Sweeping vacuum gravitational waves under the rug

Negro, A.

Sweeping Vacuum Gravitational Waves Under The Rug

Proefschrift

ter verkrijging van
de graad van doctor aan de Universiteit Leiden,
op gezag van rector magnificus prof.dr.ir. H. Bijl,
volgens besluit van het college voor promoties
te verdedigen op dinsdag 1 oktober 2024
klokke 14:30 uur

door

Anna Negro

geboren te Treviso, Italië
in 1996

Promotor:
Prof.dr. A. Achúcarro

Co-promotor:
Dr. S.P. Patil

Promotiecommissie:
Dr Y. Akrami (Universidad Autonoma de Madrid)
Prof.dr. C. Burgess (McMaster University, Canada)
Prof.dr. K.E. Schalm
Dr. A. Silvestri
Prof.dr.ir. S.J. van der Molen
Prof.dr. I.B. van Vulpen

The cover shows vacuum gravitational waves propagating through the Cosmic Microwave Background (credit: ESA and the Planck Collaboration) and the first galaxy clusters (credit: NASA, ESA, CSA, and STScI). Vacuum gravitational waves are expected to have been present since the very beginning of our universe, as they are a consequence of the inclusion of primordial vacuum quantum perturbations. In turn, these are represented by the loop Feynman diagrams "swept" under the Cosmic Microwave Background "rug", in the bottom right corner of the front cover.

Glossary

n Neutron.
p Proton.
γ Photon.
ν Neutrino.
ΛCDM Standard Cosmological model.
G_N Newtonian constant.
$M_{\rm pl}$ Plank mass.
$e^{+/-}$ Positron/Electron.

BBN Big Bang Nucleosynthesis.

CMB Cosmic Microwave Background.

DoF Degree of Freedom.
dS de Sitter.

EOM Equation of Motion.

FLRW Friedmann-Lemaitre-Robertson-Walker.

GR General Relativity.
GUT Grand Unification Theory.
GWs Gravitational Waves.

IR Infrared.

LSS Large Scale Structures.

QCD Quantum Chromodynamics.
QED Quantum Electrodynamics.
QFT Quantum Field Theory.

RD Radiation Dominated.

SSB Spontaneous Symmetry Breaking.
SVT Scalar Vector Tensor (decomposition).

UV Ultraviolet.

Contents

CHAPTER 1

Introduction

All quantities that can be extracted by experimental measurements and/or observations are called *observables.* Observables, are measured at some fixed temporal and spatial location and involve either direct or indirect exchange of energy or momentum among propagating degrees of freedom and a detector. Unfortunately, however, it is not always the case that we can compute what we directly observe. What we can actually compute, instead, are *correlation functions*; weighted averages of correlations of certain quantities in space and time. Thankfully, certain observables can be expressed as the coincidence limit of correlation functions – the limit at which correlations are evaluated at the same spacetime point – and/or their derivatives. That said, in the coincidence limit of quantum field theory (QFT), correlation functions contain by construction [217], ultra-violet (UV) divergent contributions. In order to obtain well-defined coincidence limits, and by extension physical correlation functions that can be translated observation, UV divergences must be consistently reabsorbed through the renormalization procedure.

One of the most important correlation functions in physics, and especially in cosmology, is the *energy density.* The energy density describes "how much energy" there is at each point in spacetime, due to the matter fields living in it. One of the most interesting contributions to the energy density content of the primordial universe, is that of gravitational waves (GWs). Indeed, primordial GWs offer the possibility of constraining epochs that are not accessible with electromagnetic signals and, considering the recent successful developments on GWs detection, the possibility of opening this new window on the early universe is becoming more and more realistic. Although there are numerous mechanisms that can produce them, in this thesis we focus on GWs produced by the *vacuum*; they are the tensorial counterpart of primordial scalar quantum fluctuations that seeded galaxies. The reason behind this choice is that, importantly, their *renormalized* energy density, is connected to an observable, i.e., the measured abundance of the first light elements produced in the radiation dominated phase of the universe. Consequently, this link presents us with a promising pathway to constrain vacuum GWs without the need for their direct detection.

However, as we already mentioned, in order exploit such connection, the correlation function of interest, $\rho_{\rm gw}$, must be well-defined, and therefore renormalized. In contrast, and as we shall delineate in more detail in chapters that follow, the current state of the art for the energy density of vacuum GWs

$$\rho_{\rm gw} \simeq \frac{A_t}{32\pi G_N}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\frac{1}{2n_t}\frac{1}{(a\tau)^2}, \tag{1.1}$$

is not the "finished product"; a well-defined quantity we can connect to observables. Although the energy density in 1.1 may not appear to be divergent, the infinities have been swept under the the rug of an arbitrary *UV cutoff*. And even though $k_{\rm UV}$ can in fact appear in intermediate steps of the renormalization procedure (e.g., during regularization), physically observable, i.e., fully renormalized, quantities *do not* depend on such cutoffs.

In this thesis, by carefully renormalizing the divergences arising in computing $\rho_{\rm gw}$, we not only obtain a finite, regulator-independent result, but we also qualitatively change the interpretation of current constraints on vacuum GWs. Our analysis reveals that vacuum GWs, by their very nature, only serve to renormalize background quantities and do not enter as an additional effective radiation-like species. We stress that a fundamental role in reaching this conclusion is played by properly following trough the renormalization procedure, which is not a way to *sweep divergences under the rug* by introducing a hard cutoff in divergent integrals. As we will see in what follows, it is instead a well-defined recipe to absorb UV-divergent contributions and extract meaningful physical predictions.

1.1 Big Bang cosmology

This section is intended as a non-technical introduction to motivate the study of vacuum quantum tensor fluctuations, which is the main focus of this thesis. We start by presenting the chronology of the evolution of the universe to contextualize our work, we then review the foundations of the standard cosmological model and we motivate why we expect the presence of vacuum quantum fluctuations in the early stages of our universe.

1.1.1 Thermal history

The Big Bang theory describes the evolution of the universe as the cooling and expansion of the initial plasma. In its very early stages the universe was a highly dense and hot plasma made of relativistic particles forming a thermal bath. Before entering into the details of the formalism used to describe the cosmological model, we review the thermal history of the universe following Kolb and Turner in [134], focusing on the earliest stages up to the dark age.

- **Plank epoch** (Time t corresponding to $t < 10^{43}$ s, temperature T corresponding to T$> 10^{32}$ K / T$> 10^{19}$ GeV): in the Plank epoch, referring to the first instants of cosmic time, currently established laws of physics may not have applied and the evolution of the plasma is assumed to be dominated by quantum effects of gravity.

- **GUT** ($t < 10^{36}$ s, T> 10^{29} K / T> 10^{16} GeV): during this earliest period a number of spontaneous symmetry breaking (SSB) phase transitions take place, which imply that the full symmetry of the theory is broken to a lower symmetry. The first SSB corresponds to the grand unification phase transition (GUT) after which the three forces of the Standard Model are not unified anymore.

- **Inflation** ($t < 10^{32}$ s, T> 10^{22} K / T> 10^{9} GeV): the universe is then expected to go through an inflationary phase during which it expands by a factor of the order of 10^{26}. As a consequence, the universe is supercooled, a second SSB occurs and strong interactions become distinct from electroweak interactions.

- **Electroweak epoch** ($t < 10^{12}$ s, T> 10^{15} K / T> 150 GeV) and **Quark epoch** ($t < 10^{5}$ s, T> 10^{12} K / T> 150 MeV): elementary particles acquire mass via the Higgs mechanism and strongly interacting particles are present in the universe as color-singlet-quark-triplet states (baryons) and color-singlet-quark- antiquark states (mesons).

- **Hadron epoch** ($t < 1$ s, T> 10^{10} K / T> 1 MeV): quarks are bound into hadrons and a fraction of baryons survive from the baryon-antibaryon annihilation due to a slight matter-antimatter asymmetry ([214, 135]). As the universe's temperature cool down various particles decouple from the thermal bath.

- **Neutrino** (ν) **decoupling** ($t \approx 1$ s, T≈ 10^{10} K/ T≈ 1 MeV): neutrinos decouple from the cosmic thermal bath and the neutron (n) to proton (p) ratio freezes.

- **Big Bang Nucleosynthesis** (BBN) ($t \approx 10^{3}$ s, T≈ 10^{7} K / T≈ 1 keV): protons and neutrons are bound into primordial atomic nuclei such as Hydrogen, Helium and Lithium so that the first light elements are formed. The primordial abundance predictions, which are in agreement with current experimental evidence ([78]), are one of the two main successes of the Big Bang theory.

- **Lepton epoch** ($t < 10$ s, T> 10^{9} K / T> 100 keV) to **photon** (γ) **epoch** ($t < 370 \cdot 10^{3}$ years, T> 4000 K / T> 0.4 eV): during the lepton epoch the universe is hot enough to keep leptons and anti-leptons in thermal equilibrium but, as the temperature drops below the mass of the electron, high-energy photons can no longer produce electron-positron ($e^{-/+}$) pairs.

- **Recombination** ($18 \cdot 10^{3}$ years $< t < 370 \cdot 10^{3}$ years, T≈ 4000 K / T≈ 0.4 keV): light nuclei and electrons become bound to form neutral atoms. The interactions keeping photons in thermal equilibrium with the primordial plasma becomes less and less effective and the universe becomes more and more transparent.

- **Cosmic Microwave Background** (CMB) ($t \approx 370 \cdot 10^{3}$ years, T≈ 4000 K / T≈ 0.4 eV): when photons are no longer in thermal equilibrium with matter, they travel freely through space and the CMB radiation is emitted. Its experimental detection by Penzias and Wilson [181] is the second main success of the Big Bang theory.

- **Dark ages** ($t < 10^9$ years, T> 60 K): after photons decoupled from the primordial plasma, the dark age begins.

Later on, the large scale structure (LSS) of the universe began to form, followed by galaxies, culminating in the structure of the universe as it is today. The work presented in this thesis investigates the early stages of the evolution of the universe, from the initial moments of the inflationary phase to the Cosmic Microwave Background (CMB). As illustrated in Fig. 1.1, which represents the evolution of the universe, this period is just a small fraction of the Big Bang expansion. However, as we try to show in the following sections, it is enormously rich and fascinating.

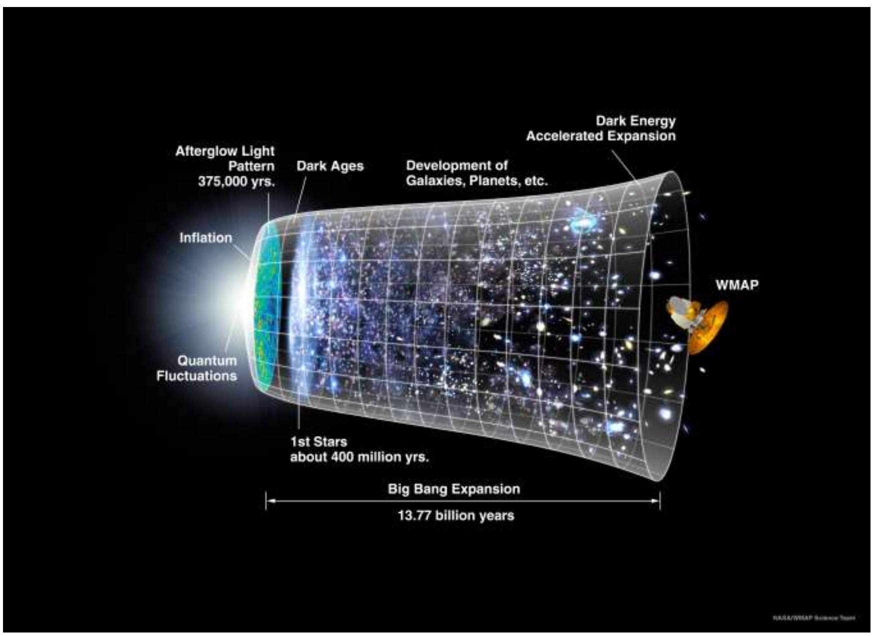


Figure 1.1: Representation of the evolution of the universe

The picture is a schematic representation of the Big Bang theory over 13.77 billion years, from left to right, inflationary evolution of the universe characterized by accelerated expansion, radiation/matter equality and dark ages.
Credit: NASA / WMAP Science Team.

1.1.2 Standard cosmological model

In this section, we aim to present the most important concepts of the standard cosmological model following the reviews by [88, 163].

The standard cosmological model is based on the cosmological principle, which can be stated as: *on sufficiently large scales the universe appears the same for all observers*, or, quoting Liddle [141]: *"the cosmological principle [means that] the universe looks the same whoever and wherever you are"*. Cosmological observations at sufficiently large scales (> 100 Mpc) validate the statement *the universe looks the same wherever you are* and, assuming that we are not special observers, validate the statement *the universe looks the same whoever you are.* This apparently simple statement implies that, at sufficiently large scales, our universe can be assumed homogeneous and isotropic. Requiring a homogeneous and isotropic solution for the Einstein equations (working in natural units in which $c = \hbar = 1$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}, \tag{1.2}$$

where $g_{\mu\nu}$ is the metric, $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar respectively, G_N is the Newtonian constant and $T_{\mu\nu}$ is the stress energy tensor that we will better define in what follows, results in the so-called Friedmann-Lemaitre-Robertson-Walker (FLRW) metric ([103, 139, 187, 212])

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j. \tag{1.3}$$

In the result for the FLRW metric in Eq. 1.3, $a(t)$ is the scale factor which encodes the time evolution of the universe and, as a result of the cosmological principle, it is a function of the cosmic time t only, x^μ are the comoving coordinates which are coordinates that move along with the overall expansion of the universe and γ_{ij} is the tensor corresponding to the spatial metric which, in spherical coordinates results[1]

$$\gamma_{ij}dx^i dx^j = \frac{dr^2}{1-\kappa r^2} + r^2 d\phi^2 + r^2 \sin\theta^2 d\theta^2 \tag{1.4}$$

where κ is the intrinsic curvature of the spatial surfaces and, modulo a rescaling the coordinates, represents flat ($\kappa = 0$), negatively curved ($\kappa = -1$) or positively curved ($\kappa = 1$) spatial slices.

To describe the dynamic of the universe, one only has to solve Eq. 1.2 for the scale factor and, to do so, it is necessary to specify the stress energy tensor $T_{\mu\nu}$ of the field sourcing the universe. To be consistent with the symmetries of the metric, $T_{\mu\nu}$ must be diagonal and the spatial components must be equal. The simplest realization of such requirements is to assume that the universe is sourced by a perfect fluid characterized by a stress energy tensor of the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{1.5}$$

where ρ is the energy density, p the pressure and u_μ is the four velocity (which is $u^\mu = (1, 0, 0, 0)$ in a comoving coordinate system). If the fluid is assumed to be

[1] Greek indices take the value $(0, 1, 2, 3)$ and Latin indices take the value $(1, 2, 3)$.

barotropic, we can express the pressure in terms of the energy density trough the equation of state

$$p = p(\rho). \tag{1.6}$$

In the standard cosmological model the equation of state is a linear relation of the form

$$p = w\rho \tag{1.7}$$

where w is a dimensionless parameter. Considering the the conservation laws associated with the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1.8}$$

we obtain that the spatial components are trivially satisfied while the zero-component results in the continuity equation

$$\dot{\rho} = -3H(\rho + p) \tag{1.9}$$

where we define $\dot{\rho} := \frac{d\rho}{dt}$ and the Hubble parameter $H(t) := \frac{\dot{a}}{a}$. Considering Eq. 1.7, the continuity equation results

$$\rho a^{3(1+w)} = \text{constant} \tag{1.10}$$

and, specifying the value of w we obtain

$$\begin{aligned} \text{MATTER :} \qquad & w = 0 \;\Rightarrow\; \rho_m a^3 = \text{constant} \\ \text{RADIATION :} \qquad & w = \frac{1}{3} \;\Rightarrow\; \rho_r a^4 = \text{constant} \\ \text{COSMOLOGICAL CONST :} \qquad & w = -1 \;\Rightarrow\; \rho_\Lambda = \text{constant}. \end{aligned} \tag{1.11}$$

In the standard ΛCDM model one assumes that the universe is mostly dominated by the so-called dark energy Λ, corresponding to $w = -1$, and cold dark matter, corresponding to $w = 0$. Before that, the early universe is assumed to be mostly composed by radiation (the so-called radiation dominated (RD) era) which follows a primordial vacuum-dominated era (inflation). By using the results of Eq. 1.11, we can express ρ and p in terms of the scale factor and straightforwardly solve the Einstein equations in Eq. 1.2.

The (00) component and any of the non-zero (ij) components of Eq. 1.2 result[2]

$$\begin{aligned} -3\frac{\ddot{a}}{a} &= 4\pi G_N(\rho + 3P) \\ \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{\kappa}{a^2} &= 4\pi G_N(\rho - P) \end{aligned} \tag{1.12}$$

and, by combining the two equations, we obtain the so-called Friedmann equations [102]

$$\begin{aligned} H^2 &= \frac{\rho}{3M_{\rm pl}^2} - \frac{\kappa}{a^2} \\ \dot{H} + H^2 &= -\frac{1}{6M_P^2}(\rho + 3p), \end{aligned} \tag{1.13}$$

[2]As a consequence of isotropy there are only two independent equations by specifying Eq. 1.2 in components.

where we define the reduced Planck mass as $M_{\rm pl} = \sqrt{\frac{1}{8\pi G_N}}$. Combining Eqs. 1.13 with the continuity equation[3] in Eq. 1.9 we can solve for the scale factor. By defining the critical density $\rho_{\rm c}$ and the parameter Ω as

$$\rho_{\rm c} := 3M_{\rm pl}^2 H_0, \quad \Omega_i := \frac{\rho_i}{\rho_{\rm c}}, \tag{1.14}$$

where we denote as H_0 the Hubble parameter at present time, we can re-write the first Friedmann equation as

$$H^2 = H_0^2 \left[\Omega_r \left(\frac{1}{a}\right)^4 + \Omega_m \left(\frac{1}{a}\right)^3 - \frac{\kappa}{H_0^2}\left(\frac{1}{a}\right)^2 + \Omega_\Lambda \right]. \tag{1.15}$$

We then consider that by CMB observations the universe appears to be flat (i.e., we assume $\kappa = 0$, [7]) to integrate Eq. 1.15 and obtain the scale factor as a function of the cosmic time t. We obtain that in the case of a RD, matter dominated and Λ-dominated universe, the scale factor results respectively

$$\dot{a} \sim a^{\frac{1}{2}(1+3w)} \Rightarrow a(t) = \begin{cases} t^{\frac{1}{2}} & \text{RADIATION} \quad (w = \frac{1}{3}) \\ t^{\frac{2}{3}} & \text{MATTER} \quad (w = 0) \\ e^{Ht} & \text{COSMOLOGICAL} \quad \text{CONST} \quad (w = -1). \end{cases} \tag{1.16}$$

For convenience, we define the conformal time τ trough the following relation $d\tau = \frac{dt}{a}$. In conformal coordinates, the FLRW metric in Eq. 1.3 can be rewritten as

$$ds^2 = a^2(\tau)(-d\tau^2 + \gamma_{ij}dx^i dx^j) = a^2(\tau)(-d\tau^2 + \delta_{ij}dx^i dx^j), \tag{1.17}$$

where in the second equality we assume $\kappa = 0$ to highlight that τ is called conformal time as the FLRW line element results conformal to the Minkowski line element, describing a static 4-dimensional hypersurface.
As in what follow we mostly use the conformal time, we recall that the relation between the Hubble parameter in cosmic time and conformal time results $H = \frac{\mathcal{H}}{a}$ and the first Friedmann equation and the continuity equation result respectively

$$\begin{aligned} \mathcal{H}^2 &= \frac{\rho a^2}{3M_{\rm pl}^2} - \kappa \\ \rho' &= -3\mathcal{H}(\rho + p), \end{aligned} \tag{1.18}$$

where we denote the derivative with respect to conformal time as $a' := \frac{da(\tau)}{d\tau}$ and the Hubble parameter in conformal time as $\mathcal{H} = \frac{a'}{a}$. Therefore, repeating the previous steps we find that in the RD era the scale factor results $a(\tau) \sim \tau$ and in the matter dominated era the scale factor results $a(\tau) \sim \tau^2$.

[3]Note that the continuity equation can be also derived by combining Friedmann equations, as a consequence of the Bianchi identities.

This apparently simple description allows us to explore many early universe phenomena with a single parameter: $a(t)$. Furthermore, as we derived how the scale factor evolves depending on the content of the universe, by knowing the values of the density parameters today (Ω_m, Ω_r and Ω_Λ), we can infer the expansion of the universe throughout its evolution. On one hand, the description reviewed in this section is successfully confirmed by the experimental evidences of CMB radiation. On the other hand, CMB brings to light some of the biggest puzzles of the modern cosmology (see [127, 144, 118, 75, 180, 175] for some examples). Assuming that the universe is homogeneous and isotropic would not account for the the formation of the structures that we see nowadays in the universe, such as galaxies, stars and planets. In the next section we review the so-called "primordial seeds": vacuum quantum perturbations.

1.1.3 Perturbation theory and vacuum perturbations

From CMB observations, as shown in Fig. 1.2, it is evident that the early universe was very nearly uniform. However, it is thanks to the word 'nearly' that we can explain our existence.

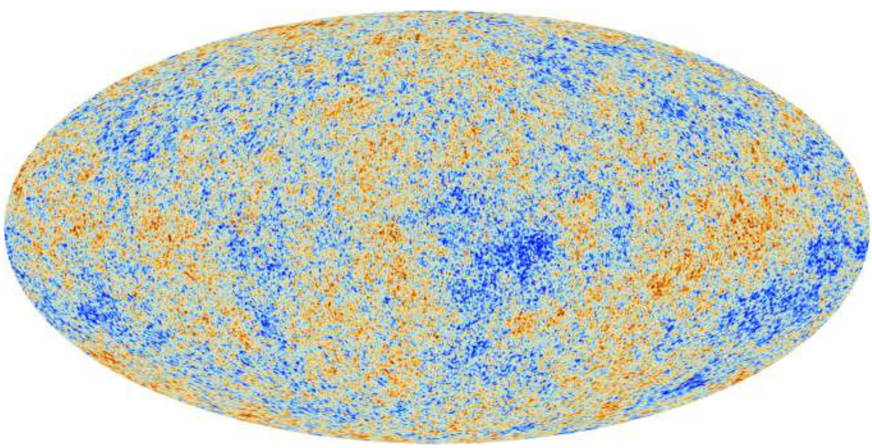


Figure 1.2: Cosmic Microwave Background

The CMB is the first picture of our universe that can be detected. The early universe (just 380,000 years old) appears surprisingly homogeneous and isotropic. Temperature fluctuations of the order of 10^{-5} are the the observational confirmation of the "primordial seeds".
Credit: ESA, Planck Collaboration.

The presence of vacuum quantum fluctuations, which are primordial departures from the homogeneous and isotropic configuration, is our current understanding of LSS formation of the universe. Such small perturbations can be treated in the context of perturbation theory up to the matter dominated era when, thanks to gravitational instabilities, some seeds grew to originate the structures that we observe ([179]). In what follows we review the so-called cosmological perturbation theory following [132, 153].

We consider a perturbation $h_{\mu\nu}$ around the FLRW solution

$$g_{\mu\nu} = g_{\mu\nu}^{\text{FLRW}} + h_{\mu\nu} \tag{1.19}$$

which, in conformal time can be rewritten as

$$g_{\mu\nu} = a^2\eta_{\mu\nu} + h_{\mu\nu}, \tag{1.20}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. As the FLRW background is under spatial rotations, one can break down the metric perturbation $h_{\mu\nu}$ into irreducible representations of Euclidian rotations[4]. This is the so-called scalar vector tensor (SVT) decomposition, first introduced by Lifshitz [143]. As a consequence, the perturbed line element can be written as

$$ds^2 = a^2\left[-(1+2\psi)d\tau^2 + 2w_i d\tau dx^i + (1+2\phi)\delta_{ij}dx^i dx^j + 2S_{ij}dx^i dx^j\right], \tag{1.21}$$

where ψ and ϕ are two scalar gravitational potentials, w_i is a vector and S_{ij} is traceless under the spatial part of the background metric.
Similarly to the metric, the stress energy tensor is decomposed as a background part $\bar{T}_{\mu\nu}$, which sources the background metric, and perturbation $\delta T_{\mu\nu}$. In this way, the components of the stress energy tensor result

$$\begin{aligned}
T_0^0 &= -\left(\bar{\rho} + \delta\rho\right), \\
T_i^0 &= \left(\bar{\rho} + \bar{p}\right)\left(v_i + w_i\right), \\
T_0^i &= \left(\bar{\rho} + \bar{p}\right)v^i i, \\
T_j^i &= \bar{p}\delta_j^i + \left(\delta p\delta_j^i + \Pi_j^i\right),
\end{aligned} \tag{1.22}$$

where background quantities $(\bar{\rho}, \bar{p})$ are functions of the time only, Π_j^i accounts for velocity gradients and v^i is the spatial perturbation of the peculiar velocity.
The last ingredient that one has to consider before tackling the study of cosmological perturbations is that, under an infinitesimal change of coordinates

$$x'^\mu = x^\mu + \xi^\mu, \tag{1.23}$$

the metric perturbation transforms as

$$h'_{\mu\nu} = h_{\mu\nu} - \bar{\nabla}_\mu\xi_\nu - \bar{\nabla}_\nu\xi_\mu \tag{1.24}$$

where $\bar{\nabla}_\nu$ is the covariant derivative built with the FLRW background metric. General relativity (GR) is invariant under change of coordinates, which implies that the change of coordinates in Eq. 1.23 does not effect observable results. As an important consequence, we obtain that the transformation in Eq. 1.24, called gauge

[4] In representation theory, this corresponds to decomposing perturbations under the group of spatial rotations.

transformation, does not have physical implications, it highlights a redundancy of our description and the need of fixing a gauge.
At this point, one can consider the perturbed line element in Eq. 1.21, the perturbed stress energy tensor in Eq. 1.22 and, being mindful of fixing the gauge, describe the evolution of primordial perturbations by deriving the perturbed Einstein equations in Eq. 1.2 (among the many references on perturbation theory and the observable consequences of including vacuum quantum perturbations, see [46, 153, 165, 32, 6, 192, 39, 5, 186, 19, 119, 105, 113] for some examples). In the next chapters we will review the details of this procedure, focusing our attention on tensor vacuum perturbations, or vacuum GWs.

1.2 Quantum corrections and infinities

What are quantum corrections? How do we quantify them? How do they correct the classical answer? The answers to these questions, that we now review, are the guideline of the work presented in this thesis.

We consider the cross section of the e^+/e^- scattering process[5] σ_e, which is a natural quantity to be measured experimentally. In the context of second quantization we can derive the Feynman diagrams contributing to the process at arbitrarily high order in $\hbar$. The diagrams of order $\hbar^0$, the so-called tree diagrams, represent the classical answer and graphs corresponding to higher orders in $\hbar$, loop diagrams, represent the quantum corrections to the classical answer (See Fig. 1.3 for examples of tree diagrams and the first order loop corrections).
Once we include loop diagrams in the estimate of σ_e, infinities arise in the short distance limit, which take the name of UV divergences. Infinities often arise when computing physical observables in any quantum or statistical field theoretic framework, and this naturally rise the question of how to deal with such apparently meaningless results. As a consequence, the renormalization procedure has been developed so that quantum corrections can be consistently included in the estimate of observables.
Following trough the renormalization procedure, one realizes that the diagrammatic splitting into tree diagrams and loop diagrams can be misleading as calculations of quantum corrections tell us nothing of loops absolute values. Even if certain properties of observables can be explained as coming uniquely from quantum effects[6], in measuring the scattering amplitude σ_e one cannot separately measure the tree level and loop corrections and, in general, all physical observations are necessarily of fully dressed quantities; as a result, the tree diagram around which one computes the quantum corrections is only to be viewed as a calculational fiction that serves as bootstrap into computing physical observables.

[5] the electron positron scattering takes the name of Bhabha scattering ([47]).
[6] It is the case of the Casimir effect ([69, 137]), anomalies ([11, 159]) or running of coupling constants ([116, 117]).

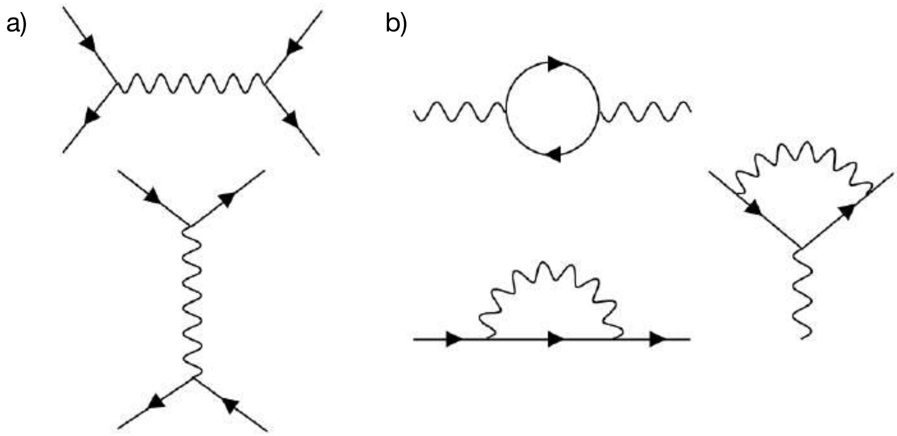


Figure 1.3: Examples tree and 1-loop Feynman diagrams in QED.

a) Examples of tree level QED Feynman diagrams in QED (s-channel and t-channel of Bhabha scattering). **b)** 1-loop QED Feynman diagrams.

1.2.1 UV and IR divergences

Before reviewing the details of the renormalization procedure and discuss the divergences arising in computing the vacuum energy, we comment on the differences between UV divergences and infra-red (IR) divergences.

Divergences can typically be categorized into two types, each with its own distinct physical meaning. On one hand, short distance or UV divergences are cured with the well-defined prescription of renormalization, which we review in the next section[7]. The regularization of UV divergences has been developed on flat backgrounds and, as we will discuss in this thesis, a transposition of this formalism to curved backgrounds is possible, albeit with numerous caveats and subtleties that one must be mindful of depending on the context [48, 178, 164]. UV divergences arise in including quantum corrections and are the result of the mathematical idealization of probing arbitrarily small scales by admitting arbitrarily high energies in any given calculation. Once these infinities have been appropriately subtracted, causality and locality prevent such idealization [205, 26] and operators can be classed by derivative expansion into relevant, marginal and irrelevant operators (see e.g. [130, 61, 155]).

On the other hand, long wavelength or IR divergences require additional care in their interpretation. At the very least, they indicate that one has additional

[7] Among the many references introducing this topic such as [125, 218, 215, 194], see the treatment of Kleinert in [130] in the context of renormalizable theories and of Burgess in [61] in the context of effective field theories.

work to do in order to arrive at a physical quantity. IR divergences can appear in intermediate expressions in theories with gapless excitations, and cancel out when computing physical observables. As an example, this is the case of soft photons in quantum electrodynamics (QED) ([49]) where, by considering that an arbitrarily large numbers of soft photons is not detected due to limited sensitivity of the experimental apparatus, one obtains an intermediate IR divergent result for certain scattering processes. However, such IR divergencies cancel when virtual soft photons are included and all the diagrams contributing to the process are summed.
IR divergences could also appear because of the breakdown of a particular perturbative scheme. This can be cured through resummation, as is the case when one studies thermalization in an out of equilibrium context [66, 45], or in quantum chromodynamics (QCD) scattering via the factorization properties of its soft and collinear limits [136, 148].
In the most problematic scenario, IR divergences arise as a consequence of the instability of the background around which one is attempting to perturb. This is particularly relevant for cosmology, as a class of IR divergences arise perturbing around de Sitter (dS) or quasi dS bacgkrounds ([101, 162, 25, 89, 20, 122, 62, 108, 174, 126, 183, 15, 16, 21, 114]) and one cannot rely on standard background field quantization methods. Large quantum corrections to the background could invalidate the split between classical background and fluctuations, which is a limiting case of the general complication of doing perturbation theory around a time evolving background that must satisfy the tadpole condition at any given order in $\hbar$. As a consequence, stochastic inflation and its background statistical field theory limit is used to tackle this problem ([202, 33, 185, 110, 168, 166, 167, 161, 189, 145, 203, 99, 95, 100, 182, 211, 29, 96, 23, 31, 209, 77]) as well as dynamical renormalization group and open effective field theory approaches [52, 64, 63].
However, as reviewed [35, 197], in situations where one can presume the validity of background field quantization, which is the assumption for this thesis, it is possible to extract meaningful results and well defined observables in spite of the appearence of IR divergences in inflationary cosmology.

In what follows we will put extra care in commenting the appearance of IR divergences; however, as we aim to focus on the study of UV divergences in early universe observables, in the next section we review the details of subtracting and renormalizing UV divergent quantities.

1.2.2 Renormalization

In this section we introduce the renormalization procedure by computing the QED vacuum polarization diagram (see [194, 218] for more details). We aim to review the recipe that leads to a finite, physical result and highlight the steps that one needs to follow in order to consistently reabsorb UV divergences arising from including quantum corrections.

We consider the Lagrangian of QED

$$\mathcal{L}_{\text{QED}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}(\partial\!\!\!/ + ieA\!\!\!/)\psi - m\bar{\psi}\psi, \tag{1.25}$$

where $F^{\mu\nu}$ is the electromagnetic tensor that, as a function of the electromagnetic potential[8] A_μ, results $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\psi/\bar{\psi}$ is are fermionic fields that represent the electron/positron, m and e represent the mass and the charge of the electron. From the Lagrangian in Eq. 1.25 one can derive the Feynman rules that allow to straightforwardly obtain the following result

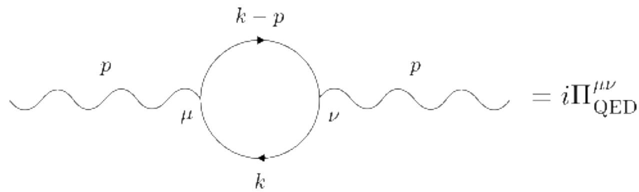


Figure 1.4: Self energy of photon

Feynman diagram representing the vacuum polarization loop diagram in QED.

$$i\Pi^{\mu\nu}_{\text{QED}} = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(i)^2\,\text{Tr}\left[\gamma^\mu(k\!\!\!/ - p\!\!\!/ + m)\gamma^\nu(k\!\!\!/ + m)\right]}{[(p^\alpha - k^\alpha)^2 - m^2 + i\epsilon][k^\alpha k_\alpha - m^2 + i\epsilon]}. \tag{1.26}$$

As a consequence, by computing the integral in Eq. 1.26 we can quantify the contribution of the loop diagram represented in the Feynman graph in Fig. 1.4. The trace of the gamma matrices results (See Appendix A of [194] for more details)

$$\text{Tr}\left[\gamma^\mu(k\!\!\!/ - p\!\!\!/ + m)\gamma^\nu(k\!\!\!/ + m)\right] = 4[-p^\mu k^\nu - k^\mu p^\nu + 2k^\mu k^\nu + \eta^{\mu\nu}(p^\alpha k_\alpha - k^\alpha k_\alpha + m^2)] \tag{1.27}$$

and, dropping the p^μ and p^ν terms as they do not contribute for symmetry reasons, we obtain

$$i\Pi^{\mu\nu}_{\text{QED}} = -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu + \eta^{\mu\nu}(p^\alpha k_\alpha - k^\alpha k_\alpha + m^2)}{[(p^\alpha - k^\alpha)^2 - m^2 + i\epsilon][k^\alpha k_\alpha - m^2 + i\epsilon]}. \tag{1.28}$$

The loop contribution in Eq. 1.28 is unsurprisingly divergent in the $k \to \infty$ limit. In what follows, we review the procedure that allows to extract a physical meaning from divergences arising in including quantum corrections.

1) Explicit the appearance of the divergent structure (regularization)

The first step is to isolate the divergent structure of Eq. 1.28 with the help of a regulator. This procedure is called regularization and can be done in different ways,

[8] We use the Feynman slash notation, where slashed quantities are covariant quantities contracted with gamma matrices: $A\!\!\!/ = A_\mu\gamma^\mu$.

using different regularization methods. A regulator is a nonphysical parameter that one inserts to factor the divergent structure and, by definition, it cannot appear in the final result.

To introduce the regularization methods that will be used in this thesis, we review the regulators of dimensional regularization, hard cutoffs and Hadamard regularization. In the case of dimensional regularization, the dimension of the measure of the integral is generalized to $D = 4 - \delta$ dimensions, δ represents the regulator and divergences are written as poles in δ: $\lim_{\delta\to 0} \frac{1}{\delta}(...)$. In the case of hard cutoffs, the regulator can be a cutoff energy scale $\Lambda_{\rm UV}$ and divergences are written as functions that diverge in the limit $\Lambda_{\rm UV} \to \infty$. Lastly, in the case of Hadamard regularization, the regulator is the geodesic distance from two points σ^μ and divergences are rewritten as poles in the coincidence limit.
To dimensionally regularize Eq. 1.28, we use the Feynman parameter trick:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B - A)x]^2} \tag{1.29}$$

and we shift the integration variable as $k^\mu \to k^\mu + p^\mu(1 - x)$, to obtain[9]

$$\Pi^{\mu\nu}_{\rm QED} = 4ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p^2x(1 - x) - m^2)}{[k^2 + p^2x(1 - x) - m^2 + i\epsilon]^2}. \tag{1.30}$$

We then generalize the measure of the integral to $D = 4 - \delta$ dimensions and, after performing a Wick rotation to recover the Euclidean signature, we use the following result (See Appendix B of [194] for more details)

$$\int \frac{d^Dk}{(2\pi)^D} \frac{(k^2)^a}{(k^2 - \Delta)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{\Delta^{b-a-\frac{D}{2}}} \frac{\Gamma\left(a + \frac{D}{2}\right)\Gamma\left(b - a - \frac{D}{2}\right)}{\Gamma(b)\Gamma\left(\frac{D}{2}\right)} \tag{1.31}$$

to obtain

$$\begin{aligned}\Pi^{\mu\nu}_{\rm QED} &= -8p^2\eta^{\mu\nu} \lim_{D\to 4} \frac{e^2\mu^{4-D}}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx x(1 - x) \left(\frac{1}{m^2 - p^2x(1 - x)}\right)^{2-\frac{D}{2}} \\ &= -\frac{e^2}{2\pi^2} p^2\eta^{\mu\nu} \lim_{\delta\to 0} \int_0^1 dx\, x(1 - x) \left[\frac{2}{\delta} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2x(1 - x)}\right) + \mathcal{O}(\delta)\right].\end{aligned} \tag{1.32}$$

In Eq 1.32 we we use $k^\mu k^\nu \to \frac{1}{D}\eta^{\mu\nu}k^2$ in D dimensions, to rewrite the $k^\mu k^\nu$ term and we define $\tilde{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E}$ where γ_E is the Euler-Mascheroni constant.
In conclusion, we can rewrite Eq. 1.28 as

$$i\Pi^{\mu\nu}_{\rm QED} = ie^2\left(-p^2g^{\mu\nu} + p^\mu p^\nu\right)\Pi_{\rm QED}\left(p^2\right), \tag{1.33}$$

where

$$\Pi_{\rm QED}\left(p^2\right) = \frac{1}{2\pi^2} \lim_{\delta\to 0} \int_0^1 dx\, x(1 - x) \left[\frac{2}{\delta} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2x(1 - x)}\right)\right]. \tag{1.34}$$

[9] Just for this section we define $p^2 = p^\alpha p_\alpha$

Considering the $p^2 \to 0$ limit, for which the integral over x is straightforward to compute, $\Pi_{\rm QED}\left(p^2\right)$ results

$$\Pi_{\rm QED}\left(0\right) = \frac{1}{12\pi^2}\lim_{\delta\to 0}\left[\frac{2}{\delta} + \ln\left(\frac{\tilde{\mu}^2}{m^2}\right)\right] \tag{1.35}$$

and we obtain, as expected, that the divergence now appears as a pole in δ and depends on an energy scale μ, introduced to have a dimensionless argument of the logarithm.

2) Renormalize the divergence by reabsorbing it in the coupling constants

The second step consists in defining the renormalized coupling constants in such a way that the divergences are reabsorbed.

We consider again the Lagrangian of QED

$$\mathcal{L}_{\rm QED} = \frac{1}{4}F_{b,\mu\nu}F_b^{\mu\nu} + i\bar{\psi}_b(\not{\partial} + ie_b\not{A}_b)\psi_b - m_b\bar{\psi}_b\psi_b, \tag{1.36}$$

where now we make explicit the appearance of bare quantities with the subscript b. To absorb the divergence arising in computing the loop in Fig. 1.4, we have to define the renormalized electric charge as

$$e_{\rm R}^2 := e_b^2 - e_b^4\Pi_{\rm QED}\left(p_0^2\right), \tag{1.37}$$

where p_0 is a reference scale[10], so that, solving for the bare electric charge e_b, we find

$$e_b^2 = e_{\rm R}^2 + e_{\rm R}^4\Pi_{\rm QED}\left(p_0^2\right). \tag{1.38}$$

As a result, the bare electric charge appearing in the Lagrangian is now divergent but this is not a problem as it is not the observed charge. More importantly, such divergence cancels the divergence appearing once we include the quantum correction in Fig. 1.4.
In conclusion, the Lagrangian of QED is given by the sum of the renormalized Lagrangian $\mathcal{L}_{\rm QED}^{\rm R}$ and the counterterms $\mathcal{L}_{\rm QED}^{\rm CT}$

$$\mathcal{L}_{\rm QED} = \mathcal{L}_{\rm QED}^{\rm R} + \mathcal{L}_{\rm QED}^{\rm CT}, \tag{1.39}$$

where the counterterms are defined in such a way that they cancel the divergences arising in adding the contribution of quantum corrections.

3) Fix the finite leftover with experiments (renormalization conditions)

The last step consists in imposing the renormalization conditions. In order to obtain a predictive theory, one has to fix the finite renormalized results with the

[10] Differences in the choice of p_0 result in differences in the finite leftover that has to be fixed by imposing renormalization conditions.

outcome of independent observations.

In the case under analysis, the finite part of $\Pi_{\text{QED}}\left(p^2\right)$ depends on the regularization method adopted in order to regularize the divergence. As a consequence, it has to be fixed by imposing a renormalization condition. If we were to repeat the computation using the Feynman rules derived from the Lagrangian in Eq. 1.39, we would have to compute the graphs represented in Fig. 1.5, where we add the contribution of $\mathcal{L}^{\text{CT}}_{\text{QED}}$ to the loop.

Figure 1.5: Renormalized self energy of photon

Feynman diagrams representing, from left to right, the counterterm and the vacuum polarization. The sum is finite as the counterterm cancels the divergences arising in the loop diagram.

As expected, we obtain that the contribution of the counterterm reabsorbs the divergence of the loop integral and $i\Pi^{\mu\nu}_{\text{QED,fin}}$ results

$$\begin{aligned} i\Pi^{\mu\nu}_{\text{QED,fin}} &= ie_R^2\left(-p^2 g^{\mu\nu} + p^\mu p^\nu\right)\left[\Pi_{\text{QED}}\left(p^2\right) - \Pi_{\text{QED}}\left(p_0^2\right)\right] \\ &= \frac{ie_R^2}{2\pi^2}\left(-p^2 g^{\mu\nu} + p^\mu p^\nu\right)\lim_{\delta\to 0}\int_0^1 dx\, x(1-x)\left[\ln\left(\frac{m^2 - p_0^2 x(1-x)}{m^2 - p^2 x(1-x)}\right)\right], \end{aligned} \tag{1.40}$$

where in the second line we use the result in Eq. 1.34. Even if the second line of Eq. 1.40 is finite and we can safely take the limit $\delta \to 0$, it is not uniquely defined. In conclusion, a renormalization condition has to be imposed in order to get rid of this scheme-dependence.

1.2.3 Infinite vacuum energy?

Let's consider the action for a massless scalar field $\hat{\phi}$ in Minkowski space

$$S_\phi = \int d^4x\, \partial_\mu\hat{\phi}\partial^\mu\hat{\phi} \tag{1.41}$$

and write the field in terms of the mode function u_k and creation and annihilation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}_{\vec{k}} u_k + \hat{a}^\dagger_{\vec{k}} u_k^*\right], \tag{1.42}$$

where the operators $\hat{a}$ and $\hat{a}^\dagger$ obey the commutation relations

$$\begin{aligned} \left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] &= (2\pi)^3\delta^3(\vec{k} - \vec{k}'), \\ \left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] &= 0, \\ \left[\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] &= 0. \end{aligned} \tag{1.43}$$

By considering the stress energy tensor derived from the action in Eq. 1.41

$$\hat{T}^{\phi}_{\mu\nu} = \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha \hat{\phi} \partial^\alpha \hat{\phi} \tag{1.44}$$

the results for the vacuum energy is divergent[11]:

$$\rho := -\langle 0|\hat{T}^{\phi}{}_0{}^0|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{w_k}{2} \sim k^4 \sim \infty, \tag{1.45}$$

where we assume that solving the equation of motion (EOM), the mode function results $u_k = \frac{e^{-ik_\mu x^\mu}}{\sqrt{2w_k}}$ and we define $k^\mu = (w_k, k)$, $w_k = |k^2|$. In Minkowski space the divergence is simply discarded, for example by the use of normal ordering. This can be done because in flat spacetime just the difference of energies has a physical meaning, and the energy of the vacuum can be simply rescaled to zero. In curved spacetime this procedure is more subtle as, simply put, being energy the source of gravity, one has to be careful in reabsorbing the energy of the vacuum (See section 6.1 of [48] for more details). As a consequence, such divergence cannot be simply thrown away and we are not free to rescale the zero point of energy. Instead, the renormalization procedure allows to make sense to such divergence by reabsorbing it via the redefinition of the action of the background metric.
In order to better understand the difference between Minkowski and curved spacetimes, we repeat the computation of the energy of a massless scalar field considering now a spatially flat FLRW background described by the scale factor ([48])

$$a(t) = \sqrt{1 - A^2 t^2}, \tag{1.46}$$

where A is a constant. In four dimensions the mode function in conformal time results

$$u_k = \frac{1}{a(t)}\frac{1}{\sqrt{2}(k^2 + A^2)^{\frac{1}{4}}} e^{-ik\cdot x + i\sqrt{k^2+A^2}\tau} \tag{1.47}$$

from which we can recover the result in Minkowski in the limit $a(t) = 1$, $A = 0$. The energy density results (we recall that H is the Hubble parameter)

$$\rho = -\langle 0|T^{\phi}{}_0{}^0|0\rangle = \frac{1}{32\pi^3 a(t)^4}\int d^3k \left[\sqrt{k^2 + A^2} + \frac{k^2 + H^2}{\sqrt{k^2 + A^2}}\right] \tag{1.48}$$

which is, as expected, UV-divergent. by introducing the UV-cutoff $e^{\epsilon\sqrt{k^2+A^2}}$ and expanding the integral in powers of ϵ, we obtain the regularized result

$$\rho = \frac{1}{32\pi^2 a(t)^4}\lim_{\epsilon\to 0}\left[\frac{48}{\epsilon^4} + \frac{4H^2 - 8A^2}{\epsilon^2} + A^2(2H^2 - A^2)\log\epsilon\right] + \mathcal{O}(\epsilon^0). \tag{1.49}$$

From the result in Eq. (1.49) we not only recover the Minkowski-like quartic divergence, but also divergences that depend on the scale factor. We indeed obtain that

[11] In what follows we drop the vector notation and with k we refer to the spatial components.

the quadratic and log divergence in Eq. (1.49) redshift and, even if we discard the zero-point divergence that we would obtain in Minkowski space, the energy density is still divergent in the limit $\epsilon \to 0$. As a consequence, the usual normal ordering prescription is not enough to get rid of the vacuum energy divergences in curved spacetime and one has to refer to the renormalization procedure to obtain a finite result.
As we will see in detail in this thesis, to obtain a finite answer one has to follow the procedure reviewed in Section 1.2.2 to reabsorb the divergences by redefining the bare constants of the background action.

1.3 Work in this thesis

This thesis is divided in two parts, in each of which the goal is to tackle the renormalization of divergences arising in including quantum vacuum tensor pertutbations on a curved spacetime. In the first part we adopt a foliation specific description as we aim to study divergent quantities on backgrounds of cosmological interest. In the second part, in which we work in a fully covariant formalism, we retrace and expand the results of the first part.

Part I: Foliation specific formulation

- **Chapter 2:** we provide the details of vacuum GWs constraints from N_{eff} bounds and motivate the work in this thesis. We review how primordial vacuum tensor fluctuations are included in estimating N_{eff} by examining the derivation of the stress energy tensor for GWs first derived by Isaacson in ([123]). We motivate on why it is appealing to include vacuum GWs in the radiation-like content of the universe by reminding the role of BBN in constraining N_{eff}. To conclude, we highlight the caveats that motivated our work, such as the appearance of an UV regulator in what should be a physical result and the regime of validity of energy density of GWs, namely sufficiently high frequency signals for which the curvature of the background can be neglected.

- **Chapter 3:** considering that many cosmological observables derive from primordial vacuum fluctuations evolved to late times, we present the process of renormalization by studying a massless spectator scalar field on a FLRW background. This apparently simple working example brings to light many relevant results. We first demonstrate that infinities arise in observables that represent statistical draws from some underlying quantum field theoretic framework. We then elaborate on how infinities can be regularized: we show that in spite of the ubiquity of scaleless integrals, UV divergences can still be meaningfully extracted using dimensional regularization and that the coefficients of the logarithm divergence do not depend on the regularization method. We both analyze the divergences arising in computing the two point function in different backgrounds and in computing the components of the stress energy tensor. In studying the latter, we conclude that regularization methods which preserve general covariance are to be preferred. By studying backgrounds that transition from finite duration inflation to radiation domination, we show that UV and IR scales corresponding to the beginning and end of inflation do not appear as UV cutoffs and motivate that observables cannot depend on the latter, although will certainly depend on the former. Furthermore, it appears that IR divergences are an artifact of the dS limit and are cured for finite duration inflation.

- **Chapter 4:** we address the caveats highlighted in Chapter 2. By undoing the steps that relied on a high frequency signal for which the effect of curvature can be neglected, we derive an improved stress tensor that does not presume a prior scale separation. To understand whether one can meaningfully constrain vacuum GWs from $N_{\rm eff}$ bounds, we specify the improved formula of the energy density on backgrounds that transition from finite duration inflation to radiation domination. We then comment on the physical interpretation of the power spectral density of vacuum GWs on a FLRW background, and we find that the regularized energy density is consistent with the results of the scalar case studied in Chapter 3 (scheme independence of UV logarithmic divergences and need of using schemes that preserve general covariance to obtain counterterms constructed from geometric invariants). After the subtraction of the divergences arising in the regularized energy density, renormalization conditions must be imposed by measurements at some scale, mindful of scheme and background dependence. We review this process considering first the energy density of a scalar field derived in Chapter 3, and then the improved energy density of vacuum GWs, obtaining a final result that does not depend on UV regulators. We conclude by highlighting the inextricable connection between inferring $N_{\rm eff}$ bounds from vacuum tensor perturbations and the process of background renormalization.

  Chapters 3 and 4 are based on [169]:
  An Étude on the Regularization and Renormalization of Divergences in Primordial Observables
  A. Negro and S. P. Patil, *Riv.Nuovo Cim.* 47 (2024) 3, 179-228.

Part II: Covariant formulation

- **Chapter 5:** we review the formalism needed to study vacuum GWs as a massless spin-2 particle on curved spacetime. Considering that we aim to address the caveats reviewed in Chapter 2 in a fully covariant formulation, we introduce the fundamental concepts of QFT in curved spacetime to define the renormalized stress energy tensor for GWs as the variation of the effective action with respect to the background metric. After reviewing the background field method to obtain the 1-loop effective action, we remind the Faddeev-Popov procedure to fix the gauge at the level of the action. Lastly, we derive the Lagrangian formulation of a radiation-like fluid in the context of $P(X)$ theories.

- **Chapter 6:** the derivation, regularization and renormalization of the graviton stress tensor is addressed in a fully covariant formalism. We obtain the regularized graviton action by gauge-fixing via the Faddeev-Popov procedure and adopting Hadamard regularization techniques to isolate UV divergences. After defining the counterterms at the level of the action, from which the renormalized stress tensor can be derived by variating with respect to the backgorund metric, we comment on the finite contributions which are absorbed in the process of imposing renormalization conditions. We then proceed to specify our otherwise general results in a RD universe, to reexamine and connect with the results of Chapter 4. We discuss the imposition of renormalization conditions via a physical measurement at some fixed scale, which we retrace for primordial GWs sourced from vacuum fluctuations through direct or indirect observation. In agreement with our conclusions of Chapter 4, we obtain that one cannot constrain vacuum tensor perturbations from $N_{\rm eff}$ bounds: the net effect of including vacuum tensor perturbations is to shift the normalization of otherwise unobservable bare quantities. In addition, from the covariant description we find divergences that require higher order corrections to the Einstein Hilbert action to be reabsorbed and that the regularized stress tensor associated with the shifted radiation-like content is no longer traceless, features that were not evident in the foliation dependent description.

  Chapter 6 is based on [170]:
  Hadamard Regularization of the Graviton Stress Tensor
  A. Negro and S. P. Patil, arXiv:2403.16806, (2024).

Part I

Foliation specific formulation

CHAPTER 2

Possibility of constraining primordial GWs with $N_{\rm eff}$ bounds

2.1 Introductory remarks

In this chapter, we aim to both thoroughly explain the possibility of constraining primordial GWs with $N_{\rm eff}$ bounds and motivate the work described in Chapters 3 and 4. After reviewing the definition of the energy density of GWs and the link between BBN and the effective number of relativisitc species, we elaborate on the widely used relation between $N_{\rm eff}$ and $\rho_{\rm gw}$. By doing this, we both provide the context upon which our work is structured and highlight relevant discrepancies to motivate our improvements.

In Section 2.2 we start by recalling the most relevant results in the literature upon which the formula for the energy density used to infer bounds on primordial GWs relies. We review the formula for the stress energy tensor of GWs that was first derived by Isaacson, in which GWs are assumed to be high frequency signals propagating in an effectively flat background. From this definition of the stress energy tensor, we review the way the energy density for primordial GWs at the time of CMB is determined. In doing so, we review the SVT decomposition and the most commonly used gauge-fixing in this context, the TT-gauge.
We then give in Section 2.3 more details on the process that explains the formation of light nuclei in the early universe: BBN. We study the link between BBN and the effective number of relativistic species and we explain how the latter can be constrained by CMB observations. Finally, we review how the energy density derived in the first section can be linked with the effective number of relativistic species at the time of BBN and we present how such link is used to infer constraints on the amount of primordial GWs. In reviewing the results used in the literature for the estimate of $N_{\rm eff}$ that include the effects of primordial GWs, we show how a hard UV cutoff appears in the derivation.
We then conclude in Section 2.4 by commenting on both the reliability of using Isaacson's stress-energy tensor in the context of primordial vacuum fluctuations and the presence of a hard UV cutoff in the final result for estimating $N_{\rm eff}$.

2.2 Stress energy tensor: Isaacson's definition

The Isaacson form of the stress tensor $T_{\mu\nu}$ [124] (see also [151, 149, 54, 160]) is defined as the averaged high frequency part of the second order expansion of Einstein equations

$$T_{\mu\nu}{}^{\rm gw,hf} := -\frac{1}{8\pi G_N}\left\langle \delta^2 R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g_{\alpha\beta}\delta^2 R^{\alpha\beta}\right\rangle_{\rm BH}, \tag{2.1}$$

where $g_{\mu\nu}$ is the background metric (defined as $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ where $\tilde{g}_{\mu\nu}$ is the total metric and $h_{\mu\nu}$ is the perturbation). The averaging scheme used in [124] is the Brill-Hartle (BH) averaging scheme [54] which, as a corollary, allows one to neglect divergences, freely integrate by parts within expectation values, and commute covariant derivatives. The BH averaging scheme is well justified only if one is interested in studying GWs with a spectrum of bounded support and comprised wavelengths and periods much shorter than the background curvature scale[1]. This is a good assumption for GWs of astrophysical origin, for which one can indeed assume that the curvature scale of the background is much smaller than the frequencies of interest. Assuming that the BH averaging scheme is legitimated and using the expansions of the metric and Ricci tensor up to second order in $h_{\mu\nu}$ (see [151] for details)

$$\begin{aligned}
&\delta^1 g_{\mu\nu} = h_{\mu\nu} \qquad \delta^1 g^{\mu\nu} = -h^{\mu\nu} \qquad \delta^2 g^{\mu\nu} = h^{\mu\alpha}h^{\nu}_{\alpha}\\
&\delta^1 R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}D_\nu D_\mu h + \frac{1}{2}D^\rho D_\mu h_{\nu\rho} + \frac{1}{2}D^\rho D_\nu h_{\mu\rho}\\
&\delta^2 R_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}g^{\alpha\beta}\left[\frac{1}{2}D_\mu h_{\rho\alpha}D_\nu h_{\sigma\beta} + (D_\rho h_{\nu\alpha})\left(D_\sigma h_{\mu\beta} - D_\beta h_{\mu\sigma}\right)\right.\\
&\qquad + h_{\rho\alpha}\left(D_\nu D_\mu h_{\sigma\beta} + D_\beta D_\sigma h_{\mu\nu} - D_\beta D_\nu h_{\mu\sigma} - D_\beta D_\mu h_{\nu\sigma}\right)\\
&\qquad \left. + \left(\frac{1}{2}D_\alpha h_{\rho\sigma} - D_\rho h_{\alpha\sigma}\right)\left(D_\nu h_{\mu\beta} + D_\mu h_{\nu\beta} - D_\beta h_{\mu\nu}\right)\right],
\end{aligned} \tag{2.2}$$

one obtain the simplified result

$$T^{\rm gw,hf}_{\mu\nu} = \frac{1}{32\pi G_N}\left\langle \bar{h}_{\alpha\beta;\mu}\bar{h}^{\alpha\beta}{}_{;\nu} - \frac{1}{2}\bar{h}_{;\mu}\bar{h}_{;\nu} - \bar{h}^{\alpha\beta}{}_{;\beta}\bar{h}_{\alpha\mu;\nu} - \bar{h}^{\alpha\beta}{}_{;\beta}\bar{h}_{\alpha\nu;\mu}\right\rangle_{\rm BH} \tag{2.3}$$

where we have defined

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h. \tag{2.4}$$

Eq. 2.3 is covariantly conserved and gauge-invariant up to correction terms that are negligible for all wavelengths shorter than the averaging scale.
After fixing the gauge as $D_\rho h^{\rho\nu} = 0$, $h_\mu{}^\mu = h = 0$, the resulting stress energy tensor is given by

$$T_{\mu\nu}{}^{\rm gw,hf} = \frac{1}{32\pi G_N}\left\langle \nabla_\mu h_{\rho\sigma}\nabla_\nu h^{\rho\sigma}\right\rangle_{\rm BH}. \tag{2.5}$$

[1] That is, when $\lambda, \omega^{-1} \ll 2\pi\mathcal{R}$, where $\mathcal{R}^{-2}$ is the typical magnitude of the non-vanishing components of the background Riemann tensor.

As we review in the following (see [68] for more details), the above expression is widely taken as the starting point for determining the energy density associated with stochastic backgrounds of primordial origin in terms of the primordial tensor power spectrum.
Under the assumptions of a high frequency signal (so that $\mathcal{H}$ terms can be neglected) propagating on a effectively flat background (null Riemann tensor), the EOM for the tranverse traceless part of the metric results

$$\partial_\mu \partial^\mu h_{ij}(\tau, k) = 0, \tag{2.6}$$

one can find a relation among the amplitudes

$$h'^2_{ij}(\tau, k) \sim k^2 h^2_{ij}(\tau, k) \tag{2.7}$$

and rewrite the (00)-component of Eq. (2.5) as

$$\rho_{\rm gw} = \frac{1}{32\pi G_N a^2} \delta^{im} \delta^{j\ell} \left\langle h'_{ij}(\tau, k) h'_{lm}(\tau, k) \right\rangle. \tag{2.8}$$

Focusing on the signal produced by quantum fluctuations and assuming that primordial GWs are a stochastic field, the tensor modes of the metric are quantized and the averaging $\langle ... \rangle$ in Eq. (2.8) is promoted to vacuum expectation value (see [68] for details). In this way, $\rho_{\rm gw}$ can be related to the tensor power spectrum $P_t(k)$ and computed as the integral over all the frequencies

$$\rho_{\rm gw} = \frac{1}{32\pi G_N a^2} \int d\log k \, k^2 P_t(k). \tag{2.9}$$

2.2.1 Gauge-fixing: SVT decomposition and TT-gauge

As in this section we are studying metric perturbations of a FLRW background, it is useful to work with the SVT decomposition of the metric (introduced in Section 1.1.3). We recall that the SVT decomposition is a decomposition of perturbations into components according to their transformations under spatial rotations and it allows to split the general metric perturbation $h_{\mu\nu}$ into a 3+1 splitting

$$h_{00} \equiv -2\psi, \quad h_{0i} \equiv w_i, \quad h_{ij} = 2\left(\phi g_{ij} + S_{ij}\right). \tag{2.10}$$

The vector w_i and the tensor S_{ij} are further decomposed as

$$\begin{aligned} w_i &= w_i^{\parallel} + w_i^{\perp} \quad \text{where } \vec{\nabla} \times \vec{w}^{\parallel} = \vec{\nabla} \cdot \vec{w}^{\perp} = 0 \\ S_{ij} &= S_{ij}^{\parallel} + S_{ij}^{\perp} + S_{ij}^{T} \quad \text{where } g^{jk} \nabla_k S_{ij} = g^{jk} \nabla_k S_{ij}^{\parallel} + g^{jk} \nabla_k S_{ij}^{\perp}. \end{aligned} \tag{2.11}$$

By rewriting $w_i^{\parallel}$ and $S_{ij}^{\parallel}$ as

$$\begin{aligned} w_i^{\parallel} &= \nabla_i \phi_w \\ S_{ij}^{\parallel} &= \left(\nabla_i \nabla_j - \frac{1}{3}\gamma_{ij} \nabla^2\right) \phi_S \end{aligned} \tag{2.12}$$

and $S^{\perp}_{ij}$ as

$$S^{\perp}_{ij} = \nabla_i S^{\perp}_j + \nabla_j S^{\perp}_i , \tag{2.13}$$

we obtain that the perturbation $h_{\mu\nu}$, which is a symmetric 4x4 matrix and thus has 10 degrees of freedom (DoFs), can be split into 4 scalars (ψ, ϕ, ϕ_w, ϕ_S), 2 divergence-free vectors ($w^{\perp}_i$, $S^{\perp}_i$) and one transverse traceless tensor (S^T_{ij}). What makes the SVT-decomposition so commonly used in cosmology is the fact that it is a very convenient way of splitting of the perturbations. Indeed, at linear order the Einstein equations for scalars, vectors and tensors do not mix at linear order and can therefore be treated separately.

As GR is invariant under coordinate transformations, in order to identify the two physical DoFs of a massless spin-2 particle from the 10 DoFs of $h_{\mu\nu}$, we have to fix the gauge and get rid of the unphysical DoFs. This can be done in different ways (see Chapter 6 for an example of gauge-fixing via the Faddeev–Popov method); however, since physical observables are independent from both the choice of the gauge and the method adopted to fix the gauge, every gauge-fixing method has to lead to the same final result.
In this Section, it is convenient to adopt the so-called TT-gauge (see [68] for more details) in which the propagating DoFs are parametrized with the transverse traceless part of the metric $h^T_{ij} = 2S^T_{ij}$. In this way, we easily refer and compare to most of the results in the literature[2].

2.3 Big Bang Nucleosinthesis

As reviewed in Section 1.1.1, when the temperature of the universe was of order an $\sim$ MeV, there were no neutral atoms or bound nuclei: any atom or nucleus produced would have been immediately destroyed by the overabundance of high-energy photons. As the universe cooled well below the binding energies of typical nuclei, light elements began to form, this process, which we review in detail in this section, goes by the name of Big Bang Nucleosynthesis ([173, 190, 213]).

Nuclear binding energies are typically in the MeV range, which explains why BBN occurs at temperatures ~ 1 MeV even though nuclear masses are in the GeV range. At ~ 1 MeV the universe was radiation dominated and the cosmic plasma consists of

- Relativistic particles in equilibrium (γ, e^+ and e^-). Photons, electrons and positrons are kept in thermal equilibrium by electromagnetic interactions such as $e^+ + e^- \rightleftharpoons \gamma\gamma$.
- Decoupled relativistic particles (ν). At high temperatures neutrinos are coupled with electrons via electroweak interactions; however, as explained in more details in the following, when the temperature drops at around ~ 1 MeV, the

[2]Throughout the thesis we specify when we are working in the TT-gauge but we drop the superscript "T" in denoting the transverse traceless part of the metric.

rate of such processes drops beneath the expansion rate and neutrino decouple form the relativistic plasma.

- Non relativistic particles (n and p). According to the Big Bang model, due to the asymmetry of the initial number of baryons, the rate of baryons present in the universe remains constant throughout the expansion. As at ~ 1 MeV baryon/anti-baryon annihilation processes are very effective, all anti-baryons have been annihilated and the presence of baryons in the cosmic plasma is only due to the initial baryon/anti-baryon asymmetry.

With these ingredients, the production of primordial light elements occurs in two stages: the decoupling of the weak interactions and the sequence of nuclear reactions which build up the light nuclei. At temperatures higher than ~ 1 MeV, the following weak interactions are effective

$$\mathrm{n} + \nu_\mathrm{e} \rightleftharpoons \mathrm{p} + \mathrm{e}^-, \quad \mathrm{n} + \mathrm{e}^+ \rightleftharpoons \mathrm{p} + \nu_\mathrm{e}, \quad \mathrm{n} \rightleftharpoons \mathrm{p} + \mathrm{e}^- + \nu_\mathrm{e} \tag{2.14}$$

and the balance between neutrons and protons is maintained. Since the interaction rate of the weak interactions in Eq. 2.14 depends the temperature, as soon as the universe cools down and interaction rate falls below the expansion rate, weak interactions can no longer keep up with the expansion of the universe. At this point, neutrinos decouple from the cosmic plasma, the neutron-to-proton ratio is frozen and the second stage of nucleosynthesis can begin.
The first element produced in the BBN chain is Deuterium ($D =^2$He). Deuterium is produced through the process

$$\mathrm{n} + \mathrm{p} \rightarrow D + \gamma \tag{2.15}$$

which, due to the binding energy of the Deuterium, should become effective around right after neutrino decoupling. However, Deuterium production is slowed down by the over-density of photons and the production of light elements starts only at about 0.07 Mev. After overcoming the Deuterium bottleneck, so that the synthesis of ^{3}He becomes effective, light elements such as Helium and Lithium are produced. Heavier nuclei do not form in any significant quantity because of the absence of a stable nucleus with 8 or 5 nucleons.

2.3.1 N_eff bounds

The BBN bounds is one of the tight constraints that can be inferred from the CMB (see [78] for a review). As standard BBN relies on the Standard Model in fixing the prediction of the number of neutrino flavors to three, BBN constraints are often used to test models beyond the standard model by allowing the neutrino flavors to vary. In this way, BBN bounds are an important tool to test the Standard Model as the theoretical description of microphysics in the early universe. Furthermore, the comparison of the baryon density predictions from BBN and the constraints inferred by the CMB is a fundamental test for Big Bang cosmology, and its underlying assumptions (such as a homogeneity and isotropy, GR as the theory describing gravity).

From the CMB anisotropies we can quantify the amount of light elements produced in the early universe and, as reviewed above, among all the processes involved in the BBN description, such amount depends on the freeze-out temperature of neutrinos. As a consequence, by measuring the amount of light elements from the CMB, we can infer the neutrino freeze-out temperature T_f. From the latter, we then constrain the expansion rate as this is equal to the interaction rate $\Gamma_\nu(T)$ at the time of freeze-out

$$\Gamma_\nu(T_f) = H(T_f)\,. \tag{2.16}$$

Such constraint can be formulated as an upper bound for the total energy density ρ_{tot} at the time of BBN, as ρ_{tot} is related to the expansion rate of the universe through the Friedman equation

$$H^2 = \frac{8\pi G_N}{3}\rho_{\text{tot}}. \tag{2.17}$$

As in the standard BBN we assume that the universe is dominated by photons at the time of BBN, only neutrinos and photons contribute to the total energy density

$$\rho_{\text{tot}}^{\text{BBN}} = \rho_\gamma + \rho_\nu \tag{2.18}$$

and equation above is usually written in terms of the effective number of DoFs, N_{eff}, defined as

$$\rho_{\text{tot}}^{\text{BBN}} = \rho_\gamma\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}} N_{\text{eff}}\right). \tag{2.19}$$

In conclusion, the so-called BBN's constraint is given as an upper bound for N_{eff} (see [8] for a review of Planck constraints of N_{eff}).

2.3.2 N_{eff} bounds and primordial GWs

As we reviewed in Section 1.1.3, even in the context of the standard ΛCDM model in equation (2.18) we are neglecting the contribution of the stochastic GWs background produced by the primordial quantum fluctuations of inflation (see [201, 188, 115, 119] for a review). Indeed, one of the features of the stochastic GWs background is that it behaves as a free-streaming gas of massless particles, thus contributing to the total radiation energy density of the universe. Among the many cosmological mechanisms that can produce stochastic GWs backgrounds in the early universe ([17]), such as primordial black holes formation ([142]), reheating ([129]) and phase transitions ([195]), we certainly expect that the GWs produced by primordial quantum fluctuations contribute to the radiation energy density at the time of BBN.
By including the contribution of the energy density of GWs ρ_{gw} in equation (2.18), and using the definition for N_{eff} in equation (2.19), the BBN bounds for N_{eff} become

relevant in constraining the amount of GWs produced by quantum fluctuations

$$\frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}} N_{\rm eff} = \frac{\rho_\nu}{\rho_\gamma} + \frac{\rho_{\rm gw}}{\rho_\gamma}. \tag{2.20}$$

To further make explicit the relation between $N_{\rm eff}$ and the amount of GWs produced by tensor vacuum perturbations, we follow [158]. We start from the result for the energy density of GWs reviewed in Section 2.2

$$\rho_{\rm gw} = \frac{1}{32\pi G a^2}\delta^{im}\delta^{j\ell}\left\langle \hat{h}'_{ij}(\tau,k)\hat{h}'_{lm}(\tau,k)\right\rangle \tag{2.21}$$

where the averaging $\langle ... \rangle$ refers to computing the vacuum expectation value. By writing the two-point function of GWs during RD era in terms of the tensor power spectrum $P_t(k) = A_t\left(\frac{k}{k_*}\right)^{n_t}$

$$\left\langle \hat{h}_{ij}(\tau,\mathbf{x})\hat{h}^{ij}(\tau,\mathbf{x})\right\rangle \equiv \int d\log k P_t(k)[\mathcal{T}(\tau,k)]^2 \tag{2.22}$$

where $\mathcal{T}(\tau,k)$ is the transfer function during radiation domination ($\mathcal{T}'(\tau,k) = -kj_1(k\tau)$), $\rho_{\rm gw}$ is computed as the integral over all the frequencies

$$\rho_{\rm gw} = \frac{1}{32\pi G a^2}\int d\log k P_t(k)\left[\mathcal{T}'(\tau,k)\right]^2. \tag{2.23}$$

However, the integral domain in equation (2.23) is then constrained to a limited range of frequencies ($[k_{\rm IR}, k_{\rm UV}]$)

$$\rho_{\rm gw} = \frac{A_t}{32\pi G a^2}\int_{k_{\rm IR}}^{k_{\rm UV}} kdk\left(\frac{k}{k_*}\right)^{n_t} j_1^2(k\tau) \tag{2.24}$$

to avoid the divergences. The integral is computed by introducing a new integration variable $q = \frac{k}{k_{\rm UV}}$ and expanding in $\tilde{\epsilon} = \frac{k_{\rm IR}}{k_{\rm UV}} \to 0$ to get rid of the IR cutoff. For positive n_t, the integral results

$$k_{\rm UV}^2\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\int_0^1 q^{n_t+1}dq j_1^2\left(qk_{\rm UV}\tau\right) = \left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\frac{1}{2n_t}\frac{1}{\tau^2} + \mathcal{O}\left(\frac{1}{k_{\rm UV}\tau}\right). \tag{2.25}$$

Hence, for the gravitational wave energy density we find (up to corrections of order $\tilde{\epsilon}$)

$$\rho_{\rm gw} \simeq \frac{A_t}{32\pi G_N}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\frac{1}{2n_t}\frac{1}{(a\tau)^2}. \tag{2.26}$$

Using equation (2.26) in equation (2.20), considering that the standard model predicts $N_{\rm eff,th} = 3.046$ and that $\frac{1}{(a\tau)^2} = H = \frac{8\pi G}{3}\rho_{\rm tot}$ in the RD era, we obtain

$$\frac{\rho_{\rm gw}}{\rho_{\rm tot}} = \frac{A_t}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t} = \frac{\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\left(N_{\rm eff} - 3.046\right)\rho_\gamma}{\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}N_{\rm eff}\right)\rho_\gamma}. \tag{2.27}$$

Solving the equation above for $N_{\rm eff}$, we obtain the final result used in the literature[3] to link the primordial power spectrum of GWs and the measured value of $N_{\rm eff}$ (see e.g., [158, 150, 106, 200, 53, 147, 121, 65, 146, 176, 140, 43, 44, 107])

$$N_{\rm eff} = \frac{\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\left[\frac{A_t}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right] + 3.046}{1-\left[\frac{A_t}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right]}. \tag{2.29}$$

In order to better understand the effects of $k_{\rm UV}$, we expand Eq 2.29 in the limit in which GWs are a small fraction of radiation-like species compared with the total amount of radiation (see [158] for details) and obtain

$$N_{eff} \sim 3.046 + \left(3.046 + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\right)\left(\frac{H_*^2}{12\pi^2 M_{\rm pl}^2 n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right). \tag{2.30}$$

The ratio $\frac{k_{\rm UV}}{k_*}$ is usually estimated by considering the duration of inflation ($k_{\rm UV}$ is usually associated to the smallest scale undergoing to a super-horizon phase during inflation) and is of order $\sim 10^{20-25}$. As a consequence, as one can see from Eq. 2.30, $N_{\rm eff}$ bounds would be particularly stringent for $n_t > 0$ (blue-tilted) power spectra.

2.4 Comments and motivations for our work

Although the steps reviewed in this chapter constitute an important progress in the direction of constraining the tensor power spectrum using the measurements of $N_{\rm eff}$, a careful formal examination of these results allowed us to bring to light fundamental discrepancies.

The first discrepancy is the dependence of the observable $N_{\rm eff}$ on the regularization parameter $k_{\rm UV}$. As studied in [107], the dependence on $k_{\rm UV}$ in equation Eq. (2.29) induces a strong constraint on blue-tilted power spectra and it is largely studied to infer constraints or rule out mechanisms that produce such power spectra. However, $k_{\rm UV}$ is an arbitrary regulator used to avoid the divergences in Eq. (2.23) and cannot be considered to infer any physical analysis. Having a divergent expectation value as in Eq. (2.23), entails the need to follow the procedure reviewed in 1.2.2 and renormalize the divergences before obtaining a predictive result to be constrained by observations. Indeed, if these divergences are inconsistently subtracted, one will end up with regularization artifacts in the final answer, such as hard cut-off

[3] In some works (see [151] as an example) instead of explicating the effective number of species as in equation (2.29), the BBN's constraint results in an upper bound of the energy density in equation (2.23), written as the integral over all the frequencies

$$\int_{f=0}^{f=\infty} d(\log f) h_0^2 \Omega_{\rm gw}(f) \leq 5.6\times 10^{-6}\,(N_{\rm eff}-3.046) \tag{2.28}$$

where $\Omega_{\rm gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{\rm gw}}{d\log f}$ and $0.5 < h_0 < 0.85$. In this derivation, the integral domain is equivalently constrained by inserting an UV-cutoff to avoid divergencies.

dependencies that cannot feature in any observables, which nevertheless feature in Eq. (2.29).
Furthermore, we stress that UV cutoffs cannot be confused with scales corresponding to the beginning and end of inflation. While the latter are physical scales, that can be in principle observed or enter in formulas of observables, the former is merely parametrizing our ignorance in computing the theoretical estimate of an observable[4].

Secondly, the derivation of the formula for the energy density of GWs should be modified in order to be used in the context of the CGWsB. By following [124] and defining the stress energy tensor as the high-frequency part of the second order expansion of the Einstein equations, as in equation eq. (2.1), we are assuming that the CGWsB is a high-frequency signal. This would be a realistic assumption if we were to study astrophysical sourced GWs. However, it is expected that the CGWsB is a signal produced by long-wavelength GWs (especially when compared with the horizon size at BBN). Then, Eq. (2.1) should be modified in order to include all the terms of the second order expansion of Einstein equations. More importantly, as pointed out in the previous paragraph, one must renormalize the divergences naturally appearing in computing the stress energy tensor. Thus, one cannot start from derivation that relies on a scale separation, as any computation at any loop order implicit integrates over all scales and will eventually run afoul of this approximation. Furthermore, by commuting the derivatives in the averaging procedure, we implicitly assume that the background is flat compared to the signal: to obtain Eq. (2.5) we are assuming that the background metric is Minkowski instead of FLRW. This would be again a good assumption if we were interested in studyng, for example, a signal coming from a black holes merging happening far from the detector, but it fails to be a good approssimation in the case under analysis. In [124], the averaging is introduced in order to neglect the fine details of the very high-frequency oscillations of the GWs and, due to the quantum nature of the CGWsB, this can be done by computing the expectation value. However, simply interpreting the average as a vacuum expectation value is not enough, as we have to re-introduce the effects arising from expanding around a curved background.

In Chapter 4 we will present our work and solve the discrepancies presented in this section. Furthermore, we will show that an improved formal description of the problem leads to a substantial difference in the interpretation of BBN bounds as a constraint for tensor primordial perturbations.

[4]For more details and explicit examples on the difference between UV/IR cutoffs and physical scales corresponding to the beginning and end of inflation see Sections 3.3.3 and 4.3

CHAPTER 3

Handbook for divergences in cosmology – scalar case

This chapter is based on:
An Étude on the Regularization and Renormalization of Divergences in Primordial Observables
Anna Negro, and Subodh P. Patil, *Riv.Nuovo Cim.* 47 (2024) 3, 179-228.

3.1 Introductory remarks

This chapter is intended as a practical tour for the regularization of nominally divergent quantities on backgrounds of cosmological interest. Before studying vacuum tensor perturbations and the possibility to infer constrains from BBN bounds, we focus on the simplest possible applications one could think of, given the richness of the problems already encountered in cosmology, that of minimally coupled non-interacting test scalar fields. Moreover, we work with a formalism that should be familiar to most practicing cosmologists: two-point functions, power spectra and spectral densities, all constructed through the intermediary of mode functions derived on a FRLW slicing.

In the following, we start with a recap on how UV and IR divergences arise in the computations of standard cosmological observables. In Section 3.2, we show that UV and IR divergences naturally appear in both the stress energy tensor and the two-point function. Taking a cue from non-relativistic QED and QCD, we demonstrate how to split scaleless integrals into UV and IR contributions that would otherwise cancel in dimensional regularization. We derive the regularized power spectrum of a massless scalar field on dS regularizing both using dimensional regularization and physics momenta cutoff and we verify that the logarithmic running of physical quantities does not depend on the regularization method.
In Section 3.3, we focus on non-interacting test scalar fields and the UV log-divergent parts to make sense of results in the literature that impose hard cutoffs in physical momenta. We compare the coefficients for the UV divergent logs, regularizing both using dimensional regularization and physical cutoffs. We find identical coefficients for the power spectrum of a massless scalar field on dS background and massless and light scalar field on a quasi dS background. We then study the divergences

arising in the stress energy tensor and we point out that it is not a straightforward extension of the two-point function. We explicitly show that by using hard cutoffs, one fails to construct a counterterm that subtracts the UV divergences from background geometric invariants. Lastly, we consider the process of regularization on backgrounds that transition in and out of inflation to radiation domination. Although scales corresponding to the beginning and end of inflation can be shown to merely parametrize rather than regulate UV divergences, nominally IR divergent contributions on a dS background are cured in the examples we consider.
We conclude in Section 3.4 by sumarizing the implications of our results.

3.2 Regularization of stress tensors and correlation functions

All physical observations entail the exchange of energy or momentum among interacting components and detectors or tracers, whether through direct or indirect means. Correlation functions serve as useful computational tools bridging the gap between any given (effective) description valid during the primordial epoch and observables at late times, yet they are not necessarily directly observable by themselves. They can, however, be related to observations at later times by acting upon them with the appropriate derivatives to extract energy and momentum, and subsequently convolving them with transfer functions that encapsulate how they are processed by the intervening cosmological evolution (see e.g., [70] for a review).
Correlation functions are typically computed in Fourier space, whereas the observations they relate to are made at some fixed temporal and spatial location. Furthermore, correlation functions in real space involve the coincident limits of bilinear or higher point functions, necessitating subtraction of UV divergences associated with this limit. Well defined coincident limits also implicitly depend on the IR behavior of correlation functions, with the difference that any IR divergence appearing in computing physical quantities must cancel between all contributions in convolution with the transfer function (see Section 1.2.1 for more details). This must be the case for any well defined physical observables in any self-consistent calculational setup.
In this chapter we elaborate on the process of extracting these divergences, regularizing quantities of interest on different backgrounds by following the procedure reviewed in Section 1.2.2. In order to do this, in this section we establish some preliminary facts and conventions.

The energy momentum tensor for a non-interacting, minimally coupled test scalar field ϕ on an FRLW background is given by

$$T^{\mu}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}\delta^{\mu}_{\nu}\left(g^{\lambda\beta}\partial_{\lambda}\phi\partial_{\beta}\phi + m^2\phi^2\right), \tag{3.1}$$

from which we can extract

$$-T^{0}{}_{0} := \rho = \frac{\phi'^2}{2a^2} + \frac{(\nabla\phi)^2}{2a^2} + \frac{m^2}{2}\phi^2, \tag{3.2}$$

where we work in conformal time, and primes denote derivatives with respect to conformal time. The above is a local density that we can rewrite as the coincident limit of a bilinear form as

$$\rho := \lim_{y\to x} \rho(\tau; x, y), \tag{3.3}$$

where

$$\rho(\tau; x, y) := \frac{1}{2a^2}\left[\phi'(\tau,x)\phi'(\tau,y) + \nabla_x\phi(\tau,x)\cdot\nabla_y\phi(\tau,y) + m^2a^2\phi(\tau,x)\phi(\tau,y)\right]. \tag{3.4}$$

Given the Fourier transform convention for the corresponding field operator $\hat{\phi}$

$$\hat{\phi}(\tau,x) = \int \frac{d^3k}{(2\pi)^3}\hat{\phi}(\tau,k)e^{ik\cdot x}, \tag{3.5}$$

where the argument distinguishes the field operator in position space from its Fourier component, and the definition[1]

$$\frac{k^3}{2\pi^2}\langle\hat{\phi}(\tau,k)\hat{\phi}(\tau,k')\rangle := (2\pi)^3\delta^3(k+k')\mathcal{P}_\phi(\tau,k), \tag{3.6}$$

we can express the two point correlation function as

$$\begin{aligned}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,y)\rangle &= \int \frac{d^3k}{4\pi}\frac{\mathcal{P}_\phi(\tau,k)}{k^3}e^{ik\cdot(x-y)},\\ &= \int\frac{dk}{k}\mathcal{P}_\phi(\tau,k)\frac{\sin(kr)}{kr},\end{aligned} \tag{3.7}$$

where we have assumed statistical isotropy to perform the angular integrals, and $r := |x-y|$, and will work with the adiabatic vacuum associated with a given background in what follows. For inflationary spacetimes, this will be the usual Bunch Davies vacuum state. The coincident limit of Eq. 3.7 can be expressed as

$$\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \int_0^\infty \frac{dk}{k}\mathcal{P}_\phi(\tau,k), \tag{3.8}$$

and the coincident limit of Eq. 3.4 can similarly be expressed as

$$\lim_{y\to x}\rho(\tau;x,y) = \frac{1}{2a^2}\int_0^\infty\frac{dk}{k}\left[\left(k^2+a^2m^2\right)\mathcal{P}_\phi(\tau,k) + \mathcal{P}_{\phi'}(\tau,k)\right], \tag{3.9}$$

where we have used the shorthand

$$\frac{k^3}{2\pi^2}\langle\hat{\phi}'(\tau,k)\hat{\phi}'(\tau,k')\rangle := (2\pi)^3\delta^3(k+k')\mathcal{P}_{\phi'}(\tau,k). \tag{3.10}$$

In terms of canonically normalized mode functions $\phi_k(\tau)$ of the appropriate adiabatic vacuum, the free field operator admits the expansion

$$\hat{\phi}(\tau,k) = \hat{a}_k\phi_k(\tau) + \hat{a}^\dagger_{-k}\phi^*_k(\tau), \tag{3.11}$$

[1] Below and in the rest of what follows, the equal time expectation values are short hand for in-in correlation functions. For non-interacting test scalar fields evaluated in the adiabatic vacuum, this reduces to Eq. 3.6 and the following.

where $[\hat{a}_k, \hat{a}^\dagger_{-k'}] = (2\pi)^3\delta^3(k+k')$, so that

$$\mathcal{P}_\phi(\tau,k) = \frac{k^3}{2\pi^2}|\phi_k(\tau)|^2, \tag{3.12}$$

and so that Eq. 3.9 can be expressed as

$$\lim_{y\to x}\rho(\tau;x,y) = \frac{1}{4\pi^2a^2}\int_0^\infty k^2\,dk\left[(k^2+a^2m^2)|\phi_k(\tau)|^2+|\phi'_k(\tau)|^2\right]. \tag{3.13}$$

On a purely dS background, the mode functions for a massless field are given by

$$|\phi_k(\tau)|^2 = \frac{H^2}{2k^3}(1+k^2\tau^2), \qquad |\phi'_k(\tau)|^2 = \frac{H^2}{2k^3}k^4\tau^2, \tag{3.14}$$

so that the late time power spectrum for a massless test scalar is exactly scale invariant with amplitude determined from the Bunch Davies vacuum as $\mathcal{P}_\phi = \left(\frac{H}{2\pi}\right)^2$, where H is the Hubble factor that defines the dS background, so that Eq. 3.8 reduces to

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H}{2\pi}\right)^2\int_0^\infty\frac{dk}{k}. \tag{3.15}$$

This above is evidently divergent in both the UV and IR. Similar qualitative divergences arise when constructing the corresponding energy-momentum tensor, as we explore in the subsequent sections. Before delving into that, we take a detour on the different approaches one might adopt to regularize expressions like the above where important caveats immediately arise.

3.2.1 Regularization scheme (in)dependence

Given the absence of any scale in the integrand in Eq. 3.15, the integral nominally vanishes in mass independent regularization schemes such as dimensional or zeta function regularization (see e.g. the relevant chapters of [130]). Nevertheless, in what follows we show that it is still possible to cleanly separate UV and IR divergent contributions (using techniques borrowed from matching calculations in non-relativistic QCD and QED [61, 154]), and compare these to the contributions one would have calculated in mass dependent regularization schemes, such as through imposing hard cutoffs in physical momenta. This is useful, as UV and IR divergences have distinct physical interpretations – as discussed in Section 1.2.1, whereas UV divergences are to be subtracted with local counterterms, IR divergences cancel among themselves once physical observables are computed[2].
If instead one attempts to regularize the divergences by imposing hard cutoffs in both the UV and the IR, one is immediately presented with a choice as to precisely how. For instance, Burgess et al. [64] adopts cutoffs in terms of physical momenta on the basis that since IR divergences must cancel in all physical processes, the IR

[2] With the caveat that well defined observables can be identified, which presume that some of the IR divergences in question are not signaling the invalidity of the particular background quantization.

scales associated with these must be expressed in terms of physical scales. On the other hand, Baumgart and Sundrum [41] argue that it is more practical to adopt cutoffs in terms of comoving momenta for IR divergences given that the existence of a pre-inflationary phase ought to serve as a natural regulator where the distinction becomes less relevant (in the sense that it leads to only sub-leading corrections)[3]. In what follows we aim to show that using hard cutoffs in physical momenta works particularly straightforwardly for matching with the logarithmic UV divergences that one can extract from scaleless integrals. The expected scheme dependence drops out of physical quantities once renormalization conditions are imposed consistently[4], and both hard cutoffs in physical momenta and dimensional regularization lead to straightforward identification of the requisite counterterms. Simply put, even if one finds scheme dependence for the coefficients of nominal power law divergences, we show that one finds identical coefficients for UV divergent logarithms whether one dimensionally regularizes or uses hard cutoffs in physical momenta.

3.2.2 UV and IR divergences in mass (in)dependent schemes

Reconsider the scaleless integral Eq. 3.15, which can formally be rewritten as

$$\int_0^\infty \frac{dk}{k} = \int_0^\infty \frac{k^3 dk}{k^4} = \frac{1}{2\pi^2}\int_{-\infty}^\infty \frac{d^4k}{k^4}, \tag{3.16}$$

where the formal manipulations that result in the above render this integral to be Euclidean by default. We see that Eq.3.16 is a specific case of the more general form [61]

$$I_D(m^2) = \int \frac{d^Dk}{(2\pi)^D}\frac{k^{2A}}{(k^2+m^2)^B}. \tag{3.17}$$

Formally, Eq. 3.17 evaluates to[5]

$$I_D(m^2) = \frac{\Gamma\left(A+\frac{D}{2}\right)\Gamma\left(B-A-\frac{D}{2}\right)}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}\right)\Gamma(B)}\left(m^2\right)^{A-B+D/2}, \tag{3.18}$$

---

[3]Logarithmic divergences are special in that they involve the ratios of scales for some observable at some time, where the distinction between comoving and physical cutoffs becomes irrelevant.

[4]That is to say, a suitable counterterm can be subtracted that does not invalidate the background field quantization prescription one has presumed.

[5]The integral evaluates to $\left|\frac{x^{A+D/2}}{2A+D}\,{}_2F_1[B,\,A+D/2,\,1+A+D/2,\,-x]\right|_0^\infty$ with Eq. 3.18 coming from the upper limit along with additional divergent contributions from the upper and lower limits if $A-B+D/2>0$ or if $A+D/2<0$, which are subleading to the divergences in Eq. 3.18 for integer values of A and B as $D\to 4$.

which permits analytic continuation to non-integer values of D. Re-expressing the two point function Eq. 3.15 via Eq. 3.16 as[6]

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \frac{H^2}{8\pi^4}\int_{-\infty}^{\infty}\frac{d^4k}{k^4} = \frac{H^2}{8\pi^4}\int_{-\infty}^{\infty}d^4k\left[\frac{1}{k^2(k^2+m^2)}+\frac{m^2}{k^4(k^2+m^2)}\right], \tag{3.19}$$

we see via Eq. 3.18 that this evaluates in $D = 4-\delta$ dimensions to two equal and opposite contributions of the form

$$\pm\frac{H^2}{4\pi^2}\left[\frac{1}{\delta}-\frac{1}{2}\left(\log\frac{m^2}{4\pi\mu^2}+\gamma_E-1\right)\right], \tag{3.20}$$

where γ_E is the Euler-Mascheroni constant, and μ is some arbitrary mass scale necessitated by dimensional deformation. The sum of the two contributions can be written as

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H}{2\pi}\right)^2\left[\frac{1}{\delta_{\rm UV}}-\frac{1}{\delta_{\rm IR}}+\log\frac{\mu_{\rm UV}}{\mu_{\rm IR}}\right], \tag{3.21}$$

where we have artificially given a separate label to the dimensional deformation parameters δ and μ from the two terms to distinguish the UV and IR divergent contributions in Eq. 3.19. We do so to illustrate how dimensional regularization works in canceling two separately divergent contributions by default.
It is informative to compare this result with the outcome of imposing hard cutoffs. Whether we chose to do so in Eq. 3.15 in terms of physical momenta (where $\Lambda_{\rm UV/IR} = a\, k_{\rm UV/IR}$) or in terms of comoving momenta (where $\Lambda_{\rm UV/IR} = k_{\rm UV/IR}$) is immaterial in the context of logarithmic divergences which only sees ratios of these scales:

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H}{2\pi}\right)^2\log\left(\frac{k_{\rm UV}}{k_{\rm IR}}\right). \tag{3.22}$$

From this, we see that in spite of the fact that scaleless integrals vanish under dimensional regularization, factorizing them into a sum of scaleful integrals allows one to match the coefficients of the UV divergent logarithms in Eq. 3.21 with those in Eq. 3.22 obtained by imposing hard cutoffs in physical momenta. As we show in subsequent sections, this will be true no matter the background.

3.3 Non-interacting test scalar fields

In this section, we examine divergences in the two point correlation function and energy momentum tensor for non-interacting, minimally coupled massless test

[6]Dimensional regularization requires us to deform the dimension of every quantity that depends on spacetime dimension, including the mode functions on the background in question. This is especially crucial in the context of loop corrections in the presence of interactions [199, 84, 40, 59], where spuriously large loop corrections might otherwise be inferred. However, because we only consider bubble diagrams of non-interacting fields in what follows (with no external momenta), the result of doing so contributes only additional finite contributions and will be elided in what follows.

scalar fields on dS and quasi dS backgrounds, as well as that for sufficiently light massive scalar fields on a dS background. All divergent integrals turn out to be scaleless, a feature that persists even when one considers a background that transitions from a pre-inflationary epoch through to inflation and exiting to a terminal phase of radiation domination. Nevertheless, one can extract UV divergences via the techniques elaborated in section 3.2.2 and compare with what one would have obtained with a hard cutoff in physical momenta, recovering identical coefficients for the logarithmic divergences.

3.3.1 Regularization – (quasi) dS backgrounds

On backgrounds that deviate from dS, one has to be careful to factor in the difference in the scale factor from the pure dS form of $a(\tau) = -1/(H\tau)$. For constant but non-zero $\epsilon := -\dot{H}/H^2$, as is the case during power law inflation for instance, one has $a(\tau) = 1/[-\tau H_0]^{\nu-1/2}$, where $\nu = \frac{3-\epsilon}{2(1-\epsilon)}$ (and where the integration constant H_0 corresponds to the value of H at $t = 0$ in cosmological time – cf. [70]). The net result for the coincident limit of the two point function for a massless scalar field on a background corresponding to constant ϵ is given by

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H_0}{2\pi}\right)^2 \frac{\Gamma^2(\nu)}{\pi} 2^{2\nu-1} \int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{3-2\nu}\left[1+\frac{k^2\tau^2}{2(\nu-1)}+...\right], \tag{3.23}$$

or more generally,

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle \sim \left(\frac{H_0}{2\pi}\right)^2 \int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{n_s-1}\left[1+\left(\frac{k}{aH_0}\right)^2\frac{a^{n_s-1}}{2-n_s}+...\right] \tag{3.24}$$

where the ellipses denote terms that vanish in the $\tau \to 0$ limit and the $\sim$ indicates a numerical pre-factor that is close to unity for $n_s \approx 1$ (and tends to it in the dS limit).

In the dS limit, we find the relatively simple expression

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \left(\frac{H_0}{2\pi}\right)^2 \int_0^\infty \frac{dk}{k}\left[1+\left(\frac{k}{aH_0}\right)^2\right], \tag{3.25}$$

which supplements Eq. 3.15 with a sub-leading correction in the late time limit, notable relative to the analogous expression on quasi dS Eq. 3.24 in that is Eq. 3.25 exact for all times. The scaleless nature of Eq. 3.24 implies that it vanishes when regularized in any mass independent scheme. Instead, one might contemplate imposing a hard UV or IR cutoff in physical momenta by setting $k_{\rm UV/IR} = a\Lambda_{\rm UV/IR}$, and subtracting the UV divergence with the necessary counterterm. Doing so for the expression Eq. 3.25, one finds (see e.g. [64])

$$\begin{aligned}\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle &= \left(\frac{H_0}{2\pi}\right)^2\left[\int_{a\Lambda_{\rm IR}}^{a\Lambda_{\rm UV}}\frac{dk}{k}\left(1+\left(\frac{k}{aH_0}\right)^2\right)\right.\\ &\quad\left.-\int_{a\mu}^{a\Lambda_{\rm UV}}\frac{dk}{k}\left(1+\left(\frac{k}{aH_0}\right)^2\right)\right]\\ &= \left(\frac{H_0}{2\pi}\right)^2\left\{\log\left(\frac{\mu}{\Lambda_{\rm IR}}\right)+\frac{1}{2H_0^2}\left(\mu^2-\Lambda_{\rm IR}^2\right)\right\}, \qquad (n_s=1)\end{aligned} \tag{3.26}$$

where μ is some arbitrary renormalization scale. We note that the counterterm that subtracts the UV divergence in Eq. 3.26 is given by

$$\text{c.t.} = \left(\frac{H_0}{2\pi}\right)^2\left[\log\left(\frac{\mu}{\Lambda_{\rm UV}}\right)+\frac{1}{2H_0^2}\left(\mu^2-\Lambda_{\rm UV}^2\right)\right], \tag{3.27}$$

corresponding to a scheme dependent renormalization of the cosmological constant[7]. That these expressions are independent of time is a consequence of having imposed a physical cutoff on a dS invariant background. The remaining IR divergence is indicative of the fact that we have not yet arrived at something that can be processed into an observable quantity. It could end up being canceled by other contributions when we compute a well defined observable, or it could herald the need for resummation in which case it will also eventually get canceled. It could also indicate that the background we are attempting quantization around is not what it seems [76], and additional physical inputs are required in order to proceed. As we will show explicitly in what follows, were we to abandon the assumption that inflation was past dS eternal, then all IR divergences cancel among themselves and are in effect regulated by the scale corresponding to the beginning of inflation.

It is informative to compare what we would have obtained if we had dimensionally regularized Eq. 3.25 instead. We first note that the scaleless nature of the integral permits us to change variables to $q := k/(aH_0)$ for any fixed time. Hence the integral we need to evaluate can be factorized as in Eq. 3.19 as:

$$\frac{H_0^2}{8\pi^4}\int_{-\infty}^{\infty}\frac{d^4q}{q^4}(1+q^2) = \frac{H_0^2}{8\pi^4}\int_{-\infty}^{\infty}d^4q\left[\frac{1}{(q^2+\widetilde{m}^2)}+\frac{1+\widetilde{m}^2}{q^2(q^2+\widetilde{m}^2)}+\frac{\widetilde{m}^2}{q^4(q^2+\widetilde{m}^2)}\right] \tag{3.28}$$

where $\widetilde{m} := \mu/H_0$ is some auxiliary dimensionless mass scale. The first and second terms above are power law and log divergent in the UV, respectively, whereas the third term is IR divergent. The UV divergences must of course, be subtracted by an appropriate counterterm. One finds, after cancellations between the contributions

[7] We note that imposing a comoving cutoff in the limits of Eq. 3.26 instead of a physical cutoff would have resulted in a time dependence in the required counterterms (at odds with the dS invariance of the vacuum state) and would have necessitated further gymnastics to arrive at any physically meaningful quantity.

from the first and second terms of Eq. 3.28, a remaining UV divergence

$$\text{c.t.} = \left(\frac{H_0}{2\pi}\right)^2 \left\{\log\left(\frac{\mu}{H_0}\right) - \frac{1}{\delta_{\text{UV}}} + \frac{1}{2}\left(\gamma_E - 1 - \log 4\pi\right)\right\}, \tag{3.29}$$

which is to be compared to Eq. 3.27.

The situation for quasi dS is more complicated if one were to regularize it in a mass dependent scheme, as is evident from Eq. 3.24. Nevertheless, it is still worth elaborating upon, provided we simplify matters by further specifying that the background corresponds to that of eternal power-law inflation, defined by a constant but non-zero ϵ, so that $n_s - 1 = -2\epsilon - 2\epsilon^2 + \dots$. We will address the more realistic case of finite duration inflation next. One finds from inspection of the integrand of Eq. 3.24:

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{n_s-1}\left[1 + \left(\frac{k}{aH_0}\right)^2 \frac{a^{n_s-1}}{2-n_s}\right] \tag{3.30}$$

that the first term results

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{n_s-1} = \frac{1}{|n_s-1|}\left(\frac{a\Lambda}{H_0}\right)^{n_s-1}, \tag{3.31}$$

and the second term results

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{aH_0}\right)^{n_s+1}\frac{a^{2(n_s-1)}}{2-n_s} = \frac{a^{2(n_s-1)}}{(2-n_s)(n_s+1)}\left(\frac{\Lambda}{H_0}\right)^{n_s+1} \tag{3.32}$$

where whether Λ is an IR or UV cutoff depends on the value of n_s. By rewriting Eq. 3.24 as a function of the slow roll parameter ϵ

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle \sim \left(\frac{H_0}{2\pi}\right)^2 \int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{-2\epsilon}\left[1 + \left(\frac{k}{aH_0}\right)^2\frac{a^{-2\epsilon}}{1+2\epsilon}\right] \tag{3.33}$$

we find that in the quasi dS limit ($\epsilon \ll 1$) the first term gives a logarithmic divergent contribution

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{-2\epsilon} = \frac{1}{2\epsilon} + \log\frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}} + \mathcal{O}(\epsilon) \tag{3.34}$$

and the second term a power law UV divergence

$$\int_0^\infty \frac{dk}{k}\left(\frac{k}{aH_0}\right)^{2(1-\epsilon)}\frac{a^{-4\epsilon}}{1+2\epsilon} = \frac{a^{-4\epsilon}}{2}\left(\frac{\Lambda_{\text{UV}}}{H_0}\right)^2. \tag{3.35}$$

The interpretation of the IR divergences is as discussed above, while the UV divergence must be subtracted with a local counterterm if we are to obtain physically meaningful quantities. For small ϵ, the counterterm is given by

$$\begin{aligned} \text{c.t.} &= \lim_{\tau\to 0}\left(\frac{H_0}{2\pi}\right)^2\left[\frac{1}{2\epsilon}+\log\frac{\mu}{\Lambda_{\rm UV}}+\frac{a^{-4\epsilon}}{2}\left(\left(\frac{\mu}{H_0}\right)^2-\left(\frac{\Lambda_{\rm UV}}{H_0}\right)^2\right)+\mathcal{O}(\epsilon)\right] \\ &= \lim_{\tau\to 0}\left(\frac{H_0}{2\pi}\right)^2\left[\frac{1}{2\epsilon}+\log\frac{\mu}{\Lambda_{\rm UV}}-\frac{a^{-4\epsilon}}{2}\frac{\Lambda_{\rm UV}^2}{H_0^2}+\mathcal{O}(\epsilon)\right] \end{aligned} \tag{3.36}$$

where again μ is some renormalization scale and given that $\Lambda_{\rm UV} \gg \mu$, we have neglected the sub-leading dependence on μ in the second equality.
We first note that the required counterterm is time dependent, which should not come as a surprise, given that the background is no longer maximally symmetric. It behooves us to elaborate on the specific form of the local counterterm that could possibly have this particular time dependence. Noting that on the specified power law inflating background, $\dot{H}, H^2 \sim a^{-2\epsilon}$, so that curvature squared invariants evaluated on this background have the required secular dependence. That is, Eq. 3.36 derives from

$$\text{c.t.} \subset \int d^4x\sqrt{-g}\left[c_1(\mu)R^2+c_2(\mu)R_{\mu\nu}R^{\mu\nu}\right], \tag{3.37}$$

which are the usual leading curvature squared counterterms encountered in the effective field theory treatment of gravity [90, 91, 92, 60].

As before, we can retrace the same computation using dimensional regularization. We first consider the separate contributions to the integrand Eq. 3.33. The logarithmic divergent contribution can be re-expressed as

$$\begin{aligned} \int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{-2\epsilon} &= \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{d^4q}{q^4}q^{-2\epsilon} \\ &= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d^4q\left[\frac{q^{-2\epsilon}}{q^2(q^2+\widetilde{m}^2)}+\frac{\widetilde{m}^2 q^{-2\epsilon}}{q^4(q^2+\widetilde{m}^2)}\right], \end{aligned} \tag{3.38}$$

where we have switched to the dimensionless variable $q := k/H_0$, and similarly for $\widetilde{m} := \mu/H_0$. By isolating the UV pole, we find that the presence of the non-integer exponents eliminates the δ poles and we obtain that in the quasi dS limit the first term results [8]

$$\left(\frac{H_0}{2\pi}\right)^2\int_0^\infty\frac{dk}{k}\left(\frac{k}{H_0}\right)^{-2\epsilon} = \left(\frac{H_0}{2\pi}\right)^2\left[\frac{1}{2\epsilon}-\log\frac{\mu}{H_0}+\frac{\epsilon}{12}\left(\pi^2+24\log\frac{\mu}{H_0}\right)\right]. \tag{3.39}$$

[8] Note that we are required to take the limits in the order $\delta \to 0$ and then $\epsilon \to 0$. To do the opposite would mean to have regularized on a dS background first and then deformed the background in the hopes that no new divergences will have appeared through this process. One can show that this is not justified on general grounds [48], and is seen directly from the proliferation of higher order δ poles encountered when taking the limits in the opposite order.

The second term of Eq. 3.33 can be rewritten as

$$\begin{aligned}\frac{a^{-4\epsilon}}{1+2\epsilon}\int_0^\infty \frac{dk}{k}\left(\frac{k}{aH_0}\right)^{2(1-\epsilon)} &= \frac{a^{-4\epsilon}}{1+2\epsilon}\frac{1}{2\pi^2}\int_{-\infty}^\infty \frac{d^4q}{q^4}q^{2(1-\epsilon)} \\ &= \frac{a^{-4\epsilon}}{1+2\epsilon}\frac{1}{2\pi^2}\int_{-\infty}^\infty d^4q\left[\frac{q^{2(1-\epsilon)}}{q^2(q^2+\widetilde{m}^2)}+\frac{\widetilde{m}^2q^{2(1-\epsilon)}}{q^4(q^2+\widetilde{m}^2)}\right],\end{aligned} \tag{3.40}$$

where just as we did on dS space, we work with dimensionless variables $q := k/aH_0$ and $\widetilde{m} := \mu/H_0$. Using Eq. 3.18 we see that power law UV divergences cancel among themselves, and so do not necessitate any counterterms. In conclusion, we find that the counterterm in the quasi dS limit results

$$\text{c.t.} = \lim_{\tau\to 0}\left(\frac{H_0}{2\pi}\right)^2\left[\frac{1}{2\epsilon}+\log\frac{\mu}{H_0}+\mathcal{O}(\epsilon)\right]. \tag{3.41}$$

The generalization to massive fields is straightforward on dS backgrounds, where analytic expressions for the mode functions can be obtained:

$$|\phi_k(\tau)|^2 = \frac{\pi}{4}H_0^2(-\tau)^3|H^{(1)}_{\nu_m}(-k\tau)|^2, \tag{3.42}$$

where

$$\nu_m^2 := \frac{9}{4}-\frac{m^2}{H_0^2}, \tag{3.43}$$

and where $H^{(1)}_{\nu_m}$ is the corresponding Hankel function with degree ν_m. Presuming $0 < m^2 \leq 9H_0^2/4$ so that ν_m is still real, the coincident limit of the two point function at late times can then be computed as

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \frac{H_0^2 2^{2\nu_m}\Gamma^2(\nu_m)}{8\pi^3}\int_0^\infty\frac{dk}{k}\left[\left(\frac{k}{aH_0}\right)^{3-2\nu_m}+\frac{(k/aH_0)^{5-2\nu_m}}{2\,(\nu_m-1)}\right]. \tag{3.44}$$

It is notable that in spite of the presence of the additional mass scale m, the integral remains scaleless[9]. Here also, we can make the change of variable to $q = k/(aH)$, so that the above can be recast as

$$\lim_{\tau\to 0}\langle\hat{\phi}(\tau,x)\hat{\phi}(\tau,x)\rangle = \frac{H_0^2}{16\pi^5}2^{2\nu_m}\Gamma^2(\nu_m)\int_{-\infty}^\infty\frac{d^4q}{q^4}q^{3-2\nu_m}\left[1+\frac{q^2}{2(\nu_m-1)}\right], \tag{3.45}$$

where the first thing to note is that the IR divergence encountered for a non-interacting massless scalar on dS is eliminated given that $\nu_m < 3/2$[10]. Proceeding exactly as before, we can compare the UV divergence obtained from imposing a hard

[9]This can be understood from the fact that on dS, the quantity nominally labeled mass in Eq. 3.43 is constructed from eigenvalues of the Casimir operators corresponding to conformal dimension and spin expressed in units of the dS radius, which remains the only scale in the problem.

[10]A short-lived conclusion on dS backgrounds the minute any interactions are incorporated.

cutoff in physical momenta to that obtained by factorizing the scaleless integral in dimensional regularization. In the limit $m^2/H_0^2 \ll 1$, one finds that the required counterterms are given by

$$\text{c.t.} = \frac{H_0^2}{8\pi^2}\frac{2^{2\nu_m}\Gamma^2(\nu_m)}{\pi}\left[\frac{3H_0^2}{m^2} + \log\frac{\mu}{\Lambda_{\rm UV}} - \frac{\Lambda_{\rm UV}^2}{4H_0^2} + ...\right] \qquad \text{(physical cutoff)} \quad (3.46)$$

and

$$\text{c.t.} = \frac{H_0^2}{8\pi^2}\frac{2^{2\nu_m}\Gamma^2(\nu_m)}{\pi}\left[\frac{3H_0^2}{m^2} + \log\frac{\mu}{H_0} + ...\right] \qquad \text{(dim reg)} \quad (3.47)$$

where again, one takes the limit $\delta \to 0$ to regularize the integrals before taking the limit $m^2/H_0^2 \to 0$. In both cases, the required counterterm corresponds to a renormalization of the cosmological constant.

3.3.2 Regularization – stress energy tensor

Having labored over the details of regularizing and renormalizing divergences for the two point function of test scalar fields, repeating the exercise for the associated stress tensor may seem a straightforward extension[11]. However, we immediately encounter a difficulty: it is not always possible to construct a counterterm that subtracts the encountered UV divergences from background geometric invariants. This appear to be the case if the regularization scheme itself does not respect this symmetry. This is not to a concern to be casually dismissed[12], as persisting with this regularization scheme could lead us to the erroneous conclusion that the background we attempted to quantize around is transmuting into something else under renormalization if not worked through to the end. It is here that dimensional and related zeta function regularization techniques distinguish themselves in theories with general covariance, as they effortlessly preserve the symmetries of the background even if the precise mechanism by which this occurs can seem non-trivial. For this reason, it is worth explicitly tracing through this process in detail[13].

Consider a minimally coupled non-interacting test scalar field on dS space in the Bunch Davies vacuum. The rotational invariance of the vacuum state allow us to write the non-vanishing components as

$$\rho = \frac{1}{4\pi^2 a^2}\int_0^\infty k^2\, dk\left[\left(k^2 + a^2m^2\right)|\phi_k(\tau)|^2 + |\phi_k'(\tau)|^2\right], \qquad (3.48)$$

and

$$p = \frac{1}{4\pi^2 a^2}\int_0^\infty k^2\, dk\left[-\left(\frac{k^2}{3} + a^2m^2\right)|\phi_k(\tau)|^2 + |\phi_k'(\tau)|^2\right], \qquad (3.49)$$

[11] See Section 1.2.3 for an introduction on divergences arising in computing the stress energy tensor.
[12] An issue that did not arise for the two-point function given that its nature as a scalar bilinear.
[13] Nevertheless, it remains true that the coefficients of the UV divergent logarithms would match regardless of whether one dimensionally regularizes or one imposes hard cutoffs in physical momenta.

with all other components of the energy momentum tensor vanishing. From the asymptotic forms of Eq. 3.42, one can infer that both integrals are UV divergent. The energy density in the massless limit is straightforwardly computed via Eq. 3.14, and found to be

$$\rho = \frac{H^4}{8\pi^2}\int_0^\infty \frac{dk}{k}\left[\left(\frac{k}{aH}\right)^2 + 2\left(\frac{k}{aH}\right)^4\right], \tag{3.50}$$

and the corresponding pressure is found to be

$$p = \frac{H^4}{8\pi^2}\int_0^\infty \frac{dk}{k}\left[-\frac{1}{3}\left(\frac{k}{aH}\right)^2 + 2\left(\frac{k}{aH}\right)^4\right]. \tag{3.51}$$

Despite one can regularize both components through imposing physical cutoffs, one is immediately confronted by the fact that the required counterterm for the leading divergences will not be proportional to the metric nor the Ricci tensor on the presumed dS background (which would correspond either to renormalizations of the cosmological and Newton's constants). This is not the case when one implements dimensional regularization. Although the fully covariant formalism to regularize stress tensors using mass independent schemes is largely studied in the literature (as elaborated upon in [48, 178]), it is nevertheless informative to trace through this process.

We consider the example of a massive scalar field on an FRLW background, and consider the analogs of Eqs. 3.48 and 3.49 in cosmic time where we can neglect the effects of background expansion. This is in order to be facilitate working with transparent analytic expressions that, moreover, are non-vanishing (as would be the case for a massless scalar field), with the further justification that as we are only interested in computing the relevant counterterms to subtract UV divergences and can thus be forgiven for this approximation for illustrative purposes. Details of how one dimensional regularizes energy momentum tensors for more realistic examples in a fully covariant approach can be found in e.g. [48, 178]. What follows below closely tracks the treatment of [74, 133].
Working in $D = 4 - \delta$ dimensions, where the background metric is of the FRLW form in cosmic time, one finds the following expressions for the vacuum expectation values of the stress tensor:

$$\begin{aligned}\langle \hat{T}_{00}\rangle &= \frac{\mu^{4-D}}{2a^{D-1}(2\pi)^{D-1}}\int \mathrm{d}^{D-1}k\sqrt{m^2+\frac{k^2}{a^2}}\\ \langle \hat{T}_{ii}\rangle &= \frac{\mu^{4-D}}{2a^{D-1}(2\pi)^{D-1}}\frac{1}{D-1}\int \mathrm{d}^{D-1}k\frac{k^2}{\sqrt{m^2+\frac{k^2}{a^2}}}.\end{aligned} \tag{3.52}$$

Both components have the same degree of divergence in the UV, but with different coefficients. Imposing hard cutoffs in physical momenta would necessitate a counterterm that cannot be constructed from background geometric invariants. Instead, we proceed via Eq. 3.18, with $A = 0$, $B = -\frac{1}{2}$ in the first integral above, and $A = 1$,

$B = \frac{1}{2}$ for the second integral, to find

$$\int \mathrm{d}^{D-1}k\sqrt{m^2 + \frac{k^2}{a^2}} = \frac{(ma)^D}{a}\frac{(2\pi)^{D-1}}{(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}$$
$$\int \mathrm{d}^{D-1}k\frac{k^2}{\sqrt{m^2 + \frac{k^2}{a^2}}} = a\,(ma)^D\frac{(2\pi)^{D-1}}{(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{1}{2}+\frac{D}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}, \tag{3.53}$$

so that the dimensionally regularized components of the stress energy tensor become

$$\begin{aligned}\langle\hat{T}_{00}\rangle &= \frac{\mu^{4-D}}{2a^{D-1}(2\pi)^{D-1}}\frac{(ma)^D}{a}\frac{(2\pi)^{D-1}}{(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}\\ &= \frac{\mu^{4-D}}{2(4\pi)^{\frac{D-1}{2}}}\frac{(ma)^D}{a^D}\frac{\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)}\\ &= \frac{\mu^4}{2(4\pi)^{\frac{D-1}{2}}}\left(\frac{m}{\mu}\right)^D\frac{\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)},\end{aligned} \tag{3.54}$$

for the energy density, and

$$\begin{aligned}\langle\hat{T}_{ii}\rangle &= \frac{\mu^{4-D}}{2a^{D-1}(2\pi)^{D-1}}\frac{1}{D-1}a\,(ma)^D\frac{(2\pi)^{D-1}}{(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{1}{2}+\frac{D}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}\\ &= \frac{\mu^{4-D}}{2(4\pi)^{\frac{D-1}{2}}}\frac{1}{D-1}\frac{a\,(ma)^D}{a^{D-1}}\frac{\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}\\ &= -\frac{\mu^4a^2}{2(4\pi)^{\frac{D-1}{2}}}\left(\frac{m}{\mu}\right)^D\frac{\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)},\end{aligned} \tag{3.55}$$

for the pressure components, where in the last equality we have used $\Gamma(x+1) = x\Gamma(x)$ to re-express

$$\frac{\Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)} = -\frac{D-1}{\Gamma\left(-\frac{1}{2}\right)}. \tag{3.56}$$

From this, we see that the stress tensor has a divergence of the form

$$\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{div}} = -g_{\mu\nu}\frac{m^4}{64\pi^2}\left\{\frac{2}{\delta} - \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma_E - \frac{3}{2}\right\}, \tag{3.57}$$

which can straightforwardly be subtracted by a cosmological constant-like countert-erm. For this reason, regularization schemes that preserve general covariance such as dimensional or zeta function regularization are preferred in the context of renor-malizing the stress energy tensor. In spite of this caveat, however, one can still be forgiven for using hard cutoffs in physical momenta when restricted to calculating logarithmic divergences, as one can show the result to be identical to what would have been obtained in a mass independent scheme. We will return to this point in regards the regularization of the stress tensor for GWs in the next chapter.

3.3.3 Regularization – finite duration inflation

Having acquainted ourselves with the basics of regularizing and renormalizing divergences for familiar quantities, one would like to generalize this to backgrounds that realistically model the universe we inhabit. In particular, one might wonder if the fact that we only obtained scaleless integrals in the previous subsections is an artifact of considering the unrealistic scenario of infinite duration inflation. We show that this situation persists even when we consider a cosmology that transitions between different epochs.

In order for inflation to have finite duration by definition, we must be considering an epoch before and after inflation, which we choose to be radiation domination in both cases. We consider the scale factor to evolve as

$$\begin{aligned} a(\tau) &= a_{\rm R}\left(2-\frac{\tau}{\tau_{\rm I}}\right)e^{-\mathcal{N}_{\rm tot}} \qquad \tau<\tau_{\rm I} \\ &= a_{\rm R}\left(\frac{\tau_{\rm I}}{\tau}\right)e^{-\mathcal{N}_{\rm tot}} \qquad \tau_{\rm I}<\tau<\tau_{\rm R} \\ &= a_{\rm R}\left(2-\frac{\tau}{\tau_{\rm R}}\right) \qquad \tau_{\rm R}<\tau \end{aligned} \tag{3.58}$$

where $a_{\rm R}$ is the scale factor at reheating, and $\tau_{\rm I}$ and $\tau_{\rm R}$ correspond to the (negative) conformal time at the start and end of inflation[14] respectively, and

$$\mathcal{N}_{\rm tot} = \log(a_{\rm R}/a_{\rm I}) = \log(\tau_{\rm I}/\tau_{\rm R}) \tag{3.59}$$

is the total amount e-folds of inflation. The domain of τ is $(-\infty,\infty)$ with inflation occurring during negative conformal time. With these definitions, the Hubble rate during inflation is given by

$$H = -\frac{1}{a_{\rm R}\tau_{\rm R}}. \tag{3.60}$$

In what follows, we keep the normalization $a_{\rm R}$ arbitrary, although one can readily set $a_{\rm R}\equiv 1$ for convenience in what follows. Here again, we focus on a massless, minimally coupled non-interacting scalar field for illustrative purposes. The mode functions during the terminal stage of radiation domination can be rewritten for the purposes of a matching to the end of inflation as

$$\begin{aligned} \phi_k^{\rm RD} &= \frac{1}{a}\frac{1}{\sqrt{2k}}\left[\alpha_k^{\rm R}e^{-ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)} + \beta_k^{\rm R}e^{ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\right] \\ \phi_k'^{\mathcal{RD}} &= \frac{a_{\rm R}}{a^2\tau_{\rm R}\sqrt{2k}}\left[\alpha_k^{\rm R}e^{-ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1-i\frac{ak\tau_{\rm R}}{a_{\rm R}}\right) + \beta_k^{\rm R}e^{ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1+i\frac{ak\tau_{\rm R}}{a_{\rm R}}\right)\right]. \end{aligned} \tag{3.61}$$

[14] Note that this puts the initial singularity at $\tau = 2\tau_{\rm I}$, however, the only manner in which the pre-inflationary phase bears on late time observables is through the choice to begin with the adiabatic vacuum evolved to the start of the inflationary epoch at $\tau_{\rm I}$.

The mode functions during inflation are given by

$$\begin{aligned}\phi_k^{\rm I} &= \frac{H}{\sqrt{2k^3}}\left[\alpha_k^{\rm I} e^{i\frac{k}{aH}}\left(1-\frac{ik}{aH}\right)+\beta_k^{\rm I} e^{-i\frac{k}{aH}}\left(1+\frac{ik}{aH}\right)\right]\\ \phi_k'^{\mathcal{I}} &= -\frac{H}{\sqrt{2k^3}}\left[\alpha_k^{\rm I} e^{i\frac{k}{aH}}\frac{k^2}{aH}+\beta_k^{\rm I} e^{-i\frac{k}{aH}}\frac{k^2}{aH}\right],\end{aligned} \tag{3.62}$$

where the presence of non-trivial Bogoliubov coefficients come from having matched to a radiation dominated pre-inflationary phase initiated in the adiabatic vacuum, where the mode functions are given by

$$\begin{aligned}\phi_k^{\rm PI} &= \frac{1}{a}\frac{1}{\sqrt{2k}}e^{-ik\tau_{\rm I}\left(2-\frac{a}{a_{\rm R}}e^{\mathcal{N}_{\rm tot}}\right)}\\ \phi_k'^{\mathcal{PI}} &= -\frac{1}{a^2\sqrt{2k}}\frac{a_{\rm I}}{\tau_{\rm I}}e^{-ik\tau_{\rm I}\left(2-\frac{a}{a_{\rm R}}e^{\mathcal{N}_{\rm tot}}\right)}\left(-1+ik\tau_{\rm I}\frac{a}{a_{\rm I}}\right).\end{aligned} \tag{3.63}$$

One first obtains $\alpha_k^{\rm I}$ and $\beta_k^{\rm I}$ by matching to the pre-inflationary RD epoch, so that

$$\begin{aligned}\alpha_k^{\rm I} &= \left(i-i\frac{a_{\rm I}^2H^2}{2k^2}+\frac{a_{\rm I}H}{k}\right)\\ \beta_k^{\rm I} &= i\frac{a_{\rm I}^2H^2}{2k^2}e^{2i\frac{k}{a_{\rm I}H}}.\end{aligned} \tag{3.64}$$

Similarly, after the matching between inflation and RD era, one finds that $\alpha_k^{\rm R}$ and $\beta_k^{\rm R}$ are given by

$$\begin{aligned}\alpha_k^{\rm R} &= \alpha_k^{\rm I}\left(-i+\frac{a_{\rm R}H}{k}+i\frac{a_{\rm R}^2H^2}{2k^2}\right)+i\beta_k^{\rm I}\frac{a_{\rm R}^2H^2}{2k^2}e^{-2i\frac{k}{a_{\rm R}H}}\\ \beta_k^{\rm R} &= -i\alpha_k^{\rm I}\frac{a_{\rm R}^2H^2}{2k^2}e^{2i\frac{k}{a_{\rm R}H}}+\beta_k^{\rm I}\left(i+\frac{a_{\rm R}H}{k}-i\frac{a_{\rm R}^2H^2}{2k^2}\right).\end{aligned} \tag{3.65}$$

We now use these results to find expressions for the coincident limits of the two point function and ρ during the terminal stage of radiation domination. The two point correlation function coincidence limit can be expressed as

$$\begin{aligned}\lim_{x\to y}\langle\hat\phi(\tau,x)\hat\phi(\tau,y)\rangle &= \frac{1}{2\pi^2a^2}\int_0^\infty\frac{dk}{k}\frac{k^2}{2}\left[1+2|\beta_k^{\rm R}|^2\right.\\ &\quad\left.+\alpha_k^{\rm R}\beta_k^{{\rm R}*}e^{\frac{2ik}{a_{\rm R}H}\left(2-\frac{a}{a_{\rm R}}\right)}+\alpha_k^{{\rm R}*}\beta_k^{\rm R}e^{-\frac{2ik}{a_{\rm R}H}\left(2-\frac{a}{a_{\rm R}}\right)}\right]\\ &= \frac{1}{2\pi^2a^2}\int_0^\infty\frac{dk}{k}\frac{k^2}{2}\left[1+2|\beta_k^{\rm R}|^2_{\rm power}+\{\rm osc\}\right]\end{aligned} \tag{3.66}$$

where the first line has used $|\alpha_k^{\rm I}|^2-|\beta_k^{\rm I}|^2=|\alpha_k^{\rm R}|^2-|\beta_k^{\rm R}|^2=1$, and the second line splits the integrand into strictly power law contributions and oscillatory contributions. From Eqs. 3.64 and 3.65, we find that

$$|\beta_k^{\rm R}|^2_{\rm power}=\frac{a_{\rm R}^4H^4+a_{\rm I}^4H^4}{4k^4}+\frac{a_{\rm I}^4a_{\rm R}^4H^8}{8k^8}, \tag{3.67}$$

which nominally appears to have aggravated the IR divergence of the two point function. Although the oscillatory terms can indeed be neglected for the purposes of extracting UV divergences, the oscillations freeze in the IR and exactly cancel the contributions from $|\beta_k^{\rm R}|^2_{\rm power}$. Specifically, one finds that

$$\lim_{k\to 0}\{{\rm osc}\} = -\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4} - \frac{a_{\rm I}^4 a_{\rm R}^4 H^8}{4k^8} - 1 + \frac{(3a_{\rm I}^3 a_{\rm R} + 2a(a_{\rm R}^3 - a_{\rm I}^3))^2}{9a_{\rm I}^2 a_{\rm R}^6}, \tag{3.68}$$

which cancels the contributions from $1 + 2|\beta_k^{\rm R}|^2_{\rm power}$. The UV divergence resulting from the final term of Eq. 3.68 to reckon with:

$$\lim_{x\to y}\langle\hat\phi(\tau,x)\hat\phi(\tau,y)\rangle_{\rm div} = e^{-4\mathcal{N}_{\rm tot}}\left(3 + \frac{2a}{a_{\rm R}}\left[e^{3\mathcal{N}_{\rm tot}-1}\right]\right)^2 \frac{1}{36\pi^2 a^2}\int_0^\infty dk\, k, \tag{3.69}$$

where as before, $e^{-\mathcal{N}_{\rm tot}} = a_{\rm I}/a_{\rm R}$. Two things are to be immediately noted. Firstly, that the physical UV and IR scales $k_{\rm IR/UV} = a_{\rm I/R}H$ associated with the beginning and end of inflation do not by themselves regulate the UV divergences. Instead, they merely parametrize the divergence through their ratio $k_{\rm IR}/k_{\rm UV} = e^{-\mathcal{N}_{\rm tot}}$ in the pre-factor above. Furthermore, the pre-factor itself diverges as $\mathcal{N}_{\rm tot}\to\infty$, indicative of the restoration of the divergences arising considering a past infinite dS space in that limit, where, moreover, a logarithmic IR divergence reappears. This is perhaps best illustrated by Fig. 3.1 where we plot all contributions of the power spectrum for finite duration inflation (identified through the logarithmic integrand of Eq. 3.66) along with what would have resulted for a past-infinite dS cosmology matched to a terminal phase of radiation domination[15]. The UV divergence is unsurprisingly unchanged, and can be subtracted with a cosmological constant counterterm.

In computing the energy density ρ, we begin with

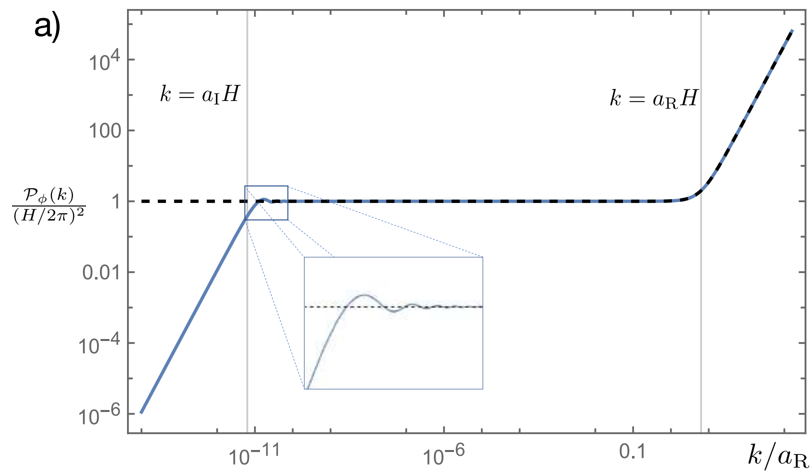
$$\lim_{y\to x}\rho(\tau;x,y) = \frac{1}{4\pi^2 a^2}\int_0^\infty k^2\, dk\left[k^2|\phi_k^{\rm RD}(\tau)|^2 + |\phi_k^{\rm RD\prime}(\tau)|^2\right] \tag{3.70}$$

and use Eq. 3.61 to express the energy density during radiation domination as

$$\begin{aligned}\rho &= \frac{1}{8\pi^2 a^4}\int_0^\infty \frac{dk}{k}\left[k^4\left(2 + \frac{a_{\rm R}^4 H^2}{a^2k^2}\right)\left(1 + 2|\beta_k^{\rm R}|^2_{\rm power}\right) + \{{\rm osc}\}\right]\\ &:= \int_0^\infty \frac{dk}{k}\left[\Omega^\phi_{\rm power}(k) + \Omega^\phi_{\rm osc}(k)\right]\end{aligned} \tag{3.71}$$

where the integrated contribution of $\Omega^\phi_{\rm osc}$, defined from $\rho_{\rm osc}$ that includes the oscillatory contributions coming from $\alpha_k^{\rm R*}\beta_k^{\rm R}$ as well as from computing the modulus

[15] The limit of a pre-inflationary phase of infinite duration inflation can straightforwardly be read off from inserting Eq. 3.65 into Eq. 3.64 with $a_{\rm I}\equiv 0$.



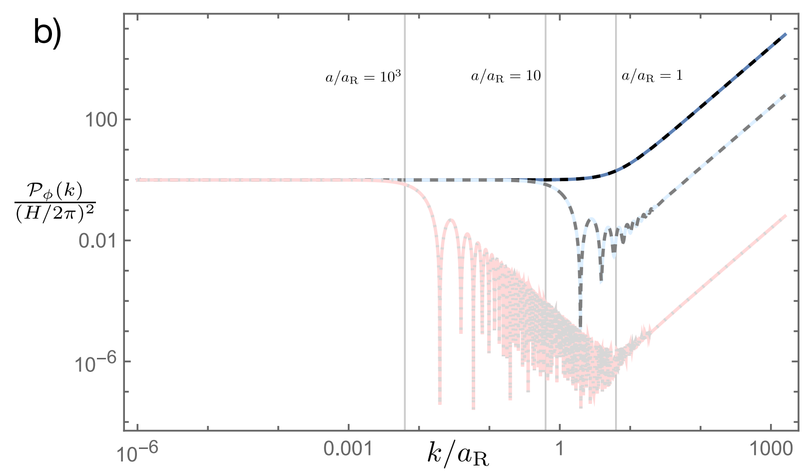


Figure 3.1: Comparison power spectra massless test scalar field

The graphs show the comparison between the power spectrum of a massless test scalar field with past infinite dS inflation (dashed line) vs finite duration dS inflation (bold line) matched to a terminal stage of radiation domination. Finite duration inflation cures the would be IR divergences, leaving only UV divergences to attend to.
a): Power spectra evaluated at reheating $a = a_{\rm R}$, where $a_{\rm I} = 10^{-12} a_{\rm R}$ in units where H is set to 2π. **b)**: Power spectra evaluated at different times during radiation domination.

square of $|\beta_k^{\rm R}|^2$, is given by

$$\begin{aligned}\rho_{\rm osc} =& \frac{1}{4\pi^2 a^4}\int_0^\infty dk k^2 \left(k+\frac{a_{\rm R}^4 H^2}{a^2 2k}\right)\left[-i\alpha_k^{\rm I}\beta_k^{\rm I*}e^{2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2 H^2}{k^2}\left(-i+\frac{a_{\rm R}H}{k}+i\frac{a_{\rm R}^2H^2}{2k^2}\right)\right.\\ &\left.+i\alpha_k^{\rm I*}\beta_k^{\rm I}e^{-2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2}\left(i+\frac{a_{\rm R}H}{k}-i\frac{a_{\rm R}^2H^2}{2k^2}\right)\right]\\ &+\frac{1}{4\pi^2a^4}\int_0^\infty dk\, k^2\left[\alpha_k^{\rm R}\beta_k^{\rm R*}e^{-2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(\frac{k}{2}+\frac{a_{\rm R}^4H^2}{a^2 2k}\left(1+i\frac{ka}{a_{\rm R}^2H}\right)^2\right)\right.\\ &\left.+\alpha_k^{\rm R*}\beta^{\rm R}e^{2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(\frac{k}{2}+\frac{a_{\rm R}^4H^2}{a^2 2k}\left(1-i\frac{ka}{a_{\rm R}^2H}\right)^2\right)\right].\end{aligned} \tag{3.72}$$

Making use of Eqs. 3.64 and 3.65, one can integrate the various terms and expand the result in the limit $k\to\infty$, upon which we obtain

$$\begin{aligned}\rho_{\rm osc} =& \frac{1}{4\pi^2a^4}\lim_{k\to\infty}\left[\left(e^{\left(-\frac{2ik(a_{\rm I}-a_{\rm R})}{Ha_{\rm R}^2}\right)}+e^{\left(\frac{2ik(a_{\rm I}-a_{\rm R})}{Ha_{\rm R}^2}\right)}\right)\left(-\frac{H^4a_{\rm R}^6}{4a^2-4aa_{\rm R}}\right)\right.\\ &\left.+\left(e^{\left(-\frac{2ik(aa_{\rm I}-2a_{\rm I}a_{\rm R}+a_{\rm R}^2)}{Ha_{\rm I}a_{\rm R}^2}\right)}+e^{\left(\frac{2ik(aa_{\rm I}2a_{\rm I}a_{\rm R}+a_{\rm R}^2)}{Ha_{\rm I}a_{\rm R}^2}\right)}\right)\left(\frac{H^4a_{\rm I}^3a_{\rm R}^4}{4a\left(aa_{\rm I}-2a_{\rm I}a_{\rm R}+a_{\rm R}^2\right)}\right)\right]\end{aligned} \tag{3.73}$$

up to terms of order $\sim\frac{1}{k}$. We note that this contribution is manifestly finite, and oscillatory in the upper limit in a manner that would be eliminated via the analog of the $i\epsilon$ prescription for the Minkowski space limit of all the two point propagators in the in-in formalism. Then, as was the case with the two point function, the oscillatory contributions are negligible in the UV and can be neglected in the UV-regularization. On the other hand, they become relevant when the oscillations freeze as $k\to 0$, also softening the IR behavior for the spectral power density relative to what would have been on a past infinite dS background. Indeed, the contributions form Eq. 3.67 that nominally appear to aggravate IR divergences in the context of Eq. 3.71 are canceled by contributions from the oscillatory terms that freeze out in the IR. One can expand the sum of all contributions in the IR to find the IR safe scaling $\Omega^\phi_{\rm tot}\propto k^4$, to be compared to what one would have obtained if there had been no pre-inflationary phase, where $\Omega^\phi_{\rm tot}\propto k^2$. We illustrate this behavior and the processing of the logarithmic spectra power density as we go deeper into the radiation dominated regime in Fig. 3.2. The sub-horizon decay compensates for the k^2 scaling for the modes that exited the Hubble horizon during inflation to produce a scale invariant spectral density during radiation domination. We further note the additional long wavelength suppression of the spectral density for a pre-inflationary phase relative to past eternal dS.

Returning to Eq. 3.71, we see that upon inserting Eq. 3.67, the only divergences that need to be regulated are given by the contributions

$$\rho_{\rm div} = \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k\left(1+\frac{H^4a_{\rm R}^4+H^4a_{\rm I}^4}{2k^4}\right)+\frac{1}{8\pi^4 a^6}\int_{-\infty}^{\infty} d^4k\frac{a_{\rm R}^4H^2}{2k^2}, \tag{3.74}$$

where again we recast the logarithmic integration into the formally equivalent 4D Euclidean form as per Eq. 3.16, and change integration variables to physical momenta $q = k/(a_{\rm R}H)$. We stress that all factors of H in the expressions above correspond to the Hubble parameter during the intermediate phase on inflation, which is presumed dS and therefore fixed as per Eq. 3.60.

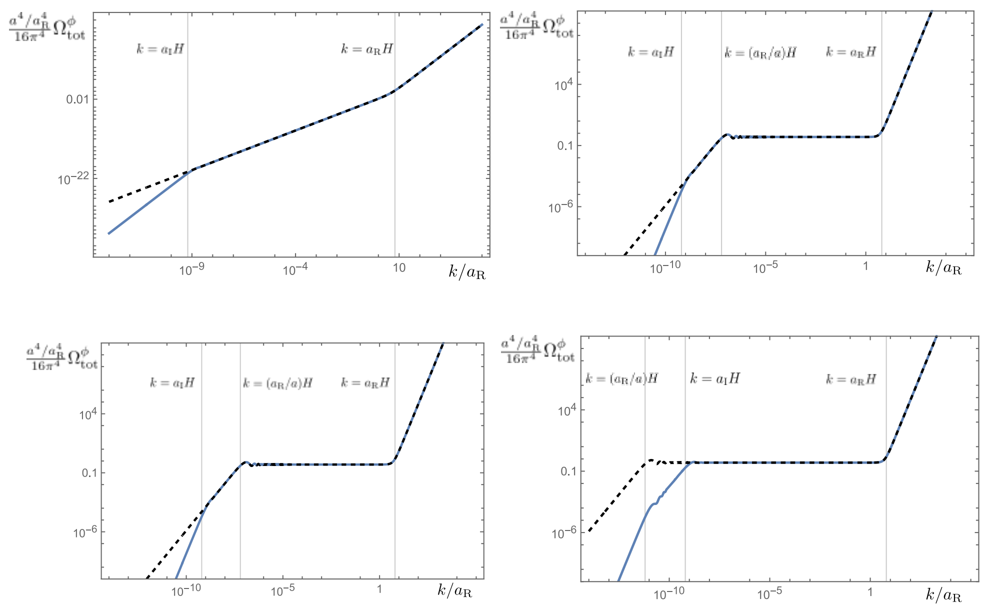


Figure 3.2: Comparison power spectra massless test scalar field

The graphs show the comparison between the power spectrum of a massless test scalar field with past infinite dS inflation (dashed line) vs finite duration dS inflation (bold line) matched to a terminal stage of radiation domination. The panels are evaluated at subsequent times during radiation domination, with $a_{\rm I} = 10^{-10}a_{\rm R}$, and with $a/a_{\rm R} = 1, 10^{-8}, 10^{-10}$, and 10^{-12}, respectively, in units where $H = 2\pi$.

One can proceed from here as we did in the previous subsection in comparing the results from imposing cutoffs in physical momenta with that of dimensionally regularizing factorized scaleless integrals for the energy density (with the caveat that the counterterms are strictly speaking to be identified only through a scheme that preserves diffeomorphism invariance as per the discussion at the end of the previous subsection). By imposing cutoffs in physical momenta: $k = a\Lambda_{\rm UV}$, we see that the first term in Eq. 3.74 corresponds to a divergence of the form

$$\frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k = \frac{1}{4\pi^2 a^4}\int_0^{a\Lambda_{\rm UV}} k^3 dk = \frac{\Lambda_{\rm UV}^4}{16\pi^2}, \tag{3.75}$$

and the third term corresponds to a divergence of the form

$$\frac{1}{8\pi^4 a^6}\int_{-\infty}^{\infty} d^4k \frac{a_{\rm R}^4 H^2}{2k^2} = \frac{a_{\rm R}^4 H^2}{8\pi^2 a^6}\int_0^{a\Lambda_{\rm UV}} kdk = \frac{\Lambda_{\rm UV}^2}{16\pi^2}\frac{H^2}{(a/a_{\rm R})^4}, \tag{3.76}$$

which would nominally be subtracted by a counterterm proportional to R were we to insist on hard cutoff cutoff regularization. The former, when varied with respect to the metric yields the contribution $R_{\mu\nu} \sim 1/a^4$ and thus corresponds to a renormalization of Newton's constant. Finally, the second term in Eq. 3.74 results in a divergence of the form

$$\begin{aligned}\frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4}\right) &= \frac{1}{8\pi^2 a^4}\int_{a\Lambda_{\rm IR}}^{a\Lambda_{\rm UV}} \frac{dk}{k}\left(a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4\right) \\ &= \frac{H^4(1+e^{-4\mathcal{N}_{\rm tot}})}{8\pi^2 (a/a_{\rm R})^4}\log\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}},\end{aligned} \tag{3.77}$$

where the scale $\Lambda_{\rm IR}$ does not have the same significance as before, given the IR safe behavior of the spectral density, and where the above also corresponds to a renormalization of Newton's constant. We note that the scales corresponding to the beginning and end of inflation appear in the coefficient of the logarithm, whereas the UV scale corresponding to the unknown completion the theory appears inside the logarithm.

Were we to now dimensionally regularize the divergences as advised, we can perform a similar factorization of the scaleless integrals as in the previous subsection. The nominally power law divergent parts can be isolated as

$$\begin{aligned}\rho_{\rm div}^{\rm UV} &= \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(1 + \frac{a_{\rm R}^4 H^2}{2a^2 k^2}\right) \\ &= \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left[\frac{k^2+m^2}{(k^2+m^2)} + \frac{a_{\rm R}^4 H^2}{2a^2}\left(\frac{1}{(k^2+m^2)} + \frac{m^2}{k^2(k^2+m^2)}\right)\right],\end{aligned} \tag{3.78}$$

where we see that power law UV divergences cancel among themselves, and so do not necessitate any counterterms. On the other hand, the UV divergent logarithmic term can be isolated and factorized as

$$\begin{aligned}\rho_{\rm div}^{\rm log} &= \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4}\right) \\ &= \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(\frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2}\right)\left(\frac{1}{k^2(k^2+m^2)} + \frac{m^2}{k^4(k^2+m^2)}\right).\end{aligned} \tag{3.79}$$

By isolating the logarithmic UV poles, we find

$$\rho_{\text{div}}^{\log} = \frac{H^4(1+e^{-4\mathcal{N}_{\text{tot}}})}{8\pi^2(a/a_{\text{R}})^4}\left[\frac{1}{\delta_{\text{UV}}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right] \tag{3.80}$$

from which we read off an identical coefficient as computed in Eq. 3.77[16].

3.4 Conclusions

We stress that the results of this chapter, even if applied to an example of pure academic interest, brought to light relevant results in the context of regularizing and renormalizing divergences in primordial observables.
We elaborated on how it is possible to extract logarithmic divergences from scaleless divergent integrals using dimensional regularization and we analyzed various simple examples to show the scheme independence of logarithmic divergences. We then showed that regularizing the divergence of just one particular component of the stress tensor (such as the energy density) is not sufficient. By separately regularizing the energy density and the pressure using physical cutoffs, the required counterterm cannot be constructed from background quantities in a particular regularization scheme. As a consequence, all subsequent conclusions will necessarily be scheme-dependent, and therefore unphysical. We then conclude that regularization schemes that preserve general covariance are preferred in this context.
Furthermore, we explicitly verified that the scales corresponding to the beginning and end of inflation must in principle be separated from any UV and IR scales parametrizing the unknown completion of our theory and well defined observable quantities. Although physical quantities cannot depend on the latter, they may certainly depend on the former. We found that all IR divergences encountered in the simplified settings we worked in were regulated by the existence of a pre-inflationary phase, and therefore were an artifact of the approximation of a past infinite dS phase. Whether this generalizes to a wider class of IR divergences, especially when higher point interactions are included, is an important question that demands to be followed up on.
Unsurprisingly, UV divergences persist for backgrounds corresponding to finite duration inflation, and the scale corresponding to the end of inflation parametrizes their coefficients rather than regulating them. Upon subtraction of these divergences with local counterterms and the imposition of renormalization conditions[17], a discussion that we postpone to the next chapter, one is able to draw meaningful physical conclusions. The two-point function and stress tensor of a test scalar field are of relatively academic interest given the assumed negligibility of the scalar field

[16] One can also show that the analogous computation for the pressure component also necessitates a counterterm corresponding to a renormalization of G_N (i.e. resulting in a divergent contribution proportional to the spatial components of the Ricci tensor when varied with respect to the background metric) resulting in a renormalized stress tensor that is traceless on a radiation domination background. We do so explicitly for the case of tensor modes in Section 4.3.

[17] Which, as reviewed in Section 1.2.2, are fundamental steps to obtain meaningful results from divergences arising in including quantum corrections.

background, and therefore inability to fix renormalization conditions from observation. The analogous set of questions for vacuum tensor perturbations presents a much more interesting application given the assumption of an evolving background gravitational field (that of FRLW cosmology), perturbations around which represent GWs, which is where we will turn our attention towards next.

CHAPTER 4

(Im)possibility of constraining primordial GWs with N_{eff} bounds

This chapter is based on:
An Étude on the Regularization and Renormalization of Divergences in Primordial Observables
Anna Negro, and Subodh P. Patil, *Riv.Nuovo Cim.* 47 (2024) 3, 179-228.

4.1 Introductory remarks

In this chapter we question the validity of the results reviewed in Chapter 2 and show the implications of both re-deriving the stress energy tensor of GWs and renormalizing the energy density of GWs following the renormalization procedure reviewed in Section 1.2.2. We apply what we learned in the previous chapter to an example of great interest in cosmology: the contribution of primordial vacuum tensor fluctuation to N_{eff}, being the latter tightly constrained by BBN bounds.

In Section 4.2 we improve the definition of the stress energy tensor of GWs being mindful of avoiding the scale separation on which the result derived in Section 2.2 relies. We obtain an original result that is valid independently from the frequency of the signal and takes into account the effects of curvature. Before proceeding with the renormalization of the energy density, we comment on the physical meaning of the improved definition.
In Section 4.3 we follow trough the procedure studied in the previous chapter in order to regularize the energy density of vacuum GWs. We compare the result obtained by regularizing using physical hard cutoffs and dimensional regularization and, as we learned in the scalar case, we show that even if the coefficients of the log-divergences agree in the two regularization schemes, dimensional regularization is to be preferred to obtain meaningful results.
We proceed in Section 4.4 with the second step of renormalization that consists in imposing renormalization conditions. We stress that this is a fundamental step that, even if it might appear of secondary importance, as it is often overlooked in the literature of renormalization of divergences, it is essential to understand the role of BBN bounds on constraining primordial GWs. By carefully imposing the renormalization conditions to fix the finite, scheme-dependent leftover of the renormalized result, we demonstrate that the BBN constraints fail to constrain the

production of primordial vacuum tensor perturbations.
We conclude in Section 4.5 and comment on the possibility of constraining vacuum GWs with N_{eff} bounds.

4.2 Stress energy tensor (re)definition

As reviewed in Section 2.2, the formula for the stress energy tensor of GWs[1]

$$\rho_{\text{gw}}^{\text{Isc}} = \frac{1}{32\pi a^2 G_N} \left\langle \hat{h}'_{ij}(\tau, k)\hat{h}'^{ij}(\tau, k) \right\rangle \tag{4.1}$$

was derived in the context of astrophysically sourced GWs. It is nevertheless extensively referenced in computations of the stress energy tensor of cosmological gravitational wave backgrounds[2]. However, retracing the steps used in its derivation, it becomes immediately clear that one needs to proceed with extra care for applications in a cosmological context, in particular when considering questions of renormalized stress tensors in prescriptions that involve the integral over all momenta[3]. This is because, as reviewed in detail in Section 2.2, the Isaacson form presumes a prior scale separation between fast and slow frequencies defined relative to the curvature scale of the background (a criterion that is time dependent), and repeatedly relies on this scale separation along with the implicit averaging prescription to facilitate various approximations resulting in the simplified final form Eq. 4.1. In the context of an expanding spacetime, with modes of observational relevance that might have crossed any a-priori defined scale more than once over cosmic evolution, undoing the steps that relied on these approximations is warranted. Therefore, it behooves us to re-examine the derivation of the stress tensor for GWs without resorting to approximations that may be at odds with the need to incorporate wavelengths beyond this approximation in any intermediate steps in following through the process of renormalization on cosmological spacetimes. In the following we re-derive Eq. 4.1 without the assumptions of having a high-frequency signal propagating on a flat background.

We compute the stress energy tensor of GWs $\hat{T}^{\text{gw}}_{\mu\nu}$ following the derivation of Section 2.2, where the stress energy tensor of GWs is defined as the averaged second

[1] Implicit in the expectation value of Eq. 4.1 is an additional (Brill-Hartle) spatial and temporal averaging prescription discussed in Section 2.2. We presume this as implicit when discussing the Isaacson form of the stress tensor in what follows.
[2] e.g. leading to Eq. 2.26. See also [68] and references therein.
[3] A feature that can be bypassed entirely in the context of Hadamard regularization [18, 171].

order perturbed Einstein equations[4]

$$\begin{aligned}\langle \hat{T}^{\rm gw}{}_{\mu}{}^{\nu}\rangle &:= -\frac{1}{8\pi G_N}\langle \delta^2 G^{\nu}_{\mu}\rangle \\ &= -\frac{1}{8\pi G_N}\Big\langle \delta^2 g^{\nu\alpha}R_{\mu\alpha}+\delta^1 g^{\nu\alpha}\delta^1 R_{\mu\alpha}+g^{\nu\alpha}\delta^2 R_{\mu\alpha} \\ &\quad -\frac{1}{2}\delta^{\nu}_{\mu}\left(\delta^2 g^{\alpha\beta}R_{\alpha\beta}+\delta^1 g^{\alpha\beta}\delta^1 R_{\alpha\beta}+g^{\alpha\beta}\delta^2 R_{\alpha\beta}\right)\Big\rangle . \qquad (4.2)\end{aligned}$$

Using the expansions in Eqs. 2.2 and fixing the gauge ($D_\rho \hat{h}^{\rho\nu}=0$, $\hat{h}_\mu{}^\mu=\hat{h}=0$) the stress energy tensor in (4.2) results

$$\begin{aligned}\langle \hat{T}^{\rm gw}{}_{\mu}{}^{\nu}\rangle = -\frac{1}{8\pi G_N}\Big\langle &-\frac{1}{2}\hat{h}^{\nu}{}_{\alpha}\hat{h}_{\mu}{}^{\sigma}R^{\alpha}{}_{\sigma}+\frac{1}{2}\hat{h}^{\rho}{}_{\alpha}\hat{h}_{\rho}{}^{\nu}R^{\alpha}{}_{\mu}+\hat{h}_{\alpha}{}^{\nu}\hat{h}_{\sigma}{}^{\rho}R^{\sigma\alpha}{}_{\rho\mu}+\frac{1}{2}\hat{h}^{\nu}{}_{\alpha}\Box\hat{h}^{\alpha}{}_{\mu} \\ &+\frac{1}{4}D_\mu\hat{h}^{\beta}{}_{\alpha}D^{\nu}\hat{h}_{\beta}{}^{\alpha}+\frac{1}{2}D_\sigma\hat{h}^{\nu}{}_{\alpha}D^{\sigma}\hat{h}_{\mu}{}^{\alpha}-\frac{1}{2}D_\sigma\hat{h}^{\nu}{}_{\alpha}D^{\alpha}\hat{h}_{\mu}{}^{\sigma}+\frac{1}{2}\hat{h}^{\sigma}{}_{\alpha}D_\mu D^{\nu}\hat{h}_{\sigma}{}^{\alpha} \\ &+\frac{1}{2}\hat{h}^{\sigma}{}_{\alpha}D^{\alpha}D_{\sigma}\hat{h}_{\mu}{}^{\nu}-\frac{1}{2}\hat{h}^{\sigma}{}_{\alpha}D^{\alpha}D_{\mu}\hat{h}_{\sigma}{}^{\nu}-\frac{1}{2}\hat{h}^{\alpha}{}_{\sigma}D_{\alpha}D^{\nu}\hat{h}_{\mu}{}^{\sigma} \\ &-\frac{\delta^{\nu}{}_{\mu}}{2}\left(\hat{h}_{\beta}{}^{\alpha}\hat{h}_{\sigma}{}^{\rho}R^{\sigma\beta}{}_{\rho\alpha}+\hat{h}^{\beta}{}_{\alpha}\Box\hat{h}^{\alpha}{}_{\beta}+\frac{3}{4}D_\rho\hat{h}^{\alpha}{}_{\beta}D^{\rho}\hat{h}_{\alpha}{}^{\beta}-\frac{1}{2}D_\sigma\hat{h}^{\beta}{}_{\alpha}D^{\alpha}\hat{h}_{\beta}{}^{\sigma}\right)\Big\rangle . \qquad (4.3)\end{aligned}$$

At this stage, Brill-Hartle averaging would result in the Maccallum-Taub averaged stress tensor [149], which upon further integrations by parts within the spatial and temporal averaged integrals in addition to commuting covariant derivatives would bring the latter into the Isaacson form Eq. 2.5. Instead, we persist with Eq. 4.3 as it is. To find the energy density of GWs $\rho_{\rm gw}$, we have to specify the covariant derivatives in Eq 4.3. Using that in conformal time, the only non-vanishing Christoffel symbols on a FLRW universe are $\Gamma^0_{00}=\frac{a'}{a}$, $\Gamma^i_{0j}=\frac{a'}{a}\delta_j{}^i$ and $\Gamma^0_{ij}=\frac{a'}{a}\delta_{ij}$, and considering only the transverse traceless part of the metric as the propagating DoFs (see Section 2.2.1 for more details), $\hat{T}_0{}^0$ then results

$$\langle \hat{T}_0{}^0\rangle = \frac{-1}{64\pi a^2 G_N}\Big\langle \hat{h}'_j{}^i\hat{h}'_i{}^j-3\partial_k\hat{h}_j{}^i\partial^k\hat{h}_i{}^j-4\hat{h}_j{}^i\partial_k\partial^k\hat{h}_i{}^j+8\mathcal{H}\hat{h}_j{}^i\hat{h}'_i{}^j+2\partial_k\hat{h}_j{}^i\partial^j\hat{h}_i{}^k\Big\rangle . \qquad (4.4)$$

Consequently, in the coincidence limit ($\rho_{\rm gw}:=\lim_{y\to x}\rho_{\rm gw}(\tau;x,y)$) the energy density of GWs can be expressed as:

$$\rho^{\rm Imp}_{\rm gw}=\frac{1}{64\pi a^2 G_N}\Big\langle \hat{h}'_{ij}\hat{h}'^{ij}-3\partial_k\hat{h}_{ij}\partial^k\hat{h}^{ij}-4\hat{h}_{ij}\partial_k\partial^k\hat{h}^{ij}+2\partial_k\hat{h}_{ij}\partial^j\hat{h}^{ik}+8\mathcal{H}\hat{h}_j{}^i\hat{h}'_i{}^j\Big\rangle \qquad (4.5)$$

where $\mathcal{H}:=a'/a$. In spite of not invoking any additional averaging prescriptions, the expectation value featuring in the improved stress tensor above is still doing a lot of heavy lifting. We stress that this is purely a quantum expectation value (strictly,

[4] One can also independently derive what follows by expanding the action to second order in perturbations and varying with respect to the background metric, taking care to properly address gauge-fixing and ghost terms that are necessary in the context of quantum expectation values, the net result of which will be Eq 4.3 [171].

an in-in correlation function) at some fixed time with no extra spatial or temporal averaging invoked. Furthermore, the expectation value presumes a density matrix, which we take to be Bunch Davies vacuum. A far more interesting story beyond the scope of the present work commences once one incorporates the effects of non-trivial density matrices, and interactions induced by gravitational non-linearities, as well as those induced by matter couplings.

It is worth stressing that Eq. 4.5 is the temporal component of a covariantly conserved tensor, and it is under no obligation to be conserved in isolation. This is a corollary of the fact that on FRLW backgrounds, a locally conserved energy cannot be defined. Moreover, it is also under no obligation to be positive definite. In fact, were one to insist of constructing the spectral density associated with the vacuum expectation value of Eq. 4.5, one can show that it crosses zero at a comoving scale corresponding to the horizon scale at any given time. This is consistent with the operational ambiguity of associating a background/fluctuation split for wavelengths commensurate with the background curvature. However, none of this is of any operational concern for our purposes, as we stress that Eq. 4.5 is to be viewed as the components of a covariantly conserved tensor corresponding to a massless spin-2 excitation whose role in renormalization of background quantities is well understood in the covariant context, but obscured and widely conflated for a physical contribution to the number of relativistic species in foliation-specific computations as we elaborate upon next. The specific computation we are interested in performing is the renormalization of the graviton stress tensor on a background corresponding to radiation domination preceded by a finite period of inflation. In doing so, we will compare and contrast the results reviewed in Chapter 2, obtained from both the Isaacson and improved forms of the stress tensor, before revisiting the question of $N_{\rm eff}$ bounds from vacuum tensor modes and concluding.

Before proceeding, an important aside is due: although one might find statements in the literature that questions the utility of even defining GWs with wavelengths grater than the background curvature scale[5], this would nominally be at odds with the premise of many computations. It is also add odds with observations: GWs from mergers of binary black hole systems have been observed [4, 2, 3]. Moreover, the search for primordial GWs is premised on the fact that they induce local quadrupolar anisotropies in the density field of the primordial plasma at all scales, resulting in a signature B-mode polarization pattern [128, 198]. Both situations feature GWs with wavelengths comparable to or greater than the background curvature radius at some point – the peak frequency emitted from a merger corresponding to wavelengths commensurate to the Schwarzschild radius, and primordial tensor fluctuations having crossed the Hubble radius before sourcing local anisotropies.
Clearly, nature tells us that the notion of tensor perturbations with wavelengths

[5] In section 35.7 of [160] for example, one finds the statement that "One must always have $\mathcal{A} \ll 1$ as well as $\lambda \ll 2\pi\mathcal{R}$ if the concept of gravitational wave is to make any sense", where $\mathcal{A}$ is the dimensionless amplitude of the gravitational wave and $\mathcal{R}^{-2}$ is as defined in footnote 1 (see also [152, 149, 204, 68] for more detailed discussions of this point).

longer than than the background curvature radius has to make sense. By general covariance and the Bianchi identities that follow as a corollary, one must also be able to identify a conserved rank two tensor that plays the role of a stress tensor from direct perturbation of the equations of motion. That one can do so with minimal fuss by simply undoing the averaging prescription as done in this section. It is also the premise of the computation in [18] and the covariant formulation of Chapter 6 where we parametrize vaccum GWs a massless spin-2 DoFs on a curved background, whose regularization and renormalization proceeds via established prescriptions.

Isaacson limit

As a consistency check, we show how, by reintroducing Isaacson's assumptions, we can reduce the improved energy density in Eq. 4.5 to Eq. 4.1.
In order to obtain Eq. 4.1, we have to assume that the curvature is negligible ($k \gg \mathcal{H}$) and that the average scheme allows to integrate by parts both in time and space. Applying these assumptions to Eq. 4.5 we obtain that

- The second and third term can be rewritten as $-3\partial_k \hat{h}_{ij}\partial^k \hat{h}^{ij} - 4\hat{h}_{ij}\partial_k\partial^k \hat{h}^{ij} = 3\hat{h}_{ij}\partial_k\partial^k \hat{h}^{ij} - 4\hat{h}_{ij}\partial_k\partial^k \hat{h}^{ij} = -\hat{h}_{ij}\partial_k\partial^k \hat{h}^{ij}$.
- The EOM results: $\Box \hat{h}_{ij} = 0 \quad \Rightarrow \quad \partial_k \partial^k \hat{h}_{ij} = \hat{h}''_{ij}$.
- The last therm is negligible $\mathcal{H} h_{ij} \dot{h}^{ij} \sim 0$.

In conclusion[6], (4.5) results

$$\rho_{\mathrm{gw}}^{\mathrm{Imp}} = \frac{1}{64\pi G_N a^2}\left\langle \hat{h}'_{ij}\hat{h}'^{ij} - \hat{h}_{ij}\hat{h}''_{ij}\right\rangle = \frac{1}{32\pi G a^2}\left\langle \hat{h}'_{ij}\hat{h}'^{ij}\right\rangle = \rho_{\mathrm{gw}}^{\mathrm{Isc}} \tag{4.6}$$

where in the second equality we integrate by parts in time.

4.3 Regularization – Finite inflation

Before delving into the substance of this section, it is informative to contrast the results typically found in the literature. As reviewed in Section 2.3.2, imposing hard cutoffs the the energy density of GWs results in expressions of the form:

$$\rho_{\mathrm{gw}} \simeq \frac{A_t}{32\pi G_N}\left(\frac{k_{\mathrm{UV}}}{k_*}\right)^{n_t}\frac{1}{2n_t}\frac{1}{a^4} \propto \frac{1}{a^4}\left[\frac{1}{n_t} + \log\frac{k_{\mathrm{UV}}}{k_*}\right], \tag{4.7}$$

where k_* is some reference IR scale, and the approximation is only valid when $n_t \to 0$ [158]. We deconstruct and rederive the energy density for vacuum tensor perturbations in what follows, but before doing so, it is useful to compare what one would have obtained in retracing the steps leading to Eq. 4.7 for a massless test scalar field. In doing so, one would obtain the expression

$$\rho \simeq \lim_{n_s\to 1}\frac{A_s}{32\pi G_N}\left(\frac{k_{\mathrm{UV}}}{k_*}\right)^{n_s-1}\frac{1}{2(n_s-1)}\frac{1}{a^4} \propto \frac{1}{a^4}\left[\frac{1}{n_s-1} + \log\frac{k_{\mathrm{UV}}}{k_*}\right], \tag{4.8}$$

[6] The second last term of Eq. 4.5 does not contribute as once we compute the expectation value in our gauge choice results in $k^i\epsilon_{ij}(k) = 0$ (see Section 4.3 for more details).

which can directly be compared to the divergences computed in Eqs. 3.75 - 3.77 from hard cutoffs in physical momenta and Eq. 3.80 from dimensional regularization, all of which are understood to be intermediate expressions that are to be subtracted and renormalized.
An immediate reservation one might express for expressions such as Eq. 4.7 is the appearance of cutoffs in what should be a physical result. Although one might argue that the effects of UV and IR modes are negligible relative to what can be extracted from Eq. 4.7, its form nevertheless suggests that the expression above is an intermediate result on the way to computing a physical energy density. Moreover, one might be concerned in applying the formula Eq. 4.7 to the case of blue tilted spectra (i.e. for positive $n_t \sim \mathcal{O}(1)$, so that $\rho_{\rm GW} \sim (k_{\rm UV}/k_*)^{n_t}$ as derived in e.g. [158]), that some part of this expression is nothing other than the Fourier transform of a UV divergence that gets subtracted in the usual way.
In the following we continue this process of regularizing and renormalizing the energy density of GWs on a background that transitions from a pre-inflationary RD era to a pure dS inflation and finally back to RD era. This is an example of how one should proceed to ensure that well defined physical observables are been computed.

Going back to the study of finite inflation and tensor vacuum perturbations, we work in TT-gauge and we expand the tensor perturbations as

$$\hat{h}_{ij}(\tau,x) = \sum_{r=+,x} \int \frac{d^3k}{M_{\rm pl}\,(2\pi)^3} e^{ix\cdot k}\left[\epsilon^r_{ij}(k)\hat{a}_k\gamma_k(\tau) + \epsilon^{r*}_{ij}(-k)\hat{a}^\dagger_{-k}\gamma^*_k(\tau)\right]. \qquad (4.9)$$

The polarization tensors are normalized as $\epsilon^s_{ij}\epsilon^{r*}_{ij} = 4\delta^{rs}$, $k^i\epsilon^r_{ij} = 0$. The normalizations are chosen so that the relevant mode functions during inflation and the pre and post-inflationary phases of radiation domination are given for each polarization just as for massless minimally coupled scalars[7]. These are:

$$\begin{aligned} \gamma^{\rm PI}_k(\tau) &= \frac{1}{a}\frac{1}{\sqrt{2k}} e^{-ik\tau_{\rm I}\left(2-\frac{a}{a_{\rm R}}e^{\mathcal{N}_{\rm tot}}\right)} \\ \gamma'^{\rm PI}_k(\tau) &= -\frac{1}{a^2\sqrt{2k}}\frac{a_{\rm I}}{\tau_{\rm I}} e^{-ik\tau_{\rm I}\left(2-\frac{a}{a_{\rm R}}e^{\mathcal{N}_{\rm tot}}\right)}\left(-1+ik\tau_{\rm I}\frac{a}{a_{\rm I}}\right), \end{aligned} \qquad (4.10)$$

during the pre-inflationary phase,

$$\begin{aligned} \gamma^{\rm I}_k(\tau) &= \frac{H}{\sqrt{2k^3}}\left[\alpha^{\rm I}_k e^{i\frac{k}{aH}}\left(1-\frac{ik}{aH}\right) + \beta^{\rm I}_k e^{-i\frac{k}{aH}}\left(1+\frac{ik}{aH}\right)\right] \\ \gamma'^{\rm I}_k(\tau) &= -\frac{H}{\sqrt{2k^3}}\left[\alpha^{\rm I}_k e^{i\frac{k}{aH}}\frac{k^2}{aH} + \beta^{\rm I}_k e^{-i\frac{k}{aH}}\frac{k^2}{aH}\right], \end{aligned} \qquad (4.11)$$

[7]The EOM for h^j_i in TT-gauge can be derived from the first order expansion of Einstein equation: $\hat{h}''^j_i + 2\mathcal{H}\hat{h}'^j_i - \partial^2_k\hat{h}^j_i = 0$.

during inflation, and

$$\gamma_k^{\rm RD}(\tau) = \frac{1}{a}\frac{1}{\sqrt{2k}}\left[\alpha_k^{\rm R} e^{-ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)} + \beta_k^{\rm R} e^{ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\right]$$
$$\gamma_k'^{\rm RD}(\tau) = \frac{a_{\rm R}}{a^2\tau_{\rm R}\sqrt{2k}}\left[\alpha_k^{\rm R} e^{-ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1-\frac{iak\tau_R}{a_{\rm R}}\right) + \beta_k^{\rm R} e^{ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1+\frac{ik\tau_{\rm R}a}{a_{\rm R}}\right)\right] \tag{4.12}$$

during the terminal RD phase. The corresponding Bogoliubov coefficients are the same as those given in Eq. 3.64 and Eq. 3.65.

An immediate corollary of the above is that the two point correlation function of each graviton polarization is identical to that of a massless, minimally coupled scalar. Therefore, as illustrated in Figs. 3.1, one finds that the IR divergences exhibited on a past infinite dS background are also cured for the graviton two point function on a background corresponding to finite duration inflation.

Using Eq. 4.9, the energy density of GWs in Eq. 4.1 results (with the reduced Planck mass defined as $M_{\rm pl}^2 = \frac{1}{8\pi G_N}$) in:

$$\rho_{\rm gw}^{\rm Isc} = \lim_{y\to x}\rho_{\rm gw}(\tau;x,y) = \frac{1}{\pi^2 a^2}\int_0^\infty dk\; k^2\left[\gamma_k'^{\rm RD}\gamma_k'^{\rm RD*}\right] \tag{4.13}$$

as the tensor mode counterpart of Eq. 3.70, after having summed the contributions from the two independent polarizations. Using the mode functions specified by Eqs. 4.12, the relations 3.65, and tracing through the steps of the previous section, we find

$$\begin{aligned}\rho_{\rm gw}^{\rm Isc} = {} & \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty}\frac{d^4k}{k}\left[k+\frac{a_{\rm R}^4 H^2}{a^2 k}\right]\left\{1+\frac{a_{\rm R}^4H^4}{2k^4}+2\left|\beta_k^{\rm I}\right|^2\left(1+\frac{a_{\rm R}^4H^4}{2k^4}\right)\right. \\ & -i\alpha_k^{\rm I}\beta_k^{\rm I*}e^{2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2}\left(-i+\frac{a_{\rm R}H}{k}+i\frac{a_{\rm R}^2H^2}{2k^2}\right)+i\alpha_k^{\rm I*}\beta_k^{\rm I}e^{-2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2H^2}{k^2} \\ & \left.\left(i+\frac{a_{\rm R}H}{k}-i\frac{a_{\rm R}^2H^2}{2k^2}\right)\right\}+\frac{1}{4\pi^4a^4}\int_{-\infty}^{\infty}\frac{d^4k}{k}\left(\frac{a_{\rm R}^4H^2}{a^2k}\right)\left[\alpha_k^{\rm R}\beta_k^{\rm R*}e^{-2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\right. \\ & \left.\left(1+i\frac{ka}{a_{\rm R}^2H}\right)^2+\alpha_k^{\rm R*}\beta^{\rm R}e^{2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\left(1-i\frac{ka}{a_{\rm R}^2H}\right)^2\right],\end{aligned} \tag{4.14}$$

which again can be split into a power law contribution and oscillating but finite contributions in the UV-limit that become relevant when the oscillations freeze as $k\to 0$, softening the IR behavior. As we did for the scalar case, we similarly separate the spectral density into oscillatory and power law contributions

$$\begin{aligned}\rho_{\rm gw}^{\rm Isc} &= \frac{1}{2\pi^2a^4}\int_0^\infty\frac{dk}{k}\,k^4\left[\left(1+\frac{a_{\rm R}^4H^2}{a^2k^2}\right)\left(1+2|\beta_k^{\rm R}|^2_{\rm power}\right)+\{\rm osc\}\right] \\ &:= \int_0^\infty\frac{dk}{k}\left[\Omega_{\rm power}^{\rm gw}(k,\tau)+\Omega_{\rm osc}^{\rm gw}(k,\tau)\right]\end{aligned} \tag{4.15}$$

and neglect $\Omega_{\rm osc}^{\rm gw}(k,\tau)$ in regularizing the UV divergences. Even though the Isaacson form of the stress tensor is not strictly valid when incorporating wavelengths greater than the Hubble scale at any given time, naïvely persisting with it for all wavelengths would show an IR regular spectral density $\Omega^{\rm gw}_{\rm total} \propto k^2$. Consequently, one only has the resulting UV divergent contributions to regulate:

$$\rho^{\rm Isc}_{\rm gw,div} = \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(1 + \frac{a_{\rm R}^4 H^4 + a_{\rm I}^4 H^4}{2k^4} + \frac{a_{\rm R}^4 H^2}{a^2 k^2}\right), \tag{4.16}$$

which up to overall factors, is formally identical to the contributions Eq. 3.74, and whose subtraction proceeds along the same lines.

Were we to consider the improved stress tensor Eq. 4.5 (which does not invoke a scale separation or time averaging prescription), we would obtain

$$\rho^{\rm Imp}_{\rm gw} = \frac{1}{2\pi^2 a^2}\int_0^{\infty} dk\, k^2 \left[\gamma_k'^{\rm RD}\gamma_k'^{\rm RD*} + k^2\gamma_k^{\rm RD}\gamma_k^{\rm RD*} + 4\mathcal{H}\left(\gamma_k'^{\rm RD}\gamma_k^{\rm RD*} + \gamma_k^{\rm RD}\gamma_k'^{\rm RD*}\right)\right] \tag{4.17}$$

as the counterpart of Eq. 4.13 that is now valid for for all wavelengths. Using the mode functions and normalizations specified above, we find

$$\begin{aligned}\rho^{\rm Imp}_{\rm gw} =& \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} \frac{d^4k}{k}\left[k - \frac{7a_{\rm R}^4 H^2}{a^2 2k}\right]\left\{1 + \frac{a_{\rm R}^4 H^4}{2k^4} + 2\left|\beta_k^{\rm I}\right|^2\left(1 + \frac{a_{\rm R}^4 H^4}{2k^4}\right)\right. \\ & -i\alpha_k^{\rm I}\beta_k^{\rm I*}e^{2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2 H^2}{k^2}\left(-i + \frac{a_{\rm R}H}{k} + i\frac{a_{\rm R}^2 H^2}{2k^2}\right) + i\alpha_k^{\rm I*}\beta_k^{\rm I}e^{-2i\frac{k}{a_{\rm R}H}}\frac{a_{\rm R}^2 H^2}{k^2} \\ & \left.\left(i + \frac{a_{\rm R}H}{k} - i\frac{a_{\rm R}^2 H^2}{2k^2}\right)\right\} + \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty}\frac{d^4k}{k}\left[\alpha_k^{\rm R}\beta_k^{\rm R*}e^{-2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)}\right. \\ & \left(\frac{k}{2} + \frac{a_{\rm R}^4 H^2}{a^2 2k}\left(1 + i\frac{ka}{a_{\rm R}^2 H}\right)^2 - 4\frac{a_{\rm R}^4 H^2}{a^2 k}\left(1 + i\frac{ka}{a_{\rm R}^2 H}\right)\right) + \alpha_k^{\rm R*}\beta^{\rm R}e^{2ik\tau_{\rm R}\left(2-\frac{a}{a_{\rm R}}\right)} \\ & \left.\left(\frac{k}{2} + \frac{a_{\rm R}^4 H^2}{a^2 2k}\left(1 - i\frac{ka}{a_{\rm R}^2 H}\right)^2 - 4\frac{a_{\rm R}^4 H^2}{a^2 k}\left(1 - i\frac{ka}{a_{\rm R}^2 H}\right)\right)\right]\end{aligned} \tag{4.18}$$

and we can again separate the spectral density[8] into oscillatory and power law contributions

$$\begin{aligned}\rho^{\rm Imp}_{\rm gw} &= \frac{1}{2\pi^2 a^4}\int_0^{\infty}\frac{dk}{k}\,k^4\left[\left(1 - \frac{7a_{\rm R}^4 H^2}{2a^2 k^2}\right)\left(1 + 2|\beta_k^{\rm R}|^2_{\rm power}\right) + \{\rm osc\}\right], \\ &:= \int_0^{\infty}\frac{dk}{k}\left[\Omega^{\rm gw}_{\rm power}(k,\tau) + \Omega^{\rm gw}_{\rm osc}(k,\tau)\right].\end{aligned} \tag{4.19}$$

We again find that the spectral density summing all contributions in the IR goes as $\Omega^{\rm gw}_{\rm total} \propto k^2$, albeit with a negative overall coefficient (cf. the discussion below Eq. 4.5) with the following divergent contributions in the UV that necessitate

[8] We stress that the spectral density as defined in Eq. 4.19 is to be viewed as only a calculational definition for the purposes of comparison to the literature, and not be to viewed as the power spectral density of vacuum fluctuations without additional caveats.

subtraction

$$\rho^{\text{Imp}}_{\text{gw,div}} = \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(1 + \frac{a_{\text{R}}^4 H^4 + a_{\text{I}}^4 H^4}{2k^4} - \frac{7\, a_{\text{R}}^4 H^2}{2a^2k^2}\right). \tag{4.20}$$

In proceeding to regularizing divergences, we can identify the various UV divergences is different schemes, finding identical coefficients for the log divergences in all of them. By imposing cutoffs in physical momenta ($k = a\Lambda_{\text{UV}}$) on the identifiably UV-divergent terms above, we obtain

$$\frac{1}{2\pi^2 a^4}\int_0^{a\Lambda_{\text{UV}}} dk\, k^3 - \frac{7}{2}\frac{a_{\text{R}}^4 H^2}{2\pi^2 a^6}\int_0^{a\Lambda_{\text{UV}}} dk\, k = \frac{\Lambda_{\text{UV}}^4}{8\pi^2} - \frac{7H^2\Lambda_{\text{UV}}^2}{8\pi^2 (a/a_{\text{R}})^4} \tag{4.21}$$

and

$$\frac{1}{4\pi^2 a^4}\int_{a\Lambda_{\text{IR}}}^{a\Lambda_{\text{UV}}} \frac{dk}{k}\left(a_{\text{R}}^4 H^4 + a_{\text{I}}^4 H^4\right) = \frac{H^4\left(1 + e^{-4\mathcal{N}_{\text{tot}}}\right)}{4\pi^2 (a/a_R)^4}\log\frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}. \tag{4.22}$$

If instead we were to now isolate the power law UV divergent parts and dimensionally regularize, we would obtain

$$\begin{aligned}\rho^{\text{Imp}}_{\text{gw,div}} \supset& \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k - \frac{7a_{\text{R}}^2}{4\pi^4 a^6}\int_{-\infty}^{\infty} d^4k \left(\frac{a_{\text{R}}^2 H^2}{2k^2}\right)\\ =& \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left[\frac{k^2+m^2}{(k^2+m^2)} - \frac{7}{2}\frac{a_{\text{R}}^4 H^2}{2a^2}\left(\frac{1}{(k^2+m^2)} + \frac{m^2}{k^2(k^2+m^2)}\right)\right]\end{aligned} \tag{4.23}$$

which, as for the dimensionally regularized scalar case Eq. 3.78, has the divergent contributions canceling among themselves and not necessitating any counterterms. The logarithmic divergence can be expressed as

$$\begin{aligned}\rho^{\text{Imp}}_{\text{gw,div}} \supset& \frac{1}{4\pi^4 a^4}\int_{-\infty}^{\infty} d^4k \left(\frac{a_{\text{R}}^4 H^4 + a_{\text{I}}^4 H^4}{2k^4}\right)\\ =& \frac{H^4\left(1 + e^{-4\mathcal{N}_{\text{tot}}}\right)}{4\pi^2 (a/a_{\text{R}})^4}\left[\frac{1}{\delta_{\text{UV}}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right].\end{aligned} \tag{4.24}$$

The results of the regularized improved energy density can be collected and summarized as

$$\rho^{\text{Imp}}_{\text{gw,div}} = \lim_{\Lambda_{\text{UV}}\to\infty}\left\{\frac{1}{2\pi^2}\frac{\Lambda_{\text{UV}}^4}{4} - \frac{7}{2}\frac{a_{\text{R}}^4 H^2}{2\pi^2 a^4}\frac{\Lambda_{\text{UV}}^2}{2} + \frac{H^4\left(1 + e^{-4\mathcal{N}_{\text{tot}}}\right)}{4\pi^2 (a/a_{\text{R}})^4}\log\frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}\right\}, \text{(cutoff)} \tag{4.25}$$

and also compared to the result obtained via dimensional regularization

$$\rho^{\text{Imp}}_{\text{gw,div}} = \lim_{\delta_{\text{UV}}\to 0}\left\{\frac{H^4\left(1 + e^{-4\mathcal{N}_{\text{tot}}}\right)}{4\pi^2 (a/a_R)^4}\left[\frac{1}{\delta_{\text{UV}}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right]\right\}. \qquad \text{(dim − reg)} \tag{4.26}$$

Before commenting on the regularized results in Eqs. 4.25 and 4.26, for illustrative purposes, we plot the power spectral densities of the (time unaveraged) Isaacson and improved forms of the stress tensors in Fig. 4.1. For sub-horizon modes, where a positive spectral density results, the improved stress tensor evaluated at any given time does not exhibit oscillations[9].

Going back to the rengularized results in Eqs. 4.25 and 4.26, as cautioned in the previous chapter, one may not be able to consistently subtract such divergences unless one uses regularization schemes that preserve the symmetries of the background. In section 3.3.2 we showed how hard cutoffs fail to give a counterterm for the leading divergences that is proportional to the metric nor the Ricci tensor. Similarly, we follow through the same logic to study the divergences appearing in computing the energy density and pressure of GWs. Indeed, we show that if we were to regularize the energy density and pressure of GWs using hard cutoffs, it appears that the equation of state would not be satisfied. Following the procedure described in section 4.2, we derive the energy density and pressure of GWs during RD era that result respectively

$$\begin{aligned}\rho_{\rm gw} &= \frac{1}{64\pi a^2 G_N}\left\langle h'^{\,i}_{j} h'^{\,j}_{i} - 3\partial_k h_{ij}\partial^k h^{ij} - 4h_{ij}\partial_k\partial^k h^{ij} + 8\mathcal{H} h_j{}^i h'^{\,j}_{i} + 2\partial_k h_{ij}\partial^j h^{ik}\right\rangle \\ P_{\rm gw} &= -\frac{1}{3}\frac{1}{64\pi a^2 G_N}\left\langle 5h'^{\,i}_{j} h'^{\,j}_{i} - 3\partial_k h_{ij}\partial^k h^{ij} - 4h_{ij}\partial_k\partial^k h^{ij} + 16\mathcal{H} h_j{}^i h'^{\,j}_{i} + 8h_j{}^i h''^{\,j}_{i}\right\rangle \\ &= \frac{1}{3}\frac{1}{8\pi a^2 G_N}\left\langle -5h'^{\,i}_{j} h'^{\,j}_{i} + 3\partial_k h_{ij}\partial^k h^{ij} - 4h_{ij}\partial_k\partial^k h^{ij}\right\rangle\end{aligned} \tag{4.27}$$

where in the second equality of the pressure we use the EOM. From the result above it appears that the equation of state is not satisfied. Specifying the result in the case of finite inflation and proceeding as before, we find that by using physical cutoff and dimensional regularization respectively, the regularized pressure results

$$P_{\rm gw,div} = \frac{1}{3}\lim_{\Lambda_{\rm UV}\to\infty}\left\{\frac{\Lambda_{\rm UV}^4}{8\pi^2} - \frac{5}{2}\frac{a_{\rm R}^4 H^2}{2\pi^2 a^4}\frac{\Lambda_{\rm UV}^2}{2} + \frac{H^4\left(1+e^{-4\mathcal{N}_{\rm tot}}\right)}{4\pi^2 (a/a_{\rm R})^4}\log\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}\right\}, \quad \text{(cutoff)} \tag{4.28}$$

$$P_{\rm gw,div} = \frac{1}{3}\lim_{\delta_{\rm UV}\to 0}\left\{\frac{H^4\left(1+e^{-4\mathcal{N}_{\rm tot}}\right)}{4\pi^2 (a/a_R)^4}\left[\frac{1}{\delta_{\rm UV}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right]\right\}. \quad \text{(dim}-\text{reg)} \tag{4.29}$$

By recalling the result for the regularized energy density which are given in Eqs 4.25 and 4.26, we can see that only the result regularized using dimensional regularization gives a traceless stress energy tensor and satisfies the equation of state for radiation-like species. We then find that even if at the operator level this is not evident, once one regularize using dimensional regualrization the equation of state ($P = \frac{1}{3}\rho$ in the case under analysis) is satisfied. Moreover, using dimensional regualarization the

[9]The oscillatory behavior of the time unaveraged Isaacson stress tensor can be understood from the fact that the magnitude of the time derivative squared of a linearized gravitational wave is by itself not a constant of the linearized equations of motion (which requires restoring the spatial derivative contributions), unlike for the improved form.

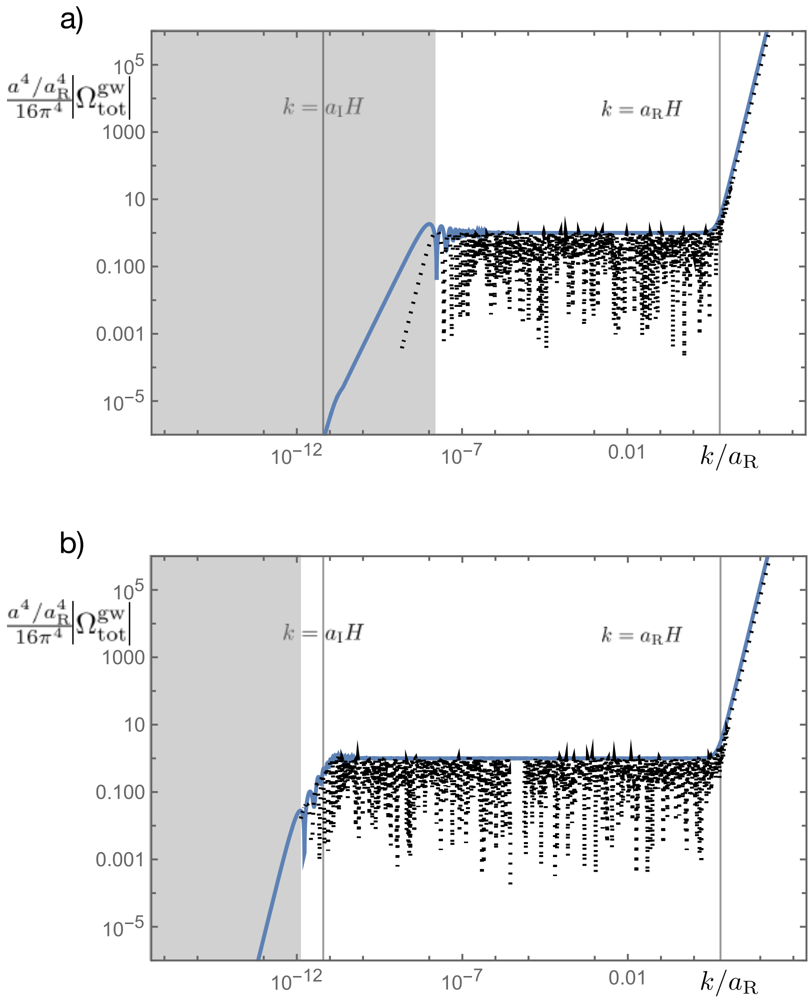


Figure 4.1: Comparison Power spectral densities vacuum GWs

The graph shows the comparison between the power spectral density of the Isaacson stress tensor (dashed lines) and the spectral density of the improved stress tensor (blue line). The gray shaded regions correspond to super-horizon scales, which are outside the domain of validity of the Isaacson stress tensor (where we note that the oscillations would not appear in its time averaged form), and also where the spectral density for the improved stress tensor nominally becomes negative (cf. discussion below Eq. 4.5).
Spectral density evaluated at $a = 10^9 a_R$ (in **a)**) and at $a = 10^{13} a_R$ (in **b)**), with $a_I = 10^{-12} a_R$ in units where H is set to 2π.

trace of the energy tensor is proportional to the Ricci scalar ($T_\mu{}^\mu = R = 0$ in the case under analysis) and the divergences can be subtracted with a counterterm that comes from varying the background curvature with respect to the metric (and thus corresponds to a renormalization of G_N). For this reason, we stick to dimensional regularization in what follows, although covariant point splitting methods also offer a practical alternative[48, 14, 72, 80, 71, 57, 58].

Before proceeding in the next section in reabsorbing the divergences appearing in Eq. 4.26, we comment on the counterterms we need to add to have a predictive theory with which we can calculate the results for any subsequent observations. To understand which counterterms are required to reabsorb the divergences in dimensional regularization, we need only to look at the scale factor dependence of the divergences in Eq. 4.26. We immediately notice that the divergences we have computed do not necessitate higher derivative counterterms beyond those already in the Einstein Hilbert action.
In order to understand why this is, we trace through a treatment that uses adiabatic regularization to renormalize the stress tensor of a non-minimally coupled test scalar field [56], where mode functions are adiabatically expanded in order to regularize divergences and identify counterterms. Higher orders in the adiabatic expansion necessitate successively higher order counterterms. The adiabatic solution of order zero is regularized by a cosmological constant counterterm, the second order solution by a curvature counterterm, with curvature squared counterterms needed to renormalize divergences that appear only at fourth order in the adiabatic expansion. In our case the adiabatic solution of order four is null and non-vanishing curvature squared counterterms would only be necessitated if we were to consider couplings to other matter fields, or by incorporating the effects of loops and higher order gravitational and matter coupling non-linearities (see [138, 109, 98, 156, 24] for corresponding studies of adiabatic expansions for fields of different spins and masses which necessitate higher order counterterms).

4.4 Renormalization and N_{eff} bounds

We turn our attention in this section to the second and most consequential step in the process of renormalization[10] – that of extracting physical observables after imposing renormalization conditions. Before studying the GWs result, we begin the discussion with the relatively academic exercise of renormalizing the stress tensor of a test scalar field on a background that transitions in and out of inflation (considering the results obtained in section 3.3.3). By virtue of not having a classically evolving background and energy density, the only effects of a test scalar will be in renormalizing background couplings in the context of the effective theory of gravity [90, 91, 92, 60]. We return to the more interesting and physically relevant case of primordial GWs next.

Our starting point is the bare matter and gravitational actions, along with the

[10] See Section 1.2.2 for more details.

requisite counterterms:

$$S = S_{\text{EH}} + S_{\text{bg}} + S_\phi + S_{\text{ct}}. \tag{4.30}$$

In order to impose renormalization conditions after having regularized divergences, we first consider the background equations of motion

$$\frac{1}{8\pi G_B}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = T^{\text{bg}}_{\mu\nu} + \langle \hat{T}^\phi_{\mu\nu}\rangle + \langle \hat{T}^{\text{ct}}_{\mu\nu}\rangle, \tag{4.31}$$

where we stress that the couplings that appear in Eq. 4.30 are to be understood as bare couplings, and the presence of the counterterm shifts the tadpole condition in a manner that we will shortly make precise. Given the consistency of dimensional regularization with general covariance, we can proceed by considering the above for any given component. For the 00 component, we have

$$-\frac{{R_0}^0}{8\pi G_B} = \rho^{\text{cl}}_{\text{bg}} + \rho_\phi + \rho_{\text{ct}}, \tag{4.32}$$

where $\rho^{\text{cl}}_{\text{bg}}$ is the to be renormalized classical background energy density that sources the expanding geometry around which we have computed the stress tensor for the test scalar field. The corresponding energy density ρ_ϕ is defined as

$$\rho_\phi := \rho^{\text{cl}}_\phi - \langle {\hat{T}^0}_0\rangle \equiv -\langle {\hat{T}^0}_0\rangle, \tag{4.33}$$

where $\rho^{\text{cl}}_\phi \equiv 0$ for a test scalar by assumption. The background satisfies

$$-{R_0}^0 = -\frac{3}{a^2}\left(\frac{a'}{a}\right)' = \frac{3H^2}{(a/a_{\text{R}})^4} \tag{4.34}$$

via Eq. 3.58 during the terminal RD phase, where it is to be stressed again that H is the Hubble constant during the intermediate phase of inflation that enters the definition of the scale factor during radiation domination via Eq. 3.58. We need to identify the counterterm that absorbs the divergence exhibited in Eq. 3.80:

$$\frac{1}{8\pi G_B}\frac{3H^2}{(a/a_R)^4} = \rho^{\text{cl}}_{\text{bg}} + \frac{H^4(1+e^{-4\mathcal{N}_{\text{tot}}})}{8\pi^2(a/a_{\text{R}})^4}\left[\frac{1}{\delta_{\text{UV}}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right] + \rho_{\text{ct}} + \rho_{\phi,\text{finite}}, \tag{4.35}$$

where $\rho_{\text{finite}} \equiv \rho_\phi - \rho_{\phi,\text{div}}$. We are immediately presented with a choice here – after subtracting the pole, do we proceed to (multiplicatively) renormalize Newton's constant, or (additively) renormalize the background whose unshifted value is determined by $\rho^{\text{cl}}_{\text{bg}}$? It turns out that this choice is rendered moot by the tadpole condition that determines the background equations of motion at any given order in $\hbar$, in that whichever choice we make will lead us to the same shifted tadpole condition. We thus proceed by reabsorbing the pole by adding a counterterm that multiplies the Ricci scalar in the Einstein Hilbert action with coefficient B defined as

$$\rho_{\text{ct}} = \frac{3H^2}{(a/a_R)^4}\left(\frac{B_{-1}}{\delta_{\text{UV}}} + B_0\right). \tag{4.36}$$

By assigning

$$B_{-1} = -\frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{24\pi^2}, \tag{4.37}$$

one subtracts the pole contribution. By defining a scale dependent gravitational coupling as

$$\frac{1}{8\pi G_N(\mu)} = \frac{1}{8\pi G_B} - B_0 - \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\rm tot}})\left\{1-\gamma_E+\log\left(\frac{\mu}{H}\right)\right\}, \tag{4.38}$$

we can rewrite Eq. 4.35 as

$$\frac{1}{8\pi G_N(\mu)}\frac{3H^2}{(a/a_R)^4} = \rho^{\rm cl}_{\rm bg} + \rho_{\phi,{\rm finite}}, \tag{4.39}$$

where the finite remainder from the counterterm and $\rho_{\phi,{\rm div}}$ are absorbed by the scale dependent gravitational coupling, allowing us to start imposing renormalization conditions to fix the finite parts. We do so by determining the Newtonian constant via a measurement at some energy scale μ_*, with which we can eliminate all reference to G_B and B_0 via

$$\frac{1}{8\pi G_B} = \frac{1}{8\pi G_N(\mu_*)} + B_0 + \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\rm tot}})\left\{1-\gamma_E+\log\left(\frac{\mu_*}{H}\right)\right\}, \tag{4.40}$$

and substituting the result into Eq. 4.38 to obtain

$$\frac{1}{8\pi G_N(\mu)} = \frac{1}{8\pi G_N(\mu_*)} - \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\rm tot}})\log\left(\frac{\mu}{\mu_*}\right). \tag{4.41}$$

Picking μ_* to be some scale where we have determined Newton's constant[11] to be $8\pi G_N(\mu_*) = M_{\rm pl}^{-2}$ where $M_{\rm pl} = 2.435\times 10^{18}$ GeV, we can finally express $G_N(\mu)$ as

$$8\pi G_N(\mu) = \frac{1}{M_{\rm pl}^2}\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{24\pi^2 M_{\rm pl}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}, \tag{4.42}$$

which can be used to express Eq. 4.39 in its fully covariant form as

$$G_{\mu\nu} = \frac{1}{M_{\rm pl}^2}T^{\rm bg,shift}_{\mu\nu}\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{24\pi^2 M_{\rm pl}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}, \tag{4.43}$$

where the shifted background stress tensor is defined as the sum of tree level and finite contributions on the left hand side of Eq. 4.39. Several things are to be immediately noted here – foremost is the minuscule nature of the scale dependence of the

[11] Note that measuring the strength of the gravitational coupling can only be done via a Cavendish type experiment, typically done at laboratory scales where we have independent knowledge of the masses whose mutual gravitational force we can determine. This yet another manner in which gravity is distinguished among forces as the only force whose coupling strength we measure in the UV (i.e. mm scale) and run into the IR, rather than the other way around.

gravitational coupling, should we phrase it that way. We could also simply view it as a multiplicative renormalization of the background matter content that sources the expansion history[12]. Given that virtual effects from test scalar fields serve only to renormalize background quantities[13] and impart scale dependence in observables associated to other propagating DoFs (and that too, in a highly suppressed manner [84, 40, 59]), this is the furthest we can take this exercise.

The situation for GWs is more interesting. Retracing the steps above with the dimensionally regularized result for GWs on a finite duration background in Eq. 4.26, we end up with the renormalized background

$$G_{\mu\nu} = \frac{1}{M_{\text{pl}}^2} T_{\mu\nu}^{\text{bg,shift}} \left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\text{tot}}})}{12\pi^2 M_{\text{pl}}^2} \log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}, \tag{4.44}$$

which should be the starting point for determining any constraints on vacuum sourced primordial GWs from N_{eff} bounds. As reviewed in Section 2.3.2, this is in essence the question of how vacuum tensor perturbations renormalize the background expansion through $1/a^4$ contributions that mimic additional relativistic species. Unlike the case for virtual test scalars, however, GWs have a classically evolving background upon which they represent perturbations – the background geometry itself. Therefore, we have additional means to potentially measure the contributions from vacuum tensor modes.

We first reconsider the spectral density for GWs $\Omega^{\text{gw}}(k,\tau)$, whose amplitude on the scale invariant plateau for sub-horizon modes is well defined, and given by

$$\Omega^{\text{gw}}(k,\tau) = \frac{H^4}{16\pi^4 (a/a_{\text{R}})^4} \qquad (\text{sub}-\text{horizon}) \tag{4.45}$$

as plotted in Fig. 4.1. Let us presume that we have the means to determine the ratio H^2/M_{pl}^2 during inflation via measurement of the tensor to scalar ratio at some pivot scale via B-mode anisotropy observations, or via the measurement of the spectral density of the stochastic gravitational wave background at some fixed scale via interferometeric means, or both. In the context of Eq. 4.44, which can now be re-expressed as

$$\begin{aligned} \frac{3H^2}{(a/a_R)^4} &= \frac{1}{M_{\text{pl}}^2}\left(\rho_{\text{bg}}^{\text{cl}} + \rho_{\text{gw,finite}}\right)\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\text{tot}}})}{12\pi^2 M_{\text{pl}}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1} \\ &= \frac{\rho_{\text{bg}}^{\text{cl}}}{M_{\text{pl}}^2}(1+\delta_{\text{r}})\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\text{tot}}})}{12\pi^2 M_{\text{pl}}^2}\log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}, \end{aligned} \tag{4.46}$$

[12]This interpretation is to be preferred if one would like keep the graviton to be canonically normalized throughout all of cosmic history, something that is implicitly taken for granted for most quoted observational results.

[13]Noting that given the vanishing background energy density of the test scalar field, it contributes vanishingly to the curvature perturbation.

where $\delta_{\rm r}$ is a constant since the putative tree level background and the stochastic background of vacuum tensor modes both scale as $\propto (a/a_{\rm R})^4$. Therefore, recalling that the quantities that parametrized the background on the left hand side of Eq. 4.32 were also implicitly bare quantities (see the second step of the renormalization procedure reviewed in Section 1.2.2 for more details), one finds that the shifted tadpole condition which one obtains upon renormalization is indistinguishable from a rescaling of the scale factor normalization at reheating, or to simple shift the temperature redshift relation in an otherwise unobservable manner[14]. This result might not come with surprise after the review on the effects of including quantum corrections in Section 1.2 but leads to substantial consequences on the meaning of the BBN bounds on the amount of GWs produced by vacuum tensor perturbations. It is now evident that the bound on $N_{\rm eff}$ from CMB as presented in Section 2.3.2, cannot be used as a constraint as it is first necessary to fix the renormalization conditions trough independent observations.

4.5 Conclusions

In this chapter we analyzed the physical example of vacuum tensor perturbations and showed that divergences in primordial observables are not something cosmologists have the luxury of ignoring: every cosmological tracer corresponding to a density fluctuation samples and convolves the coincident limit of a bilinear field, necessitating subtraction.

Merely regularizing, however, is not enough. As reviewed in Section 1.2.2, the process of arriving at a physical observable is incomplete unless one follows through by constructing the requisite counterterms and fixing any finite contributions that could accompany any subtraction via the imposition of renormalization conditions. Failure to do so runs the risk of drawing unphysical conclusions that include scheme (e.g. cutoff) dependence in observables where there should not be any, or over-interpreting contributions to physical observables that are absorbed or otherwise accounted for in the process of renormalization. To carry out the renormalization of the energy density of GWs, we needed to improve the definition of the stress tensor for GWs to obtain a result that does not presume a prior scale separation in the context of cosmology. We were then allowed to apply what we learned from the study of the scalar case, perform the integral over all wavelengths and regularize the divergences appearing in computing the energy density. After reabsorbing the divergences by adding the required counterterms, we are left with a finite amount of renormalization conditions to be imposed after measuring physical observables.

Therefore, in agreement with the review on the effects of including quantum corrections in Section 1.2, we conclude that vacuum tensor fluctuations by their very nature only serve to renormalize background quantities. As a consequence, vacuum tensor fluctuations do not enter as an additional effective light species as registered

[14]We should also stress that although we kept the scale dependent factor in Eq. 4.46 for completeness, it can effectively be set to unity given the current upper bounds on the tensor to scalar ratio r_* or about $r_* \lesssim 3 \times 10^{-2}$ [67, 10, 9], so that $H^2/(8\pi^2 M_{\rm pl}^2) = \frac{r_*}{16}\Delta^2_{\mathcal{R}} \lesssim 10^{-12}$, in combination with the fact that the log of the ratio between laboratory and Hubble scales is no more than order 10^2.

by N_{eff} bounds. This should be immediately apparent from the physical nature of N_{eff} bounds as measuring the ratio of propagating light species that have undergone freeze-out relative to the entropy density of the universe, which does not apply to vacuum fluctuations. This, is of course, not true for gravitons that are physically produced by some mechanism in the early universe; however, the latter will also feature a bounded integrated spectral density.

Part II

Covariant formulation

CHAPTER 5

QFT in curved spacetime: $T_{\mu\nu}$ from the action

5.1 Introductory remarks

In this chapter, we give the context upon which our work described in Chapter 6 is structured as, by studying simple applications such as quantum scalar fields or QED, we introduce the tools used in the next chapter. Indeed, as we aim to repeat the derivations of chapter 4 in a fully covariant formulation, we introduce the basics of QFT in curved spacetime to derive the semi-classical Einstein equations, the Faddeev-Popov method as gauge-fixing method, and the $P(X)$ theory as the Lagrangian description of a perfect fluid.

In Section 5.2 we start by reviewing the semi-classical approach of QFT in curved spacetime, where the fields appearing as the source of Einstein equations are treated quantum-mechanically and the background metric is parametrized by a classical field. In doing so, we introduce the effective action of a quantum field and the link between the latter with the stress energy tensor of the field. We introduce the background field method as a way to compute the 1-loop corrected effective action and we apply the background field method to the case of a scalar field. We explicitly show that the effective action of a given background field can be computed as the sum of all one-particle-irreducible vacuum graphs where all internal lines correspond to the fluctuations around the background. We then compare the divergences appearing in computing the 1-loop effective action and the stress energy tensor and we show that in general they are not identical. We review with a specific example the differences between the regularized stress energy tensor at the regularized effective action, even if it is possible to connect both results with the divergences arising in the propagator. We then recall the Faddeev-Popov method as a gauge-fixing method which allows to covariantly fix the gauge at the level of the action. By analyzing the example of QED, we review the main features of this gauge-fixing method, such as the appearance of ghosts as unphysical DoFs that fix the remaining spurious DoFs.
Finally, in Section 5.3 we introduce the $P(X)$ theory and we show how radiation can be represented by a scalar field. We derive the stress energy tensor and, by comparing the result with the stress energy tensor of a perfect fluid, we accordingly define energy density and pressure as a function of the scalar field. We then specify

the results in the case of radiation and derive the Lagrangian that reproduces the stress energy tensor for the radiation fluid.
In Section 5.4 we conclude and motivate our work on the covariant formulation of GWs on curved spacetime presented in the following chapter.

5.2 Effective action and background field method

An alternative strategy with respect to how we proceeded in the previous chapter is to derive the stress energy tensor of GWs from the action. We follow chapter 6 of [48] to review how we can treat the computation of the stress energy tensor of a quantum field in a semi-classical approach, where the background metric is parametrized by a classical field while matter fields are treated quantum mechanically[1]. We then seek the action defined as

$$S = S_{\mathrm{g}} + S_{\mathrm{m}} \tag{5.1}$$

that, once variated with respect to the background metric $g_{\mu\nu}$, gives Einstein's field equations

$$R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = -8\pi G_N \langle \hat{T}^{\mathrm{m}}_{\mu\nu} \rangle \tag{5.2}$$

where $\langle T^{\mathrm{m}}_{\mu\nu} \rangle := \langle \mathrm{in}, 0 | \hat{T}^{\mathrm{m}}_{\mu\nu} | 0, \mathrm{in} \rangle$ is the quantum expectation value of the stress energy tensor of the matter content[2] defined as

$$\hat{T}^{\mathrm{m}}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} \tag{5.3}$$

and g is the determinant of the background metric. The first term of the action in Eq. 5.1 is the Einstein Hilbert action

$$S_{\mathrm{g}} = S_{\mathrm{EH}} = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G_N} \right) \tag{5.4}$$

while the second term, that in the classical case would be the classical matter action, is now the effective action $W_{\mathrm{m}}[0]$. The effective action is defined from the generating functional

$$Z[J] = \int \mathcal{D}\phi \exp\left\{ \mathrm{i} S_{\mathrm{m}}[\phi] + \mathrm{i} \int d^4x J(x) \phi(x) \right\} \tag{5.5}$$

where from now on the notation implies a treatment for the scalar field, but the formal manipulations are identical for fields of higher spins. The external current J in Eq. 5.5 causes the initial vacuum state to be unstable and brings the production

[1] This is a generalization of semiclassical theory of electrodynamics ([111]).
[2] Note that with "matter content", we are referring to a generic quantum field that sources the background expansion. Later on, we will be interested in specifying these results for GWs; however, the derived results are valid for fields with generic spin. Furthermore in this section we drop the hat to denote the difference between operators and mode functions as matter fields which are always intended to be operators.

of particles. Taking the limit $J \to 0$ we find the vacuum partition function that in Minkowski is normalized to 1

$$Z[0] := \langle \text{out}, 0|0, \text{in}\rangle = 1. \tag{5.6}$$

Even if in curved spacetime in general $|0, \text{out}\rangle \neq |0, \text{in}\rangle$ in the limit $J \to 0$, path-integral quantization still works and J is interpreted as the current density. We then define the effective action in the limit $J \to 0$ as

$$Z[0] := e^{iW_\phi[0]} \tag{5.7}$$

which, using Eq. 5.6 gives

$$W_\phi[0] = \lim_{J\to 0}(-i \ln Z[J]) = -i \ln\langle \text{out}, 0|0, \text{in}\rangle \tag{5.8}$$

and that, using

$$\delta Z[0] = i\langle \text{out}, 0|\delta S_\text{m}|0, \text{in}\rangle, \tag{5.9}$$

can be used to rewrite Eq. 5.3 as

$$\frac{\langle \text{out}, 0|\hat{T}^\text{m}_{\mu\nu}|0, \text{in}\rangle}{\langle \text{out}, 0|0, \text{in}\rangle} = \frac{2}{\sqrt{-g}}\frac{\delta W_\phi[0]}{\delta g^{\mu\nu}}. \tag{5.10}$$

The appearance of $\langle \text{out}, 0|\hat{T}^\text{m}_{\mu\nu}|0, \text{in}\rangle$ in Eq. 5.10 instead of $\langle \text{in}, 0|\hat{T}^\text{m}_{\mu\nu}|0, \text{in}\rangle$, is related to different boundary conditions. However, as we will discuss in detail in what follows, boundary conditions do not effect the divergences arising in computing the stress energy tensor and enter only in the finite leftover of the renormalized result[3]. The latter, being scheme dependent, needs to be fixed imposing renormalization conditions. As a consequence, in the following we do not worry about the arrangement of vacuum states and proceed in describing the background field method as a procedure to calculate the 1-loop effective action.

The effective action can be determined up to 1-loop corrections in different ways. Functional techniques such as the heat kernel method (through which the DeWitt-Schwinger formalism can be implemented [196, 81, 82, 48, 178]) have the advantage of being fully covariant and therefore particularly suited for computation on arbitrary backgrounds, albeit in Euclidean signature [210].
In the next chapter we work in the context of the background field method, which we now review following [194], as it is particularly useful in computing loop corrections in gauge field theories. The background field method, introduced by DeWitt in [81] in a formalism applicable to one-loop processes ([94, 112, 27, 191, 131]), has the benefit of quantizing gauge field theories without losing explicit gauge invariance. The background field method has been then extended to multi– loop calculations ([207, 86, 51, 1] and it is extensively used in gravity and supergravity theories (i.e., [85, 208, 206]).
We previously reviewed the definition of $W_\text{m}[0]$ in order to derive the stress energy

[3]See Section 1.2.2 for more details.

tensor of a quantum field. We now reintroduce the source J to briefly review the background field method in the simple example of a scalar field (see [1] for more details and applications to gauge theories and [90] for a more recent review) to derive the effective action given by the Legendre transform[4]

$$\Gamma[\phi] = W_\phi[J_\phi] - \int d^4x J_\phi \phi \tag{5.11}$$

where J_ϕ satisfies

$$\left.\frac{\partial W_\phi[J]}{\partial J}\right|_{J_\phi} = \phi. \tag{5.12}$$

Starting from the definition of the generating function in Eq. 5.5, we split the field ϕ into a non-dynamical, but arbitrary background-field $\bar{\phi}$ and a quantum correction φ:

$$\phi = \bar{\phi} + \varphi \tag{5.13}$$

and rewrite the action as $S_\mathrm{m}[\bar{\phi} + \varphi]$ so that the generating functional results

$$\tilde{Z}[J] := e^{iW[\bar{\phi},J]} = \int \mathcal{D}\varphi \exp\left\{\mathrm{i}S_\mathrm{m}[\bar{\phi} + \varphi] + \mathrm{i}\int d^4x\; J(x)\varphi(x)\right\}. \tag{5.14}$$

After shifting the integrated field as $\varphi \to \varphi - \bar{\phi}$ we obtain

$$W[\bar{\phi}, J] = W_\varphi[J] - \int d^4x J\bar{\phi} \tag{5.15}$$

and $\Gamma[\bar{\phi}, \varphi]$, computed trough the Legendre transform as in 5.11, results

$$\Gamma[\bar{\phi}, \varphi] = W[\bar{\phi}, J_{\bar{\phi}}] - \int d^4x J_{\bar{\phi}}\varphi \tag{5.16}$$

which has the additional dependence on the background field. Using Eq. 5.15 we obtain

$$\Gamma[\bar{\phi}, \varphi] = W_\varphi[J_{\bar{\phi}}] - \int d^4x J_{\bar{\phi}}(\bar{\phi} + \varphi). \tag{5.17}$$

We then define $\bar{\phi}_J$ as an implicit functional of J satisfying

$$\frac{\partial \Gamma[\bar{\phi}]}{\partial \bar{\phi}} = J, \tag{5.18}$$

which gives the inverse Legendre transform

$$W[\bar{\phi}, J] = \Gamma[\bar{\phi}_J] + \int d^4x J\bar{\phi}_J \tag{5.19}$$

that once variated with respect to J results in

$$\frac{\partial W[\bar{\phi}, J]}{\partial J} = \bar{\phi}_J. \tag{5.20}$$

[4]Note that Eq. 5.11 reduces to Eq. 5.8 in the limit $J \to 0$.

Repeating the same exercise starting from Eq. 5.11 we obtain $\frac{\partial W_\phi[J]}{\partial J} = \phi_J$. In this way, the variation of Eq. 5.15 with respect to J can be rewritten as

$$\bar{\phi}_J = \varphi_J - \bar{\phi}. \tag{5.21}$$

Then, taking $\varphi = \bar{\phi}_J$, $J_{\bar{\phi}_J} = J$ and the result in Eq. 5.21, Eq. 5.17 becomes

$$\Gamma[\bar{\phi}, \bar{\phi}_J] = W_\varphi[J] - \int d^4x J\varphi_J = \Gamma[\varphi_J] = \Gamma[\bar{\phi}_J + \bar{\phi}]. \tag{5.22}$$

Since this is true for any current J, we obtain the final result $\Gamma[\bar{\phi}, \varphi] = \Gamma[\bar{\phi} + \varphi]$, which implies $\Gamma[\bar{\phi}, 0] = \Gamma[\bar{\phi}]$. This means that one can compute the effective action as the sum of all one-particle-irreducible vacuum graphs in the presence of a given background, with all internal lines corresponding to fluctuations around this background.

Going back to the derivation of the stress energy tensor of a quantum field in curved spacetime, there is one last caveat we want to review, before studying the graviton example in the next chapter. Generalizing the derivation in flat spacetime (see [48] for more details) it is possible to connect the effective action $W_\phi[0]$ to the Feynman propagator $G_{\mathrm{F}}(x, x')$

$$W_\phi[0] = -\frac{i}{2} \lim_{x \to x'} \mathrm{Tr} \ln(-G_{\mathrm{F}}(x, x')) \tag{5.23}$$

where $G_{\mathrm{F}}(x, x')$ is to be interpreted in position space normalized by

$$\langle x | x' \rangle = \frac{1}{\sqrt{-g}} \delta^4(x - x') \tag{5.24}$$

so that

$$G_{\mathrm{F}}(x, x') = \langle x | G_{\mathrm{F}} | x' \rangle \tag{5.25}$$

and where trace is defined as

$$\mathrm{Tr} M = \int d^4x \sqrt{-g} \langle x | M | x \rangle. \tag{5.26}$$

In computing the coincidence limit in Eq 5.23, divergences arise and, considering Eq. 5.10, such divergences affect the stress energy tensor as well[5]. As the divergences arise because of the UV behaviour of the field modes, they are independent on the quantum state and can be expressed in terms of the background geometry only. By defining the effective Lagrangian $\mathcal{L}^{\mathrm{m}}_{\mathrm{eff}}$ as

$$W_{\mathrm{m}}[0] = \int d^4x \sqrt{-g} \mathcal{L}^{\mathrm{m}}_{\mathrm{eff}}, \tag{5.27}$$

[5] Many examples of the study of such divergences can be found in the literature (see [56, 79, 104, 93, 22, 13, 57, 71, 12, 42, 18] for some examples and [48] for an overview and comparison of different regularization methods), where both fields of different spins and different renormalization methods are investigated.

this implies that the divergences of $\mathcal{L}^{\mathrm{m}}_{\mathrm{eff}}$ are purely geometrical and can be expressed solely as a function the background metric, even if they arise from the action of the quantum field[6]. The divergencies arising in $\mathcal{L}^{\mathrm{m}}_{\mathrm{eff}}$ are then reabsorbed by redefining the coupling constant of the background action: after regularizing, to rewrite the divergences in the form

$$\mathcal{L}^{\mathrm{m}}_{\mathrm{eff,div}} = c_0 + c_1 R + c_2 R^2 + c_3 R_{\mu\nu} R^{\mu\nu} + c_4 \Box R, \tag{5.28}$$

where c_0, c_1, c_2, c_3 and c_4 are divergent in the coincidence limit. Then, by adding higher order corrections to the background action in Eq. 5.1, we can absorb the divergences arising in the effective action of the quantum content by redefining the coupling constants of S_{g} (cosmological constant, Newtonian constant and prefactors of higher order corrections of the Einstein-Hilbert action) and obtain

$$S = S_{\mathrm{g,ren}} + W_{\mathrm{m,ren}} \tag{5.29}$$

where $W_{\mathrm{m,ren}}$ is now finite. By variating with respect to the background metric, we obtain the renormalized semi-classical Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + a H^{(1)}_{\mu\nu} + b H^{(2)}_{\mu\nu} = -8\pi G \frac{\langle \mathrm{out}, 0 | \hat{T}^{\mathrm{m}}_{\mu\nu} | 0, \mathrm{in} \rangle_{\mathrm{ren}}}{\langle \mathrm{out}, 0 | 0, \mathrm{in} \rangle} \tag{5.30}$$

where $H^{(1)}_{\mu\nu}$ and$H^{(2)}_{\mu\nu}$ are the variation of $\int d^4x \sqrt{-g} R^2$ and $\int d^4x \sqrt{-g} R_{\mu\nu} R^{\mu\nu}$ respectively and Λ, G, a and b must be fixed by renormalization conditions.
However, even if the divergences arising in computing the stress energy tensor of a quantum field are connected with those arising in computing the 1-loop effective action, they are not always identical. Indeed, as we show in the following, both the divergences of the tress energy tensor and the effective action can be derived from the regularized propagator $G_{\mathrm{F}}(x, x')$. Regardless of this, subtracting the divergences arising in computing $G_{\mathrm{F}}(x, x')$ is in general not enough to obtain a finite result for the stress energy tensor derived from Eq. 5.10. Considering the DeWitt-Schwinger representation of the 1-loop effective action ([196, 81])

$$W_{\mathrm{m}}[0] = \frac{i}{2} \int \mathrm{d}^n x [-g(x)]^{\frac{1}{2}} \lim_{x' \to x} \int_0^\infty \mathrm{d}m^2 G_F(x, x'), \tag{5.31}$$

we obtain that the regularized form of the effective Lagrangian in the DeWitt-Schwinger expansion results

$$\begin{aligned} \mathcal{L}^{\mathrm{m}}_{\mathrm{eff,div}} = -\lim_{x' \to x} \frac{\Delta^{\frac{1}{2}}(x, x')}{32\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s^3} \mathrm{e}^{-i\left(m^2 s - \sigma/2s\right)} \left[a_0(x, x') + a_1(x, x')\, \mathrm{i}s \right.\\ \left. + a_2(x, x') (\mathrm{i}s)^2\right] \end{aligned} \tag{5.32}$$

where Δ is the Van Vleck-Morette determinant and $a_0(x, x')$, $a_1(x, x')$ and $a_2(x, x')$ depend on the spin of the quantum field we are considering and are functions of

[6]Note that this is not the case for the finite contribution to $\mathcal{L}^{\mathrm{m}}_{\mathrm{eff}}$, which depends on the quantum state, it corresponds to the long wavelength part and can probe the large scale structure of the manifold.

the background metric only (see [210] for examples of effective Lagrangian in the DeWitt-Schwinger expansion for fields of different spins). On the other hand, the result for the regularized stress energy tensor is given not only by the DeWitt-Schwinger representation of $G_{\rm F}(x,x')$ as for the effective action, but also by the divergences obtained by differentiating the DeWitt-Schwinger representation of $G_{\rm F}(x,x')$. Only in the case of conformally trivial systems, in which both the background metric and the quantum field are conformally invariant, the divergences arising in the stress energy tensor are identical to those of the 1-loop renormalized action[7]. However, in general such "short cut" is not available, and it is necessary to separately compute the divergent contribution to the stress energy tensor. As an example, we review the results of Bunch et al. in [57] to show the difference between the regularized effective action and stress energy tensor. Considering a massive scalar field in curved spacetime, the stress energy tensor results

$$\begin{aligned} T^{\phi}_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{2}g_{\mu\nu}m^2\phi^2 \\ &= \lim_{x\to x'}\left(\nabla_{\mu'}\nabla_\nu G^{\phi}_{\rm F}(x,x') - \frac{1}{2}g_{\mu\nu}\nabla_{\alpha'}\nabla^\alpha G^{\phi}_{\rm F}(x,x') + \frac{1}{2}g_{\mu\nu}m^2 G^{\phi}_{\rm F}(x,x')\right) \end{aligned} \tag{5.33}$$

where we use the notation in which primed indices refer to the derivative with respect to x'. As shown in [57], the divergent contribution to the stress energy tensor is given by the sum of the following terms

$$\begin{aligned} \lim_{x\to x'}\left(\nabla_{\mu'}\nabla_\nu G^{\phi}(x,x') - \frac{1}{2}g_{\mu\nu}\nabla_{\alpha'}\nabla^\alpha G^{\phi}(x,x')\right) = \lim_{\sigma\to 0}(2\pi)^{-1}\left[\left(\frac{1}{\sigma} - \frac{1}{4}m^2 + \frac{1}{24}R\right)\right. \\ \left.\left(g_{\mu\nu} - 2\frac{\sigma_\mu\sigma_\nu}{\sigma_{\alpha^2}\sigma^\alpha}\right) + \frac{1}{60}m^{-2}\left(R_{;\mu\nu} - \frac{1}{2}\Box R g_{\mu\nu}\right) + O\left(m^{-4}\right) + O\left(\sigma^{1/2}\right)\right] \end{aligned} \tag{5.34}$$

$$\begin{aligned} \lim_{x\to x'}\frac{1}{2}m^2 g_{\mu\nu}G^{\phi}(x,x') = -\lim_{\sigma\to 0}\frac{g_{\mu\nu}}{2\pi}\left[\frac{m^2}{4}\ln\left|\frac{m^2\sigma}{2}\right| - \frac{1}{24}R - \frac{1}{240m^2}\left(R^2 + 2\Box R\right)\right. \\ \left. + O\left(m^{-4}\right) + O\left(\sigma^{1/2}\right)\right]. \end{aligned} \tag{5.35}$$

As a consequence, regularizing only $G^{\phi}_{\rm F}(x,x')$ is not sufficient to obtain the regularized result for the stress energy tensor. Indeed, looking at Eq. (5.34), we notice that there is an extra pole arising in the derivative of $G^{\phi}(x,x')$ that would not appear if we were regularizing the effective action.

5.2.1 Gauge-fixing: Faddeev-Popov method

As mentioned in section 2.2.1, in order to reduce the 10 DoFs of the symmetric rank two tensor $h_{\mu\nu}$ to the two physical DoFs representing the graviton, it is necessary to fix the gauge to get rid of the spurious DoFs. In the following chapter we

[7] See [48] for more details or [73] for some examples. Note that this is not the case for the graviton, see Chapter 6 for more details.

will do so using the so-called Faddeev-Popov method [97], which is reviewed in this section (see [194, 125, 184] for more details).

The Faddeev-Popov method is based on the fact that, in defining the partition function as the integral over all the field configurations, in gauge theories one is including configurations that are equivalent up to a gauge transformation. As a consequence, one is integrating over an infinite set of copies of just one configuration. The Faddeev-Popov method provides a procedure for isolating and computing the integral over the unphysical redundant configurations. We review how this is done in the context of QED, even if, as we will see in the following, QED is a trivial example that does not require Faddeev-Popov ghosts to subtract the spurious DoFs. A more instructive example will be studied in the next chapter, where we use the Faddeev-Popov method to fix the gauge for the graviton action.
We consider the path integral of the electromagnetic field A_μ

$$\int \mathcal{D}A_\mu \, e^{-i\int d^4x \, \mathcal{L}_{\rm EM}[A_\mu]} \tag{5.36}$$

which is invariant under the gauge transformation

$$A_\mu \to A_\mu^\pi = A_\mu + \partial_\mu \pi := \mathcal{G}[A_\mu^\pi]. \tag{5.37}$$

Defining the Faddeev-Popov determinant $\Delta_{\mathcal{G}}[A_\mu] := \det\left(\frac{\partial \mathcal{G}}{\partial \pi}\right)$ such that

$$1 = \Delta_{\mathcal{G}}[A_\mu] \int \mathcal{D}\pi \, \delta(\mathcal{G}[A_\mu^\pi]), \tag{5.38}$$

we can equivalently rewrite Eq. 5.36 as

$$\int \mathcal{D}A_\mu \, \Delta_{\mathcal{G}}[A_\mu] \int \mathcal{D}\pi \, \delta(\mathcal{G}[A_\mu^\pi]) e^{-i\int d^4x \, \mathcal{L}_{\rm EM}[A_\mu]}. \tag{5.39}$$

We perform in the integrand the gauge transformation from A_μ^π to A_μ and obtain

$$\int \mathcal{D}\pi \int \mathcal{D}A_\mu \, \Delta_{\mathcal{G}}[A_\mu] \delta(\mathcal{G}[A_\mu]) e^{-i\int d^4x \, \mathcal{L}_{\rm EM}[A_\mu]} \tag{5.40}$$

where we took advantage of the fact that $\mathcal{L}[A_\mu]$, the Faddeev-Popov determinant and the measure are invariant under gauge transformations. As a result, since in Eq. 5.40 nothing depends on π anymore, the integral over π results in multiplying the partition function of an overall (infinite) constant and we obtain

$$\int \mathcal{D}A_\mu \, \Delta_{\mathcal{G}}[A_\mu] \delta(\mathcal{G}[A_\mu]) e^{-i\int d^4x \, \mathcal{L}_{\rm EM}[A_\mu]}. \tag{5.41}$$

We now note that we can equivalently rewrite the result in Eq. 5.41 by considering the average over a Gaussian-weighted selection of shifts[8] of the gauge condition

[8]We consider that nothing would change if we shift $\mathcal{G}[A_\mu]$ by a constant and that we can multiply by the unity obtained from the Gaussian integral: $1 = N(\xi) \int \mathcal{D}\chi e^{-\frac{i}{2\xi}\int d^4x \chi^2}$, where $N(\xi)$ is a normalization constant.

$\mathcal{G}[A_\mu]$

$$\int \mathcal{D}\chi \, e^{-i\int d^4x \frac{\chi^2}{2\xi}} \delta\left(\mathcal{G}[A_\mu] - \chi\right) = e^{-i\int d^4x \frac{1}{2\xi}\mathcal{G}[A_\mu]^2} \tag{5.42}$$

and obtain

$$\int \mathcal{D}A_\mu \, \Delta_{\mathcal{G}}[A_\mu] e^{-i\int d^4x \left(\mathcal{L}_{\rm EM}[A_\mu] + \frac{1}{2\xi}\mathcal{G}[A_\mu]^2\right)}. \tag{5.43}$$

Lastly, we consider that the Faddeev–Popov determinant can be expressed as the integral over an artificial fermion field η

$$\Delta_{\mathcal{G}}[A_\mu] = \det\left(\frac{\partial \mathcal{G}}{\partial \pi}\right) = \int \mathcal{D}\eta \mathcal{D}\bar{\eta} \, e^{i\int d^4x \bar{\eta}\frac{\partial \mathcal{G}}{\partial \pi}\eta} \tag{5.44}$$

to obtain the final result for the gauge-fixed partition function

$$\int \mathcal{D}A_\mu \mathcal{D}\eta \mathcal{D}\bar{\eta} \, e^{-i\int d^4x \left(\mathcal{L}_{\rm EM}[A_\mu] + \frac{1}{2\xi}\mathcal{G}[A_\mu]^2 - \bar{\eta}\frac{\partial \mathcal{G}}{\partial \pi}\eta\right)}. \tag{5.45}$$

We now comment on the terms appearing in the gauge-fixed action of Eq 5.45

$$S_{\rm EM} = \int d^4x \, \mathcal{L}_{\rm EM}[A_\mu] + S_{\rm gb} + S_{\rm gh} \tag{5.46}$$

where we defined the so-called gauge breaking action and the Faddeev–Popov ghost action respectively as

$$S_{\rm gb} = \frac{1}{2\xi}\int d^4x \, \mathcal{G}[A_\mu]^2 \tag{5.47}$$

and

$$S_{\rm gh} = -\int d^4x \, \bar{\eta}\frac{\partial \mathcal{G}}{\partial \pi}\eta. \tag{5.48}$$

We notice that the gauge breaking term fixes the gauge condition $\mathcal{G}[A_\mu]$ and, as a consequence, the resulting action is gauge dependent. The fermion field η, called Faddeev–Popov ghost, is an anticommuting Lorentz scalar and, even if the ghost term does not contribute in the case of QED[9], this new unphysical[10] field is the responsible of subtracting the remaining spurious DoFs in non-abelian gauge theories.

5.3 $P(X)$ theory and radiation-like species

In this section, we review the Lagrangian formulation of a perfect fluid ([157, 28]). We follow the notation of Boubekeur et al. in [50] to review the $P(X)$ theory, where the physics of a perfect fluid is derived from a unique scalar field and radiation can be represented by the scalar field ψ.

[9] Since $\frac{\partial \mathcal{G}}{\partial \pi}$ does not depend on the electromagnectic field, the Faddeev–Popov determinant is just a constant and can be dropped.

[10] Faddeev–Popov ghosts violate the spin statisitc theorem, however, since they appear only as internal lines, this is not a problem in computing physical quantities.

We consider the Lagrangian density

$$\mathcal{L} = P(X) = X^2 \quad \text{where} \quad X := -g_{\mu\nu}\partial_\mu\psi\partial^\nu\psi, \tag{5.49}$$

so that the stress energy tensor results[11]

$$T_{\mu\nu} = \mathcal{L}g_{\mu\nu} - \frac{\partial\mathcal{L}}{\partial\partial^\mu\psi}\partial_\nu\psi = 2P'(X)\partial_\mu\psi\partial_\nu\psi + P(X)g_{\mu\nu} \tag{5.50}$$

and the EOM results

$$\partial_\mu\left[P'(X)\partial^\mu\psi\right] = 0. \tag{5.51}$$

Given that the stress energy tensor of a perfect and barotropic fluid is of the form

$$T^{\text{pf}}_{\mu\nu} = (\rho + p(\rho))u_\mu u_\nu + p(\rho)g_{\mu\nu}, \tag{5.52}$$

where u^μ is the 4-velocity of the fluid, we compare with Eq. 5.50 to identify the energy density, the pressure and the 4-velocity as

$$\rho = 2P'(X)X - P(X), \quad p = P(X), \quad u_\mu = \frac{\partial_\mu\psi}{\sqrt{X}} \tag{5.53}$$

and impose that $\partial_\mu\psi$ is everywhere timelike and future directed. In this way, we obtain a consistent Lagrangian formulation of a perfect fluid. Note that in projecting the conservation equation $\partial_\mu T^{\mu\nu}$ along and orthogonal to the fluid flux we obtain the conservation of energy, which results in the EOM in Eq. 5.51, and Euler equation respectively. Furthermore, one can derive the speed of sound c_s that results

$$c_s = \frac{dp}{d\rho} = \frac{P'(X)}{P'(X) + 2P''(X)X}. \tag{5.54}$$

As in the next chapter we use this formalism to describe radiation, we consider the equation of state $p = w\rho$ with w constant. Using the definitions in Eq. 5.53, we then find

$$P(X) = X^{\frac{1+w}{2w}} \tag{5.55}$$

up to a proportionality constant which is irrelevant for the classical theory. In order to find the Lagrangian density for the radiation fluid we consider $w = \frac{1}{3}$, so that the Lagrangian density reduces to

$$\mathcal{L} = X^2 = \left(-\nabla_\mu\psi\nabla^\mu\psi\right)^2. \tag{5.56}$$

Furthermore, it is straightforward to verify that the speed of sound in Eq. 5.54 results $c_s = w = \frac{1}{3}$, the stress energy tensor in Eq. 5.50 is traceless and the EOM results

$$\partial_\mu\left[-2\partial_\nu\psi\partial^\nu\psi\partial^\mu\psi\right] = 0. \tag{5.57}$$

[11] we define $P'(X) = \frac{\partial P(X)}{\partial X}$.

5.4 Comments and motivations for our work

In this chapter, which is to be intended as a continuation of Chapter 2, we introduced the formalism that we will use in the next chapter to study vacuum GWs in a covariant formulation. As we aim to study the caveats introduced in Chapter 2, we introduced the definition of the stress energy tensor of a quantum field in curved spacetime as the variation of the effective action with respect to the background metric and we reviewed the tools needed to parametrize GWs as a massless spin-2 particle on a RD spacetime. This includes the background field method, the Faddeev-Popov method and $P(X)$ theory.

In Chapter 6 we present our work and re-derive in a fully covariant way the results in Chapter 4. In studying the case of the graviton, we derive the regularized Lagrangian that has the form of Eq. 5.28, we reabsorb the divergences by redefining the coupling constants of the background action and we comment on the finite part and on the renormalization conditions to be imposed.
As we will show in the next chapter, the covariant formulation is not only confirming the conclusion of the previous chapter, but also bringing to light features that were hidden in the foliation dependent treatment.

CHAPTER 6

Hadamard regularization of the graviton stress tensor

This chapter is based on:
Hadamard Regularization of the Graviton Stress Tensor
Anna Negro, and Subodh P. Patil.
(March, 2024), arXiv:2403.16806 .

6.1 Introductory remarks

In this chapter we reexamine what was done in Chapter 4 in a fully covariant way. Having reviewed in the previous chapter the necessary tools to study GWs as a massless spin-2 particle on a curved spacetime, we revisit the regularization and renormalization of vacuum tensor perturbations. This allows us to highlight that the FRLW foliation formulation studied in the previous chapters does not feature all the subtleties that one can encounter in renormalizing the stress energy tensor of a quantum field on a curved spacetime. Thus, even if in fixing the renormalization conditions we refer to the study of vacuum GWs on a RD universe in order to compare our findings with the previous results, this chapter is intended as a generalization of Chapter 4.

In the following we first derive the effective action for GWs in Section 6.2. To do so we expand the Einstein Hilbert action up to second order in the perturbation $h_{\mu\nu}$ and we fix the de Donder gauge using the Faddeev-Popov method.
Once we have obtained the action that, if variated with respect to the background metric, gives the stress energy tensor of a massless spin-2 particle on a generic background, we proceed to regularize in Section 6.3 using Hadamard regularization. We first introduce Hadamard regularization by reviewing the definition of the regulator σ^{μ}, the Hadamard form of the Feynman propagator and the link with the Hadamard representation of Green's function. We then rewrite both the action of GWs and the action of the Faddeev-Popov ghosts in terms of the Hadamard representation of Green's function and we compute the counterterms needed to reabsorb the divergencies appearing in computing the stress energy tensor.
In Section 6.4 we then proceed in renormalizing the divergences by fixing the renormalization conditions. Before doing so we comment on the finite contributions

of the regularized result, stressing that the finite leftovers of renormalization are not uniquely defined and need to be fixed by experiments. The renormalization conditions are then imposed by specifying the background metric, left unspecified throughout the previous sections. In this way, by using the $P(X)$ theory to derive the action that reproduces the stress tensor for the background radiation fluid, we can reconnect Chapter 4 with and study the effects of a massless spin-2 particle on a RD universe.
We conclude by commenting the results of the fully covariant description of this chapter in Section 6.5 and we leave the details of the Hadamard regularization of the GWs action in the Appendix A.

6.2 Vacuum stress energy tensor from the effective action

Our treatment proceeds from the standard effective action via the background field method (see Chapter 5 for more details). As in Chapter 2, we define $h_{\mu\nu}$ as the perturbations around some background, defined as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{6.1}$$

where $g_{\mu\nu}$ is the background metric, which we leave unspecified for the time being, and the field $h_{\mu\nu}$ represents a massless spin-2 particle. We then consider the effective action

$$S_{\rm cl} := S_{\rm EH} + S_{\rm M} + S_{\rm ct}, \tag{6.2}$$

where $S_{\rm EH}$, which represents the Einstein Hilbert action, and $S_{\rm M}$, representing the matter content that sources the background expansion, constitute the tree level action for the classical background and $S_{\rm ct}$ represents the counterterms needed to subtract the UV divergences that arise at any given loop order, with finite remainders that are to be fixed through renormalization conditions.
We expand the classical Einstein-Hilbert action to quadratic order in perturbations $h_{\mu\nu}$ and obtain [18, 34]

$$\begin{aligned} S^{(2)} = \frac{\kappa^2}{2} \int d^4x \sqrt{-g} \Big[& \frac{1}{2} h_{\rho\sigma} \Box h^{\rho\sigma} + h \nabla^\rho \nabla^\sigma h_{\sigma\rho} + \nabla^\alpha h_\alpha{}^\rho \nabla^\sigma h_{\sigma\rho} - \frac{1}{2} h \Box h \\ & + R^\beta{}_{\rho\alpha\sigma} h_\beta{}^\alpha h^{\rho\sigma} + h_\alpha{}^\rho h^{\alpha\sigma} R_{\rho\sigma} - h h^{\rho\sigma} R_{\rho\sigma} - \frac{1}{2} h^{\rho\sigma} h_{\rho\sigma} R + \frac{1}{4} h h R \Big], \end{aligned} \tag{6.3}$$

where we have defined $\kappa^2 \equiv \frac{1}{8\pi G_N}$. The stress tensor for GWs is obtained by variation with respect to the background metric – a process that is equivalent to perturbing the Einstein equations to second order and bringing the quadratic terms over to the other side to act as a source for the background.

Before proceeding, we recall that if one is interested in studying GWs for which there is a prior scale separation, a number of approximations and simplifications are possible. A detailed overview of such simplifications can be found in Section 2.2, where we study the derivation of the stress energy tensor, that this is valid for spectra of bounded support sourced by some physical production mechanism (and hence sub-horizon). However, taking the result of such derivation in the context of cosmological stochastic backgrounds ought to be treated with caution. This caution should be amplified when one encounters divergences that need to be regularized, as is the case under analysis. Consequently, we define the stress energy tensor for GWs as the variation of the graviton action with respect to the background metric:

$$T^{\rm gw}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left\langle \frac{\delta\left(-S_{\rm gw}\right)}{\delta g^{\mu\nu}}\right\rangle_{\rm in,in} \tag{6.4}$$

where the in-in expectation value implicitly traces over some initial density matrix. When this state is taken as the adiabatic vacuum, one obtains the stress tensor for vacuum tensor perturbations.

In order to define the action of GWs $S_{\rm gw}$, we must first consider the process of gauge-fixing, for which de Donder gauge presents a particularly efficient choice. We proceed via the Faddeev-Popov method [97] (see Section 5.2.1 for more details) and add a gauge breaking term which fixes the chosen gauge condition to Eq. 6.3, defined as $\nabla_\mu h^{\mu\nu} = \frac{1}{2}\nabla_\nu h$, along with a ghost term that accounts for the measure factor induced by gauge-fixing:

$$S_{\rm gb} = -\frac{\kappa^2}{2}\int d^4x\sqrt{-g}\left(\nabla^\mu h_{\mu\nu} - \frac{1}{2}\nabla_\nu h\right)\left(\nabla^\alpha h_\alpha{}^\nu - \frac{1}{2}\nabla^\nu h\right), \tag{6.5}$$

$$S_{\rm gh} = \frac{\kappa^2}{2}\int d^4x\sqrt{-g}\left[\bar{\eta}^\mu\left(g_{\mu\nu}\Box - R_{\mu\nu}\right)\eta^\nu\right]. \tag{6.6}$$

In the above, η^ρ represents the ghost field that accounts for the residual gauge freedom by subtracting the spurious DoFs from the action $S^{(2)}$. Consequently, the gauge-fixed action for the gravitational sector is given by:

$$\begin{aligned} S_{\rm gw} =& S^{(2)} + S_{\rm gb} + S_{\rm gh} \\ =& \frac{\kappa^2}{2}\int d^4x\sqrt{-g}\left[\frac{1}{2}h_{\rho\sigma}\Box h^{\rho\sigma} - \frac{1}{4}h\Box h + R^\beta{}_{\rho\alpha\sigma}h_\beta{}^\alpha h^{\rho\sigma} + h_\alpha{}^\rho h^{\alpha\sigma}R_{\rho\sigma} - hh^{\rho\sigma}R_{\rho\sigma}\right. \\ & \left. -\frac{1}{2}h^{\rho\sigma}h_{\rho\sigma}R + \frac{1}{4}hhR + \bar{\eta}^\mu\left(g_{\mu\nu}\Box - R_{\mu\nu}\right)\eta^\nu\right]. \end{aligned} \tag{6.7}$$

In this way, starting with the action Eq. 6.3 for a rank-2 symmetric tensor field nominally consisting of ten DoFs, we obtain the action of a massless spin-2 particle $S_{\rm gw}$ with only two propagating DoFs. From Eq. 6.4, the stress energy tensor of vacuum tensor perturbations is given by the sum of contributions

$$T^{\rm gw}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left\langle \frac{\delta}{\delta g^{\mu\nu}}\left(-S_{\rm gr} - S_{\rm gh}\right)\right\rangle_{\rm in,in} = T^{\rm gr}_{\mu\nu} + T^{\rm gh}_{\mu\nu}, \tag{6.8}$$

where we have defined $S_{\rm gr} \equiv S^{(2)} + S_{\rm gb}$. The angled brackets above denote the time ordered in-in correlation function $\langle ...\rangle := \langle {\rm in, vac}|T[...]|{\rm in, vac}\rangle$, which inevitably exhibits divergences for field bilinears in the coincident limit, the regularization of which will be studied in the next section.

Before proceeding with the regularization and renormalization of the effective action, a final and no less consequential comment is necessitated by the question of whether we are obliged to work with the on-, or off-shell formulation of the effective action in our computations. To 1-loop, the former can be obtained by expanding the action to quadratic order in fluctuations and evaluating the resulting functional determinant as a function of the background:

$$\Gamma_1 = \frac{i}{2} \ln \det \left\{ \frac{\delta^2 S_{\rm cl}[\bar{\phi}^a]}{\delta\varphi^b \delta\varphi^c} \right\}, \tag{6.9}$$

where, referring to the notation of Section 5.2, $\bar{\phi}^a$ is the relevant background field, and φ^a denote fluctuations around it. The result of differentiating the above with respect to the background yields the radiation currents Eqs. 6.18 and 6.19, whose contractions make explicit that we are differentiating the 1-loop 1PI vacuum graph [1]. Unlike the classical action, however, Eq. 6.9 is not a scalar and is dependent on how one parametrizes field space in addition to also depending on the background field gauge in the presence of gauge symmetries.
Instead, an effective action that does not suffer from these drawbacks was arrived at by Vilkovisky and DeWitt ([36, 37]) by working covariantly in field space and writing down the equivalent of the functional determinant of the field covariant second variational derivative of the action:

$$\Gamma_1^{\rm VdW} = \frac{i}{2} \ln \det \left\{ \frac{\delta^2 S_{\rm cl}[\varphi^a]}{\delta\varphi^b \delta\varphi^c} - \Gamma^d_{bc}[\bar{\phi}^a] \frac{\delta S_{\rm cl}[\bar{\phi}^a]}{\delta\varphi^d} \right\}, \tag{6.10}$$

where Γ^d_{bc} is the connection on field space. When the background field φ^a minimizes $S_{\rm cl}$ (i.e. one is working on shell) the two forms are equivalent. However, the two forms will in general differ for any quantity obtained from differentiating the effective action when the field space connection is non-vanishing even when evaluated on shell. Therefore, the renormalized stress tensor obtained from the Vilkovisky-DeWitt effective action will have additional contributions relative to the stress tensor obtained from the ‘standard theory’. However, the difference is only in terms of additional finite contributions, which moreover vanish for vacuum contributions on maximally symmetric spacetimes and for one-particle states on Minkowski space [18]. On a general FRLW background, with a homogeneous and isotropic fluid sourcing the background expansion, these are non-vanishing and provide additional finite contributions that depend on the quantum state.
Nevertheless, the procedure we follow allows us to work with the standard form of the effective action, as the divergences that need to be regularized are unaffected by the Vilkovisky-DeWitt correction term, and any finite contributions are absorbed by the process of fixing renormalization conditions. What is crucial for the subtraction process, however, is the identification of the scale factor dependence of the various finite and state dependent terms, for which we determine recursion relations with initial coefficients that can be fixed with a sufficient number of measurements at the renormalization scale.

6.3 Regularization

The regularization of the stress tensor for any propagating DoFs on a general background must proceed with care, all the more so when gauge redundancies are present. Differently from what we did in Capter 4, where we worked directly at the level of the stress tensor obtained by variation of Eq. 6.3, gauge-fixing by hand, and then imposing the SVT decomposition to extract the stress tensor for the propagating spin-2 polarizations when evaluated as an expectation value, in this section we proceed to do so in a covariant manner. This procedure can be related to the covariant method detailed below by a series of Ward identities that we shall return to further on. In what follows, we proceed with the first step of the renormalization procedure reviewed in Section 1.2.2 and we regularize the divergences encountered in the evaluation of Eq. 6.8 by adopting Hadamard regularization techniques [18, 120, 193], which are an extension of the covariant point-splitting method[1].

6.3.1 Hadamard point splitting

The point-split version of a tensor $U^{\mu\nu}(x)$ is defined as the coincidence limit of the bitensor $U^{\mu\nu'}(x,x')$ defined in a neighbourhood of x^μ

$$U^{\mu\nu}(x) = \lim_{\sigma^\mu \to 0} U^{\mu\nu'}(x,x'), \tag{6.11}$$

where primed indices refer to the point $x^{\mu'}$ and σ^μ is the geodesic distance between x^μ, and $x^{\mu'}$. Doing so allows us to isolate the divergent from finite contributions to Eq. 6.8 with σ^μ as the UV regulator. Hadamard regularization of the effective action proceeds through the intermediary of the Feynman propagators (in the notation of [18]):

$$\begin{aligned} G^{\mu\nu\alpha'\beta'}(x,x') &= \frac{i}{32\pi G_N} \frac{\langle\psi| T\left(h^{\mu\nu}(x) h^{\alpha'\beta'}(x')\right)|\psi\rangle}{\langle\psi \mid \psi\rangle} \\ &= \frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma + i\varepsilon}\left(g^{\alpha'(\mu} g^{\nu)\beta'}\right) + V^{\mu\nu\alpha'\beta'} \ln(\Lambda^2(\sigma + i\varepsilon)) + W^{\mu\nu\alpha'\beta'}\right] \\ \tilde{G}^{\mu\alpha'}(x,x') &= \frac{i}{32\pi G_N} \frac{\langle\psi| T\left(\bar{\eta}^{\mu}(x) \eta^{\alpha'}(x')\right)|\psi\rangle}{\langle\psi \mid \psi\rangle} \\ &= \frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma + i\varepsilon} g^{\mu\alpha'} + \tilde{V}^{\mu\alpha'} \ln(\Lambda^2(\sigma + i\varepsilon)) + \tilde{W}^{\mu\alpha'}\right], \end{aligned} \tag{6.12}$$

where Λ is some arbitrary mass scale so that the argument of the logarithms are dimensionless, and primed indices are geodesically transported from $x^{\mu'}$ to x^μ by using the bivector of parallel displacement $g^{\alpha'}{}_{\alpha}$, defined by the differential equation

[1] cf. [196, 172] for applications of point splitting to gauge theories on flat space, and [72, 55, 30, 83, 42] for applications of Hadamard techniques to scalar, vector and fermionic DoFs on curved backgrounds.

[81, 87]

$$\nabla_\rho g_{\alpha'\beta} \nabla^\rho \sigma = 0, \tag{6.13}$$

with the boundary condition

$$\lim_{x \to x'} g_{\alpha'\beta}(x, x') = g_{\alpha\beta}(x). \tag{6.14}$$

The $i\varepsilon$ appearing above is characteristic of the Feynman propagator. We note that one should in general be careful in distinguishing and extracting quantities relevant to the Cauchy problem, namely, in-in currents and expectation values as opposed to the corresponding in-out quantities, both of which can be extracted from the Euclidean effective action through different choices of boundary conditions [36, 37, 38]. Using the notation of the former references, we first consider the following definitions for the effective background:[2]

$$\varphi^a_{\mathrm{F}} = \frac{\langle \mathrm{out}, \mathrm{vac} | \varphi^a | \mathrm{in}, \mathrm{vac} \rangle}{\langle \mathrm{out}, \mathrm{vac} | \mathrm{in}, \mathrm{vac} \rangle}, \tag{6.15}$$

which is of interest in applications when in and out states can be defined (e.g., scattering problems), and

$$\varphi^a_{\mathrm{IN}} = \langle \mathrm{in}, \mathrm{vac} | \varphi^a | \mathrm{in}, \mathrm{vac} \rangle, \tag{6.16}$$

which is of primary relevance to problems where only the initial state is specified. Both fields can be obtained from the effective equations of motion, which can be brought into the form:

$$\frac{\delta S_{\mathrm{cl}}}{\delta \varphi^a_{\mathrm{F}}} + J^{\mathrm{F}}_a = 0, \qquad \frac{\delta S_{\mathrm{cl}}}{\delta \varphi^a_{\mathrm{IN}}} + J^{\mathrm{IN}}_a = 0, \tag{6.17}$$

where S_{cl} is the 'classical action' and the J_a are the so-called radiation currents. Different diagrammatic rules apply when attempting to determine J^{F}_a or J^{IN}_a. The radiation current J^{F}_a can be obtained via techniques relevant to the computation of transition amplitudes, and is given to 1-loop given by [36]:

$$J^{\mathrm{F}}_a = -\frac{i}{2} \frac{\delta^3 S_{\mathrm{cl}}}{\delta \varphi^a_{\mathrm{F}} \delta \varphi^b_{\mathrm{F}} \delta \varphi^c_{\mathrm{F}}} G^{cb}_{\mathrm{F}}, \qquad G^{cb}_{\mathrm{F}} = i \frac{\langle \mathrm{out}, \mathrm{vac} | T[\varphi^c \varphi^b] | \mathrm{in}, \mathrm{vac} \rangle}{\langle \mathrm{out}, \mathrm{vac} | \mathrm{in}, \mathrm{vac} \rangle}, \tag{6.18}$$

where G^{cb}_{F} is the Feynman propagator. Similarly, the current J^{IN}_a is given to 1-loop by

$$J^{\mathrm{IN}}_a = -\frac{i}{2} \frac{\delta^3 S_{\mathrm{cl}}}{\delta \varphi^a_{\mathrm{IN}} \delta \varphi^b_{\mathrm{IN}} \delta \varphi^c_{\mathrm{IN}}} G^{cb}_{\mathrm{IN}}, \qquad G^{cb}_{\mathrm{IN}} = i \langle \mathrm{in}, \mathrm{vac} | T[\varphi^c \varphi^b] | \mathrm{in}, \mathrm{vac} \rangle, \tag{6.19}$$

where the latter can be evaluated as it appears, or with the full regalia of the Schwinger-Keldysh formalism. G^{cb}_{F} and G^{cb}_{IN} differ in terms of their boundary conditions: although in the specific case of future and past asymptotic flatness one has

[2]Where we also adopt DeWitt's condensed notation, and to avoid a proliferation of indices, the composite index a can also be taken to denote a pair of spacetime indices $a := \{\mu, \nu\}$.

$|\mathrm{out,vac}\rangle = |\mathrm{in,vac}\rangle$ so that $G_{\mathrm{F}}^{cb} \equiv G_{\mathrm{IN}}^{cb}$, in general $G_{\mathrm{F}}^{cb} \neq G_{\mathrm{IN}}^{cb}$. Nevertheless, for the purposes of regularization, only the short distance divergence structure of G_{IN}^{cb} is relevant, which is identical to that of G_{F}^{cb}. The reason for this can be inferred from the fact that if the two Green's functions differ only in their boundary conditions, completeness dictates that the short distance modes of the two vacua must be related to each other by a Bogoliubov rotation that tends to zero for short wavelengths, otherwise one would represent an infinite energy excitation relative to the other (see also [48] for an expanded discussion on this point). The long wavelength behavior of G_{IN}^{cb} will certainly differ from that of G_{F}^{cb}; however, the difference will manifest as finite and non-local terms that will be absorbed in the process of fixing renormalization conditions.
We stress this point as it offers the possibility to adapt computations that make use of Feynman Green's functions for the purposes of identifying the local counterterms necessitated by the subtraction procedure (as done in [18]). However, since within the present context it is more convenient to work with the Hadamard Green's functions, we follow [42] in deriving their representations from the Hadamard form of the Feynman propagator. By using the identities

$$\frac{1}{\sigma + i\varepsilon} = \mathcal{P}\frac{1}{\sigma} - i\pi\delta(\sigma), \quad \ln(\sigma + i\varepsilon) = \ln|\sigma| + i\pi\Theta(-\sigma), \tag{6.20}$$

where $\mathcal{P}$ and Θ denote the Cauchy principal value and the Heaviside theta function respectively, we can rewrite Eq. 6.12 as:

$$G_F^{ab}(x,x') = G_A^{ab}(x,x') + \frac{i}{2}G^{ab}(x,x'), \tag{6.21}$$

where[3]

$$\begin{aligned} G_A^{\mu\nu\alpha'\beta'}(x,x') &= \frac{1}{8\pi}\left[\Delta^{1/2}\left(g^{\alpha'(\mu}g^{\nu)\beta'}\right)\delta(\sigma) - V^{\mu\nu\alpha'\beta'}\Theta(-\sigma)\right] \\ \tilde{G}_A^{\mu\alpha'}(x,x') &= \frac{1}{8\pi}\left[\Delta^{1/2}g^{\mu\alpha'}\delta(\sigma) - \tilde{V}^{\mu\alpha'}\Theta(-\sigma)\right], \end{aligned} \tag{6.22}$$

are the average of the advanced and retarded Green's functions, and

$$\begin{aligned} G^{\mu\nu\alpha'\beta'}(x,x') &= \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}\left(g^{\alpha'(\mu}g^{\nu)\beta'}\right) + V^{\mu\nu\alpha'\beta'}\ln(\Lambda^2\sigma) + W^{\mu\nu\alpha'\beta'}\right] \\ \tilde{G}^{\mu\alpha'}(x,x') &= \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}g^{\mu\alpha'} + \tilde{V}^{\mu\alpha'}\ln(\Lambda^2\sigma) + \tilde{W}^{\mu\alpha'}\right], \end{aligned} \tag{6.23}$$

are the Hadamard Green's functions.

The state $|\psi\rangle$ appearing in Eq. 6.23 is somewhat circularly defined as any quantum state – the Hadamard state – such that the short distance divergence structure is of the forms indicated in the square brackets, where $\sigma = \frac{1}{2}\sigma_\mu\sigma^\mu$ denotes the square

[3]Round/square parenthesis denote symmetrization/anti-symmetrization of indices.

of the geodesic distance between x^μ and $x^{\mu'}$, Δ is the Van Vleck-Morette determinant and the bitensors $V^{\mu\nu\alpha'\beta'}$, $W^{\mu\nu\alpha'\beta'}$, $\tilde{V}^{\mu\alpha'}$ and $\tilde{W}^{\mu\alpha'}$ are smooth functions in the limit $\sigma \to 0$ of the form:

$$V^{\mu\nu\alpha'\beta'} = \sum_{n=0}^{\infty} V_n^{\mu\nu\alpha'\beta'} \sigma^n \qquad W^{\mu\nu\alpha'\beta'} = \sum_{n=0}^{\infty} W_n^{\mu\nu\alpha'\beta'} \sigma^n$$
$$\tilde{V}^{\mu\alpha'} = \sum_{n=0}^{\infty} \tilde{V}_n^{\mu\alpha'} \sigma^n \qquad \tilde{W}^{\mu\alpha'} = \sum_{n=0}^{\infty} \tilde{W}_n^{\mu\alpha'} \sigma^n. \tag{6.24}$$

It is to be stressed that the bitensors $V_n^{\mu\nu\alpha'\beta'}$ and $\tilde{V}_n^{\mu\alpha'}$ depend only on the local geometry, whereas the bitensors $W_n^{\mu\nu\alpha'\beta'}$ and $\tilde{W}_n^{\mu\alpha'}$ depend on the boundary conditions and the precise choice of the state $|\psi\rangle$. The finite contributions are what is misinterpreted in the literature and treated as the theoretical estimate of observables. On the contrary, as we will show in in Section 6.4, those are the part that is absorbed by renormalizing the couplings in the effective action through the process of imposing renormalization conditions.

Each of the $V_n^{\mu\nu\alpha'\beta'}$, $W_n^{\mu\nu\alpha'\beta'}$, $\tilde{V}_n^{\mu\alpha'}$ and $\tilde{W}_n^{\mu\alpha'}$ bitensors can be rewritten in the form of a covariant Taylor expansion for x^μ in the neighbourhood of $x^{\mu'}$:

$$\begin{aligned} V_n^{\mu\nu\alpha'\beta'} &= g^{\alpha'}{}_{\alpha} g^{\beta'}{}_{\beta} \left[v_n^{\mu\nu\alpha\beta} + v_n^{\mu\nu\alpha\beta}{}_{\gamma}\sigma^\gamma + \frac{1}{2} v_n^{\mu\nu\alpha\beta}{}_{\gamma\tau}\sigma^\gamma\sigma^\tau + ... \right], \\ \tilde{V}_n^{\mu\alpha'} &= g^{\alpha'}{}_{\alpha} \left[\tilde{v}_n^{\mu\alpha} + \tilde{v}_n^{\mu\alpha}{}_{\gamma}\sigma^\gamma + \frac{1}{2} \tilde{v}_n^{\mu\alpha}{}_{\gamma\tau}\sigma^\gamma\sigma^\tau + ... \right], \end{aligned} \tag{6.25}$$

and similarly for $W_n^{\mu\nu\alpha'\beta'}$ and $\tilde{W}_n^{\mu\alpha'}$. By expanding the Hadamard Green's function of Eq. 6.23 using the Taylor expansions in Eq. 6.25, we obtain

$$\begin{aligned} G^{\rho\sigma\alpha'\beta'}(x,x') =& \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}\left(g^{\alpha'(\rho}g^{\sigma)\beta'} \right) + V^{\rho\sigma\alpha'\beta'}\ln(\mu^2\sigma) + W^{\rho\sigma\alpha'\beta'} \right] \\ =& \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{2\sigma}\left(g^{\alpha'\rho}g^{\sigma\beta'} + g^{\alpha'\sigma}g^{\rho\beta'} \right) + g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}v_0^{\rho\sigma\alpha\beta}\ln(\mu^2\sigma) \right. \\ &+ g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}v_0^{\rho\sigma\alpha\beta}{}_{\gamma}\sigma^\gamma\ln(\mu^2\sigma) + \frac{1}{2}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}v_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}\sigma^\gamma\sigma^\varepsilon\ln(\mu^2\sigma) \\ &+ \frac{1}{2}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}v_1^{\rho\sigma\alpha\beta}\sigma_\gamma\sigma^\gamma\ln(\mu^2\sigma) + g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}w_0^{\rho\sigma\alpha\beta} \\ &+ g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}w_0^{\rho\sigma\alpha\beta}{}_{\gamma}\sigma^\gamma + \frac{1}{2}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}w_0^{\rho\sigma\alpha\beta}{}_{\gamma\tau}\sigma^\gamma\sigma^\tau \\ &\left. + \frac{1}{2}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}w_1^{\rho\sigma\alpha\beta}\sigma^\gamma\sigma_\gamma \right], \end{aligned} \tag{6.26}$$

$$
\begin{aligned}
\tilde{G}^{\mu\alpha'}(x,x') =& \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}g^{\mu\alpha'} + \tilde{V}^{\mu\alpha'}\ln(\mu^2\sigma) + \tilde{W}^{\mu\alpha'}\right] \\
=& \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma}g^{\mu\alpha'} + g^{\alpha'}{}_{\alpha}\tilde{v}_0^{\mu\alpha}\ln(\mu^2\sigma) + g^{\alpha'}{}_{\alpha}\tilde{v}_0^{\mu\alpha}{}_{\gamma}\sigma^{\gamma}\ln(\mu^2\sigma)\right. \\
&+\frac{1}{2}g^{\alpha'}{}_{\alpha}\tilde{v}_0^{\mu\alpha}{}_{\gamma\varepsilon}\sigma^{\gamma}\sigma^{\varepsilon}\ln(\mu^2\sigma) + \frac{1}{2}g^{\alpha'}{}_{\alpha}\sigma_{\gamma}\sigma^{\gamma}\tilde{v}_1^{\mu\alpha}\ln(\mu^2\sigma) + g^{\alpha'}{}_{\alpha}\tilde{w}_0^{\mu\alpha} \\
&\left.+g^{\alpha'}{}_{\alpha}\tilde{w}_0^{\mu\alpha}{}_{\gamma}\sigma^{\gamma} + \frac{1}{2}g^{\alpha'}{}_{\alpha}\tilde{w}_0^{\mu\alpha}{}_{\gamma\tau}\sigma^{\gamma}\sigma^{\tau} + \frac{1}{2}g^{\alpha'}{}_{\alpha}\tilde{w}_1^{\mu\alpha}\sigma^{\gamma}\sigma_{\gamma}\right],
\end{aligned}
\tag{6.27}
$$

where higher orders in powers of σ vanish in the limit $\sigma^{\mu} \to 0$, and the tensors contributing to the divergent part ($v_0^{\rho\sigma\alpha\beta}$, $v_0^{\rho\sigma\alpha\beta}{}_{\gamma}$, $v_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}$, $v_1^{\rho\sigma\alpha\beta}$, $\tilde{v}_0^{\rho\alpha}$, $\tilde{v}_0^{\rho\alpha}{}_{\gamma}$, $\tilde{v}_0^{\rho\alpha}{}_{\gamma\varepsilon}$ and $\tilde{v}_1^{\rho\alpha}$) can be found in the appendix 6.6.

The Hadamard Green's functions as expressed in Eqs. 6.26 and 6.27 facilitate the regularization of Eq. 6.7 in that they can be viewed as the point split expression:

$$
\begin{aligned}
\langle S_{\text{gr}}\rangle = \lim_{\sigma^{\mu}\to 0}\Bigg\{\int d^4x\sqrt{-g}\Bigg[&\left(-\frac{1}{2}g_{\rho\alpha'}g_{\sigma\beta'} + \frac{1}{4}g_{\rho\sigma}g_{\alpha'\beta'}\right)\nabla_{\tau}\nabla^{\tau'}G^{\rho\sigma\alpha'\beta'} \\
+&\left(R_{\alpha'\rho\beta'\sigma} + g_{\beta'\sigma}R_{\rho\alpha'} - g_{\alpha'\beta'}R_{\rho\sigma} - \frac{1}{2}R\, g_{\rho\alpha'}g_{\sigma\beta'} + \frac{1}{4}R\, g_{\rho\sigma}g_{\alpha'\beta'}\right)G^{\rho\sigma\alpha'\beta'}\Bigg]\Bigg\}
\end{aligned}
\tag{6.28}
$$

$$
\langle S_{\text{gh}}\rangle = \lim_{\sigma^{\mu}\to 0}\left\{\int d^4x\sqrt{-g}\left[-g_{\mu\alpha'}\nabla_{\tau}\nabla^{\tau'}\tilde{G}^{\mu\alpha'} - R_{\mu\alpha'}\tilde{G}^{\mu\alpha'}\right]\right\}. \tag{6.29}
$$

Recalling that the divergence structure of the Hadamard Green's functions is completely captured by the terms containing the Van Vleck-Morette determinant Δ, and the bitensors $V^{\mu\nu\alpha'\beta'}$ and $\tilde{V}^{\mu\alpha'}$, the divergent part of the gravitational sector of the effective action Eq. 6.7 must be of the form

$$
\langle S_{\text{gw}}\rangle_{\text{div}} \sim \lim_{\sigma^{\mu}\to 0}\int d^4x\sqrt{-g}\left[\gamma_1(\sigma)R + \gamma_2(\sigma)R^2 + \gamma_3(\sigma)R^{\mu\nu}R_{\mu\nu} + \gamma_4(\sigma)\Box R\right]. \tag{6.30}
$$

We compute the coefficients $\gamma_1(\sigma)$, $\gamma_2(\sigma)$, $\gamma_3(\sigma)$, and $\gamma_4(\sigma)$ in the next subsection, from which one can immediately identify the counterterms required to subtract them.

6.3.2 Counterterms

The process of determining the counterterms needed to regularize the effective action begins with rewriting the coincidence limits as:

$$
\lim_{\sigma^{\mu}\to 0}R_{\mu\alpha'}\tilde{G}^{\mu\alpha'} = R_{\mu\alpha}\lim_{\sigma^{\mu}\to 0}g_{\alpha'}{}^{\alpha}\tilde{G}^{\mu\alpha'} \tag{6.31}
$$

so that the tensors contracted with the Hadamard Green's functions are tensors in x^{μ} and scalars in $x^{\mu'}$. Furthermore, as the coincidence limit depends on the path by which σ^μ approaches zero, it is necessary to specify a path-averaging procedure. Following [14], we use the so-called *elementary averaging procedure*, whereby one makes the replacements:

$$\begin{aligned}
\sigma_\lambda\sigma_\mu &\to \frac{1}{4}g_{\lambda\mu}\sigma_\rho\sigma^\rho = \tfrac{1}{2}g_{\lambda\mu}\sigma, \\
\sigma_\lambda\sigma_\mu\sigma_\gamma\sigma_\delta &\to \frac{\sigma^2}{6}\left(g_{\gamma\delta}g_{\lambda\mu} + g_{\gamma\lambda}g_{\delta\mu} + g_{\gamma\mu}g_{\delta\lambda}\right), \\
\sigma_\alpha\sigma_\beta\sigma_\lambda\sigma_\mu\sigma_\gamma\sigma_\delta &\to \frac{\sigma^3}{24}\left[g_{\alpha\beta}\left(g_{\lambda\mu}g_{\nu\delta} + g_{\lambda\gamma}g_{\mu\delta} + g_{\lambda\delta}g_{\mu\nu}\right) + g_{\alpha\lambda}\left(g_{\beta\mu}g_{\gamma\delta} + g_{\beta\gamma}g_{\mu\delta}\right.\right. \\
&\quad \left. + g_{\beta\delta}g_{\mu\gamma}\right) + g_{\alpha\mu}\left(g_{\beta\lambda}g_{\gamma\delta} + g_{\beta\nu}g_{\lambda\delta} + g_{\beta\delta}g_{\lambda\gamma}\right) + g_{\alpha\gamma}\left(g_{\beta\lambda}g_{\mu\delta}\right. \\
&\quad \left.\left. + g_{\beta\mu}g_{\lambda\delta} + g_{\beta\delta}g_{\lambda\mu}\right) + g_{\alpha\delta}\left(g_{\beta\lambda}g_{\mu\nu} + g_{\beta\mu}g_{\lambda\gamma} + g_{\beta\gamma}g_{\lambda\mu}\right)\right] \quad (6.32)
\end{aligned}$$

The singularity structure is then extracted by expanding at the endpoints and iteratively solving the equation of motion of the propagator to find the Taylor coefficients of $V^{\mu\nu\alpha'\beta'}$ and $\tilde{V}^{\mu\alpha'}$ (see appendix 6.6 for more details). From this, the divergent contributions to $S_{\rm gh}$ are found to be

$$\langle S_{\rm gh}\rangle_{\rm div} = \frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[-\frac{3}{\sigma}R + \ln(\Lambda^2\sigma)\left(R_{\mu\nu}R^{\mu\nu} + \frac{1}{6}R^2 - \frac{1}{24}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{5}{12}\Box R\right)\right], \quad (6.33)$$

whereas the divergent contributions to S_{gw} are given by:

$$\langle S_{\rm gr}\rangle_{\rm div} = \frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[-\frac{7}{6\sigma}R + \ln(\Lambda^2\sigma)\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{4}R^2 - \frac{13}{24}\Box R\right)\right]. \quad (6.34)$$

By use of the Bianchi identity and the Gauss-Bonnet theorem, the divergent contribution of the action of vacuum tensor fluctuations results from the difference of Eq. 6.34 and Eq. 6.33 (accounting for the statistics of the ghost contributions), and is given by:

$$\langle S_{\rm gw}\rangle_{\rm div} = \frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[\frac{11}{6\sigma}R + \ln(\Lambda^2\sigma)\left(\frac{1}{6}R_{\mu\nu}R^{\mu\nu} - \frac{11}{24}R^2 - \frac{23}{24}\Box R\right)\right], \quad (6.35)$$

which is of the form of Eq. 6.30, with the $\gamma_i(\sigma)$ coefficients straightforwardly read off from the above.

6.4 Renormalization

Having regularized the divergent contributions of the effective action, that once variated with respect to the background metric leads to the divergent contribution to the stress energy tensor, allows us to proceed with the remaining steps of the renormalization procedure (see Section 1.2.2 for more details). In what follows, we first subtract the divergent contributions with the appropriate counterterms and, as widely discussed in the previous chapters, we then have to fix the finite contribution of the regularized action by imposing renormalization conditions before being able to infer constraints from experiments. This leads to a qualitatively different result in the interpretations of existing constraints, such as the $N_{\rm eff}$ bounds. In this section, we first comment on the finite contribution of the renormalized action and then we proceed to specify our otherwise general results in a RD universe to subtract the divergent contributions, impose renormalization conditions and find results that can be compared with Section 4.4.

We begin by considering all contributions to the action (cf. Eqs. 6.2 and 6.7):

$$\langle S\rangle = S_{\rm EH} + S_{\rm RD} + S_{\rm ct} + \langle S_{\rm gw}\rangle\,, \tag{6.36}$$

where $S_{\rm M}$ is the matter content that sources the background expansion and where we have defined

$$S_{\rm ct} = \frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\Big[\alpha_1(\sigma,\mu)R + \alpha_2(\sigma,\mu)R_{\mu\nu}R^{\mu\nu} + \alpha_3(\sigma,\mu)R^2 + \alpha_4(\sigma,\mu)\Box R\Big], \tag{6.37}$$

where μ is an arbitrary mass scale whose meaning will become clear shortly. The divergent contributions in Eq. 6.35 can be subtracted with the following choices for the α_i:

$$\begin{aligned}
\alpha_1(\mu,\sigma) &= -\frac{11}{6}\frac{1}{\sigma} + \alpha_1^{\rm F}(\mu),\\
\alpha_2(\mu,\sigma) &= -\frac{1}{6}\log(\mu^2\sigma) + \alpha_2^{\rm F}(\mu),\\
\alpha_3(\mu,\sigma) &= \frac{11}{24}\log(\mu^2\sigma) + \alpha_3^{\rm F}(\mu),\\
\alpha_4(\mu,\sigma) &= \frac{23}{24}\log(\mu^2\sigma) + \alpha_4^{\rm F}(\mu),
\end{aligned} \tag{6.38}$$

where the $\alpha_i^{\rm F}$ are finite contributions that we leave unspecified for now. With this the action can be expressed as:

$$\begin{aligned}
\langle S\rangle = &\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G(\mu)}R + \bar\alpha_2(\mu)R_{\mu\nu}R^{\mu\nu} + \bar\alpha_3(\mu)R^2 + \bar\alpha_4(\mu)\Box R\right]\\
&+ S_{\rm RD} + \langle S_{\rm gw}\rangle_{\rm fin}\,,
\end{aligned} \tag{6.39}$$

where $\langle S_{\rm gw}\rangle_{\rm fin} := \langle S_{\rm gw}\rangle - \langle S_{\rm gw}\rangle_{\rm div}$ and where

$$\begin{aligned}
\frac{1}{16\pi G(\mu)} &= \frac{1}{16\pi G_B} + \frac{\alpha_1^{\rm F}(\mu)}{4\pi^2},\\
\bar\alpha_2(\mu) &= \frac{1}{4\pi^2}\left[\frac{1}{6}\log\frac{\Lambda^2}{\mu^2} + \alpha_2^{\rm F}(\mu)\right],\\
\bar\alpha_3(\mu) &= \frac{1}{4\pi^2}\left[-\frac{11}{24}\log\frac{\Lambda^2}{\mu^2} + \alpha_3^{\rm F}(\mu)\right],\\
\bar\alpha_4(\mu) &= \frac{1}{4\pi^2}\left[-\frac{23}{24}\log\frac{\Lambda^2}{\mu^2} + \alpha_4^{\rm F}(\mu)\right].
\end{aligned} \tag{6.40}$$

The subscripts B appearing above are to denote bare quantities.

6.4.1 Finite contributions

The finite remainder given by $\langle S_{\rm gw}\rangle_{\rm fin}$ can be separated into parts that are uniquely determined by the background geometry – i.e. the terms determined by the bitensors $V^{\mu\nu\alpha'\beta'}$ and $\tilde{V}^{\mu\alpha'}$ which we denote as $\langle S_{\rm gw}\rangle^{\rm bg}_{\rm fin}$ – and those that depend on the state, determined by the bitensors $W^{\mu\nu\alpha'\beta'}$ and $\tilde{W}^{\mu\alpha'}$, which we denote as $\langle S_{\rm gw}\rangle^{\rm sd}_{\rm fin}$.

Computing $\langle S_{\rm gw}\rangle^{\rm sd}_{\rm fin}$, can be reduced to express the following four terms

$$\text{(I)}: \qquad R_{\mu\alpha}\lim_{\sigma^\mu\to 0} g_{\alpha'}{}^{\alpha}\tilde{G}^{\mu\alpha'} \tag{6.41}$$

$$\text{(II)}: \qquad g_{\rho}{}^{\gamma}\lim_{\sigma^\mu\to 0} g_{\tau'}{}^{\tau}g_{\alpha'\gamma}\nabla_\tau\nabla^{\tau'}\tilde{G}^{\rho\alpha'} \tag{6.42}$$

$$\text{(III)}: \qquad \mathcal{P}_{\mu\nu\alpha\beta}\lim_{\sigma^\mu\to 0} g_{\alpha'}{}^{\alpha}g_{\beta'}{}^{\beta}G^{\mu\nu\alpha'\beta'} \tag{6.43}$$

$$\text{(IV)}: \qquad \mathcal{Q}_{\mu\nu}{}^{\gamma\delta}\lim_{\sigma^\mu\to 0} g_{\tau'}{}^{\tau}g_{\alpha'\gamma}g_{\beta'\delta}\nabla_\tau\nabla^{\tau'}G^{\mu\nu\alpha'\beta'}, \tag{6.44}$$

where we defined

$$\begin{aligned}
\mathcal{P}_{\mu\nu\gamma\delta} &\equiv R_{\gamma\mu\delta\nu} + g_{\delta\nu}R_{\mu\gamma} - g_{\gamma\delta}R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\gamma}g_{\nu\delta} + \frac{1}{4}R\,g_{\mu\nu}g_{\gamma\delta},\\
\mathcal{Q}_{\mu\nu\gamma\delta} &\equiv -\frac{1}{2}g_{\mu\gamma}g_{\nu\delta} + \frac{1}{4}g_{\mu\nu}g_{\gamma\delta},
\end{aligned} \tag{6.45}$$

as a function of the state-dependent terms of the Taylor expansions of Eqs. 6.26 and 6.27 – $w_0^{\rho\sigma\alpha\beta}$, $w_0^{\rho\sigma\alpha\beta}{}_{\gamma}$, $w_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}$, $w_1^{\rho\sigma\alpha\beta}$, $\tilde{w}_0^{\rho\alpha}$, $\tilde{w}_0^{\rho\alpha}{}_{\gamma}$, $\tilde{w}_0^{\rho\alpha}{}_{\gamma\varepsilon}$ and $\tilde{w}_1^{\rho\alpha}$. Considering only the state-dependent part of Eqs. 6.26 and 6.27, one can straightforwardly isolate the state-dependent part of Eqs. 6.41 and 6.43 which results

$$\begin{aligned}
\text{(I)}^{\rm sd}_{\rm fin}: &\qquad \frac{i}{8\pi^2}R_{\mu\alpha}\tilde{w}_0^{\mu\alpha}\\
\text{(III)}^{\rm sd}_{\rm fin}: &\qquad \frac{i}{8\pi^2}\mathcal{P}_{\mu\nu\alpha\beta}w_0^{\mu\nu\alpha\beta}.
\end{aligned} \tag{6.46}$$

Extracting the state-dependent part of Eqs. 6.42 and 6.44 is less straightforward: following the procedure of the appendix 6.6 we obtain

$$\begin{aligned}
\text{(II)}^{\rm sd}_{\rm fin}: &\qquad \frac{i}{8\pi^2}\left[-\nabla_\tau\tilde{w}_{0\,\mu}^{\mu}{}^{\tau} - \tilde{w}_{0\,\mu}^{\mu}{}^{\tau}{}_{\tau} - 4\tilde{w}_{1\,\mu}^{\mu}\right]\\
\text{(IV)}^{\rm sd}_{\rm fin}: &\qquad \frac{i}{8\pi^2}\left[\mathcal{Q}_{\mu\nu}{}^{\gamma\delta}\left(-\nabla_\tau w_0^{\mu\nu}{}_{\gamma\delta}{}^{\tau} - w_0^{\mu\nu}{}_{\gamma\delta}{}^{\tau}{}_{\tau} - 4w_1^{\mu\nu}{}_{\gamma\delta}\right)\right].
\end{aligned} \tag{6.47}$$

Finally, $\langle S_{\rm gw}\rangle^{\rm sd}_{\rm fin}$ can be expressed as

$$\begin{aligned}\langle S\rangle^{\rm sd}_{\rm fin} &= \int d^4x\sqrt{-g}\,\Big[\Big((\mathrm{III})^{\rm sd}_{\rm fin} + (\mathrm{IV})^{\rm sd}_{\rm fin}\Big) - \Big(-(\mathrm{I})^{\rm sd}_{\rm fin} - (\mathrm{II})^{\rm sd}_{\rm fin}\Big)\Big] \\ &= \frac{1}{4\pi^2}\int d^4x\sqrt{-g}\,\big[\mathcal{Q}_{\mu\nu}{}^{\gamma\delta}\,(-\nabla_\tau w_0^{\mu\nu}{}_{\gamma\delta}{}^{\tau} - w_0^{\mu\nu}{}_{\gamma\delta}{}^{\tau}{}_{\tau} - 4w_1^{\mu\nu}{}_{\gamma\delta}) \\ &\quad + \mathcal{P}_{\mu\nu\alpha\beta}w_0^{\mu\nu\alpha\beta} + R_{\mu\alpha}\tilde{w}_0^{\mu\alpha} - \nabla_\tau\tilde{w}_0{}^{\mu}{}_{\mu}{}^{\tau} - \tilde{w}_0{}^{\mu}{}_{\mu}{}^{\tau}{}_{\tau} - 4\tilde{w}_1{}^{\mu}{}_{\mu}\Big]\,. \end{aligned} \tag{6.48}$$

There is one more step that can be done to simplify the form of $\langle S_{\rm gw}\rangle^{\rm sd}_{\rm fin}$. We note that we can obtain the Taylor coefficients of $W_1^{\mu\nu\alpha'\beta'}$ in terms of the Taylor coefficients of $W_0^{\mu\nu\alpha'\beta'}$ (we focus on the graviton contribution, but a similar procedure will give us the analogous Taylor coefficients for the ghost contributions). By iteratively solving order by order in σ^μ in the equation of motion for the propagator, we find[4] (cf. [18] for more details):

$$\begin{aligned} &n(n+1)W_n^{\mu\nu}{}_{\alpha'\beta'} + nW_n^{\mu\nu}{}_{\alpha'\beta';\rho}\sigma^\rho - nW_n^{\mu\nu}{}_{\alpha'\beta'}\Delta^{-1/2}\Delta^{1/2}_{;\rho}\sigma^\rho + (2n+1)V_n^{\mu\nu}{}_{\alpha'\beta'} \\ &+V_n^{\mu\nu}{}_{\alpha'\beta';\rho}\sigma^\rho - V_n^{\mu\nu}{}_{\alpha'\beta'}\Delta^{-1/2}\Delta^{1/2}_{;\rho}\sigma^\rho + \frac{1}{2}D_{\rho\sigma}{}^{\mu\nu}\,W^{\rho\sigma}_{n-1\,\alpha'\beta'} = 0, \end{aligned} \tag{6.49}$$

where

$$\begin{aligned} D_{\mu\nu}{}^{\alpha\beta} &= \Box g_\mu^{(\alpha}\,g_\nu^{\beta)} - P_{\mu\nu}{}^{\alpha\beta}, \\ P_{\mu\nu}{}^{\alpha\beta} &= -2R_{(\mu}{}^{\alpha}{}_{\nu)}{}^{\beta} + \frac{1}{2}g_{\mu\nu}R^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}g^{\alpha\beta} + \frac{1}{2}Rg_{(\mu}{}^{\alpha}g_{\nu)}{}^{\beta}. \end{aligned} \tag{6.50}$$

By specifying the recursion relation Eq. 6.49 for $n=1$ and expanding at the 0^{th} order in σ we obtain $w_1^{\mu\nu}{}_{\alpha\beta}$ as a function of the Taylor coefficients of $W_0^{\mu\nu\alpha'\beta'}$

$$2w_1^{\mu\nu}{}_{\alpha\beta} = -3v_1^{\mu\nu}{}_{\alpha\beta} - \frac{1}{2}g_\rho^{(\mu}g_\sigma^{\nu)}\left[\Box w_0^{\rho\sigma}{}_{\alpha\beta} + \nabla_\tau w_0^{\rho\sigma}{}_{\alpha\beta}{}^{\tau} + \frac{1}{2}w_0^{\rho\sigma}{}_{\alpha\beta}{}^{\tau}{}_{\tau}\right] + \frac{1}{2}P^{\mu\nu}{}_{\rho\sigma}w_0^{\rho\sigma}{}_{\alpha\beta}. \tag{6.51}$$

In the above, $w_0^{\rho\sigma\alpha\beta}$, $w_0^{\rho\sigma\alpha\beta}{}_{\gamma}$ and $w_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}$ are the 'initial' inputs for the recursion relations corresponding to the specifics of the state.

6.4.2 Renormalization conditions

In this section we fix the finite contributions and impose the renormalization conditions. In doing so we specify the background expansion as, although the treatment that follows generalizes to any background, we aim to compare our results with the FLRW foliation studied in Chapter 4.

To connect with the findings of Chapter 4, $S_{\rm M}$ is now specified by $S_{\rm RD}$ to denote radiation domination. For concreteness, we work with the action formulation for a

[4] Semi-colons denote covariant derivatives with respect to the background metric.

barotropic fluid expressed in terms of a derivatively coupled scalar [50, 28], so that

$$S_{\rm RD} = \int d^4x\sqrt{-g}P^{\rm bg}(X), \tag{6.52}$$

where $P^{\rm bg}(X) = X^2$, with $X := -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$, reproduces the stress tensor for the background radiation fluid[5] (See section 5.3 for more details). In order to proceed and fix the finite contributions, we are obliged to make use of leading order field equations to eliminate redundant higher order correction terms containing second derivatives and time derivatives of what were auxiliary fields in the tree level action [216, 61]. That is, one can substitute the tree level equations of motion $R = -8\pi G(\mu)T^{\rm bg}$ and $R_{\mu\nu} = 8\pi G(\mu)[T^{\rm bg}{}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\rm bg}]$ into the above, where

$$T^{\rm bg}{}_\nu{}^\mu(X_B) = \delta^\mu{}_\nu P^{\rm bg} - P^{\rm bg}{}_{,X}\partial^\mu\psi_B\partial_\nu\psi_B \tag{6.53}$$

is obtained from Eq. 6.52 through variation with the background metric. The functional form $P^{\rm bg}(X_B) = X_B^2$ ensures that $T^{\rm bg} \equiv 0$, so that the regularized action takes the form:

$$\langle S\rangle = \int d^4x\sqrt{-g}\left[\frac{1}{2}M^2(\mu)R + \widetilde{P}^{\rm bg}(X_B,\mu)\right] + \langle S_{\rm gw}\rangle_{\rm fin}\,, \tag{6.54}$$

where we have dropped total derivatives and defined $M^{-2}(\mu) := 8\pi G(\mu)$, which we use interchangeably in what follows, and where

$$\widetilde{P}^{\rm bg}(X_B,\mu) := X_B^2 + 12X_B^4\frac{\bar\alpha_2(\mu)}{M^4(\mu)}, \tag{6.55}$$

with $\bar\alpha_2$ defined as in Eq. 6.40. We immediately notice that the stress tensor associated with the shifted matter sector $\widetilde{P}(X_B,\mu)$ is no longer traceless:

$$\widetilde{T}^{\rm bg}{}_\mu{}^\mu = -48X_B^4\frac{\bar\alpha_2(\mu)}{M^4(\mu)}. \tag{6.56}$$

This is because Einstein gravity is not conformally invariant, and therefore neither are the field equations governing GWs, even if the background is conformally flat [177]. Hence, the stress tensor for GWs will not be exactly traceless unless $\bar\alpha_2 \equiv 0$, and this feature gets imported into the matter sector via operator redundancy at 1-loop, a point which we will return to shortly.

Before proceeding with the renormalization conditions, we comment on how one could compare the finite remainder $\langle S_{\rm gw}\rangle_{\rm fin}$ with the TT-gauge-fixed, foliation specific treatment of 4.4.

We recall that in fixing de Donder gauge with the Faddeev Popov method, we considered the action for a rank-2 symmetric tensor field (describing 10 DoFs) and we fixed the gauge by adding the gauge breaking term in Eq. 6.5 (which fixes 4 DoFs)

[5] A feature that remains true to all orders in perturbation theory [50].

and the ghost term (which subtracts the remaining 4 spurious DoFs). The ghost term and gauge breaking term have the same properties as the spurious eight DoFs present in the initial action, but with fermionic statistics that subtracts them from all on shell quantities. In order to obtain a fully gauge-fixed final result of the action in terms of the graviton only (equivalently for stress energy tensor, once one variates with respect to the background metric) we have to determine the ghost propagator in terms of the graviton propagator. This is done by using the generalization of the Ward identities discussed in [18], but now evaluated on a background that does not correspond to a vacuum spacetime. While addressing this matter falls outside the current investigation's scope, fully completing it, considering the remaining symmetries of FRLW spacetimes, would represent a valuable and practical computation worth pursuing further.

We now proceed to fix the finite parts of the relevant couplings via renormalization conditions. We recall the shifted tadpole condition, which is defined by the requirement that the background effective field $g_{\mu\nu}$ must be put on shell in all final expressions. That is, we demand that

$$\frac{1}{8\pi G(\mu)}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=\widetilde{T}^{\rm bg}_{\mu\nu}+\langle T^{\rm gw,fin}_{\mu\nu}\rangle. \tag{6.57}$$

We first note that Newton's constant can only be fixed via a Cavendish-type experiment, where we have knowledge of the masses whose strength of gravitational interactions we are attempting to fix. Assuming that this is done at mm scales, we impose the renormalization condition $[8\pi G(\mu)]^{-1}\equiv M^2(\mu_*)\equiv M^2_{\rm pl}$ where the latter is given by the reduced Planck mass $M^2_{\rm pl}=2.435\times 10^{18}$ GeV. From Eq. 6.40 it therefore folows that

$$8\pi G_N(\mu)=\frac{1}{M^2_{\rm pl}}\left[1+\frac{\alpha_1^{\rm F}(\mu)-\alpha_1^{\rm F}(\mu_*)}{2\pi^2 M^2_{\rm pl}}\right]^{-1}, \tag{6.58}$$

which can be used to express Eq. 6.57 in covariant form as

$$G_{\mu\nu}=\frac{1}{M^2_{\rm pl}}\left[\widetilde{T}^{\rm bg}_{\mu\nu}+\langle T^{\rm gw,fin}_{\mu\nu}\rangle\right]\left[1+\frac{\alpha_1^{\rm F}(\mu)-\alpha_1^{\rm F}(\mu_*)}{2\pi^2 M^2_{\rm pl}}\right]^{-1}. \tag{6.59}$$

We see from the above how additive renormalization of Newton's constant is equivalent to the multiplicative renormalization of the matter sector via the tadpole condition.

In order to fix the remaining finite contribution appearing in $\bar{\alpha}_2$ in combination with those coming from $\langle T^{\rm gw,fin}_{\mu\nu}\rangle$, we have to appeal to additional observations. Before we do so, we note that $\langle T^{\rm gw,fin}_{\mu\nu}\rangle$ has a series of contributions that can be recursively obtained. We recall that if on one hand the background dependent finite contributions can be uniquely determined by the recursion relations of the EOM, this is not the case for the state-dependent part (see Section 6.4.1). However, the adiabatic vacuum by definition is invariant under the symmetries of the background geometry, and so all the initial state-dependent inputs must themselves be constructed out of geometric invariants. This implies that both the background and the state-dependent contributions to the finite part will result in the generation of a handful of terms that redshift as radiation, along with a series of additional slow quenching terms that decay much faster, as we will se in the following. We also note that the example studied in Chapter 4 implied that the contribution Eq. 6.56 is canceled by a compensating term from the state-dependent part of $\langle T_{\mu\nu}^{\rm gw,fin}\rangle$ in the adiabatic vacuum. Regardless of this fact, in addition to admitting the possibility of other operators in the effective action from the presence of additional DoFs, one simply extracts that part of the renormalized expression that scales as radiation and proceeds accordingly.

Consider for example the measurement of the equation of state parameter w during what we presume to be radiation domination. In principle, any other measurement of a dimensionless ratio will do (e.g. $H^2/M_{\rm pl}^2$), as what follows transcribes straightforwardly. Making such a measurement at the scale $\mu_{\rm R}$ results in the renormalization condition:

$$3H_{\rm R}^2(3\omega_{\rm R}-1) = -48X_B^4\frac{\bar{\alpha}_2(\mu_{\rm R})}{M_{\rm pl}^6} + \frac{\beta^{\rm F}(\mu_{\rm R})}{M_{\rm pl}^2}, \tag{6.60}$$

where $\beta^{\rm F}(\mu_{\rm R}) := \langle T^{\rm gw,fin}\rangle$. Regardless of how suppressed the right hand side may appear, it in principle fixes the remaining finite remainder in $\bar{\alpha}_2$ defined in Eq. 6.40, up to the state dependence of the terms that appear in $\beta^{\rm F}(\mu_{\rm R})$. If the right hand side cancels exactly, the effective background corresponds to RD expansion. If the right hand side does not cancel, it would still correspond to RD expansion up to suppressed slow quenching terms[6], which, moreover, dilute much faster than radiation. Hence, the most one can conclude from the shifted tadpole condition Eq. 6.59 is:

$$3H^2 = \frac{1}{M_{\rm pl}^2}\left(\widetilde{\rho}_{\rm bg} + \rho_{\rm gw,fin}^{\rm rd}\right)\left[1 + \frac{\alpha_1^{\rm F}(\mu) - \alpha_1^{\rm F}(\mu_*)}{2\pi^2 M_{\rm pl}^2}\right]^{-1} \approx \frac{\widetilde{\rho}_{\rm bg}}{M_{\rm pl}^2}\left(1+\delta_{\rm Z}\right), \tag{6.61}$$

whith $\widetilde{\rho}_{\rm bg}$ corresponding to the time-time component of $\widetilde{T}_{\mu\nu}^{\rm bg}$, $\rho_{\rm gw,fin}^{\rm rd}$ denoting any (possibly vanishing) state-dependent contributions from $\langle T_{\mu\nu}^{\rm gw,fin}\rangle$ that scales as radiation, and $\delta_{\rm Z}$ is some constant by this definition. The latter defines the wavefunction renormalization of the otherwise unobservable bare thermal potential ψ_B:

$$\psi := \left(1+\delta_{\rm Z}\right)^{1/4}\psi_B, \tag{6.62}$$

where now ψ denotes a dressed quantity. Therefore one concludes that the net effect of the renormalization procedure is to simply mimic shifts in the definition of otherwise unaccessible quantities[7] a conclusion that is in agreement with the results of Section 4.4.

[6]In practice, the best accuracy with which we can ever hope to constrain the left hand side of Eq. 6.60 means that the right hand side can only be taken in practice to be consistent with zero.
[7]Which is consistent with the results reviewed in Section 1.2.

6.5 Conclusions

Motivated by the results of Chapter 4, in which we concluded that one has to follow through the renormalization process to completion in order to make contact with cosmological observations, in this chapter we generalized the previously studied FLRW foliation derivations to a generic background metric in a fully covariant formulation. In doing so we confirmed the findings of Chapter 4, namely that any attempts to extract N_{eff} bounds from vacuum tensor perturbations is inextricable from the process of background renormalization, but also we highlighted subtleties that were not evident in studying a specific foliation.

Differently from the previous results, we find that higher order corrections to the Einstein Hilbert action are required in order to define the counterterms that subtract the divergences arising in computing the stress energy tensor of GWs. Furthermore, we obtain that the regularized stress energy tensor including the contribution of both radiation and GWs is no longer traceless, due to the fact that a massless spin-2 particle is not conformally invariant. This leads to additional slow quenching terms that decay much faster than radiation. We then showed that $\sim R^2$ corrections, together with the anomalous trace, vanish once one imposes a radiation-like solution, which is in agreement with the result of Chapter 4.

6.6 Appendix A: Details of Hadamard regularization

In this appendix, we present further details as to how one can obtain the divergent contributions in Eq. 6.35.

Regularizing the contributions of Eqs. 6.28 and 6.29 in order to obtain Eqs. 6.33 and 6.34, comes down to to regulating the four terms defined in Section 6.4.1

$$\text{(I)}: \qquad R_{\mu\alpha} \lim_{\sigma^\mu \to 0} g_{\alpha'}{}^{\alpha} \tilde{G}^{\mu\alpha'} \tag{6.63}$$

$$\text{(II)}: \qquad g_\rho{}^\gamma \lim_{\sigma^\mu \to 0} g_{\tau'}{}^{\tau} g_{\alpha'\gamma} \nabla_\tau \nabla^{\tau'} \tilde{G}^{\rho\alpha'} \tag{6.64}$$

$$\text{(III)}: \qquad \mathcal{P}_{\mu\nu\alpha\beta} \lim_{\sigma^\mu \to 0} g_{\alpha'}{}^{\alpha} g_{\beta'}{}^{\beta} G^{\mu\nu\alpha'\beta'} \tag{6.65}$$

$$\text{(IV)}: \qquad \mathcal{Q}_{\mu\nu}{}^{\gamma\delta} \lim_{\sigma^\mu \to 0} g_{\tau'}{}^{\tau} g_{\alpha'\gamma} g_{\beta'\delta} \nabla_\tau \nabla^{\tau'} G^{\mu\nu\alpha'\beta'}. \tag{6.66}$$

By iteratively solving at orders σ^0, $\sigma^{\frac{1}{2}}$ and σ the equations of motion for the graviton propagator[8] we find $v_0^{\rho\sigma\alpha\beta}$, $v_0^{\rho\sigma\alpha\beta}{}_\gamma$, $v_0^{\rho\sigma\alpha\beta}{}_{\gamma\varepsilon}$ and $v_1^{\rho\sigma\alpha\beta}$([18])

$$\begin{aligned}
v_{0\mu\nu}{}^{\rho\gamma} =& -\frac{1}{12}R\left(g_{(\mu}{}^{\rho}g_{\nu)}{}^{\gamma} - \frac{1}{2}g_{\mu\nu}g^{\rho\gamma}\right) + \frac{1}{2}P_{\mu\nu}{}^{\rho\gamma} - \frac{1}{4}g^{\rho\gamma}P_{\mu\nu e}{}^{e} \\
v_{0\mu\nu}{}^{\rho\gamma}{}_{\alpha} =& -\frac{1}{2}v_{0a\nu}{}^{\nu\gamma}{}_{;\alpha} - \frac{1}{6}g_{(\mu}{}^{(\nu}\left(R_{|\alpha|\nu)}{}^{;\gamma)} - R_{|\alpha|}{}^{\gamma)}{}_{;\nu)}\right) \\
v_{0\mu\nu}{}^{\nu\gamma}_{\alpha\beta} =& \frac{1}{2}\left(v_0{}^{\nu\gamma}{}_{\mu\nu(\alpha;\beta)} - v_{0\mu\nu}{}^{\nu\gamma}_{(\alpha;\beta)}\right) + \frac{1}{6}P_{\mu\nu}{}^{\rho\gamma}_{;;(\alpha\beta)} + \frac{1}{12}P_{\mu\nu}{}^{\rho\gamma}R_{\alpha\beta} - \frac{1}{12}g^{\rho\gamma}P_{\mu\nu\sigma}{}^{\sigma}{}_{;(\alpha\beta)} \\
& - \frac{1}{24}g^{\rho\gamma}P_{\mu\nu\sigma}{}^{\sigma}R_{\alpha\beta} + \frac{1}{6}g_{(\mu}{}^{(\rho}R_{\nu)\sigma\epsilon(\alpha}R^{\gamma)\sigma\epsilon}{}_{\beta)} - \frac{1}{6}g_{\mu}{}^{(\sigma}g_{\nu}{}^{\epsilon)}R_{\alpha}{}^{\delta(\rho}R_{\beta\delta\epsilon}{}^{\gamma)} \\
& - \frac{1}{6}\left(g_{(\mu}{}^{\rho}g_{\nu)}{}^{\gamma} - \frac{1}{2}g_{\mu\nu}g^{\rho\gamma}\right)\left(\frac{1}{30}R_{\sigma\epsilon\delta\alpha}R^{\sigma\epsilon\delta}{}_{\beta} + \frac{1}{30}R_{\alpha\sigma\beta\epsilon}R^{\sigma\epsilon} - \frac{1}{15}R_{\alpha\sigma}R_{\beta}{}^{\sigma}\right. \\
& \left. + \frac{1}{12}RR_{\alpha\beta} + \frac{3}{20}R_{;\alpha\beta} + \frac{1}{20}\Box R_{\alpha\beta}\right) \\
v_{1\mu\nu}{}^{\rho\gamma} =& \frac{1}{48}\left(g_{\mu}{}^{\rho}g_{\nu}{}^{\gamma} + g_{\mu}{}^{\gamma}g_{\nu}{}^{\rho} - g_{\mu\nu}g^{\rho\gamma}\right)\left(\frac{1}{30}R_{\sigma\epsilon\delta\varsigma}R^{\sigma\epsilon\delta\varsigma} - \frac{1}{30}R_{\sigma\epsilon}R^{\sigma\epsilon} + \frac{1}{12}R^2 + \frac{1}{5}\Box R\right) \\
& - \frac{1}{24}(\Box + R)P_{\mu\nu}{}^{\rho\gamma} + \frac{1}{8}P_{\mu\nu}{}^{\sigma\epsilon}P_{\sigma\epsilon}{}^{\rho\gamma} + \frac{1}{48}g^{\rho\gamma}\left[(\Box + R)P_{\mu\nu\sigma}{}^{\sigma} - 3P_{\mu\nu}{}^{\sigma\epsilon}P_{\sigma\epsilon\delta}{}^{\delta}\right] \\
& + \frac{1}{24}\left(R_{\sigma\epsilon(\mu}{}^{(\rho}R_{\nu)}{}^{\gamma)\sigma\epsilon} - g_{(\mu}{}^{(\rho}R_{\nu)\sigma\epsilon\delta}R^{\gamma)\sigma\epsilon\delta}\right).
\end{aligned} \tag{6.67}$$

where

$$P_{\mu\nu}{}^{\rho\gamma} = -2R^{\gamma}_{(\mu\nu)}{}^{\rho} + \frac{1}{2}\left(g_{\mu\nu}R^{\rho\gamma} + g^{\rho\gamma}R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\rho\gamma}\right) + \frac{1}{2}Rg_{(\mu}{}^{\gamma}g_{\nu)}{}^{\rho}. \tag{6.68}$$

In the same way, we can find $\tilde{v}_0^{\rho\alpha}$, $\tilde{v}_0^{\rho\alpha}{}_\gamma$, $\tilde{v}_0^{\rho\alpha}{}_{\gamma\varepsilon}$ and $\tilde{v}_1^{\rho\alpha}$ by solving order by order in

[8]Higher orders in σ do not contribute to the divergent part.

σ^μ for the ghost propagator and obtain

$$\begin{aligned}
\tilde{v}_0^{\mu\nu} &= -\frac{1}{12}g^{\mu\nu}R - \frac{1}{2}R^{\mu\nu} \\
\tilde{v}_0^{\mu\nu\alpha} &= - - \frac{1}{2}\tilde{v}_0^{\mu\nu;\alpha} - \frac{1}{6}R^{\alpha[\mu;\nu]} \\
\tilde{v}_0^{\mu\nu\alpha\beta} &= -\tilde{v}_0^{[\mu\nu](\alpha;\beta)} + \frac{1}{12}R^{\mu\rho\gamma(\alpha}R^{\beta)}{}_{\gamma\rho}{}^{\nu} - \frac{1}{6}R^{\mu\nu;(\alpha\beta)} - \frac{1}{12}R^{\mu\nu}R^{\alpha\beta} \\
&\quad + g^{\mu\nu}\left(-\frac{1}{180}R^{\rho\gamma\sigma\alpha}R_{\rho\gamma\sigma}{}^{\beta} - \frac{1}{180}R_{\rho\gamma}R^{\alpha\rho\beta\gamma} + \frac{1}{90}R^{\alpha\rho}R^{\beta}{}_{\rho} - \frac{1}{72}RR^{\alpha\beta}\right. \\
&\quad \left. -\frac{1}{40}R^{;\alpha\beta} - \frac{1}{120}\Box R^{\alpha\beta}\right) \\
\tilde{v}_1^{\mu\nu} &= -\frac{1}{48}R^{\mu\rho\gamma\sigma}R_{\rho\gamma\sigma}{}^{\nu} + \frac{1}{24}\Box R^{\mu\nu} + \frac{1}{24}RR^{\mu\nu} + \frac{1}{8}R^{\mu\rho}R_{\rho}{}^{\nu} \\
&\quad + g^{\mu\nu}\left(\frac{1}{720}R^{\rho\gamma\sigma\epsilon}R_{\rho\gamma\sigma\epsilon} - \frac{1}{720}R^{\rho\gamma}R_{\rho\gamma} + \frac{1}{288}R^2 + \frac{1}{120}\Box R\right).
\end{aligned} \tag{6.69}$$

Eqs. 6.63 and 6.65 are then regularized by subtracting the divergent terms of the expansions in Eqs. 6.26 and 6.27 with the appropriate counterterms. These divergences, using Eqs. 6.67 and 6.69, are straightforwardly given by

$$\begin{aligned}
(\mathrm{I})_{\mathrm{div}}: &\quad -\frac{1}{16\pi^2}\lim_{\sigma^\mu\to 0}\left[\frac{1}{\sigma}R + \ln(\mu^2\sigma)\left(-\frac{1}{12}R^2 - \frac{1}{2}R_{\mu\nu}R^{\mu\nu}\right)\right] \\
(\mathrm{III})_{\mathrm{div}}: &\quad -\frac{1}{16\pi^2}\lim_{\sigma^\mu\to 0}\left[-3\frac{1}{\sigma}R + \ln(\mu^2\sigma)\left(\frac{3}{2}R_{\mu\nu}R^{\mu\nu} - R^2 - \frac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right.\right. \\
&\quad \left.\left. -\frac{1}{2}R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma}\right)\right],
\end{aligned} \tag{6.70}$$

and the requisite counterterms are readily identified. Regularizing Eq. 6.64 and Eq. 6.66 is less straightforward, as in order to regularize $\nabla_\tau\nabla^{\tau'}\tilde{G}^{\rho\alpha'}$ and $\nabla_\tau\nabla^{\tau'}G^{\mu\nu\alpha'\beta'}$ we need to sequentially:

1. Compute the derivative of the Hadamard Green's functions using the expansions in 6.26 and 6.27 and keeping the terms that are divergent in the limit $\sigma^\mu \to 0$.

2. Expand the result in powers of σ^μ using the endpoint expansions in [14].

3. Use the averages in Eq. 6.32 to obtain a direction independent result.

Following these steps, one obtains

$$\begin{aligned}(\mathrm{II})_{\mathrm{div}} :& -\frac{1}{16\pi^2}\lim_{\sigma^\mu\to 0}\left[\frac{2}{\sigma}R+\ln(\mu^2\sigma))\left(-\frac{1}{2}R_{\mu\nu}R^{\mu\nu}-\frac{1}{12}R^2+\frac{1}{12}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right.\right.\\ &\left.\left.-\frac{1}{12}R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma}-\frac{5}{12}\Box R\right)\right]\\ (\mathrm{IV})_{\mathrm{div}} :& -\frac{1}{16\pi^2}\lim_{\sigma^\mu\to 0}\left[\frac{11}{6}\frac{1}{\sigma}R+\ln(\mu^2\sigma)\left(-\frac{1}{2}R_{\mu\nu}R^{\mu\nu}+\frac{3}{4}R^2+\frac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right.\right.\\ &\left.\left.+\frac{1}{2}R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma}-\frac{13}{24}\Box R\right)\right].\end{aligned} \tag{6.71}$$

In summary, considering the extra minus in front of the ghost terms accounting for the different statistics, the divergent contributions in Eq. 6.35 are given by

$$\begin{aligned}\langle S\rangle_{\mathrm{div}} =& \int d^4x\sqrt{-g}\left[\Big((\mathrm{III})_{\mathrm{div}}+(\mathrm{IV})_{\mathrm{div}}\Big)-\Big(-(\mathrm{I})_{\mathrm{div}}-(\mathrm{II})_{\mathrm{div}}\Big)\right]\\ =&\frac{1}{4\pi^2}\lim_{\sigma^\mu\to 0}\int d^4x\sqrt{-g}\left[\frac{11}{6}\frac{1}{\sigma}R+\ln(\mu^2\sigma)\left(\frac{1}{6}R_{\mu\nu}R^{\mu\nu}-\frac{11}{24}R^2-\frac{23}{24}\Box R\right)\right]\end{aligned} \tag{6.72}$$

where we have used the Bianchi identity $R^{\mu\nu\rho\sigma}+R^{\mu\rho\sigma\nu}+R^{\mu\sigma\nu\rho=0}$ to obtain $2R_{\mu\nu\rho\sigma}R^{\mu\rho\nu\sigma}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and the Gauss-Bonnet theorem[9] to rewrite the Riemann squared terms in terms of the Ricci tensor and scalar.

[9] The Gauss-Bonnet theorem implies that we can add to the action the Gauss-Bonnet action $S_{GB}=\int d^4x\sqrt{-g}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-4R^{\mu\nu}R_{\mu\nu}+R^2\right)$ without consequences on the theory.

Bibliography

[1] L. F. Abbott. Introduction to the Background Field Method. *Acta Phys. Polon. B*, 13:33, 1982.

[2] R. Abbott et al. GW190521: A Binary Black Hole Merger with a Total Mass of $150 M_\odot$. *Phys. Rev. Lett.*, 125(10):101102, 2020.

[3] R. Abbott et al. Properties and Astrophysical Implications of the 150 $M_\odot$ Binary Black Hole Merger GW190521. *Astrophys. J. Lett.*, 900(1):L13, 2020.

[4] R. Abbott et al. GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo during the Second Part of the Third Observing Run. *Phys. Rev. X*, 13(4):041039, 2023.

[5] A. Achúcarro, V. Atal, and Y. Welling. On the viability of $m^2\phi^2$ and natural inflation. *JCAP*, 07:008, 2015.

[6] V. Acquaviva, N. Bartolo, S. Matarrese, and A. Riotto. Second order cosmological perturbations from inflation. *Nucl. Phys. B*, 667:119–148, 2003.

[7] P. A. R. Ade et al. Planck 2013 results. XVI. Cosmological parameters. *Astron. Astrophys.*, 571:A16, 2014.

[8] P. A. R. Ade et al. Planck 2015 results. XIII. Cosmological parameters. *Astron. Astrophys.*, 594:A13, 2016.

[9] P. A. R. Ade et al. BICEP2 / Keck Array x: Constraints on Primordial Gravitational Waves using Planck, WMAP, and New BICEP2/Keck Observations through the 2015 Season. *Phys. Rev. Lett.*, 121:221301, 2018.

[10] P. A. R. Ade et al. Improved Constraints on Primordial Gravitational Waves using Planck, WMAP, and BICEP/Keck Observations through the 2018 Observing Season. *Phys. Rev. Lett.*, 127(15):151301, 2021.

[11] S. L. Adler. Axial vector vertex in spinor electrodynamics. *Phys. Rev.*, 177:2426–2438, 1969.

[12] S. L. Adler, J. C. Collins, and A. Duncan. Energy-Momentum-Tensor Trace Anomaly in Spin 1/2 Quantum Electrodynamics. *Phys. Rev. D*, 15:1712, 1977.

[13] S. L. Adler and J. Lieberman. Trace Anomaly of the Stress - Energy Tensor for Massless Vector Particles Propagating in a General Background Metric. *Annals Phys.*, 113:294, 1978.

[14] S. L. Adler, J. Lieberman, and Y. J. Ng. Regularization of the Stress Energy Tensor for Vector and Scalar Particles Propagating in a General Background Metric. *Annals Phys.*, 106:279, 1977.

[15] E. T. Akhmedov. Physical meaning and consequences of the loop infrared divergences in global de Sitter space. *Phys. Rev. D*, 87:044049, 2013.

[16] E. T. Akhmedov, F. K. Popov, and V. M. Slepukhin. Infrared dynamics of the massive $\phi4$ theory on de Sitter space. *Phys. Rev. D*, 88:024021, 2013.

[17] B. Allen. The Stochastic gravity wave background: Sources and detection. In *Les Houches School of Physics: Astrophysical Sources of Gravitational Radiation*, pages 373–417, 4 1996.

[18] B. Allen, A. Folacci, and A. C. Ottewill. The Renormalized Graviton Stress - Energy Tensor in Curved Vacuum Space-times. *Phys. Rev. D*, 38:1069, 1988.

[19] L. Amendola and S. Tsujikawa. *Dark Energy: Theory and Observations*. Cambridge University Press, 1 2015.

[20] P. R. Anderson, C. Molina-Paris, and E. Mottola. Cosmological Horizon Modes and Linear Response in de Sitter Spacetime. *Phys. Rev. D*, 80:084005, 2009.

[21] P. R. Anderson and E. Mottola. Instability of global de Sitter space to particle creation. *Phys. Rev. D*, 89:104038, 2014.

[22] P. R. Anderson and L. Parker. Adiabatic Regularization in Closed Robertson-walker Universes. *Phys. Rev. D*, 36:2963, 1987.

[23] K. Ando and V. Vennin. Power spectrum in stochastic inflation. *JCAP*, 04:057, 2021.

[24] C. Animali, P. Conzinu, and G. Marozzi. On adiabatic renormalization with a physically motivated infrared cut-off. *JCAP*, 05(05):026, 2022.

[25] I. Antoniadis and E. Mottola. Graviton Fluctuations in De Sitter Space. *J. Math. Phys.*, 32:1037–1044, 1991.

[26] T. Appelquist and J. Carazzone. Infrared Singularities and Massive Fields. *Phys. Rev. D*, 11:2856, 1975.

[27] I. Y. Arefeva, L. D. Faddeev, and A. A. Slavnov. Generating Functional for the s Matrix in Gauge Theories. *Teor. Mat. Fiz.*, 21:311–321, 1974.

[28] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt. Essentials of k essence. *Phys. Rev. D*, 63:103510, 2001.

[29] H. Assadullahi, H. Firouzjahi, M. Noorbala, V. Vennin, and D. Wands. Multiple Fields in Stochastic Inflation. *JCAP*, 06:043, 2016.

[30] V. Balakumar and E. Winstanley. Hadamard renormalization for a charged scalar field. *Class. Quant. Grav.*, 37(6):065004, 2020.

[31] G. Ballesteros, J. Rey, M. Taoso, and A. Urbano. Stochastic inflationary dynamics beyond slow-roll and consequences for primordial black hole formation. *JCAP*, 08:043, 2020.

[32] J. M. Bardeen. Gauge Invariant Cosmological Perturbations. *Phys. Rev. D*, 22:1882–1905, 1980.

[33] J. M. Bardeen and G. J. Bublik. Quantum Fluctuations and Inflation. *Class. Quant. Grav.*, 4:573, 1987.

[34] N. H. Barth and S. M. Christensen. Quantizing Fourth Order Gravity Theories. 1. The Functional Integral. *Phys. Rev. D*, 28:1876, 1983.

[35] N. Bartolo, S. Matarrese, M. Pietroni, A. Riotto, and D. Seery. On the Physical Significance of Infra-red Corrections to Inflationary Observables. *JCAP*, 01:015, 2008.

[36] A. O. Barvinsky and G. A. Vilkovisky. Beyond the Schwinger-Dewitt Technique: Converting Loops Into Trees and In-In Currents. *Nucl. Phys. B*, 282:163–188, 1987.

[37] A. O. Barvinsky and G. A. Vilkovisky. Covariant perturbation theory. 2: Second order in the curvature. General algorithms. *Nucl. Phys. B*, 333:471–511, 1990.

[38] A. O. Barvinsky and G. A. Vilkovisky. Covariant perturbation theory. 3: Spectral representations of the third order form-factors. *Nucl. Phys. B*, 333:512–524, 1990.

[39] D. Baumann. Inflation. In *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small*, pages 523–686, 2011.

[40] M. Baumgart, J. J. Heckman, and L. Thomas. CFTs blueshift tensor fluctuations universally. *JCAP*, 07(07):034, 2022.

[41] M. Baumgart and R. Sundrum. De Sitter Diagrammar and the Resummation of Time. *JHEP*, 07:119, 2020.

[42] A. Belokogne and A. Folacci. Stueckelberg massive electromagnetism in curved spacetime: Hadamard renormalization of the stress-energy tensor and the Casimir effect. *Phys. Rev. D*, 93(4):044063, 2016.

[43] M. Benetti, L. L. Graef, and S. Vagnozzi. Primordial gravitational waves from NANOGrav: A broken power-law approach. *Phys. Rev. D*, 105(4):043520, 2022.

[44] M. Berbig and A. Ghoshal. Impact of high-scale Seesaw and Leptogenesis on inflationary tensor perturbations as detectable gravitational waves. *JHEP*, 05:172, 2023.

[45] J. Berges. Introduction to nonequilibrium quantum field theory. *AIP Conf. Proc.*, 739(1):3–62, 2004.

[46] E. Bertschinger. Cosmological dynamics: Course 1. In *Les Houches Summer School on Cosmology and Large Scale Structure (Session 60)*, pages 273–348, 8 1993.

[47] H. J. Bhabha. The scattering of positrons by electrons with exchange on Dirac's theory of the positron. *Proc. Roy. Soc. Lond. A*, 154:195–206, 1936.

[48] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK, 2 1984.

[49] F. Bloch and A. Nordsieck. Note on the Radiation Field of the electron. *Phys. Rev.*, 52:54–59, 1937.

[50] L. Boubekeur, P. Creminelli, J. Norena, and F. Vernizzi. Action approach to cosmological perturbations: the 2nd order metric in matter dominance. *JCAP*, 08:028, 2008.

[51] D. G. Boulware. Gauge Dependence of the Effective Action. *Phys. Rev. D*, 23:389, 1981.

[52] D. Boyanovsky and H. J. de Vega. Dynamical renormalization group approach to relaxation in quantum field theory. *Annals Phys.*, 307:335–371, 2003.

[53] L. A. Boyle and P. J. Steinhardt. Probing the early universe with inflationary gravitational waves. *Phys. Rev. D*, 77:063504, 2008.

[54] D. R. Brill and J. B. Hartle. Method of the Self-Consistent Field in General Relativity and its Application to the Gravitational Geon. *Phys. Rev.*, 135:B271–B278, 1964.

[55] M. R. Brown and A. C. Ottewill. Photon Propagators and the Definition and Approximation of Renormalized Stress Tensors in Curved Space-time. *Phys. Rev. D*, 34:1776–1786, 1986.

[56] T. S. Bunch. Adiabatic Regularization for Scalar Fields with Arbitrary Coupling tot he Scalar Curvature. *J. Phys. A*, 13:1297–1310, 1980.

[57] T. S. Bunch, S. M. Christensen, and S. A. Fulling. Massive Quantum Field Theory in Two-Dimensional Robertson-Walker Space-Time. *Phys. Rev. D*, 18:4435–4459, 1978.

[58] T. S. Bunch and P. C. W. Davies. Nonconformal Renormalized Stress Tensors in Robertson-Walker Space-Times. *J. Phys. A*, 11:1315–1328, 1978.

[59] C. Burgess, A. del Rio, and S. S. Patil.

[60] C. P. Burgess. Quantum gravity in everyday life: General relativity as an effective field theory. *Living Rev. Rel.*, 7:5–56, 2004.

[61] C. P. Burgess. *Introduction to Effective Field Theory.* Cambridge University Press, 12 2020.

[62] C. P. Burgess, R. Holman, L. Leblond, and S. Shandera. Breakdown of Semiclassical Methods in de Sitter Space. *JCAP*, 10:017, 2010.

[63] C. P. Burgess, R. Holman, and G. Tasinato. Open EFTs, IR effects \& late-time resummations: systematic corrections in stochastic inflation. *JHEP*, 01:153, 2016.

[64] C. P. Burgess, L. Leblond, R. Holman, and S. Shandera. Super-Hubble de Sitter Fluctuations and the Dynamical RG. *JCAP*, 03:033, 2010.

[65] G. Cabass, L. Pagano, L. Salvati, M. Gerbino, E. Giusarma, and A. Melchiorri. Updated Constraints and Forecasts on Primordial Tensor Modes. *Phys. Rev. D*, 93(6):063508, 2016.

[66] E. A. Calzetta and B.-L. B. Hu. *Nonequilibrium Quantum Field Theory.* Oxford University Press, 2009.

[67] P. Campeti and E. Komatsu. New Constraint on the Tensor-to-scalar Ratio from the Planck and BICEP/Keck Array Data Using the Profile Likelihood. *Astrophys. J.*, 941(2):110, 2022.

[68] C. Caprini and D. G. Figueroa. Cosmological Backgrounds of Gravitational Waves. *Class. Quant. Grav.*, 35(16):163001, 2018.

[69] H. B. G. Casimir and D. Polder. The Influence of retardation on the London-van der Waals forces. *Phys. Rev.*, 73:360–372, 1948.

[70] J. Chluba, J. Hamann, and S. P. Patil. Features and New Physical Scales in Primordial Observables: Theory and Observation. *Int. J. Mod. Phys. D*, 24(10):1530023, 2015.

[71] S. M. Christensen. Vacuum Expectation Value of the Stress Tensor in an Arbitrary Curved Background: The Covariant Point Separation Method. *Phys. Rev. D*, 14:2490–2501, 1976.

[72] S. M. Christensen. Regularization, Renormalization, and Covariant Geodesic Point Separation. *Phys. Rev. D*, 17:946–963, 1978.

[73] S. M. Christensen and M. J. Duff. Quantizing Gravity with a Cosmological Constant. *Nucl. Phys. B*, 170:480–506, 1980.

[74] J. C. Collins. *Renormalization*, volume 26 of *Cambridge Monographs on Mathematical Physics.* Cambridge University Press, Cambridge, 7 2023.

[75] E. J. Copeland, M. Sami, and S. Tsujikawa. Dynamics of dark energy. *Int. J. Mod. Phys. D*, 15:1753–1936, 2006.

[76] P. Creminelli, S. Dubovsky, A. Nicolis, L. Senatore, and M. Zaldarriaga. The Phase Transition to Slow-roll Eternal Inflation. *JHEP*, 09:036, 2008.

[77] D. Cruces. Review on Stochastic Approach to Inflation. *Universe*, 8(6):334, 2022.

[78] R. H. Cyburt, B. D. Fields, K. A. Olive, and T.-H. Yeh. Big Bang Nucleosynthesis: 2015. *Rev. Mod. Phys.*, 88:015004, 2016.

[79] B. N. D. The Application of Adiabatic Regularization to Calculations of Cosmological Interest. *Proc R Soc (London) A.*, 361:513–526, 1978.

[80] P. C. W. Davies, S. A. Fulling, S. M. Christensen, and T. S. Bunch. Energy Momentum Tensor of a Massless Scalar Quantum Field in a Robertson-Walker Universe. *Annals Phys.*, 109:108–142, 1977.

[81] B. De Witt. Dynamical theory of groups and fields. *Conf. Proc. C*, 630701:585–820, 1964.

[82] C. De Witt and B. De Witt. Relativité, Groupes et Topologie: Proceedings, Ecole d'été de Physique Théorique. *Session XIII, Les Houches*, France, Jul 1 - Aug 24, 1963.

[83] Y. Decanini and A. Folacci. Off-diagonal coefficients of the Dewitt-Schwinger and Hadamard representations of the Feynman propagator. *Phys. Rev. D*, 73:044027, 2006.

[84] A. del Rio, R. Durrer, and S. P. Patil. Tensor Bounds on the Hidden Universe. *JHEP*, 12:094, 2018.

[85] S. Deser, J. H. Kay, and K. S. Stelle. Renormalizability Properties of Supergravity. *Phys. Rev. Lett.*, 38:527, 1977.

[86] B. S. DeWitt. A Gauge Invariant Effective Action. In *Oxford Conference on Quantum Gravity*, 7 1980.

[87] B. S. DeWitt and R. W. Brehme. Radiation damping in a gravitational field. *Annals Phys.*, 9:220–259, 1960.

[88] S. Dodelson. *Modern Cosmology.* Academic Press, Amsterdam, 2003.

[89] A. D. Dolgov, M. B. Einhorn, and V. I. Zakharov. The Vacuum of de Sitter space. *Acta Phys. Polon. B*, 26:65–90, 1995.

[90] J. F. Donoghue. General relativity as an effective field theory: The leading quantum corrections. *Phys. Rev. D*, 50:3874–3888, 1994.

[91] J. F. Donoghue. Introduction to the effective field theory description of gravity. In *Advanced School on Effective Theories*, 6 1995.

[92] J. F. Donoghue. The effective field theory treatment of quantum gravity. *AIP Conf. Proc.*, 1483(1):73–94, 2012.

[93] J. S. Dowker and R. Critchley. Effective Lagrangian and Energy Momentum Tensor in de Sitter Space. *Phys. Rev. D*, 13:3224, 1976.

[94] G. Ecker and J. Honerkamp. Application of invariant renormalization to the nonlinear chiral invariant pion lagrangian in the one-loop approximation. *Nucl. Phys. B*, 35:481–492, 1971.

[95] K. Enqvist, S. Nurmi, D. Podolsky, and G. I. Rigopoulos. On the divergences of inflationary superhorizon perturbations. *JCAP*, 04:025, 2008.

[96] J. M. Ezquiaga, J. García-Bellido, and V. Vennin. The exponential tail of inflationary fluctuations: consequences for primordial black holes. *JCAP*, 03:029, 2020.

[97] L. D. Faddeev and V. N. Popov. Feynman Diagrams for the Yang-Mills Field. *Phys. Lett. B*, 25:29–30, 1967.

[98] A. Ferreiro and F. Torrenti. Ultraviolet-regularized power spectrum without infrared distortions in cosmological spacetimes. *Phys. Lett. B*, 840:137868, 2023.

[99] F. Finelli, G. Marozzi, A. A. Starobinsky, G. P. Vacca, and G. Venturi. Generation of fluctuations during inflation: Comparison of stochastic and field-theoretic approaches. *Phys. Rev. D*, 79:044007, 2009.

[100] F. Finelli, G. Marozzi, A. A. Starobinsky, G. P. Vacca, and G. Venturi. Stochastic growth of quantum fluctuations during slow-roll inflation. *Phys. Rev. D*, 82:064020, 2010.

[101] L. H. Ford. Quantum Instability of De Sitter Space-time. *Phys. Rev. D*, 31:710, 1985.

[102] A. Friedman. On the Curvature of Space . *General Relativity and Gravitation*, 31:1991–2000, 1999.

[103] A. Friedmann. On the Possibility of a world with constant negative curvature of space. *Z. Phys.*, 21:326–332, 1924.

[104] S. A. Fulling, L. Parker, and B. L. Hu. Conformal energy-momentum tensor in curved spacetime: Adiabatic regularization and renormalization. *Phys. Rev. D*, 10:3905–3924, 1974.

[105] J. Garcia-Bellido and E. Ruiz Morales. Primordial black holes from single field models of inflation. *Phys. Dark Univ.*, 18:47–54, 2017.

[106] M. Gasperini and G. Veneziano. String Theory and Pre-big bang Cosmology. *Nuovo Cim. C*, 38(5):160, 2016.

[107] W. Giarè, M. Forconi, E. Di Valentino, and A. Melchiorri. Towards a reliable calculation of relic radiation from primordial gravitational waves. *Mon. Not. Roy. Astron. Soc.*, 520:2, 2023.

[108] S. B. Giddings and M. S. Sloth. Semiclassical relations and IR effects in de Sitter and slow-roll space-times. *JCAP*, 01:023, 2011.

[109] D. Glavan, T. Prokopec, and T. Takahashi. Late-time quantum backreaction of a very light nonminimally coupled scalar. *Phys. Rev. D*, 94:084053, 2016.

[110] A. S. Goncharov, A. D. Linde, and V. F. Mukhanov. The Global Structure of the Inflationary Universe. *Int. J. Mod. Phys. A*, 2:561–591, 1987.

[111] E. P. Gordov and S. D. Tvorogov. Eigenfunctions of the Vector-Potential Operator of the Electromagnetic Field and Semiclassical Electrodynamics. *Phys. Rev. D*, 8:3286, 1973.

[112] M. H. Goroff and A. Sagnotti. The Ultraviolet Behavior of Einstein Gravity. *Nucl. Phys. B*, 266:709–736, 1986.

[113] A. M. Green and A. R. Liddle. Constraints on the density perturbation spectrum from primordial black holes. *Phys. Rev. D*, 56:6166–6174, 1997.

[114] D. Green and A. Premkumar. Dynamical RG and Critical Phenomena in de Sitter Space. *JHEP*, 04:064, 2020.

[115] L. P. Grishchuk. Amplification of gravitational waves in an istropic universe. *Zh. Eksp. Teor. Fiz.*, 67:825–838, 1974.

[116] G. Grunberg. Renormalization Group Improved Perturbative QCD. *Phys. Lett. B*, 95:70, 1980. [Erratum: Phys.Lett.B 110, 501 (1982)].

[117] G. Grunberg. Renormalization Scheme Independent QCD and QED: The Method of Effective Charges. *Phys. Rev. D*, 29:2315–2338, 1984.

[118] A. H. Guth. The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. *Phys. Rev. D*, 23:347–356, 1981.

[119] M. C. Guzzetti, N. Bartolo, M. Liguori, and S. Matarrese. Gravitational waves from inflation. *Riv. Nuovo Cim.*, 39(9):399–495, 2016.

[120] J. Hadamard. Le probl'eme de Cauchy et les ´equations aux d´eriv´ees partielles lin´eaires hyperboliques.

[121] S. Henrot-Versille et al. Improved constraint on the primordial gravitational-wave density using recent cosmological data and its impact on cosmic string models. *Class. Quant. Grav.*, 32(4):045003, 2015.

[122] A. Higuchi. Decay of the free-theory vacuum of scalar field theory in de Sitter spacetime in the interaction picture. *Class. Quant. Grav.*, 26:072001, 2009.

[123] R. A. Isaacson. Gravitational Radiation in the Limit of High Frequency. II. Nonlinear Terms and the Ef fective Stress Tensor. *Phys. Rev.*, 166:1272–1279, 1968.

[124] R. A. Isaacson. Gravitational Radiation in the Limit of High Frequency. II. Nonlinear Terms and the Ef fective Stress Tensor. *Phys. Rev.*, 166:1272–1279, 1968.

[125] C. Itzykson and J. B. Zuber. *Quantum Field Theory*. International Series In Pure and Applied Physics. McGraw-Hill, New York, 1980.

[126] E. O. Kahya and V. K. Onemli. Quantum Stability of a w < -1 Phase of Cosmic Acceleration. *Phys. Rev. D*, 76:043512, 2007.

[127] N. Kaiser. Theoretical implications of deviations from Hubble flow. *Mon. Not. Roy. Astron. Soc.*, 231:149, 1989.

[128] M. Kamionkowski, A. Kosowsky, and A. Stebbins. Statistics of cosmic microwave background polarization. *Phys. Rev. D*, 55:7368–7388, 1997.

[129] S. Y. Khlebnikov and I. I. Tkachev. Classical decay of inflaton. *Phys. Rev. Lett.*, 77:219–222, 1996.

[130] H. Kleinert and V. Schulte-Frohlinde. *Critical properties of phi**4-theories*. 2001.

[131] H. Kluberg-Stern and J. B. Zuber. Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 1. Green Functions. *Phys. Rev. D*, 12:482–488, 1975.

[132] H. Kodama and M. Sasaki. Cosmological Perturbation Theory. *Prog. Theor. Phys. Suppl.*, 78:1–166, 1984.

[133] J. F. Koksma and T. Prokopec. The Cosmological Constant and Lorentz Invariance of the Vacuum State. 5 2011.

[134] E. W. Kolb and M. S. Turner. *The Early Universe*, volume 69. 1990.

[135] E. W. Kolb and S. Wolfram. Baryon Number Generation in the Early Universe. *Nucl. Phys. B*, 172:224, 1980. [Erratum: Nucl.Phys.B 195, 542 (1982)].

[136] Z. Kunszt. From scattering amplitudes to cross-sections in QCD. In *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95): QCD and Beyond*, pages 515–538, 3 1996.

[137] S. K. Lamoreaux. Demonstration of the Casimir force in the 0.6 to 6 micrometers range. *Phys. Rev. Lett.*, 78:5–8, 1997. [Erratum: Phys.Rev.Lett. 81, 5475–5476 (1998)].

[138] A. Landete, J. Navarro-Salas, and F. Torrenti. Adiabatic regularization and particle creation for spin one-half fields. *Phys. Rev. D*, 89:044030, 2014.

[139] G. Lemaitre. A Homogeneous Universe of Constant Mass and Increasing Radius accounting for the Radial Velocity of Extra-galactic Nebulæ. *Mon. Not. Roy. Astron. Soc.*, 91(5):483–490, 1931.

[140] J. Li, Z.-C. Chen, and Q.-G. Huang. Measuring the tilt of primordial gravitational-wave power spectrum from observations. *Sci. China Phys. Mech. Astron.*, 62(11):110421, 2019. [Erratum: Sci.China Phys.Mech.Astron. 64, 250451 (2021)].

[141] A. R. Liddle. *An introduction to modern cosmology.* 1998.

[142] A. R. Liddle and A. M. Green. Cosmological constraints from primordial black holes. *Phys. Rept.*, 307:125–131, 1998.

[143] E. Lifshitz. Republication of: On the gravitational stability of the expanding universe. *J. Phys. (USSR)*, 10(2):116, 1946.

[144] A. D. Linde. A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems. *Phys. Lett. B*, 108:389–393, 1982.

[145] A. D. Linde, D. A. Linde, and A. Mezhlumian. From the Big Bang theory to the theory of a stationary universe. *Phys. Rev. D*, 49:1783–1826, 1994.

[146] X.-J. Liu, W. Zhao, Y. Zhang, and Z.-H. Zhu. Detecting Relic Gravitational Waves by Pulsar Timing Arrays: Effects of Cosmic Phase Transitions and Relativistic Free-Streaming Gases. *Phys. Rev. D*, 93(2):024031, 2016.

[147] J. Lizarraga, I. Sendra, and J. Urrestilla. Correlations between cosmic strings and extra relativistic species. *Phys. Rev. D*, 86:123014, 2012.

[148] G. Luisoni and S. Marzani. QCD resummation for hadronic final states. *J. Phys. G*, 42(10):103101, 2015.

[149] M. A. H. Maccallum and A. H. Taub. The averaged lagrangian and high-frequency gravitational waves. *Commun. Math. Phys.*, 30:153–169, 1973.

[150] M. Maggiore. Stochastic backgrounds of gravitational waves. *ICTP Lect. Notes Ser.*, 3:397–414, 2001.

[151] M. Maggiore. *Gravitational Waves. Vol. 1: Theory and Experiments.* Oxford University Press, 2007.

[152] M. Maggiore. *Gravitational Waves. Vol. 1: Theory and Experiments.* Oxford University Press, 2007.

[153] K. A. Malik and D. Wands. Cosmological perturbations. *Phys. Rept.*, 475:1–51, 2009.

[154] A. V. Manohar. The HQET / NRQCD Lagrangian to order alpha / m-3. *Phys. Rev. D*, 56:230–237, 1997.

[155] A. V. Manohar. Introduction to Effective Field Theories. 4 2018.

[156] F. J. Marañón González and J. Navarro-Salas. Adiabatic regularization for spin-1 fields. *Phys. Rev. D*, 108(12):125001, 2023.

[157] S. Matarrese. On the Classical and Quantum Irrotational Motions of a Relativistic Perfect Fluid. 1. Classical Theory. *Proc. Roy. Soc. Lond. A*, 401:53–66, 1985.

[158] P. D. Meerburg, R. Hložek, B. Hadzhiyska, and J. Meyers. Multiwavelength constraints on the inflationary consistency relation. *Phys. Rev. D*, 91(10):103505, 2015.

[159] D. Melikhov and B. Stech. On the $\gamma^*\gamma \to \pi(\eta, \eta')$ transition form factors. *Phys. Rev. D*, 85:051901, 2012.

[160] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation.* W. H. Freeman, San Francisco, 1973.

[161] S. Mollerach, S. Matarrese, A. Ortolan, and F. Lucchin. Stochastic inflation in a simple two field model. *Phys. Rev. D*, 44:1670–1679, 1991.

[162] E. Mottola. Thermodinamic Instability of de Sitter Space. *Phys. Rev. D*, 33:1616–1621, 1986.

[163] V. Mukhanov. *Physical Foundations of Cosmology.* Cambridge University Press, Oxford, 2005.

[164] V. Mukhanov and S. Winitzki. *Introduction to quantum effects in gravity.* Cambridge University Press, 6 2007.

[165] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger. Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions. *Phys. Rept.*, 215:203–333, 1992.

[166] K.-i. Nakao, Y. Nambu, and M. Sasaki. Stochastic Dynamics of New Inflation. *Prog. Theor. Phys.*, 80:1041, 1988.

[167] Y. Nambu. Stochastic Dynamics of an Inflationary Model and Initial Distribution of Universes. *Prog. Theor. Phys.*, 81:1037, 1989.

[168] Y. Nambu and M. Sasaki. Stochastic approach to chaotic inflation and the distribution of universes. *Phys. Lett. B*, 219:240–246, 1989.

[169] A. Negro and S. P. Patil. An Étude on the Regularization and Renormalization of Divergences in Primordial Observables. 2 2024.

[170] A. Negro and S. P. Patil. Hadamard Regularization of the Graviton Stress Tensor. 3 2024.

[171] A. Negro and S. S. Patil. .

[172] C. Newton, P. Osland, and T. T. Wu. Point splitting regularization for gauge theories: Quantum electrodynamics. *Z. Phys. C*, 61:441–448, 1994.

[173] K. A. Olive, G. Steigman, and T. P. Walker. Primordial nucleosynthesis: Theory and observations. *Phys. Rept.*, 333:389–407, 2000.

[174] V. K. Onemli and R. P. Woodard. Quantum effects can render w < -1 on cosmological scales. *Phys. Rev. D*, 70:107301, 2004.

[175] T. Padmanabhan. Cosmological constant: The Weight of the vacuum. *Phys. Rept.*, 380:235–320, 2003.

[176] L. Pagano, L. Salvati, and A. Melchiorri. New constraints on primordial gravitational waves from Planck 2015. *Phys. Lett. B*, 760:823–825, 2016.

[177] S. N. Pandey. Conformally Invariant Gravitational Waves in a Modified Gravitational Theory. *Int. J. Theor. Phys.*, 22:209–217, 1983.

[178] L. E. Parker and D. Toms. *Quantum Field Theory in Curved Spacetime: Quantized Field and Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 8 2009.

[179] P. J. Peebles. *The Large-Scale Structure of the Universe*. Princeton University Press, 11 1980.

[180] P. J. E. Peebles and B. Ratra. The Cosmological Constant and Dark Energy. *Rev. Mod. Phys.*, 75:559–606, 2003.

[181] A. A. Penzias and R. W. Wilson. A Measurement of excess antenna temperature at 4080-Mc/s. *Astrophys. J.*, 142:419–421, 1965.

[182] L. Perreault Levasseur. Lagrangian formulation of stochastic inflation: Langevin equations, one-loop corrections and a proposed recursive approach. *Phys. Rev. D*, 88(8):083537, 2013.

[183] A. M. Polyakov. Infrared instability of the de Sitter space. 9 2012.

[184] P. Ramond. *Field theory: a modern primer*, volume 51. 1981.

[185] S.-J. Rey. Dynamics of Inflationary Phase Transition. *Nucl. Phys. B*, 284:706–728, 1987.

[186] A. Riotto. Inflation and the Theory of Cosmological Perturbations. 2018.

[187] H. P. Robertson. Kinematics and World-Structure. *Astrophys. J.*, 82:284–301, 1935.

[188] V. A. Rubakov, M. V. Sazhin, and A. V. Veryaskin. Graviton Creation in the Inflationary Universe and the Grand Unification Scale. *Phys. Lett. B*, 115:189–192, 1982.

[189] D. S. Salopek and J. R. Bond. Stochastic inflation and nonlinear gravity. *Phys. Rev. D*, 43:1005–1031, 1991.

[190] S. Sarkar. Big bang nucleosynthesis and physics beyond the standard model. *Rept. Prog. Phys.*, 59:1493–1610, 1996.

[191] S. Sarkar and H. Strubbe. Anomalous Dimensions in Background Field Gauges. *Nucl. Phys. B*, 90:45–51, 1975.

[192] M. Sasaki. Large Scale Quantum Fluctuations in the Inflationary Universe. *Prog. Theor. Phys.*, 76:1036, 1986.

[193] L. Schwartz. Theorie des distributions.

[194] M. D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 3 2014.

[195] D. J. Schwarz. Evolution of gravitational waves through cosmological transitions. *Mod. Phys. Lett. A*, 13:2771–2778, 1998.

[196] J. S. Schwinger. On gauge invariance and vacuum polarization. *Phys. Rev.*, 82:664–679, 1951.

[197] D. Seery. Infrared effects in inflationary correlation functions. *Class. Quant. Grav.*, 27:124005, 2010.

[198] U. Seljak and M. Zaldarriaga. Signature of gravity waves in polarization of the microwave background. *Phys. Rev. Lett.*, 78:2054–2057, 1997.

[199] L. Senatore and M. Zaldarriaga. On Loops in Inflation. *JHEP*, 12:008, 2010.

[200] T. L. Smith, E. Pierpaoli, and M. Kamionkowski. A new cosmic microwave background constraint to primordial gravitational waves. *Phys. Rev. Lett.*, 97:021301, 2006.

[201] A. A. Starobinsky. Spectrum of relict gravitational radiation and the early state of the universe. *JETP Lett.*, 30:682–685, 1979.

[202] A. A. Starobinsky. Stochastic de Sitter (Inflationary) Stage in the Early Universe. *Lect. Notes Phys.*, 246:107–126, 1986.

[203] A. A. Starobinsky and J. Yokoyama. Equilibrium state of a selfinteracting scalar field in the De Sitter background. *Phys. Rev. D*, 50:6357–6368, 1994.

[204] L. C. Stein and N. Yunes. Effective Gravitational Wave Stress-energy Tensor in Alternative Theories of Gravity. *Phys. Rev. D*, 83:064038, 2011.

[205] K. Symanzik. Small distance behavior in field theory and power counting. *Commun. Math. Phys.*, 18:227–246, 1970.

[206] G. 't Hooft. An algorithm for the poles at dimension four in the dimensional regularization procedure. *Nucl. Phys. B*, 62:444–460, 1973.

[207] G. 't Hooft. The Background Field Method in Gauge Field Theories. In *12th Annual Winter School of Theoretical Physics*, 1975.

[208] G. 't Hooft and M. J. G. Veltman. One loop divergencies in the theory of gravitation. *Ann. Inst. H. Poincare Phys. Theor. A*, 20:69–94, 1974.

[209] Y. Tada and V. Vennin. Statistics of coarse-grained cosmological fields in stochastic inflation. *JCAP*, 02(02):021, 2022.

[210] D. V. Vassilevich. Heat kernel expansion: User's manual. *Phys. Rept.*, 388:279–360, 2003.

[211] V. Vennin and A. A. Starobinsky. Correlation Functions in Stochastic Inflation. *Eur. Phys. J. C*, 75:413, 2015.

[212] A. G. Walker. On Milne's Theory of World-Structure. *Proc. Lond. Math. Soc. s*, 2-42(1):90–127, 1937.

[213] T. P. Walker, G. Steigman, D. N. Schramm, K. A. Olive, and H.-S. Kang. Primordial nucleosynthesis redux. *Astrophys. J.*, 376:51–69, 1991.

[214] S. Weinberg. Cosmological Production of Baryons. *Phys. Rev. Lett.*, 42:850–853, 1979.

[215] S. Weinberg. *The Quantum theory of fields. Vol. 1: Foundations.* Cambridge University Press, 6 2005.

[216] S. Weinberg. Effective Field Theory for Inflation. *Phys. Rev. D*, 77:123541, 2008.

[217] A. Wightman. Hilbert's sixth problem: Mathematical treatment of the axioms of physics. *Amer. Math. Soc.*, 28:241–268, 1976.

[218] A. Zee. *Quantum Field Theory in a Nutshell: Second Edition.* Princeton University Press, 2 2010.

Summary

With the first direct detection of GWs from the Laser Interferometer Gravitational-Wave Observatory on September 14, 2015, the gravitational wave detection era started. Since then, several other signals from black holes and neutron stars events have been detected, and the possibility of studying the universe by means of GWs represents a new frontier.
How about GWs from cosmological origin? Detecting GWs from early universe phenomena would allow us to uncover mysteries that were previously inaccessible through traditional electromagnetic observations. The possibility of opening this new window on the early universe comes with challenges, not only on the experimental side, but also on the theoretical side, with the goal of having an accurate description of such early universe GWs signals. In this thesis we focus our attention on vacuum GWs: vacuum tensor quantum fluctuations which inevitably come in the form of stochastic backgrounds. It is important to be careful in distinguishing vacuum GWs from primordial gravitational waves sourced by dynamical processes involving energy and momentum transfer, such as GWs from phase transitions, particle production or decay of topological defects. The latter involve propagating gravitons sourced by some physical processes while vacuum GWs are by definition vacuum polarization effects whose imprint on various observables is no less real, but comes with important subtleties that motivate the work presented in this thesis.

To contextualize our work, in the **Introduction** we reviewed the thermal history of the universe, the foundations of the cosmological model and we motivated the need to include vacuum quantum fluctuations in the early stages of our universe. After that, we shifted our attention to introduce the tools needed to quantify the contribution of vacuum quantum fluctuations. We reviewed the basics of the renormalization procedure as a method to meaningfully reabsorb UV divergences arising in computing quantum corrections. We remarked the possibility of obtaining divergent vacuum energies and highlighted that the splitting into classical quantities and quantum corrections can be misleading: one has access to fully dressed observables and cannot separately measure quantum corrections and classical quantities.

Can vacuum GWs be constrained by N_{eff} bounds? This is the question from which our work started. **Chapter 2** is a detailed review of the typical expressions that one can find in the literature to connect the energy density of vacuum GWs to the bounds on the effective number of relativistic species, N_{eff}, at the time of Big Bang nucleosynthesis. We concluded the chapter by pointing out that whether one

can meaningfully constrain vacuum GWs with $N_{\rm eff}$ bounds is intrinsically bound to the definition of the stress energy tensor of GWs and to how the divergences that inevitably arise in computing the energy density of vacuum GWs are renormalized. The latter caveat motivated the work presented in **Chapter 3**, where we applied well-established renormalization techniques to the case of a massless, non-interacting scalar field on a Friedmann–Lemaître–Robertson–Walker background. We showed how dimensional regularization can be used to extract UV divergences from scaleless integrals, the independence of logarithm divergences on the choice of the regularization method and the need to use regularization methods that preserve covariance in order to find counterterms that are proportional to geometric invariants. Furthermore, by studying a background evolution that after a pre-inflationary radiation dominated era transitions to a pure de Sitter phase which is followed by a second post-inflationary radiation dominated era, we explicitly demonstrated that IR/UV scales connected with the beginning/end of inflation do not cure UV divergences and that IR divergences are an artifact of the idealization of a past infinite de Sitter geometry. By applying the lessons learned in the apparently simple example of a scalar field to the case of vacuum GWs, in **Chapter 4** we addressed the question whether $N_{\rm eff}$ bounds can be used to infer constraints on vacuum GWs. We started by deriving a formula for the energy density of GWs suitable for regularization, we addressed the caveats regarding the definition of the stress energy tensor of GWs and derived an improved formula that does not rely on prior scale separation. We then followed trough the renormalization procedure, we isolated the divergent structure and after having subtracted the divergences in consistently defined counterterms, we commented on the renormalization conditions that must be imposed to fix the scheme-dependent finite leftover and obtain a meaningful answer.
In the second part of the thesis we revisited the study of vacuum GWs in a covariant formulation. To do so in **Chapter 5** we introduced the formalism needed to address the study of vacuum GWs as a spin-2 particle on curved spacetime. We then applied such tools in **Chapter 6**, where the renormalized stress energy tensor for GWs is obtained from the variation of the effective action with respect to the background metric. We derived the effective action for GWs, we regularized and consistently subtracted the divergences by adding counterterms at the level of the action. Finally, we specialized our otherwise general result to a radiation dominated background to fix the renormalization conditions, connect to the foliation dependent derivation and comment on the consequences on $N_{\rm eff}$ bounds.

Among the results that we presented in tackling, both in a covariant and foliation specific formulation, the renormalization of divergences in primordial observables, we conclude that $N_{\rm eff}$ bounds cannot be used to constrain vacuum GWs. Such constraint represents an attempt to compute and measure the absolute value of quantum corrections. The latter, by definition, cannot be measured separately from the classical background, as all physical observations are necessarily of fully dressed quantities.

Samenvatting

Met de eerste directe detectie van zwaartekrachtgolven (GW's) door het Laser Interferometer Gravitational-Wave Observatory op 14 september 2015 begon het tijdperk van zwaartekrachtgolfdetectie. Sindsdien zijn er verschillende andere signalen van gebeurtenissen met zwarte gaten en neutronensterren gedetecteerd, en de mogelijkheid om het universum te bestuderen met behulp van GW's vertegenwoordigt een nieuw front van de wetenschap.
Hoe zit het met GW's van kosmologische oorsprong? Het detecteren van GW's uit verschijnselen in het vroege universum zou ons in staat stellen mysteries te ontdekken die voorheen ontoegankelijk waren via traditionele elektromagnetische waarnemingen. De mogelijkheid om dit nieuwe venster op het vroege universum te openen, brengt uitdagingen met zich mee, niet alleen aan de experimentele kant, maar ook aan de theoretische kant, met als doel een nauwkeurige beschrijving te krijgen van dergelijke GW-signalen uit het vroege universum. In dit proefschrift richten we onze aandacht op vacuüm-GW's: vacuümtensor-kwantumfluctuaties die onvermijdelijk de vorm aannemen van stochastische achtergronden. Het is belangrijk om voorzichtig te zijn bij het onderscheiden van vacuüm-GW's van oorspronkelijke zwaartekrachtgolven die afkomstig zijn van dynamische processen waarbij energie- en momentumoverdracht betrokken zijn, zoals GW's van faseovergangen, deeltjesproductie of het verval van topologische defecten. Bij dit laatste gaat het om het voortplanten van gravitonen die afkomstig zijn van bepaalde fysische processen, terwijl vacuüm-GW's per definitie vacuümpolarisatie-effecten zijn waarvan de afdruk op verschillende waarneembare zaken niet minder reëel is, maar gepaard gaat met belangrijke subtiliteiten die het werk dat in dit proefschrift wordt gepresenteerd, motiveren.

Om ons werk te contextualiseren, hebben we in de **Inleiding** de thermische geschiedenis van het universum besproken, de fundamenten van het kosmologische model uitgelegd en de noodzaak gemotiveerd om vacuümkwantumfluctuaties in de vroege stadia van ons universum op te nemen. Daarna verlegden we onze aandacht naar het introduceren van de tools die nodig zijn om de bijdrage van vacuümkwantumfluctuaties te kwantificeren. We hebben de basisprincipes van de renormalisatieprocedure besproken als een methode om op zinvolle wijze UV-verschillen te reabsorberen die ontstaan bij het berekenen van kwantumcorrecties. We merkten de mogelijkheid op om uiteenlopende vacuümenergieën te verkrijgen en benadrukten dat de opsplitsing in klassieke grootheden en kwantumcorrecties misleidend kan zijn: men heeft toegang tot volledig "dressed" waarneembare gegevens en kan kwantumcorrecties en klassieke grootheden niet afzonderlijk meten.

Kunnen vacuüm-GW's worden beperkt door N_{eff}-grenzen? Dit is de vraag waarmee ons werk is begonnen. **Hoofdstuk 2** is een gedetailleerd overzicht van de typische uitdrukkingen die men in de literatuur kan vinden om de energiedichtheid van vacuüm GWs te verbinden met de grenzen aan het effectieve aantal relativistische soorten, Neff, ten tijde van de oerknalnucleosynthese. We sloten het hoofdstuk af door erop te wijzen dat de vraag of men vacuüm-GW's zinvol kan beperken met N_{eff}-grenzen intrinsiek gebonden is aan de definitie van de spanningsenergietensor van GW's en aan de manier waarop de verschillen die onvermijdelijk ontstaan bij het berekenen van de energiedichtheid van vacuüm-GW's opnieuw worden genormaliseerd. Dit laatste voorbehoud motiveerde het werk gepresenteerd in **Hoofdstuk 3**, waar we gevestigde renormalisatietechnieken toepasten op het geval van een massaloos, niet-interacterend scalair veld op een Friedmann–Lemaître–Robertson–Walker achtergrond. We hebben laten zien hoe dimensionale regularisatie kan worden gebruikt om UV-divergenties uit schaalvrije integralen te extraheren, de onafhankelijkheid van logaritmische divergenties van de keuze van de regularisatiemethode en de noodzaak om regularisatiemethoden te gebruiken die covariantie behouden om tegentermen te vinden die proportioneel zijn aan geometrische invarianten. Door bovendien een achtergrondevolutie te bestuderen die na een pre-inflatoir stralingsgedomineerd-tijdperk overgaat naar een zuivere de Sitter-fase, gevolgd door een tweede post-inflatoir stralingsgedomineerd-tijdperk, hebben we expliciet aangetoond dat IR/UV-schalen die verband houden met het begin/einde van de inflatie UV-divergenties niet genezen en dat IR- divergenties een artefact zijn van de idealisering van een oneindige de Sitter-geometrie uit het verleden. Door de lessen uit het ogenschijnlijk eenvoudige voorbeeld van een scalair veld toe te passen op het geval van vacuüm-GW's, hebben we in **Hoofdstuk 4** de vraag beantwoord of N_{eff}-grenzen kunnen worden gebruikt om beperkingen op vacuüm-GW's af te leiden. We zijn begonnen met het afleiden van een formule voor de energiedichtheid van GW's die geschikt is voor regularisatie, we hebben de kanttekeningen bij de definitie van de spanningsenergietensor van GW's aangepakt en een verbeterde formule afgeleid die niet afhankelijk is van eerdere schaalscheiding. Vervolgens volgden we de renormalisatieprocedure, we isoleerden de divergente structuur en nadat we de verschillen in consistent gedefinieerde tegentermen hadden afgetrokken, gaven we commentaar op de renormalisatievoorwaarden die moeten worden opgelegd om de schema-afhankelijke eindige rest te fixeren en een betekenisvol antwoord te verkrijgen. In het tweede deel van het proefschrift hebben we de studie van vacuüm-GW's in een covariante formulering opnieuw bekeken. Om dit te doen, hebben we in **Hoofdstuk 5** het formalisme geïntroduceerd dat nodig is om de studie van vacuüm-GW's als een spin-2-deeltje op gekromde ruimtetijd aan te pakken. We hebben dergelijke tools vervolgens toegepast in **Hoofdstuk 6**, waar de gerenormaliseerde spanningsenergietensor voor GW's wordt verkregen uit de variatie van de effectieve actie ten opzichte van de achtergrondmetriek. We hebben de effectieve actie voor GW's afgeleid, de divergenties geregulariseerd en consequent afgetrokken door tegentermen toe te voegen op het niveau van de actie. Ten slotte hebben we ons anderszins algemene resultaat gespecialiseerd op een stralingsgedomineerd-achtergrond om de renormalisatieomstandigheden vast te stellen, verbinding te maken met de foliatie-afhankelijke afleiding en commentaar te geven op de gevolgen voor de N_{eff}-grenzen.

Onder de resultaten die we hebben gepresenteerd bij het aanpakken van de renormalisatie van divergenties in oorspronkelijke waarneembare waarden, zowel in een covariante als in een foliatie-specifieke formulering, concluderen we dat N_{eff}-grenzen niet kunnen worden gebruikt om vacuüm-GW's te beperken. Een dergelijke beperking vertegenwoordigt een poging om de absolute waarde van kwantumcorrecties te berekenen en te meten. Dit laatste kan per definitie niet los van de klassieke achtergrond worden gemeten, aangezien alle fysieke waarnemingen noodzakelijkerwijs uit volledig "dressed" hoeveelheden bestaan.

List of Publications

- *An Étude on the Regularization and Renormalization of Divergences in Primordial Observables*
  Anna Negro, and Subodh P. Patil,
  e-Print: 2402.10008, *Riv.Nuovo Cim.* 47 (2024) 3, 179-228.

- *Hadamard Regularization of the Graviton Stress Tensor*
  Anna Negro, and Subodh P. Patil,
  e-Print: 2403.16806, submitted to *Phys.Rev.D.*

Curriculum Vitae

I was born on June 29, 1996, in Treviso, Italy. Towards the end of high school, where my efforts were mainly focused on sports, I realized I wanted to become a researcher. This goal, combined with my growing fascination for physics, led me to move to Padova in 2015 to I start my bachelor in physics. Despite starting my studies with the intention of becoming an experimental particle physicist, I realized, thanks to my bachelor thesis under the supervision of Prof. Dr. S. Matarrese, that I wanted to further investigate cosmology during my master's studies. Consequently, in 2018, I enrolled in the theoretical physics master's program at the University of Padova. During the first semester of my master's, I was captivated by general relativity and intrigued by quantum field theory, which led me to choose a curriculum in the interplay of high energy physics and cosmology, drifting towards a more and more theoretical approach. I spent the second year of my master's as an Erasmus student in Paris, where I had the opportunity to take courses at École Polytechnique and work with Dr. Y. Akrami on my master's project. This first experience outside Italy sparked my curiosity to live in different countries and come in contact with diverse environments.

Soon after my graduation, in October 2020, I started my PhD at the Lorentz Institute of Leiden University, under the supervision of Dr. S. P. Patil. During my PhD, I not only had the opportunity to continue studying and start working on challenging and fascinating topics, such as quantum field theory in curved spacetime, effective field theory approaches and renormalization techniques, but also to visit, give talks, and attend PhD schools in the Netherlands, Italy, Spain, France, Canada, the United Kingdom and Germany. Furthermore, I was the teaching assistant for the courses *General Relativity* and *Effective Field Theory* and organized the *Cosmology Seminars*.

In the fall of 2024 I will start my postdoc at Case Western Reserve University, Cleveland, Ohio.

Acknowledgments

I would like to express my gratitude to my supervisor, Subodh for encouraging me to step out of my comfort zone and become a more well-rounded physicist. Furthermore, thank you for the opportunities you gave me to travel and meet other physicists; these experiences have been a fundamental drive to pursue an academic career.

I would like to thank Ana Achúcarro, Koenraad Shalm, and Alessandra Silvestri for your support and guidance. I enjoyed our scientific discussions and am grateful for your input regarding my career.

My experience at the Lorenz Institute (or more generally at the University) would not have been the same without the many people who made the university a warm environment. Thanks to Alice, Fabrizio, Aravindh, Guadalupe and the soft matter group for welcoming me to Leiden and adopting me during Covid. Thank you also to the "theory corridor" for being friendly and willing to share cakes at the coffee break. Furthermore, thanks to Norman and all the measurement hall people, who made the evenings at the university (and not only) always fun.

I would like to thank the groups that hosted me for my talks, as well as the researchers I met during the PhD schools and conferences I attended or who visited the Lorentz Institute. Thank you for the interesting discussions and for making me appreciate the academic community.
Furthermore, I would like to thank Cliff Burgess; I am grateful for the opportunity to collaborate with you and spend time at PI, which reminded me how fun physics can be. Thank you, Yashar Akrami, for helping me during the most delicate periods of my career, toward the end of both my master's and my PhD. I am looking forward to (re)starting our collaboration.

Thank you to my parents for encouraging me to pursue my career. Thanks for helping me build the strength to challenge myself day by day, making it possible for me to achieve my goals and find my path. I would also like to thank my brother for being an example to me.

Kalli, George, Mouraya, and Laura, it is hard to choose what to thank you for, as I believe that your support has been fundamental in so many ways. Thank you, Laura, for helping me with the cover of this thesis, and to all my friends outside the

physics world, your presence was fundamental in perceiving these years in Leiden as more than just a step in my career. Thank you, of course, to the bouldering people for encouraging me to be taller and stronger. I would like to express my gratitude to Fred, who supported me day by day for most of my path towards the PhD.
In addition, I want to deeply thank my friends scattered across Europe. Thank you for continuing to be part of my life, whether by visiting me or calling me. Isotta and Greta, it is difficult to express how grateful I am to know that despite my long absences, you make time in your schedules to meet my last-minute plans of "I will be here this day at this time."
Finally, I would like to thank Dimitris for always patiently helping me find the key.