{\mathrm{I}}H}{k}\right) \\ \beta_k^{\mathrm{I}} &= i\frac{a_{\mathrm{I}}^2H^2}{2k^2}e^{2i\frac{k}{a_{\mathrm{I}}H}}. \end{aligned} \tag{3.64}$$

Similarly, after the matching between inflation and RD era, one finds that $\alpha_k^{\mathrm{R}}$ and $\beta_k^{\mathrm{R}}$ are given by

$$\begin{aligned} \alpha_k^{\mathrm{R}} &= \alpha_k^{\mathrm{I}}\left(-i+\frac{a_{\mathrm{R}}H}{k}+i\frac{a_{\mathrm{R}}^2H^2}{2k^2}\right)+i\beta_k^{\mathrm{I}}\frac{a_{\mathrm{R}}^2H^2}{2k^2}e^{-2i\frac{k}{a_{\mathrm{R}}H}} \\ \beta_k^{\mathrm{R}} &= -i\alpha_k^{\mathrm{I}}\frac{a_{\mathrm{R}}^2H^2}{2k^2}e^{2i\frac{k}{a_{\mathrm{R}}H}}+\beta_k^{\mathrm{I}}\left(i+\frac{a_{\mathrm{R}}H}{k}-i\frac{a_{\mathrm{R}}^2H^2}{2k^2}\right). \end{aligned} \tag{3.65}$$

We now use these results to find expressions for the coincident limits of the two point function and $\rho$ during the terminal stage of radiation domination. The two point correlation function coincidence limit can be expressed as

$$\begin{aligned} \lim_{x\to y}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,y)\rangle &= \frac{1}{2\pi^2a^2}\int_0^\infty\frac{dk}{k}\frac{k^2}{2}\left[1+2|\beta_k^{\mathrm{R}}|^2\right. \\ &\quad \left.+\alpha_k^{\mathrm{R}}\beta_k^{\mathrm{R}*}e^{\frac{2ik}{a_{\mathrm{R}}H}\left(2-\frac{a}{a_{\mathrm{R}}}\right)}+\alpha_k^{\mathrm{R}*}\beta_k^{\mathrm{R}}e^{-\frac{2ik}{a_{\mathrm{R}}H}\left(2-\frac{a}{a_{\mathrm{R}}}\right)}\right] \\ &= \frac{1}{2\pi^2a^2}\int_0^\infty\frac{dk}{k}\frac{k^2}{2}\left[1+2|\beta_k^{\mathrm{R}}|^2_{\mathrm{power}}+\{\mathrm{osc}\}\right] \end{aligned} \tag{3.66}$$

where the first line has used $|\alpha_k^{\mathrm{I}}|^2-|\beta_k^{\mathrm{I}}|^2=|\alpha_k^{\mathrm{R}}|^2-|\beta_k^{\mathrm{R}}|^2=1$, and the second line splits the integrand into strictly power law contributions and oscillatory contributions. From Eqs. 3.64 and 3.65, we find that

$$|\beta_k^{\mathrm{R}}|^2_{\mathrm{power}}=\frac{a_{\mathrm{R}}^4H^4+a_{\mathrm{I}}^4H^4}{4k^4}+\frac{a_{\mathrm{I}}^4a_{\mathrm{R}}^4H^8}{8k^8}, \tag{3.67}$$

which nominally appears to have aggravated the IR divergence of the two point function. Although the oscillatory terms can indeed be neglected for the purposes of extracting UV divergences, the oscillations freeze in the IR and exactly cancel the contributions from $|\beta_k^{\rm R}|^2_{\rm power}$. Specifically, one finds that

$$\lim_{k\to 0}\{{\rm osc}\} = -\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4} - \frac{a_{\rm I}^4 a_{\rm R}^4 H^8}{4k^8} - 1 + \frac{(3a_{\rm I}^3 a_{\rm R} + 2a(a_{\rm R}^3 - a_{\rm I}^3))^2}{9a_{\rm I}^2 a_{\rm R}^6}, \quad (3.68)$$

which cancels the contributions from $1 + 2|\beta_k^{\rm R}|^2_{\rm power}$. The UV divergence resulting from the final term of Eq. 3.68 to reckon with:

$$\lim_{x\to y}\langle\hat\phi(\tau,x)\hat\phi(\tau,y)\rangle_{\rm div} = e^{-4\mathcal{N}_{\rm tot}}\left(3 + \frac{2a}{a_{\rm R}}\left[e^{3\mathcal{N}_{\rm tot}-1}\right]\right)^2 \frac{1}{36\pi^2 a^2}\int_0^\infty dk\, k, \quad (3.69)$$

where as before, $e^{-\mathcal{N}_{\rm tot}} = a_{\rm I}/a_{\rm R}$. Two things are to be immediately noted. Firstly, that the physical UV and IR scales $k_{\rm IR/UV} = a_{\rm I/R}H$ associated with the beginning and end of inflation do not by themselves regulate the UV divergences. Instead, they merely parametrize the divergence through their ratio $k_{\rm IR}/k_{\rm UV} = e^{-\mathcal{N}_{\rm tot}}$ in the pre-factor above. Furthermore, the pre-factor itself diverges as $\mathcal{N}_{\rm tot}\to\infty$, indicative of the restoration of the divergences arising considering a past infinite dS space in that limit, where, moreover, a logarithmic IR divergence reappears. This is perhaps best illustrated by Fig. 3.1 where we plot all contributions of the power spectrum for finite duration inflation (identified through the logarithmic integrand of Eq. 3.66) along with what would have resulted for a past-infinite dS cosmology matched to a terminal phase of radiation domination[15]. The UV divergence is unsurprisingly unchanged, and can be subtracted with a cosmological constant counterterm.

In computing the energy density $\rho$, we begin with

$$\lim_{y\to x}\rho(\tau;x,y) = \frac{1}{4\pi^2 a^2}\int_0^\infty k^2\, dk\left[k^2|\phi_k^{\rm RD}(\tau)|^2 + |\phi_k^{\rm RD\prime}(\tau)|^2\right] \quad (3.70)$$

and use Eq. 3.61 to express the energy density during radiation domination as

$$\begin{aligned}\rho &= \frac{1}{8\pi^2 a^4}\int_0^\infty \frac{dk}{k}\left[k^4\left(2 + \frac{a_{\rm R}^4 H^2}{a^2 k^2}\right)\left(1 + 2|\beta_k^{\rm R}|^2_{\rm power}\right) + \{{\rm osc}\}\right]\\ &:= \int_0^\infty \frac{dk}{k}\left[\Omega^\phi_{\rm power}(k) + \Omega^\phi_{\rm osc}(k)\right] \quad (3.71)\end{aligned}$$

where the integrated contribution of $\Omega^\phi_{\rm osc}$, defined from $\rho_{\rm osc}$ that includes the oscillatory contributions coming from $\alpha_k^{\rm R*}\beta_k^{\rm R}$ as well as from computing the modulus

[15] The limit of a pre-inflationary phase of infinite duration inflation can straightforwardly be read off from inserting Eq. 3.65 into Eq. 3.64 with $a_{\rm I}\equiv 0$.

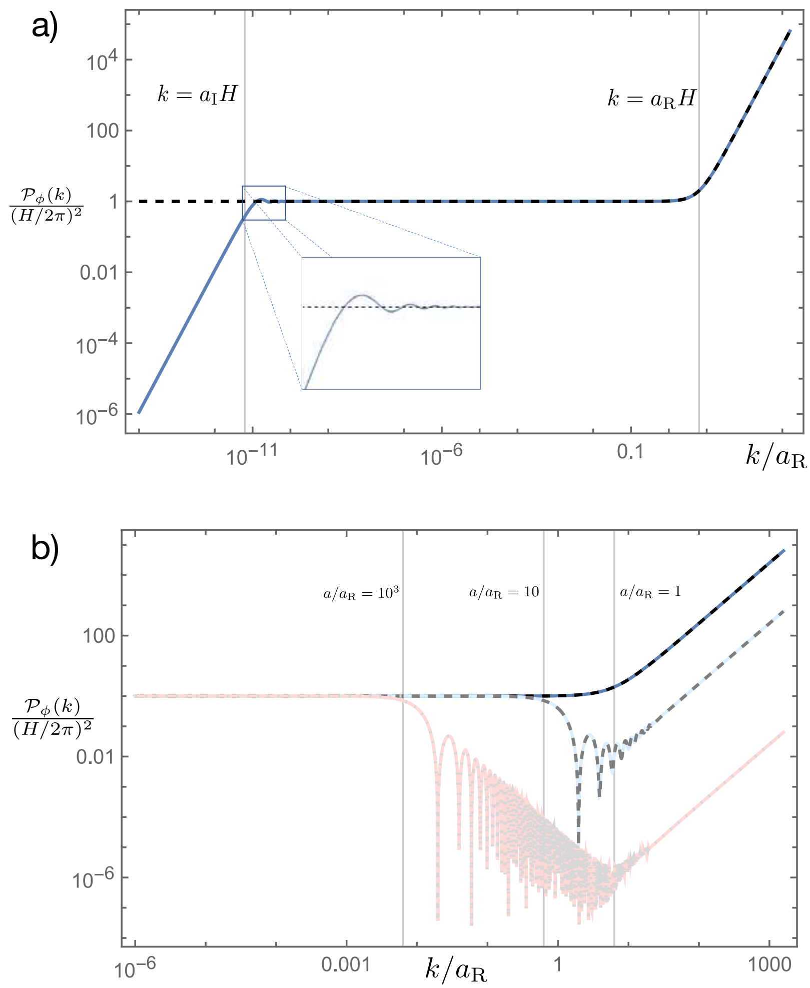


**Figure 3.1: Comparison power spectra massless test scalar field**

The graphs show the comparison between the power spectrum of a massless test scalar field with past infinite dS inflation (dashed line) vs finite duration dS inflation (bold line) matched to a terminal stage of radiation domination. Finite duration inflation cures the would be IR divergences, leaving only UV divergences to attend to.
**a)**: Power spectra evaluated at reheating $a = a_{\rm R}$, where $a_{\rm I} = 10^{-12}a_{\rm R}$ in units where $H$ is set to $2\pi$. **b)**: Power spectra evaluated at different times during radiation domination.

square of $|\beta_k^{\rm R}|^2$, is given by

$$
\begin{aligned}
\rho_{\rm osc} =& \frac{1}{4\pi^2 a^4}\int_0^\infty dk k^2\left(k+\frac{a_{\rm R}^4 H^2}{a^2 2k}\right)\left[-i\alpha_k^{\rm I}\beta_k^{\rm I*}e^{2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2}\left(-i+\frac{a_{\rm R}H}{k}+i\frac{a_{\rm R}^2H^2}{2k^2}\right)\right.\\
&\left.+i\alpha_k^{\rm I*}\beta_k^{\rm I}e^{-2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2}\left(i+\frac{a_{\rm R}H}{k}-i\frac{a_{\rm R}^2H^2}{2k^2}\right)\right]\\
&+\frac{1}{4\pi^2a^4}\int_0^\infty dk\, k^2\left[\alpha_k^{\rm R}\beta_k^{\rm R*}e^{-2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(\frac{k}{2}+\frac{a_{\rm R}^4H^2}{a^2 2k}\left(1+i\frac{ka}{a_{\rm R}^2H}\right)^2\right)\right.\\
&\left.+\alpha_k^{\rm R*}\beta^{\rm R}e^{2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(\frac{k}{2}+\frac{a_{\rm R}^4H^2}{a^2 2k}\left(1-i\frac{ka}{a_{\rm R}^2H}\right)^2\right)\right].
\end{aligned}
\tag{3.72}
$$

Making use of Eqs. 3.64 and 3.65, one can integrate the various terms and expand the result in the limit $k\to\infty$, upon which we obtain

$$
\begin{aligned}
\rho_{\rm osc} =& \frac{1}{4\pi^2a^4}\lim_{k\to\infty}\left[\left(e^{\left(-\frac{2ik(a_{\rm I}-a_{\rm R})}{Ha_{\rm R}^2}\right)}+e^{\left(\frac{2ik(a_{\rm I}-a_{\rm R})}{Ha_{\rm R}^2}\right)}\right)\left(-\frac{H^4a_{\rm R}^6}{4a^2-4aa_{\rm R}}\right)\right.\\
&\left.+\left(e^{\left(-\frac{2ik(aa_{\rm I}-2a_{\rm I}a_{\rm R}+a_{\rm R}^2)}{Ha_{\rm I}a_{\rm R}^2}\right)}+e^{\left(\frac{2ik(aa_{\rm I}2a_{\rm I}a_{\rm R}+a_{\rm R}^2)}{Ha_{\rm I}a_{\rm R}^2}\right)}\right)\left(\frac{H^4a_{\rm I}^3a_{\rm R}^4}{4a\left(aa_{\rm I}-2a_{\rm I}a_{\rm R}+a_{\rm R}^2\right)}\right)\right]
\end{aligned}
\tag{3.73}
$$

up to terms of order $\sim\frac{1}{k}$. We note that this contribution is manifestly finite, and oscillatory in the upper limit in a manner that would be eliminated via the analog of the $i\epsilon$ prescription for the Minkowski space limit of all the two point propagators in the in-in formalism. Then, as was the case with the two point function, the oscillatory contributions are negligible in the UV and can be neglected in the UV-regularization. On the other hand, they become relevant when the oscillations freeze as $k\to 0$, also softening the IR behavior for the spectral power density relative to what would have been on a past infinite dS background. Indeed, the contributions form Eq. 3.67 that nominally appear to aggravate IR divergences in the context of Eq. 3.71 are canceled by contributions from the oscillatory terms that freeze out in the IR. One can expand the sum of all contributions in the IR to find the IR safe scaling $\Omega_{\rm tot}^\phi\propto k^4$, to be compared to what one would have obtained if there had been no pre-inflationary phase, where $\Omega_{\rm tot}^\phi\propto k^2$. We illustrate this behavior and the processing of the logarithmic spectra power density as we go deeper into the radiation dominated regime in Fig. 3.2. The sub-horizon decay compensates for the $k^2$ scaling for the modes that exited the Hubble horizon during inflation to produce a scale invariant spectral density during radiation domination. We further note the additional long wavelength suppression of the spectral density for a pre-inflationary phase relative to past eternal dS.

Returning to Eq. 3.71, we see that upon inserting Eq. 3.67, the only divergences that need to be regulated are given by the contributions

$$\rho_{\rm div} = \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k\left(1+\frac{H^4 a_{\rm R}^4 + H^4 a_{\rm I}^4}{2k^4}\right) + \frac{1}{8\pi^4 a^6}\int_{-\infty}^{\infty} d^4k\frac{a_{\rm R}^4 H^2}{2k^2}, \qquad (3.74)$$

where again we recast the logarithmic integration into the formally equivalent 4D Euclidean form as per Eq. 3.16, and change integration variables to physical momenta $q = k/(a_{\rm R}H)$. We stress that all factors of $H$ in the expressions above correspond to the Hubble parameter during the intermediate phase on inflation, which is presumed dS and therefore fixed as per Eq. 3.60.

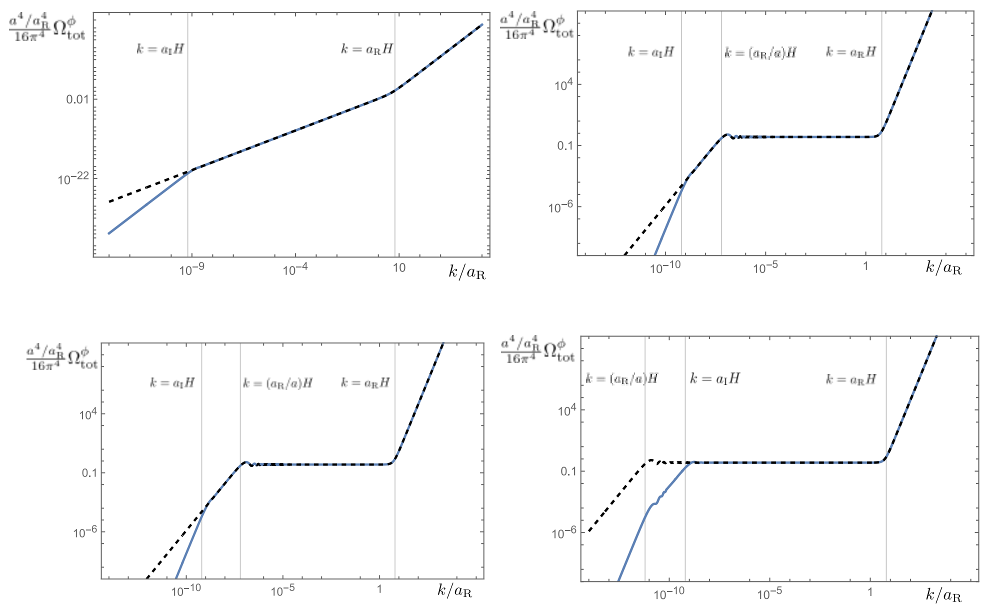


**Figure 3.2: Comparison power spectra massless test scalar field**

The graphs show the comparison between the power spectrum of a massless test scalar field with past infinite dS inflation (dashed line) vs finite duration dS inflation (bold line) matched to a terminal stage of radiation domination. The panels are evaluated at subsequent times during radiation domination, with $a_{\rm I} = 10^{-10}a_{\rm R}$, and with $a/a_{\rm R} = 1, 10^{-8}, 10^{-10}$, and $10^{-12}$, respectively, in units where $H = 2\pi$.

One can proceed from here as we did in the previous subsection in comparing the results from imposing cutoffs in physical momenta with that of dimensionally regularizing factorized scaleless integrals for the energy density (with the caveat that the counterterms are strictly speaking to be identified only through a scheme that preserves diffeomorphism invariance as per the discussion at the end of the previous subsection). By imposing cutoffs in physical momenta: $k = a\Lambda_{\rm UV}$, we see that the first term in Eq. 3.74 corresponds to a divergence of the form

$$\frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k = \frac{1}{4\pi^2 a^4}\int_0^{a\Lambda_{\rm UV}} k^3 dk = \frac{\Lambda_{\rm UV}^4}{16\pi^2}, \tag{3.75}$$

and the third term corresponds to a divergence of the form

$$\frac{1}{8\pi^4 a^6}\int_{-\infty}^{\infty} d^4k \frac{a_{\rm R}^4 H^2}{2k^2} = \frac{a_{\rm R}^4 H^2}{8\pi^2 a^6}\int_0^{a\Lambda_{\rm UV}} kdk = \frac{\Lambda_{\rm UV}^2}{16\pi^2}\frac{H^2}{(a/a_{\rm R})^4}, \tag{3.76}$$

which would nominally be subtracted by a counterterm proportional to $R$ were we to insist on hard cutoff cutoff regularization. The former, when varied with respect to the metric yields the contribution $R_{\mu\nu} \sim 1/a^4$ and thus corresponds to a renormalization of Newton's constant. Finally, the second term in Eq. 3.74 results in a divergence of the form

$$\begin{aligned}\frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4}\right) &= \frac{1}{8\pi^2 a^4}\int_{a\Lambda_{\rm IR}}^{a\Lambda_{\rm UV}} \frac{dk}{k}\left(a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4\right) \\ &= \frac{H^4(1+e^{-4\mathcal{N}_{\rm tot}})}{8\pi^2 (a/a_{\rm R})^4}\log\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}},\end{aligned} \tag{3.77}$$

where the scale $\Lambda_{\rm IR}$ does not have the same significance as before, given the IR safe behavior of the spectral density, and where the above also corresponds to a renormalization of Newton's constant. We note that the scales corresponding to the beginning and end of inflation appear in the coefficient of the logarithm, whereas the UV scale corresponding to the unknown completion the theory appears inside the logarithm.

Were we to now dimensionally regularize the divergences as advised, we can perform a similar factorization of the scaleless integrals as in the previous subsection. The nominally power law divergent parts can be isolated as

$$\begin{aligned}\rho_{\rm div}^{\rm UV} &= \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(1 + \frac{a_{\rm R}^4 H^2}{2a^2 k^2}\right) \\ &= \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left[\frac{k^2+m^2}{(k^2+m^2)} + \frac{a_{\rm R}^4 H^2}{2a^2}\left(\frac{1}{(k^2+m^2)} + \frac{m^2}{k^2(k^2+m^2)}\right)\right],\end{aligned} \tag{3.78}$$

where we see that power law UV divergences cancel among themselves, and so do not necessitate any counterterms. On the other hand, the UV divergent logarithmic term can be isolated and factorized as

$$\begin{aligned}\rho_{\rm div}^{\rm log} &= \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4}\right) \\ &= \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2}\right)\left(\frac{1}{k^2(k^2+m^2)} + \frac{m^2}{k^4(k^2+m^2)}\right).\end{aligned} \tag{3.79}$$

By isolating the logarithmic UV poles, we find

$$\rho_{\mathrm{div}}^{\log} = \frac{H^4(1+e^{-4\mathcal{N}_{\mathrm{tot}}})}{8\pi^2(a/a_{\mathrm{R}})^4}\left[\frac{1}{\delta_{\mathrm{UV}}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right] \tag{3.80}$$

from which we read off an identical coefficient as computed in Eq. 3.77[16].

## 3.4 Conclusions

We stress that the results of this chapter, even if applied to an example of pure academic interest, brought to light relevant results in the context of regularizing and renormalizing divergences in primordial observables.

We elaborated on how it is possible to extract logarithmic divergences from scaleless divergent integrals using dimensional regularization and we analyzed various simple examples to show the scheme independence of logarithmic divergences. We then showed that regularizing the divergence of just one particular component of the stress tensor (such as the energy density) is not sufficient. By separately regularizing the energy density and the pressure using physical cutoffs, the required counterterm cannot be constructed from background quantities in a particular regularization scheme. As a consequence, all subsequent conclusions will necessarily be scheme-dependent, and therefore unphysical. We then conclude that regularization schemes that preserve general covariance are preferred in this context.

Furthermore, we explicitly verified that the scales corresponding to the beginning and end of inflation must in principle be separated from any UV and IR scales parametrizing the unknown completion of our theory and well defined observable quantities. Although physical quantities cannot depend on the latter, they may certainly depend on the former. We found that all IR divergences encountered in the simplified settings we worked in were regulated by the existence of a pre-inflationary phase, and therefore were an artifact of the approximation of a past infinite dS phase. Whether this generalizes to a wider class of IR divergences, especially when higher point interactions are included, is an important question that demands to be followed up on.

Unsurprisingly, UV divergences persist for backgrounds corresponding to finite duration inflation, and the scale corresponding to the end of inflation parametrizes their coefficients rather than regulating them. Upon subtraction of these divergences with local counterterms and the imposition of renormalization conditions[17], a discussion that we postpone to the next chapter, one is able to draw meaningful physical conclusions. The two-point function and stress tensor of a test scalar field are of relatively academic interest given the assumed negligibility of the scalar field

[16] One can also show that the analogous computation for the pressure component also necessitates a counterterm corresponding to a renormalization of $G_N$ (i.e. resulting in a divergent contribution proportional to the spatial components of the Ricci tensor when varied with respect to the background metric) resulting in a renormalized stress tensor that is traceless on a radiation domination background. We do so explicitly for the case of tensor modes in Section 4.3.

[17] Which, as reviewed in Section 1.2.2, are fundamental steps to obtain meaningful results from divergences arising in including quantum corrections.

background, and therefore inability to fix renormalization conditions from observation. The analogous set of questions for vacuum tensor perturbations presents a much more interesting application given the assumption of an evolving background gravitational field (that of FRLW cosmology), perturbations around which represent GWs, which is where we will turn our attention towards next.