Sweeping vacuum gravitational waves under the rug
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Part I

Foliation specific formulation

CHAPTER 2

Possibility of constraining primordial GWs with N_{eff} bounds

2.1 Introductory remarks

In this chapter, we aim to both thoroughly explain the possibility of constraining primordial GWs with N_{eff} bounds and motivate the work described in Chapters 3 and 4. After reviewing the definition of the energy density of GWs and the link between BBN and the effective number of relativisitc species, we elaborate on the widely used relation between N_{eff} and ρ_{gw}. By doing this, we both provide the context upon which our work is structured and highlight relevant discrepancies to motivate our improvements.

In Section 2.2 we start by recalling the most relevant results in the literature upon which the formula for the energy density used to infer bounds on primordial GWs relies. We review the formula for the stress energy tensor of GWs that was first derived by Isaacson, in which GWs are assumed to be high frequency signals propagating in an effectively flat background. From this definition of the stress energy tensor, we review the way the energy density for primordial GWs at the time of CMB is determined. In doing so, we review the SVT decomposition and the most commonly used gauge-fixing in this context, the TT-gauge.
We then give in Section 2.3 more details on the process that explains the formation of light nuclei in the early universe: BBN. We study the link between BBN and the effective number of relativistic species and we explain how the latter can be constrained by CMB observations. Finally, we review how the energy density derived in the first section can be linked with the effective number of relativistic species at the time of BBN and we present how such link is used to infer constraints on the amount of primordial GWs. In reviewing the results used in the literature for the estimate of N_{eff} that include the effects of primordial GWs, we show how a hard UV cutoff appears in the derivation.
We then conclude in Section 2.4 by commenting on both the reliability of using Isaacson's stress-energy tensor in the context of primordial vacuum fluctuations and the presence of a hard UV cutoff in the final result for estimating N_{eff}.

2.2 Stress energy tensor: Isaacson's definition

The Isaacson form of the stress tensor $T_{\mu\nu}$ [124] (see also [151, 149, 54, 160]) is defined as the averaged high frequency part of the second order expansion of Einstein equations

$$T_{\mu\nu}{}^{\rm gw,hf} := -\frac{1}{8\pi G_N}\left\langle \delta^2 R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g_{\alpha\beta}\delta^2 R^{\alpha\beta}\right\rangle_{\rm BH}, \tag{2.1}$$

where $g_{\mu\nu}$ is the background metric (defined as $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ where $\tilde{g}_{\mu\nu}$ is the total metric and $h_{\mu\nu}$ is the perturbation). The averaging scheme used in [124] is the Brill-Hartle (BH) averaging scheme [54] which, as a corollary, allows one to neglect divergences, freely integrate by parts within expectation values, and commute covariant derivatives. The BH averaging scheme is well justified only if one is interested in studying GWs with a spectrum of bounded support and comprised wavelengths and periods much shorter than the background curvature scale[1]. This is a good assumption for GWs of astrophysical origin, for which one can indeed assume that the curvature scale of the background is much smaller than the frequencies of interest. Assuming that the BH averaging scheme is legitimated and using the expansions of the metric and Ricci tensor up to second order in $h_{\mu\nu}$ (see [151] for details)

$$\begin{aligned}
&\delta^1 g_{\mu\nu} = h_{\mu\nu} \qquad \delta^1 g^{\mu\nu} = -h^{\mu\nu} \qquad \delta^2 g^{\mu\nu} = h^{\mu\alpha}h^{\nu}_{\alpha}\\
&\delta^1 R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}D_\nu D_\mu h + \frac{1}{2}D^\rho D_\mu h_{\nu\rho} + \frac{1}{2}D^\rho D_\nu h_{\mu\rho}\\
&\delta^2 R_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}g^{\alpha\beta}\left[\frac{1}{2}D_\mu h_{\rho\alpha}D_\nu h_{\sigma\beta} + (D_\rho h_{\nu\alpha})(D_\sigma h_{\mu\beta} - D_\beta h_{\mu\sigma}\right)\\
&\qquad + h_{\rho\alpha}(D_\nu D_\mu h_{\sigma\beta} + D_\beta D_\sigma h_{\mu\nu} - D_\beta D_\nu h_{\mu\sigma} - D_\beta D_\mu h_{\nu\sigma})\\
&\qquad + \left(\frac{1}{2}D_\alpha h_{\rho\sigma} - D_\rho h_{\alpha\sigma}\right)(D_\nu h_{\mu\beta} + D_\mu h_{\nu\beta} - D_\beta h_{\mu\nu})\Big],
\end{aligned} \tag{2.2}$$

one obtain the simplified result

$$T^{\rm gw,hf}_{\mu\nu} = \frac{1}{32\pi G_N}\left\langle \bar{h}_{\alpha\beta;\mu}\bar{h}^{\alpha\beta}{}_{;\nu} - \frac{1}{2}\bar{h}_{;\mu}\bar{h}_{;\nu} - \bar{h}^{\alpha\beta}{}_{;\beta}\bar{h}_{\alpha\mu;\nu} - \bar{h}^{\alpha\beta}{}_{;\beta}\bar{h}_{\alpha\nu;\mu}\right\rangle_{\rm BH} \tag{2.3}$$

where we have defined

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h. \tag{2.4}$$

Eq. 2.3 is covariantly conserved and gauge-invariant up to correction terms that are negligible for all wavelengths shorter than the averaging scale.
After fixing the gauge as $D_\rho h^{\rho\nu} = 0$, $h_\mu{}^\mu = h = 0$, the resulting stress energy tensor is given by

$$T_{\mu\nu}{}^{\rm gw,hf} = \frac{1}{32\pi G_N}\langle \nabla_\mu h_{\rho\sigma}\nabla_\nu h^{\rho\sigma}\rangle_{\rm BH}. \tag{2.5}$$

[1] That is, when $\lambda, \omega^{-1} \ll 2\pi\mathcal{R}$, where $\mathcal{R}^{-2}$ is the typical magnitude of the non-vanishing components of the background Riemann tensor.

As we review in the following (see [68] for more details), the above expression is widely taken as the starting point for determining the energy density associated with stochastic backgrounds of primordial origin in terms of the primordial tensor power spectrum.
Under the assumptions of a high frequency signal (so that $\mathcal{H}$ terms can be neglected) propagating on a effectively flat background (null Riemann tensor), the EOM for the tranverse traceless part of the metric results

$$\partial_\mu \partial^\mu h_{ij}(\tau, k) = 0, \tag{2.6}$$

one can find a relation among the amplitudes

$$h'^2_{ij}(\tau, k) \sim k^2 h^2_{ij}(\tau, k) \tag{2.7}$$

and rewrite the (00)-component of Eq. (2.5) as

$$\rho_{\rm gw} = \frac{1}{32\pi G_N a^2} \delta^{im} \delta^{j\ell} \left\langle h'_{ij}(\tau, k) h'_{lm}(\tau, k) \right\rangle . \tag{2.8}$$

Focusing on the signal produced by quantum fluctuations and assuming that primordial GWs are a stochastic field, the tensor modes of the metric are quantized and the averaging $\langle ... \rangle$ in Eq. (2.8) is promoted to vacuum expectation value (see [68] for details). In this way, $\rho_{\rm gw}$ can be related to the tensor power spectrum $P_t(k)$ and computed as the integral over all the frequencies

$$\rho_{\rm gw} = \frac{1}{32\pi G_N a^2} \int d\log k \, k^2 P_t(k). \tag{2.9}$$

2.2.1 Gauge-fixing: SVT decomposition and TT-gauge

As in this section we are studying metric perturbations of a FLRW background, it is useful to work with the SVT decomposition of the metric (introduced in Section 1.1.3). We recall that the SVT decomposition is a decomposition of perturbations into components according to their transformations under spatial rotations and it allows to split the general metric perturbation $h_{\mu\nu}$ into a 3+1 splitting

$$h_{00} \equiv -2\psi, \quad h_{0i} \equiv w_i, \quad h_{ij} = 2\left(\phi g_{ij} + S_{ij}\right). \tag{2.10}$$

The vector w_i and the tensor S_{ij} are further decomposed as

$$\begin{aligned} w_i &= w_i^{\|} + w_i^{\perp} \quad \text{ where } \vec{\nabla} \times \vec{w}^{\|} = \vec{\nabla} \cdot \vec{w}^{\perp} = 0 \\ S_{ij} &= S_{ij}^{\|} + S_{ij}^{\perp} + S_{ij}^{T} \quad \text{ where } g^{jk} \nabla_k S_{ij} = g^{jk} \nabla_k S_{ij}^{\|} + g^{jk} \nabla_k S_{ij}^{\perp}. \end{aligned} \tag{2.11}$$

By rewriting $w_i^{\|}$ and $S_{ij}^{\|}$ as

$$\begin{aligned} w_i^{\|} &= \nabla_i \phi_w \\ S_{ij}^{\|} &= \left(\nabla_i \nabla_j - \frac{1}{3} \gamma_{ij} \nabla^2 \right) \phi_S \end{aligned} \tag{2.12}$$

and $S_{ij}^{\perp}$ as

$$S_{ij}^{\perp} = \nabla_i S_j^{\perp} + \nabla_j S_i^{\perp}, \tag{2.13}$$

we obtain that the perturbation $h_{\mu\nu}$, which is a symmetric 4x4 matrix and thus has 10 degrees of freedom (DoFs), can be split into 4 scalars (ψ, ϕ, ϕ_w, ϕ_S), 2 divergence-free vectors ($w_i^{\perp}$, $S_i^{\perp}$) and one transverse traceless tensor (S_{ij}^{T}). What makes the SVT-decomposition so commonly used in cosmology is the fact that it is a very convenient way of splitting of the perturbations. Indeed, at linear order the Einstein equations for scalars, vectors and tensors do not mix at linear order and can therefore be treated separately.

As GR is invariant under coordinate transformations, in order to identify the two physical DoFs of a massless spin-2 particle from the 10 DoFs of $h_{\mu\nu}$, we have to fix the gauge and get rid of the unphysical DoFs. This can be done in different ways (see Chapter 6 for an example of gauge-fixing via the Faddeev–Popov method); however, since physical observables are independent from both the choice of the gauge and the method adopted to fix the gauge, every gauge-fixing method has to lead to the same final result.
In this Section, it is convenient to adopt the so-called TT-gauge (see [68] for more details) in which the propagating DoFs are parametrized with the transverse traceless part of the metric $h_{ij}^{T} = 2S_{ij}^{T}$. In this way, we easily refer and compare to most of the results in the literature[2].

2.3 Big Bang Nucleosinthesis

As reviewed in Section 1.1.1, when the temperature of the universe was of order an $\sim$ MeV, there were no neutral atoms or bound nuclei: any atom or nucleus produced would have been immediately destroyed by the overabundance of high-energy photons. As the universe cooled well below the binding energies of typical nuclei, light elements began to form, this process, which we review in detail in this section, goes by the name of Big Bang Nucleosynthesis ([173, 190, 213]).

Nuclear binding energies are typically in the MeV range, which explains why BBN occurs at temperatures ~ 1 MeV even though nuclear masses are in the GeV range. At ~ 1 MeV the universe was radiation dominated and the cosmic plasma consists of

- Relativistic particles in equilibrium (γ, e^+ and e^-). Photons, electrons and positrons are kept in thermal equilibrium by electromagnetic interactions such as $e^+ + e^- \rightleftharpoons \gamma\gamma$.
- Decoupled relativistic particles (ν). At high temperatures neutrinos are coupled with electrons via electroweak interactions; however, as explained in more details in the following, when the temperature drops at around ~ 1 MeV, the

[2]Throughout the thesis we specify when we are working in the TT-gauge but we drop the superscript "T" in denoting the transverse traceless part of the metric.

rate of such processes drops beneath the expansion rate and neutrino decouple form the relativistic plasma.

- Non relativistic particles (n and p). According to the Big Bang model, due to the asymmetry of the initial number of baryons, the rate of baryons present in the universe remains constant throughout the expansion. As at ~ 1 MeV baryon/anti-baryon annihilation processes are very effective, all anti-baryons have been annihilated and the presence of baryons in the cosmic plasma is only due to the initial baryon/anti-baryon asymmetry.

With these ingredients, the production of primordial light elements occurs in two stages: the decoupling of the weak interactions and the sequence of nuclear reactions which build up the light nuclei. At temperatures higher than ~ 1 MeV, the following weak interactions are effective

$$\mathrm{n} + \nu_\mathrm{e} \rightleftharpoons \mathrm{p} + \mathrm{e}^-, \quad \mathrm{n} + \mathrm{e}^+ \rightleftharpoons \mathrm{p} + \nu_\mathrm{e}, \quad \mathrm{n} \rightleftharpoons \mathrm{p} + \mathrm{e}^- + \nu_\mathrm{e} \tag{2.14}$$

and the balance between neutrons and protons is maintained. Since the interaction rate of the weak interactions in Eq. 2.14 depends the temperature, as soon as the universe cools down and interaction rate falls below the expansion rate, weak interactions can no longer keep up with the expansion of the universe. At this point, neutrinos decouple from the cosmic plasma, the neutron-to-proton ratio is frozen and the second stage of nucleosynthesis can begin.
The first element produced in the BBN chain is Deuterium ($D =^2$He). Deuterium is produced through the process

$$\mathrm{n} + \mathrm{p} \rightarrow D + \gamma \tag{2.15}$$

which, due to the binding energy of the Deuterium, should become effective around right after neutrino decoupling. However, Deuterium production is slowed down by the over-density of photons and the production of light elements starts only at about 0.07 Mev. After overcoming the Deuterium bottleneck, so that the synthesis of ^{3}He becomes effective, light elements such as Helium and Lithium are produced. Heavier nuclei do not form in any significant quantity because of the absence of a stable nucleus with 8 or 5 nucleons.

2.3.1 N_{eff} bounds

The BBN bounds is one of the tight constraints that can be inferred from the CMB (see [78] for a review). As standard BBN relies on the Standard Model in fixing the prediction of the number of neutrino flavors to three, BBN constraints are often used to test models beyond the standard model by allowing the neutrino flavors to vary. In this way, BBN bounds are an important tool to test the Standard Model as the theoretical description of microphysics in the early universe. Furthermore, the comparison of the baryon density predictions from BBN and the constraints inferred by the CMB is a fundamental test for Big Bang cosmology, and its underlying assumptions (such as a homogeneity and isotropy, GR as the theory describing gravity).

From the CMB anisotropies we can quantify the amount of light elements produced in the early universe and, as reviewed above, among all the processes involved in the BBN description, such amount depends on the freeze-out temperature of neutrinos. As a consequence, by measuring the amount of light elements from the CMB, we can infer the neutrino freeze-out temperature T_f. From the latter, we then constrain the expansion rate as this is equal to the interaction rate $\Gamma_\nu(T)$ at the time of freeze-out

$$\Gamma_\nu(T_f) = H(T_f)\,. \tag{2.16}$$

Such constraint can be formulated as an upper bound for the total energy density $\rho_{\rm tot}$ at the time of BBN, as $\rho_{\rm tot}$ is related to the expansion rate of the universe through the Friedman equation

$$H^2 = \frac{8\pi G_N}{3}\rho_{\rm tot}. \tag{2.17}$$

As in the standard BBN we assume that the universe is dominated by photons at the time of BBN, only neutrinos and photons contribute to the total energy density

$$\rho_{\rm tot}^{\rm BBN} = \rho_\gamma + \rho_\nu \tag{2.18}$$

and equation above is usually written in terms of the effective number of DoFs, $N_{\rm eff}$, defined as

$$\rho_{\rm tot}^{\rm BBN} = \rho_\gamma\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}} N_{\rm eff}\right). \tag{2.19}$$

In conclusion, the so-called BBN's constraint is given as an upper bound for $N_{\rm eff}$ (see [8] for a review of Planck constraints of $N_{\rm eff}$).

2.3.2 $N_{\rm eff}$ bounds and primordial GWs

As we reviewed in Section 1.1.3, even in the context of the standard ΛCDM model in equation (2.18) we are neglecting the contribution of the stochastic GWs background produced by the primordial quantum fluctuations of inflation (see [201, 188, 115, 119] for a review). Indeed, one of the features of the stochastic GWs background is that it behaves as a free-streaming gas of massless particles, thus contributing to the total radiation energy density of the universe. Among the many cosmological mechanisms that can produce stochastic GWs backgrounds in the early universe ([17]), such as primordial black holes formation ([142]), reheating ([129]) and phase transitions ([195]), we certainly expect that the GWs produced by primordial quantum fluctuations contribute to the radiation energy density at the time of BBN.

By including the contribution of the energy density of GWs $\rho_{\rm gw}$ in equation (2.18), and using the definition for $N_{\rm eff}$ in equation (2.19), the BBN bounds for $N_{\rm eff}$ become relevant in constraining the amount of GWs produced by quantum fluctuations

$$\frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}} N_{\text{eff}} = \frac{\rho_\nu}{\rho_\gamma} + \frac{\rho_{\text{gw}}}{\rho_\gamma}. \tag{2.20}$$

To further make explicit the relation between N_{eff} and the amount of GWs produced by tensor vacuum perturbations, we follow [158]. We start from the result for the energy density of GWs reviewed in Section 2.2

$$\rho_{\text{gw}} = \frac{1}{32\pi G a^2}\delta^{im}\delta^{j\ell}\left\langle \hat{h}'_{ij}(\tau,k)\hat{h}'_{lm}(\tau,k)\right\rangle \tag{2.21}$$

where the averaging $\langle ... \rangle$ refers to computing the vacuum expectation value. By writing the two-point function of GWs during RD era in terms of the tensor power spectrum $P_t(k) = A_t\left(\frac{k}{k_*}\right)^{n_t}$

$$\left\langle \hat{h}_{ij}(\tau,\mathbf{x})\hat{h}^{ij}(\tau,\mathbf{x})\right\rangle \equiv \int d\log k P_t(k)[\mathcal{T}(\tau,k)]^2 \tag{2.22}$$

where $\mathcal{T}(\tau,k)$ is the transfer function during radiation domination ($\mathcal{T}'(\tau,k) = -kj_1(k\tau)$), ρ_{gw} is computed as the integral over all the frequencies

$$\rho_{\text{gw}} = \frac{1}{32\pi G a^2}\int d\log k P_t(k)\left[\mathcal{T}'(\tau,k)\right]^2. \tag{2.23}$$

However, the integral domain in equation (2.23) is then constrained to a limited range of frequencies ($[k_{\text{IR}}, k_{\text{UV}}]$)

$$\rho_{\text{gw}} = \frac{A_t}{32\pi G a^2}\int_{k_{\text{IR}}}^{k_{\text{UV}}} kdk\left(\frac{k}{k_*}\right)^{n_t} j_1^2(k\tau) \tag{2.24}$$

to avoid the divergences. The integral is computed by introducing a new integration variable $q = \frac{k}{k_{\text{UV}}}$ and expanding in $\tilde{\epsilon} = \frac{k_{\text{IR}}}{k_{\text{UV}}} \to 0$ to get rid of the IR cutoff. For positive n_t, the integral results

$$k_{\text{UV}}^2\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t}\int_0^1 q^{n_t+1}dq j_1^2\left(qk_{\text{UV}}\tau\right) = \left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t}\frac{1}{2n_t}\frac{1}{\tau^2} + \mathcal{O}\left(\frac{1}{k_{\text{UV}}\tau}\right). \tag{2.25}$$

Hence, for the gravitational wave energy density we find (up to corrections of order $\tilde{\epsilon}$)

$$\rho_{\text{gw}} \simeq \frac{A_t}{32\pi G_N}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t}\frac{1}{2n_t}\frac{1}{(a\tau)^2}. \tag{2.26}$$

Using equation (2.26) in equation (2.20), considering that the standard model predicts $N_{\text{eff,th}} = 3.046$ and that $\frac{1}{(a\tau)^2} = H = \frac{8\pi G}{3}\rho_{\text{tot}}$ in the RD era, we obtain

$$\frac{\rho_{\text{gw}}}{\rho_{\text{tot}}} = \frac{A_t}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t} = \frac{\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\left(N_{\text{eff}} - 3.046\right)\rho_\gamma}{\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}N_{\text{eff}}\right)\rho_\gamma}. \tag{2.27}$$

Solving the equation above for $N_{\rm eff}$, we obtain the final result used in the literature[3] to link the primordial power spectrum of GWs and the measured value of $N_{\rm eff}$ (see e.g., [158, 150, 106, 200, 53, 147, 121, 65, 146, 176, 140, 43, 44, 107])

$$N_{\rm eff} = \frac{\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\left[\frac{A_t}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right] + 3.046}{1 - \left[\frac{A_t}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right]}. \tag{2.29}$$

In order to better understand the effects of $k_{\rm UV}$, we expand Eq 2.29 in the limit in which GWs are a small fraction of radiation-like species compared with the total amount of radiation (see [158] for details) and obtain

$$N_{eff} \sim 3.046 + \left(3.046 + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\right)\left(\frac{H_*^2}{12\pi^2 M_{\rm pl}^2 n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right). \tag{2.30}$$

The ratio $\frac{k_{\rm UV}}{k_*}$ is usually estimated by considering the duration of inflation ($k_{\rm UV}$ is usually associated to the smallest scale undergoing to a super-horizon phase during inflation) and is of order $\sim 10^{20-25}$. As a consequence, as one can see from Eq. 2.30, $N_{\rm eff}$ bounds would be particularly stringent for $n_t > 0$ (blue-tilted) power spectra.

2.4 Comments and motivations for our work

Although the steps reviewed in this chapter constitute an important progress in the direction of constraining the tensor power spectrum using the measurements of $N_{\rm eff}$, a careful formal examination of these results allowed us to bring to light fundamental discrepancies.

The first discrepancy is the dependence of the observable $N_{\rm eff}$ on the regularization parameter $k_{\rm UV}$. As studied in [107], the dependence on $k_{\rm UV}$ in equation Eq. (2.29) induces a strong constraint on blue-tilted power spectra and it is largely studied to infer constraints or rule out mechanisms that produce such power spectra. However, $k_{\rm UV}$ is an arbitrary regulator used to avoid the divergences in Eq. (2.23) and cannot be considered to infer any physical analysis. Having a divergent expectation value as in Eq. (2.23), entails the need to follow the procedure reviewed in 1.2.2 and renormalize the divergences before obtaining a predictive result to be constrained by observations. Indeed, if these divergences are inconsistently subtracted, one will end up with regularization artifacts in the final answer, such as hard cut-off

[3] In some works (see [151] as an example) instead of explicating the effective number of species as in equation (2.29), the BBN's constraint results in an upper bound of the energy density in equation (2.23), written as the integral over all the frequencies

$$\int_{f=0}^{f=\infty} d(\log f) h_0^2 \Omega_{\rm gw}(f) \leq 5.6\times 10^{-6}\left(N_{\rm eff} - 3.046\right) \tag{2.28}$$

where $\Omega_{\rm gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{\rm gw}}{d\log f}$ and $0.5 < h_0 < 0.85$. In this derivation, the integral domain is equivalently constrained by inserting an UV-cutoff to avoid divergencies.

dependencies that cannot feature in any observables, which nevertheless feature in Eq. (2.29).
Furthermore, we stress that UV cutoffs cannot be confused with scales corresponding to the beginning and end of inflation. While the latter are physical scales, that can be in principle observed or enter in formulas of observables, the former is merely parametrizing our ignorance in computing the theoretical estimate of an observable[4].

Secondly, the derivation of the formula for the energy density of GWs should be modified in order to be used in the context of the CGWsB. By following [124] and defining the stress energy tensor as the high-frequency part of the second order expansion of the Einstein equations, as in equation eq. (2.1), we are assuming that the CGWsB is a high-frequency signal. This would be a realistic assumption if we were to study astrophysical sourced GWs. However, it is expected that the CGWsB is a signal produced by long-wavelength GWs (especially when compared with the horizon size at BBN). Then, Eq. (2.1) should be modified in order to include all the terms of the second order expansion of Einstein equations. More importantly, as pointed out in the previous paragraph, one must renormalize the divergences naturally appearing in computing the stress energy tensor. Thus, one cannot start from derivation that relies on a scale separation, as any computation at any loop order implicit integrates over all scales and will eventually run afoul of this approximation. Furthermore, by commuting the derivatives in the averaging procedure, we implicitly assume that the background is flat compared to the signal: to obtain Eq. (2.5) we are assuming that the background metric is Minkowski instead of FLRW. This would be again a good assumption if we were interested in studyng, for example, a signal coming from a black holes merging happening far from the detector, but it fails to be a good approssimation in the case under analysis. In [124], the averaging is introduced in order to neglect the fine details of the very high-frequency oscillations of the GWs and, due to the quantum nature of the CGWsB, this can be done by computing the expectation value. However, simply interpreting the average as a vacuum expectation value is not enough, as we have to re-introduce the effects arising from expanding around a curved background.

In Chapter 4 we will present our work and solve the discrepancies presented in this section. Furthermore, we will show that an improved formal description of the problem leads to a substantial difference in the interpretation of BBN bounds as a constraint for tensor primordial perturbations.

[4]For more details and explicit examples on the difference between UV/IR cutoffs and physical scales corresponding to the beginning and end of inflation see Sections 3.3.3 and 4.3