The wild Brauer-Manin obstruction on K3 surfaces

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Summary

This thesis is on the study of rational points on varieties. This kind of problem stems from the desire to be able to describe the rational solutions of a polynomial, i.e. given a polynomial $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, what can we say about

$$\mathrm{Z}(f)(\mathbb{Q}) := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n \mid f(\alpha_1, \ldots, \alpha_n) = 0\}?$$

A first way to study this set is by looking at the polynomial equation over the real numbers. In fact, if there is no $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $f(\alpha_1, \ldots, \alpha_n) = 0$ then $\mathrm{Z}(f)(\mathbb{Q})$ is also the empty set. The real numbers are a complete field, which makes the study of the zeros of functions defined over it much more accessible. However, $\mathbb{R}$ is just one of the possible completions of $\mathbb{Q}$, the one with respect to the euclidean metric. The p-adic metrics give for every prime p a complete field $\mathbb{Q}_p$ that contains $\mathbb{Q}$. Putting all this together gives a natural inclusion

$$\mathrm{Z}(f)(\mathbb{Q}) \hookrightarrow \prod_{p \text{ prime}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R}).$$

Moreover, the sets $\mathrm{Z}(f)(\mathbb{Q}_p)$ and $\mathrm{Z}(f)(\mathbb{R})$ are subsets of $\mathbb{Q}_p^n$ and $\mathbb{R}^n$, hence they inherit a topology coming from the topology on $\mathbb{Q}_p$ and $\mathbb{R}$ respectively. I am particularly interested in understanding the closure of the set $\mathrm{Z}(f)(\mathbb{Q})$ inside the product $\prod_{p \text{ prime}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R})$.

In the language of algebraic geometry, the zero set of the polynomial f defines a variety X over the rational numbers. The biggest advantage of looking at the image of $X(\mathbb{Q})$ inside $\prod_{p \text{ prime}} X(\mathbb{Q}_p) \times X(\mathbb{R})$ is that the latter is more accessible. In fact, both over $\mathbb{R}$ and over $\mathbb{Q}_p$, we have Newton's methods that allows to build solutions to polynomial equations. Moreover, if $X(\mathbb{Q})$ is dense in $X(\mathbb{Q}_p)$ for a given prime p, then combining this with Hensel's Lemma we get that we can lift solutions modulo p to solutions over the rational numbers. For example in the case of $\mathrm{Z}(f)$, with $f \in \mathbb{Z}[x_1, \ldots, x_n]$ a solution modulo p is an n-uple $(\beta_1, \ldots, \beta_n) \in (\mathbb{Z}/p\mathbb{Z})^n$ such that

$$f(\beta_1, \ldots, \beta_n) = 0 \mod p.$$

Unfortunately, very often it is too optimistic to hope for the density of $X(\mathbb{Q})$ in $\prod_{p \text{ prime}} X(\mathbb{Q}_p) \times X(\mathbb{R})$.

In this thesis we work with smooth and proper varieties over number fields. In this case, instead of the euclidean metric we have the archimedean places, while instead of p-adic metric we have the non-archimedean places associated to prime ideals in the ring of integers of k. We denote by Ω_k the set of places of k and for every place $\nu \in \Omega_k$, we denote by k_ν the completion of k at ν.

In 1970 Manin introduced the use of the Brauer group of a variety X, to build a subset $X(k_\Omega)^{\mathrm{Br}}$ of

$$\prod_{\nu \in \Omega_k} X(k_\nu) =: X(k_\Omega)$$

such that

$$X(k) \subseteq \overline{X(k)} \subseteq X(k_\Omega)^{\mathrm{Br}} \subseteq X(k_\Omega).$$

The key point is that this set gives a better approximation of $X(k)$ inside the product $X(k_\Omega)$ and even if its construction is more involved then the on of $X(k_\Omega)$, it is still more accessible then the set of rational points $X(k)$. Assume that the Brauer–Manin set is non-empty and that a prime $\mathfrak{p}$ is such that

$$X(k_\Omega)^{\mathrm{Br}} = X(k_\mathfrak{p}) \times Z$$

with $Z \subseteq \prod_{\nu \neq \mathfrak{p}} X(k_\nu)$. Then we say that the prime ideal $\mathfrak{p}$ **does not play a role** in the Brauer–Manin obstruction.

My research is inspired by the following question:

Question. *Assume* $\mathrm{Pic}(\bar{X})$ *to be torsion-free and finitely generated. Which primes can play a role in the Brauer–Manin obstruction to weak approximation on* X*?*

This question is inspired by a question originally asked by Swinnerton–Dyer; he asked whether under the assumption of question above the only places that can play a role in the Brauer–Manin obstruction to weak approximation are the one of bad reduction for the variety. Roughly speaking, if X is defined over the rational numbers, a prime p is of good reduction for X if we can define X by polynomial with integer coefficients whose reduction modulo p define a smooth variety over the finite field $\mathbb{F}_p$. The wish is to be able to identify all the primes that play a role in the Brauer–Manin obstruction to weak approximation in order to describe the Brauer–Manin set.

The second chapter of this thesis is devoted to exhibiting the first example of a K3 surface[1] defined over $\mathbb{Q}$ for which a prime of good reduction plays a role in the Brauer–Manin obstruction to weak approximation.

Theorem. *Let* $X \subseteq \mathbb{P}^3_{\mathbb{Q}}$ *be the projective* K3 *surface defined by the equation*

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0.$$

[1]K3 surfaces are varieties that satisfy the assumption of the question.

Then the prime 2 is of good reduction for X and plays a role in the Brauer–Manin obstruction to weak approximation.

At this point some natural questions arise:

1. In the theorem above the prime 2 is of good ordinary reduction. Is the ordinary condition needed?

2. What happens over number fields?

Answering these two questions is the aim of Chapter 3 and 4. In particular, in Chapter 3 I proved the following results.

Theorem. *Let X be a K3 surface and $\mathfrak{p}$ be a prime of good non-ordinary reduction for X with $e_{\mathfrak{p}} \leq (p-1)$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on X.*

In Chapter 4, I show that the bound $e_{\mathfrak{p}} \leq (p-1)$ appearing in the theorem above is optimal.

Theorem. *Let $\mathfrak{p}$ be a prime of good ordinary reduction for X of residue characteristic p. Assume that the special fibre Y has no non-trivial global 1-forms, $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$ and $(p-1) \nmid e_{\mathfrak{p}}$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on X.*

Moreover, in Chapter 4 I also show that the condition $(p-1) \nmid e_{\mathfrak{p}}$ in the last theorem is sufficient but not necessary.

All these results go in the direction of having a better understanding of the Brauer–Manin set and hence of the set of rational points on varieties. In my results there is an emphasis on K3 surfaces, which are one of the first kind of varieties (in terms of complexity of the geometry) for which very little is known about the arithmetic (i.e. the set of rational points).