The wild Brauer-Manin obstruction on K3 surfaces

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STELLINGEN

behorende bij het proefschrift

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1. Let $V \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the projective K3 surface defined by the equation
$$x^3y + y^3z + z^3w + w^3x + xyzw = 0.$$
The class of the quaternion algebra
$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x}\right) \in \mathrm{Br}\ \mathbb{Q}(V)$$
defines an element in $\mathrm{Br}(V)$. The evaluation map $\mathrm{ev}_{\mathcal{A}}\colon V(\mathbb{Q}_2) \to \mathrm{Br}(\mathbb{Q}_2)$ is non-constant, and therefore gives an obstruction to weak approximation on V. Finally, $V(\mathbb{Q})$ is not dense in $V(\mathbb{Q}_2)$, with respect to the analytic topology.

2. Let V be a smooth, proper and geometrically integral k-variety and $\mathcal{V}$ an $\mathcal{O}_k$-model of V. Let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Assume that the special fibre $\mathcal{V}(\mathfrak{p})$ has no non-trivial global 1-forms, $\mathrm{H}^1(\overline{\mathcal{V}(\mathfrak{p})}, \mathbb{Z}/p\mathbb{Z}) = 0$ and $(p-1) \nmid e_{\mathfrak{p}}$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.

3. Let V be K3 surface over a number field k and $\mathfrak{p}$ be a prime of good ordinary reduction for V. Then there exists a finite field extension k'/k and an element $\mathcal{A} \in \mathrm{Br}(V_{k'})[p]$ that obstructs weak approximation on $V_{k'}$.

4. Let V be a K3 surface and $\mathfrak{p}$ be a prime of good non-ordinary reduction for V with $e_{\mathfrak{p}} \leq (p-1)$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.

5. Set $X = \mathrm{Kum}(A)$, where $A = E_1 \times E_2$, with E_1, E_2 two elliptic curves over $\mathbb{Q}$ with good ordinary reduction at 2 and full 2-torsion defined over $\mathbb{Q}_2$; then every element in $\mathrm{Br}(X)[2]$ has constant evaluation map on $X(\mathbb{Q}_2)$.

6. Let $A = E \times E$, where E is the elliptic curve given by the minimal Weierstrass equation
$$y^2 + xy + y = x^3 - 7x + 5.$$
Let V be the corresponding Kummer K3 surface over $\mathbb{Q}$. Then 2 is a prime of good ordinary reduction for V that does not play a role in the Brauer–Manin obstruction to weak approximation.

7. Let V be the K3 surface over $k := \mathbb{Q}(\sqrt{2})$ defined by the equation
$$x^3y + y^3z + z^3w - w^4 + 2xyzw - \sqrt{2}xzw^2 = 0. \tag{1}$$
The class of the quaternion algebra
$$\mathcal{A} := \left(\frac{z^2 + 2xy}{z^2}, -\frac{z}{x}\right) \in \mathrm{Br}\ k(V)$$

lies in $\mathrm{Br}(V)[2]$. Let $\mathfrak{p}$ be the prime above 2 in $\mathcal{O}_k$, then the evaluation map attached to $\mathcal{A}$

$$\mathrm{ev}_{\mathcal{A}} \colon X(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$$

is non-constant.

8. Set $L = \mathbb{Q}_3(\zeta)$ with ζ primitive 3-root of unity. We define X to be the Kummer K3 surface over L attached to the abelian surface $A = E \times E$, with E the elliptic curve over L defined by the Weierstrass equation

$$y^2 = x^3 + 4x^2 + 3x + 1.$$

Then there exists $\mathcal{A} \in \mathrm{Br}(X)[3]$ such that $\mathrm{ev}_{\mathcal{A}} \colon X(L) \to \mathbb{Q}/\mathbb{Z}$ is non-constant.

9. Progress and change are always positive. But there is an exception to every rule: Snellius was a better building for the mathematical department of Leiden than Gorlaeus.

10. According to some parameters, colour analysis can be considered closer to being a science than mathematics.

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