The wild Brauer-Manin obstruction on K3 surfaces

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CHAPTER 4

Computations with the refined Swan conductor

This final chapter is divided in two parts. In the first part we develop some techniques needed to compute the refined Swan conductor of certain elements in the Brauer group. In particular, in Section 4.1, starting from a result from Bright and Newton, we prove a formula that relates the refined Swan conductor with the extension of the base field over which the variety is defined. In Section 4.2 we explain some results of Kato that allows to compute the refined Swan conductor of p-order elements when the base field contains a primitive p-root of unity. In the second part of this final chapter we provide several examples and we use them to show that Theorem 3.1.1 and 3.2.1, are optimal. In particular, we exhibit K3 surfaces V over number fields such that:

(a) V has good ordinary reduction at a prime $\mathfrak{p}$ with ramification index $e_{\mathfrak{p}} = p-1$ and there is an element $\mathcal{A} \in \mathrm{Br}(V)[p]$ whose evaluation map is non-constant on $V(k_{\mathfrak{p}})$;

(b) V has good ordinary reduction at a prime $\mathfrak{p}$ with $e_{\mathfrak{p}} = p-1$ and $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation;

(c) V has good non-ordinary reduction at a prime $\mathfrak{p}$ with $e_{\mathfrak{p}} \geq p$ and there is an element $\mathcal{A} \in \mathrm{Br}(V)[p]$ whose evaluation map is non-constant on $V(k_{\mathfrak{p}})$.

More precisely: from (a) we get that the condition $(p-1) \nmid e_{\mathfrak{p}}$ in Theorem 3.1.1 is necessary; from (b) we get that the inverse of Theorem 3.1.1 does not hold in general; from (c) we get that the bound in Theorem 3.2.1 is optimal.

Finally, as already pointed out in the introduction, we recall that in all the examples of K3 surfaces in which a prime of good reduction plays a role in the Brauer–Manin obstruction of weak approximation, the corresponding element in the Brauer group is of **transcendental** nature, i.e. it does not belong to the algebraic Brauer group, which is defined as the kernel of the natural map from $\mathrm{Br}(V)$ to $\mathrm{Br}(\bar{V})$, where $\bar{V}$ is the base change of V to an algebraic closure of k (cf. Lemma 4.3.4).

Moreover, we prove that if V is a Kummer K3 surface coming from a product of elliptic curves defined over $\mathbb{Q}$ with good ordinary reduction at the prime 2 and full 2-torsion defined over $\mathbb{Q}_2$, then $\mathrm{Br}(V)[2] = \mathrm{Ev}_{-1}\mathrm{Br}(V)[2]$ (cf. Theorem 4.5.6). This theorem proves what was already predict by Ieronymou after some computational evidence, see [Ier23, Remark 2.6].

4.1 Refined Swan conductor and extension of the base field

Let L be a p-adic field with ring of integers $\mathcal{O}_L$, uniformiser π and residue field ℓ. Let X be a proper, smooth and geometrically integral L-variety having a smooth, proper model $\mathcal{X}$ with geometrically integral fibre. We denote by Y its special fibre:

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) & \longleftarrow & \mathrm{Spec}(\ell) \end{array} \tag{4.1}$$

In this section we want to analyse what happens to the refined Swan conductor when we take a field extension L'/L of the base field L. Bright and Newton prove the following result.

Lemma 4.1.1. *Let K'/K be a finite extension of Henselian discrete valuation fields of ramification index e. Let π' be a uniformiser in K', F' be the residue field of K' and define $\bar{a} \in F'$ to be the reduction of $\pi(\pi')^{-e}$. Let $\chi \in \mathrm{fil}_n\mathrm{Br}(K)$, and let*

$$\mathrm{res}\colon \mathrm{Br}(K) \to \mathrm{Br}(K')$$

be the restriction map. Then $\mathrm{res}(\chi) \in \mathrm{fil}_{en}\mathrm{Br}(K')$ and if $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta)$, then

$$\mathrm{rsw}_{en,\pi'}(\mathrm{res}(\chi)) = (\bar{a}^{-n}(\alpha + \beta \wedge d\log(\bar{a}), \bar{a}^{-n}e\beta).$$

Proof. See [BN23, Lemma 2.16]. □

The aim of this section is to use this result to prove the following Lemma.

Lemma 4.1.2 (Base change)**.** *Let L'/L be a finite field extension, with ramification index $e_{L'/L}$. Let π' be an uniformiser in L' and ℓ' its residue field. Let $\mathcal{A} \in \mathrm{Br}(X)$ and let*

$$\mathrm{res}\colon \mathrm{Br}(X) \to \mathrm{Br}(X_{L'})$$

be the restriction map. Then $\mathrm{res}(\mathcal{A}) \in \mathrm{fil}_{e_{L'/L}n}\mathrm{Br}(X_{L'})$ *and if* $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$ *with* $(\alpha, \beta) \in \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y)$*, then*

$$\mathrm{rsw}_{e_{L'/L}n,\pi'}(\mathrm{res}(\mathcal{A})) = (\bar{a}^{-n}\alpha, \bar{a}^{-n}e_{L'/L}\beta) \in \mathrm{H}^0(Y_{\ell'}, \Omega^2_{Y_{\ell'}}) \oplus \mathrm{H}^0(Y_{\ell'}, \Omega^1_{Y_{\ell'}})$$

with $\bar{a} \in \ell'$ *reduction of* $\pi(\pi')^{-e_{L'/L}}$*.*

The refined Swan conductor of an element $\mathcal{A} \in \mathrm{Br}(X)$ is defined through the refined Swan conductor of its image in the discrete henselian valuation field K^h. Namely, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{fil}_n\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \\ \downarrow & & \downarrow \\ \mathrm{fil}_n\mathrm{Br}(K^h) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \Omega^2_F \oplus \Omega^1_F. \end{array}$$

We recall the construction of K^h: let η be the generic point of $Y \subseteq \mathcal{X}$, then we define R as henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},\eta}$ and K^h as the fraction field of R. The construction of $\mathcal{O}_{\mathcal{X},\eta}$ (and hence of K^h) is local on $\mathcal{X}$. From now on we will therefore assume $\mathcal{X} = \mathrm{Spec}(A)$, with A smooth $\mathcal{O}_L$-algebra, $Y = \mathrm{Spec}(A/\pi A)$; hence $\eta = (\pi) \in \mathrm{Spec}(A)$ and $\mathcal{O}_{\mathcal{X},\eta} = A_{(\pi)}$. We can re-write diagram (4.1) as:

$$\begin{array}{ccccc} A \otimes_{\mathcal{O}_L} L & \longleftarrow & A & \longrightarrow & A/\pi A \\ \uparrow & & \uparrow & & \uparrow \\ L & \longleftarrow & \mathcal{O}_L & \longrightarrow & \ell. \end{array} \tag{4.2}$$

Lemma 4.1.3. *The uniformiser* π *is also a uniformiser for* K^h*. Moreover,* $\mathrm{ord}_{K^h}(p) = \mathrm{ord}_L(p)$*.*

Proof. The uniformiser π is also the generator of the maximal ideal of $A_{(\pi)}$, hence of its henselisation R. The equality $(p) = (\pi)^e$ as ideals on $\mathcal{O}_L$ implies that $p \in (\pi)^e R$, hence $e_1 := \mathrm{ord}_{K^h}(p) \leq e$. The equality between the two orders follows from the fact that for every $m \geq 1$, $(\pi)^m R \cap \mathcal{O}_L = (\pi)^m \mathcal{O}_L$. □

We denote by L' a finite field extension of L, by $\mathcal{O}_{L'}$ its ring of integers with uniformiser π' and residue field ℓ'. Moreover, we denote by X', $\mathcal{X}'$ and Y' the base change of X, $\mathcal{X}$ and Y to $\mathrm{Spec}(L'), \mathrm{Spec}(\mathcal{O}_{L'})$ and $\mathrm{Spec}(\ell')$ respectively. Let $(K')^h$ be the fraction field of R', where R' is the henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X}',Y'}$. In this setting

$$\frac{A}{\pi A} \otimes_\ell \ell' = A \otimes_{\mathcal{O}_L} \ell \otimes_\ell \ell' = A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'} \otimes_{\mathcal{O}_{L'}} \ell' = \frac{A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}}{(1 \otimes \pi')}.$$

Thus, the generic point η' of Y' is the ideal generated by $1 \otimes \pi'$ and $\mathcal{O}_{\mathcal{X}',Y'}$ becomes the ring $(A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'})_{(1\otimes\pi')}$.

Lemma 4.1.4. *The field extension* $(K')^h/K^h$ *is finite with ramification index* $e_{L'/L}$*.*

Proof. We start by noticing that

$$\mathcal{O}_{\mathcal{X}',\eta'} \simeq \mathcal{O}_{\mathcal{X},\eta} \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}.$$

We have that $\mathcal{O}_{\mathcal{X},\eta} \otimes_{\mathcal{O}_L} \mathcal{O}_{L'} = S^{-1}(A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'})$, with $S = (A \setminus (\pi)) \cdot A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$, while $\mathcal{O}_{\mathcal{X}',\eta'} = T^{-1}(A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'})$, with $T = (A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}) \setminus (1 \otimes \pi')$. The isomorphism follows from the equality $(1 \otimes \pi')^{e_{L'/L}} = (1 \otimes \pi)$ together with the fact that $\mathcal{O}_{L'}$ is a free $\mathcal{O}_L$-module with basis $\{1, \pi', \dots, (\pi')^{e_{L'/L}}\}$.

As a second step we show that

$$R' \simeq R \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}.$$

The discrete valuation ring $\mathcal{O}_{L'}$ is a finite $\mathcal{O}_L$-module, hence we get that the natural map

$$R \to R \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$$

is finite; therefore [Sta, 05WS] implies that $R \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$ is henselian and therefore by [Sta, 05WP]

$$R' = R \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}.$$

As a final step, we notice that

$$(K')^h = R'\left[\frac{1}{\pi'}\right] = R'\left[\frac{1}{\pi}\right] = R\left[\frac{1}{\pi}\right] \otimes_L L' = K^h \otimes_L L'.$$

□

Proof of Lemma 4.1.2. It follows immediately from the previous lemma together with Lemma 4.1.1 and the fact that since $\bar{a} \in \ell'$, which is a finite field, $d\log(\bar{a}) = 0$. □

Corollary 4.1.5. *Assume that $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ for some $n \geq 1$ is such that $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$ with $\alpha \neq 0$, then $\mathcal{A} \notin \mathrm{Br}_1(X)$, i.e. $\mathcal{A}$ is a transcendental element in the Brauer group of X.*

Proof. Assume $\mathcal{A}$ to be in $\mathrm{Br}_1(X)$; then by definition of $\mathrm{Br}_1(X)$ there is a finite field extension L'/L such that $\mathrm{res}(\mathcal{A}) = 0$ in $\mathrm{Br}(X_{L'})$, where res is the restriction map from $\mathrm{Br}(X)$ to $\mathrm{Br}(X_{L'})$. Let $e_{L'/L}$ be the ramification index of the extension, π' be a uniformiser of L' and ℓ' its residue field. We know from Lemma 4.1.2 that

$$\mathrm{rsw}_{e(L'/L)n,\pi_{L'}}(\mathrm{res}(\mathcal{A})) = (\bar{a}^{-n} \cdot \alpha, \bar{a}^{-n} e_{L'/L} \cdot \beta)$$

where $\bar{a}^{-n} \in (\ell')^{\times}$. Hence, $\mathrm{rsw}_{e(L'/L)n,\pi_{L'}}(\mathrm{res}(\mathcal{A})) \neq (0,0)$ and therefore $\mathrm{res}(\mathcal{A})$ can not be the trivial element. □

4.2 Computations with the Refined Swan conductor on the p-torsion of the Brauer group

The aim of this section is to collect some results that will allow us to compute later in this chapter the Swan conductor and the refined Swan conductor of elements of order p in the Brauer group of K3 surfaces. In this section we work under the same setting as Section 1.2.3: K is a henselian field of characteristic 0 with ring of integers $\mathcal{O}_K$, uniformiser π and residue field F of positive characteristic p.

We work under the additional assumption that the field K contains a primitive p-root of unity ζ. It follows from the Merkurjev-Suslin Theorem [GS17, Theorem 8.6.5] that in this case $\mathrm{Br}(K)[p]$ is generated by the classes of **cyclic algebras**. In order to do computations with the refined Swan conductor, in this section we will introduce a new filtration on $\mathrm{Br}(K)[p]$, defined by Bloch and Kato in [BK86] and prove that via this filtration it is possible to compute the refined Swan conductor of cyclic algebras in $\mathrm{Br}(K)[p]$.

4.2.1 Cyclic algebras

Let $a, b \in K^\times$, then the K-algebra $(x, y)_p$ defined as

$$(x, y)_p := \langle a, b \mid a^p = x,\, b^p = y,\, ab = \zeta ba \rangle,$$

is a central simple algebra, see [GS17, Section 2.5] for more details. With abuse of notation we will denote by $(x, y)_p$ also the corresponding equivalence class in $\mathrm{Br}(K)[p]$. It is possible to realise $(x, y)_p$ also as the cup product of an element $\chi_x \in \mathrm{H}^1_p(K)$ with $\delta(y)$, where δ is the boundary map $K^\times \to \mathrm{H}^1(K, \mathbb{Z}/p\mathbb{Z}(1))$ coming from the Kummer sequence (a proof can be found in the proof of Proposition 4.7.1 [GS17]).

4.2.2 Another filtration

The map from $\mathbb{Z}/p\mathbb{Z}(1)$ to $\mathbb{Z}/p\mathbb{Z}(2)$ sending 1 to ζ induces an isomorphism

$$\mathrm{Br}(K)[p] \simeq \mathrm{H}^2\left(K, \mathbb{Z}/p\mathbb{Z}(2)\right) =: h^2(K). \tag{4.3}$$

For any two non-zero elements $x, y \in K$ we will denote by $\{x, y\} \in h^2(K)$ the cup product of $\delta(x)$ with $\delta(y)$.

Bloch and Kato [BK86] define a decreasing filtration $\{U^m h^2(K)\}_{m \geq 0}$ on $h^2(K)$ as follows: $U^0 h^2(K) = h^2(K)$ and for $m \geq 1$, $U^m h^2(K)$ is the subgroup of $h^2(K)$ generated by symbols of the form

$$\{1 + \pi^m x, y\}, \text{ with } x \in \mathcal{O}_K \text{ and } y \in K^\times.$$

In this section we are going to prove that for any $0 \leq m \leq e'$ the isomorphism (4.3) induces an isomorphism

$$U^m h^2(K) \simeq \mathrm{fil}_{e'-m} \mathrm{Br}(K)[p].$$

This isomorphism will be crucial in being able to compute the refined Swan conductor. In fact, in [BK86] Bloch and Kato describe the graded pieces of the filtration $\{U^m h^2(K^h)\}_{m\geq 0}$ on $h^2(K)$

$$\mathrm{gr}^m := \frac{U^m h^2(K^h)}{U^{m+1} h^2(K^h)}$$

in terms of differential forms on the residue field F. In [Kat89] Kato strongly relate them to the computation of the refined Swan conductor. We now state the two main results that show how it is possible to calculate the refined Swan conductor of elements of order p in $\mathrm{Br}(K)$.

Proposition 4.2.1. *We have the following description of the graded pieces* gr^m.

(1) $U^m h^2(K^h) = \{0\}$ *for* $m > e'$; $U^{e'} h^2(K)$ *coincides with the image of the injective map*

$$\begin{aligned} \lambda_\pi \colon \mathrm{H}^2_p(F) \oplus \mathrm{H}^1_p(F) &\to \mathrm{gr}^{e'} = h^2(K) \\ \delta_1 [\bar{x} \cdot d\log \bar{y}] &\mapsto \{1 + (\zeta - 1)^p x, y\} \\ \delta_1 [\bar{x}] &\mapsto \{1 + (\zeta - 1)^p x, \pi\} \end{aligned}$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K.

(2) Let $0 < m < e'$ *and* $p \nmid m$. *Then we have an isomorphism*

$$\begin{aligned} \rho_m \colon \Omega^1_F &\xrightarrow{\simeq} \mathrm{gr}^m \\ \bar{x} \cdot d\log \bar{y} &\mapsto \{1 + \pi^m x, y\} \end{aligned}$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K.

(3) Let $0 < m < e'$ *with* $p \mid m$. *Then we have an isomorphism*

$$\begin{aligned} \rho_m \colon \Omega^1_F / Z^1_F \oplus \Omega^0_F / Z^0_F &\xrightarrow{\simeq} \mathrm{gr}^m \\ ([\bar{x} \cdot d\log \bar{y}], 0) &\mapsto \{1 + \pi^m x, y\} \\ (0, [\bar{x}]) &\mapsto \{1 + \pi^m x, \pi\} \end{aligned}$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K.

(4) We have an isomorphism

$$\begin{aligned} \rho_0 \colon \Omega^2_{F,\log} \oplus \Omega^1_{F,\log} &\xrightarrow{\simeq} \mathrm{gr}^0 \\ (d\log \bar{y}_1 \wedge d\log \bar{y}_2, 0) &\mapsto \{y_1, y_2\} \\ (0, d\log \bar{y}) &\mapsto \{y, \pi\} \end{aligned}$$

where y, y_1 *and* y_2 *are any lifts of* $\bar{y}, \bar{y}_1$ *and* $\bar{y}_2$ *to* K.

Proof. See [BK86, Section 5]. More precisely, in [BK86, Lemma 5.1] Bloch and Kato prove 4.2.1.(1). However, instead of λ_π (cf. Section 1.2.2.1) they have the morphism $\rho_{e'}$, and they just prove afterwards [BK86, equation (5.15.1)] that $\rho_{e'}$ induces the map λ_π. Finally, in [BK86, Lemma 5.2 and 5.3] the rest of the proposition is proven. □

Note that since by assumption K contains a primitive p-root of unity ζ, the ramification index $e = \mathrm{ord}_K(p)$ is divisible by $(p-1)$ and hence $e' = ep(p-1)^{-1}$ is divisible by p. The key result of this section is the following proposition.

Proposition 4.2.2. *For every $0 \le m \le e'$ we have that the isomorphism 4.3 induces an isomorphism*

$$\mathrm{fil}_{e'-m}\mathrm{Br}(K)[p] \simeq U^m h^2(K).$$

Let $\bar{c}$ be the reduction modulo π of $c := \pi^{-e'}(\zeta-1)^p$. For $m < e'$ the compositions $\mathrm{rsw}_{e'-m} \circ \rho_m$ are as follows:

$$\begin{aligned}
&\mathrm{rsw}_{e',\pi}(\rho_0(\alpha,\beta)) = (\bar{c}\alpha, \bar{c}\beta) && \\
&\mathrm{rsw}_{e'-m,\pi}(\rho_m(\alpha)) = (\bar{c}d\alpha, (e'-m)\bar{c}\beta), && \textit{if } p \nmid m \\
&\mathrm{rsw}_{e'-m,\pi}(\rho_m(\alpha,\beta)) = (\bar{c}d\alpha, \bar{c}d\beta), && \textit{if } p \mid m
\end{aligned}$$

Warning: The proof of Proposition 4.2.2 is quite technical and will occupy the rest of this section.

4.2.3 Proof of Proposition 4.2.2

We divide the proof into several rather technical lemmas.

Lemma 4.2.3. *Let $a, b \in \mathcal{O}_K$ and n, m be non-negative integers, then the symbol $\{1+\pi^n a, 1+\pi^m b\}$ can be rewritten as*

$$-\left\{1+\pi^{n+m}\frac{ab}{1+\pi^n a}, 1+\pi^m b\right\} - \left\{1+\pi^{n+m}\frac{ab}{1+\pi^n a}, -\pi^n a\right\}.$$

In particular, it lies in $U^{m+n}h^2(K)$.

Proof. This lemma is a reformulation of a special case of [BK86, Lemma 4.1], for which no proof is provided. We have that

$$\begin{aligned}
&\{1+\pi^n a, 1+\pi^m b\} + \left\{1+\pi^{n+m}\frac{ab}{1+\pi^n a}, 1+\pi^m b\right\} = \\
&\{1+\pi^n a(1+\pi^m b), 1+\pi^m b\}.
\end{aligned}$$

We also have that,

$$\begin{aligned}
&\left\{1+\pi^n a(1+\pi^m b), 1+\pi^m b\right\} + \left\{1+\pi^n a(1+\pi^m b), -\pi^n a\right\} = \\
&\left\{1+\pi^n a(1+\pi^m b), -(1+\pi^n b)\pi^n a\right\} = 0
\end{aligned}$$

where the last equality follows from the fact that $\{x, y\} = 0$ if $x + y = 1$. Finally, since $\{1+\pi^n a, -\pi^n a\} = 0$, we have

$$\begin{aligned}
&\left\{1+\pi^n a(1+\pi^m b), -\pi^n a\right\} = \left\{1+\pi^n a(1+\pi^m b), -\pi^n a\right\} - \{1+\pi^n a, -\pi^n a\} = \\
&\left\{1+\pi^{n+m}\frac{ab}{1+\pi^n a}, -\pi^n a\right\}.
\end{aligned}$$

□

Lemma 4.2.4. *Let $x \in \mathcal{O}_K$, $y \in K^\times$ and n, m be two positive integers; then $\{(y, 1+\pi^m x)_p, 1+\pi^n T\}$ can be written as*

$$-\left(\chi_y \cup \left\{1+\pi^{n+m}\frac{Tx}{1+\pi^n x}, 1+\pi^n T\right\} + \chi_y \cup \left\{1+\pi^{n+m}\frac{Tx}{1+\pi^n x}, -\pi^m x\right\}\right).$$

Proof. As already anticipated at the beginning of this section, we can write $(y, 1+\pi^m x)_p$ as the cup product of $\chi_y \in \mathrm{H}^1_p(K)$ and $\delta(1+\pi^m x)$ with δ boundary map coming from the Kummer sequence, see [GS17, proof of Proposition 4.7.1]. Hence,

$$\{(y, 1+\pi^m x)_p, 1+\pi^n T\} = \chi_y \cup \{1+\pi^m x, 1+\pi^n T\}.$$

The result now follows from Lemma 4.2.3. □

Corollary 4.2.5. *If $m > 0$ and $m + n \geq e'$, we get that*

$$\{(y, 1+\pi^m x)_p, 1+\pi^n T\} = -\chi_y \cup \left\{1+\pi^{n+m}\frac{Tx}{1+\pi^n x}, -\pi^m x\right\}.$$

Proof. If follows from the previous lemma together with the fact that by Lemma 4.2.3

$$\left\{1+\pi^{n+m}\frac{Tx}{1+\pi^n x}, 1+\pi^n T\right\} \in U^{2m+n}h^2(K)$$

and $U^{2m+n}h^2(K) = 0$ from Proposition 4.2.1, since $2m + n > e'$. □

Proposition 4.2.6. *Let $e = ord_K(p)$ and $e' = ep(p-1)^{-1}$. For $0 \leq m \leq e'$ the isomorphism of equation* (4.3) *induces an inclusion*

$$U^m h^2(K) \subseteq \mathrm{fil}_{e'-m}\mathrm{Br}(K)[p].$$

Proof. This is [Kat89, Lemma 4.3(1)], we include a proof of it here. By definition $U^m h^2(K)$ is generated by symbols of the form $\{1+\pi^m x, y\}$, with $x \in \mathcal{O}_K$ and $y \in K^\times$. In order to show that the corresponding element in $\mathrm{Br}(K)[p]$ lies in $\mathrm{fil}_{e'-m}$ we need to check that

$$\{1+\pi^m x, y, 1+\pi^{e'-m+1}T\} = 0.$$

From Corollary 4.2.5 we have that $\{1+\pi^m x, y, 1+\pi^{e'-m+1}T\}$ can be written as

$$\chi_y \cup \left\{1+\pi^{e'+1}\frac{xT}{1+\pi^m x}, -x\pi^m\right\} = \left\{(y, -x\pi^m)_p, 1+\pi^{e'+1}\frac{xT}{1+\pi^m x}\right\}$$

and the latter is zero, since we know that $\mathrm{fil}_{e'}\mathrm{Br}(K)[p] = \mathrm{Br}(K)[p]$, see Section 1.2.3.1. □

We will now show how the inclusion appearing in Proposition 4.2.6 is an equality. We start by recalling the following properties that we proved in Section 1.2.3.1.

Properties 4.2.7. For any $m \geq e'$ we have $\mathrm{fil}_m\mathrm{Br}(K)[p] = \mathrm{Br}(K)[p]$. Moreover,

(1) If $p \nmid m$, then the map

$$\mathrm{fil}_m \mathrm{Br}(K)[p] \xrightarrow{\mathrm{rsw}_{m,\pi}} \Omega^2_F \oplus \Omega^1_F \xrightarrow{\mathrm{pr}_2} \Omega^1_F$$

has also kernel equal to $\mathrm{fil}_{m-1}\mathrm{Br}(K)[p]$.

(2) If $p \mid m$ and $m < e'$, then the map $\mathrm{rsw}_{m,\pi}$ takes values in $B^2_F \oplus B^1_F$.

(3) If $m = e'$, then the map $\mathrm{mult}_{\bar{c}^{-1}} \circ \mathrm{rsw}_{e',\pi}$ takes values in $\Omega^2_{F,\log} \oplus \Omega^1_{F,\log}$, with $\bar{c}$ the reduction modulo π of $\pi^{e'} \cdot (\zeta - 1)^{-p}$.

We proceed by induction on $m + 1$.

- For $m = 0$ we have by definition $U^0 h^2(k) = h^2(K)$, which implies that the inclusion $U^0 h^2(K) \subseteq \mathrm{fil}_{e'} \mathrm{Br}(K)[p]$ is indeed an equality.

- For $m = 1$ we have

$$\Omega^2_{F,\log} \oplus \Omega^1_{F,\log} \simeq \mathrm{gr}^0 h^2(K) \subseteq \mathrm{gr}_{e'} \mathrm{Br}(K)[p] \hookrightarrow \Omega^2_{F,\log} \oplus \Omega^1_{F,\log} \tag{4.4}$$

where the first isomorphism is induced by ρ_0, while the inclusion follows from property 4.2.7(3). Moreover, note that given $\alpha = d\log \bar{x} \wedge d\log \bar{y} \in \Omega^2_{F,\log}$, $\rho_0(\alpha, 0) = \{x, y\}$ and

$$\{x, y, 1 + \pi^{e'} T\} = \{x, y, 1 + (\zeta - 1)^p (cT)\} = \lambda_\pi \left(\delta_1 \left[\bar{c}T d\log \bar{x} \wedge d\log \bar{y}\right], 0\right).$$

Similarly, if we start with $\beta = d\log \bar{y} \in \Omega^1_{F,\log}$, $\rho_0(0, \beta) = \{y, \pi\}$ and

$$\{y, \pi, 1 + \pi^{e'} T\} = \{y, \pi, 1 + (\zeta - 1)^p (cT)\} = \lambda_\pi \left(0, \delta_1 \left[\bar{c}T d\log \bar{y}\right]\right).$$

Hence, $\mathrm{rsw}_{e',\pi}(\rho_0(\alpha, \beta)) = \bar{c}(\alpha, \beta)$ and therefore the chain of maps (4.4) is the identity and we have that the inclusion of $U^1 h^2(K)$ in $\mathrm{fil}_{e'-1}\mathrm{Br}(K)[p]$ is in fact an equality.

- Inductive step on $m + 1$ when $p \nmid m$ and $m < e'$. In this case we have

$$\Omega^1_F \simeq \mathrm{gr}^m \subseteq \mathrm{gr}_{e'-m} \hookrightarrow \Omega^1_F$$

where the first isomorphism is induced by ρ_m, while the inclusion is the restriction of the refined Swan conductor to Ω^1_F, which is injective because of property 4.2.7(1). In this case, given $\alpha = \bar{x} d\log \bar{y} \in \Omega^1_F$, $\rho_m(\alpha) = \{1 + \pi^m x, y\}$. From corollary 4.2.5

$$\{1 + \pi^m x, y, 1 + \pi^{e'-m} T\} = -\chi_y \cup \left\{1 + \pi^{e'} \frac{xT}{1 + \pi^m x}, -x\pi^m\right\}$$

The latter can be rewritten as

$$-\chi_y \cup \left\{1 + (\zeta - 1)^p \frac{cxT}{1 + \pi^m x}, x\right\} + m \cdot \chi_y \cup \left\{1 + (\zeta - 1)^p \frac{cxT}{1 + \pi^m x}, \pi\right\}.$$

which is equal (using isomorphism (4.3)) to

$$\left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x, y\right\} - m\left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, y, \pi\right\}. \qquad (4.5)$$

Note that $\frac{x}{1+\pi^m x}$ is a possible lift of $\bar{x} \in F$ to K. Hence, (4.5) is equal to

$$\lambda_\pi\left(\bar{c}\bar{T}\bar{x}d\log\bar{x}\wedge d\log\bar{y}, m\bar{c}\bar{T}\bar{x}d\log\bar{y}\right) = \lambda_\pi\left(\bar{c}\bar{T}d\alpha, m\bar{c}\bar{T}\alpha\right).$$

Thus, in this case the composition

$$\Omega^1_F \simeq \mathrm{gr}^m \subseteq \mathrm{gr}_{e'-m} \hookrightarrow \Omega^1_F$$

sends α to $m\bar{c}\alpha$ and therefore we get again the equality between $U^m h^2(K)$ and $\mathrm{fil}_{e'-m}\mathrm{Br}(K)[p]$. Finally, using that by induction hypothesis we have an isomorphism between $U^m h^2(K)$ and $\mathrm{fil}_{e'-m}\mathrm{Br}(K)[p]$, we get that $U^{m+1}h^2(K)$ is isomorphic to $\mathrm{fil}_{e'-(m+1)}\mathrm{Br}(K)[p]$.

- Inductive step on $m+1$ when $p \mid m$ and $m < e'$. In this case we have

$$\Omega^1_F/Z^1_F \oplus \Omega^0_F/Z^0_F \simeq \mathrm{gr}^{m-1} \hookrightarrow \mathrm{gr}_{e'-m} \hookrightarrow B^2_F \oplus B^1_F$$

where the first isomorphism is induced by ρ_m, while the inclusion comes from the refined Swan conductor property 4.2.7(2). Given $\alpha = [\bar{x}d\log\bar{y}]$ in Ω^1_F/Z^1_F, $\rho_m(\alpha, 0) = \{1+\pi^m x, y\}$ and again using an argument similar to the one used above

$$\begin{aligned}&\{1+\pi^m x, y, 1+\pi^{e'-m}T\}\\ &= \left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x, y\right\} - m\left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, y, \pi\right\}\\ &= \left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x, y\right\}\end{aligned}$$

where the last equality follows from the fact that $p \mid m$ and that we are working with groups of order p. Like before, note that $\frac{x}{1+\pi^m x}$ is a possible lift of $\bar{x} \in F$ to K. Hence,

$$\left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x, y\right\} = \lambda_\pi\left(\bar{c}\bar{T}d\alpha, 0\right)$$

With very similar computations it is possible to show that starting from $\beta = [\bar{x}] \in \Omega^0_F/Z^0_F$,

$$\{(1+\pi^m x, \pi)_p, 1+\pi^{e'-m}T\} = \lambda_\pi(0, \bar{c}\bar{T}d\beta)$$

Hence, in this case the composition

$$\Omega^1_F/Z^1_F \oplus \Omega^0_F/Z^0_F \simeq \mathrm{gr}^m \hookrightarrow \mathrm{gr}_{e'-m} \xhookrightarrow{\mathrm{rsw}_{e'-m,\pi}} B^2_F \oplus B^1_F$$

sends (α, β) to $(\bar{c}d\alpha, \bar{c}d\beta)$. Finally, using that by induction hypothesis we have an isomorphism between $U^m h^2(K)$ and $\mathrm{fil}_{e'-m}\mathrm{Br}(K)[p]$, we get an isomorphism between $U^{m+1}h^2(K)$ and $\mathrm{fil}_{e'-(m+1)}\mathrm{Br}(K)[p]$.

4.3 The case of K3 surfaces

In this thesis examples will always be about K3 surfaces with good reduction, cf. Section 2.1.3 for the definition and some properties of K3 surfaces. The special fibre of a K3 surface with good reduction is still a K3 surface, see for example [BN23, Remark 11.5]. We start by stating the following well known result, of which we include the proof as we could not find it in standard literature.

Lemma 4.3.1. *Let p be a prime number and Y a K3 surface over the finite field $\mathbb{F}_{p^n}$ for some non-negative n. Then Y is ordinary if and only if $|Y(\mathbb{F}_{p^n})| \not\equiv 1 \mod p$.*

Proof. The proof is an almost immediate consequence of [BZ09, Section 1]. Let $\bar{Y}$ be the base change of Y to an algebraic closure of $\mathbb{F}_{p^n}$ and l be a prime different from p. The Frobenius endomorphism F of $\bar{Y}$ acts by functoriality on 22-dimensional $\mathbb{Q}_l$-vector space

$$\mathrm{H}^2_{\text{ét}}(\bar{Y}, \mathbb{Q}_l) := \mathrm{H}^2_{\text{ét}}(\bar{Y}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Let λ_i with $i = 1, \ldots, 22$ be the corresponding eigenvalues. From the Lefschetz trace formula [Kat81, Section 1] we get

$$|Y(\mathbb{F}_{p^n})| = \sum (-1)^i \mathrm{Tr}(\mathrm{F}, \mathrm{H}^i_{\text{ét}}(\bar{Y}, \mathbb{Q}_l) = 1 + \sum_{i=1}^{22} \lambda_i + p^{2n}. \tag{4.6}$$

The last equality follows from the fact that for K3 surfaces both the first and the third Betti numbers are trivial and $\mathrm{H}^0_{\text{ét}}(\bar{Y}, \mathbb{Q}_l)$ and $\mathrm{H}^4_{\text{ét}}(\bar{Y}, \mathbb{Q}_l)$ are 1-dimensional $\mathbb{Q}_l$-vector spaces with Frobenius eigenvalue equal to 1 and p^{2n} respectively [Del74, Theorem 1.6].

It is proven in [BZ09, Lemma 1.1] that a K3 surface Y is ordinary if and only if $\sum_{i=1}^{22} \lambda_i$ is not divisible by p. It is therefore clear from (4.6) that

$$|Y(\mathbb{F}_{p^n})| \equiv 1 + \sum_{i=1}^{22} \lambda_i \not\equiv 1 \mod p.$$

if and only if Y is ordinary. ☐

If the K3 surface X has good ordinary reduction, then the remark that follows shows that there is a strong link between global logarithmic 2-forms on Y and p-torsion elements on X having non-constant evaluation map.

Remark 4.3.2 (K3 surfaces with good ordinary reduction)**.** Let X be a K3 surface defined over a p-adic field L having absolute ramification index divsible by $p-1$ with good ordinary reduction. Assume that there is an element $\mathcal{A} \in \mathrm{Br}(X)[p]$ that does not belong to $\mathrm{fil}_0\mathrm{Br}(X)$. Then, $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ for some $n \geq 1$. From Section 1.2.3.1 we can assume $n \leq e' = \frac{ep}{p-1} = p$, since $\mathrm{fil}_{e'}\mathrm{Br}(K)[p] = \mathrm{Br}(K)[p]$. Moreover, for $n < e' = p$ we have from Corollary 1.3.9(1) together with the fact that for K3 surfaces we do not have non-trivial global 1-forms, $\mathrm{fil}_n\mathrm{Br}(X)[p] =$

$\mathrm{fil}_0\mathrm{Br}(X)[p]$. By property 4.2.7(3) we know that there is a constant $\bar{c} \in \ell^\times$ such that

$$\mathrm{mult}_{\bar{c}}\left(\mathrm{fil}_p\mathrm{Br}(X)[p]\right) \subseteq \mathrm{H}^0(Y, \Omega^2_{Y,\log}) \subseteq \mathrm{H}^0(\bar{Y}, \Omega^2_{\bar{Y},\log}).$$

From Proposition 4.2.2 we know that the class of $\mathcal{A}$ in $\frac{\mathrm{fil}_p\mathrm{Br}(K^h)[p]}{\mathrm{fil}_{p-1}\mathrm{Br}(K^h)[p]} \simeq \mathrm{gr}^0$ has to be such that

$$[\mathcal{A}] = \rho_0(\omega, 0)$$

where $\omega \in \Omega^2_{F,\log}$ is the image in Ω^2_F of a non-trivial global logarithmic form, i.e. an element of $\mathrm{H}^0(Y, \Omega^2_{Y,\log})$. Moreover, $\mathrm{H}^0(\bar{Y}, \Omega^2_{\bar{Y},\log})$ is a 1-dimensional $\mathbb{F}_p$-vector space (cf. (3.2)).

If the K3 surface is defined by an homogeneous polynomial of degree 4, the following lemma gives us a way to write down explicitly a generator for the ℓ-vector space of global 2-forms.

Lemma 4.3.3. *Let ℓ be a field and $f(x_0, x_1, x_2, x_3) \in \ell[x_0, x_1, x_2, x_3]$ be a homogeneous polynomial of degree 4. Assume that the corresponding projective variety Y is smooth. Then Y is a K3 surface and the 1-dimensional ℓ-vector space of global 2-forms is generated by the 2-form*

$$\omega = \frac{d\left(\frac{x_1}{x_0}\right) \wedge d\left(\frac{x_2}{x_0}\right)}{\frac{1}{x_0^3} \cdot \frac{\partial f}{\partial x_3}}.$$

Proof. The first part follows from [Huy16, Example 1.3(i)]. For every permutation $\{p, q, i, j\}$ of $\{0, 1, 2, 3\}$ we define $W_{p,q} \subseteq Y$ as the open subset of Y where $x_p \cdot \frac{\partial f}{\partial x_q}$ does not vanish. We define

$$\omega_{p,q} := (-1)^{p+q+1} \cdot \frac{d\left(\frac{x_i}{x_p}\right) \wedge d\left(\frac{x_j}{x_p}\right)}{\frac{1}{x_p^3} \cdot \frac{\partial f}{\partial x_q}} \in \mathrm{H}^0(W_{p,q}, \Omega^2_Y).$$

Since Y is smooth, the open sets $\{W_{p,q}\}$ cover it. We are left to show that for every $(p, q) \neq (p', q')$, $\omega_{p,q} = \omega_{p',q'}$ on $W_{p,q} \cap W_{p',q'}$.

It is enough to show the equality in the following two cases (the complete proof follows from the symmetry among the variables):

- $(p, q) = (0, 1)$ and $(p', q') = (0, 2)$.

  Let $s_1 := x_1(x_0)^{-1}$, $s_2 := x_2(x_0)^{-1}$, $s_3 := x_3(x_0)^{-1}$ and $f_0 := f(1, s_1, s_2, s_3)$. We can then rewrite $\omega_{0,1}$ and $\omega_{0,2}$ as

  $$\omega_{0,1} = \frac{ds_2 \wedge ds_3}{\partial f_0/\partial s_1} \quad \text{and} \quad \omega_{0,2} = \frac{ds_1 \wedge ds_3}{\partial f_0/\partial s_2}.$$

  From the equation $f_0 = 0$ we get

  $$\frac{\partial f_0}{\partial s_1}ds_1 + \frac{\partial f_0}{\partial s_2}ds_2 + \frac{\partial f_0}{\partial s_3}ds_3 = 0.$$

In particular,

$$0 = \left(\frac{\partial f_0}{\partial s_1} ds_1 + \frac{\partial f_0}{\partial s_2} ds_2 + \frac{\partial f_0}{\partial s_3} ds_3\right) \wedge ds_3 = \frac{\partial f_0}{\partial s_1}(ds_1 \wedge ds_3) + \frac{\partial f_0}{\partial s_2}(ds_2 \wedge ds_3).$$

therefore, on $W_{0,1} \cap W_{0,2}$

$$\omega_{0,1} = \frac{ds_2 \wedge ds_3}{\partial f_0/\partial s_1} = (-1) \cdot \frac{ds_1 \wedge ds_3}{\partial f_0/\partial s_2} = \omega_{0,2}.$$

- $(p, q) = (0, 2)$ and $(p', q') = (1, 2)$.

  Let $t_1 := x_0(x_1)^{-1}$, $t_2 := x_2(x_1)^{-1}$, $t_3 := x_3(x_1)^{-1}$ and $f_1 := f(t_1, 1, t_2, t_3)$. On $W_{0,2} \cap W_{1,2}$ we have

$$s_1^{-1} = t_1, \quad s_2 \cdot s_1^{-1} = t_2 \quad s_3 \cdot s_1^{-1} = t_3 \quad s_1^4 \cdot f_1 = f_0.$$

  In particular,

$$\omega_{1,2} = \frac{dt_1 \wedge dt_3}{\partial f_1/\partial t_2} = \frac{d(s_1^{-1}) \wedge d(s_3 \cdot s_1^{-1})}{\partial s_1^{-4} f_0/\partial(s_2 \cdot s_1^{-1})} = (-1) \cdot \frac{s_1^{-3} ds_1 \wedge ds_3}{s_1^{-3} \partial f_0/\partial s_2} = \omega_{0,2}$$

  where the second last equality comes from the equality

$$\frac{\partial s_1^{-4} f_0}{\partial(s_2 s_1^{-1})} = s_1^{-3} \frac{\partial f_0}{\partial s_2}.$$

□

We point out that in [Ier23, Proposition 2.3] it is proven that for K3 surface an element $\mathcal{A}$ lies in $\mathrm{fil}_0\mathrm{Br}(X)$ if and only if $\mathrm{ev}_{\mathcal{A}} \colon X(L) \to \mathrm{Br}(L)$ is constant. It is already known from [BN23, Lemma 11.3] that, since for K3 surface $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$, then $\mathrm{fil}_0\mathrm{Br}(X) = \mathrm{Ev}_0\mathrm{Br}(X) = \mathrm{Ev}_{-1}\mathrm{Br}(X)$. Hence, from the result proven by Ieronymou we know that in order to detect whether $\mathcal{A}$ belongs to $\mathrm{fil}_0\mathrm{Br}(X)$ it is enough to look at the corresponding evaluation map on the L-points, $X(L)$.

Lemma 4.3.4. *Let X be a K3 surface and $\mathcal{A} \in \mathrm{Br}(X)$ be such that $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X)$. Then $\mathcal{A} \notin \mathrm{Br}_1(X)$.*

Proof. As we just pointed out, if $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X)$, then the evaluation map attached to $\mathcal{A}$ in non-constant on $X(L)$. The result now follows from the fact that in [CTS13, Proposition 2.3] Colliot-Thélène and Skorobogatov prove that for every element in the algebraic Brauer group the associated evaluation map at a prime with good reduction has to be constant. □

4.4 Example of Chapter 2 revisited

We start with recalling the following example, which is the central result of Chapter 2.

Example 4.4.1 ([Pag22]). Let $V \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the projective K3 surface defined by the equation

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \tag{4.7}$$

Then V has good ordinary reduction at 2 and the class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x}\right) \in \operatorname{Br} \mathbb{Q}(V)$$

defines an element in $\operatorname{Br}(V)$. The evaluation map $\operatorname{ev}_{\mathcal{A}} : V(\mathbb{Q}_2) \to \operatorname{Br}(\mathbb{Q}_2)$ is non-constant, and therefore gives an obstruction to weak approximation on X.

In this case, $\mathrm{H}^0(Y, \Omega^2_Y)$ is a one dimensional $\mathbb{F}_2$ vector space, let ω be the only non-trivial element, then $C(\omega) = 0$ or $C(\omega) = \omega$. However, Y being ordinary implies that $\mathrm{H}^0(Y, B^2_Y) = 0$, hence by Lemma 1.3.8 and Corollary 1.2.4 $C(\omega) = \omega$ and $\mathrm{H}^0(Y, \Omega^2_{Y,\log})$ is a 1-dimensional $\mathbb{F}_2$-vector space. From Lemma 4.3.3 we get that the non-zero global logarithmic 2-form ω can be written (locally) as:

$$\omega = \frac{d\left(\frac{z^3+w^2x+xyz}{x^3}\right)}{\left(\frac{z^3+w^2x+xyz}{x^3}\right)} \wedge \frac{d\left(\frac{z}{x}\right)}{\left(\frac{z}{x}\right)}.$$

If we denote by f and g the functions $\frac{z^3+w^2x+xyz}{x^3}$ and $\frac{z}{x}$ seen as element in the function field F of Y, then we see that the two functions appearing in the definition of $\mathcal{A}$ are lifts to characteristic 0 of f and g, and hence from Proposition 4.2.1

$$\rho_0(\omega, 0) = \left[\left\{\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x}\right\}\right] \in \mathrm{gr}^0.$$

Using Proposition 4.2.2,

$$\mathrm{rsw}_{2,\pi}(\mathcal{A}) = (\omega, 0) \neq (0, 0)$$

and $\mathcal{A} \notin \mathrm{fil}_1\mathrm{Br}(X)[2] \supseteq \mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$.

Note that, by Remark 4.3.2 we already know that since the K3 surface has good ordinary reduction at 2, the only way for the prime 2 to play a role in the Brauer–Manin obstruction to weak approximation via an 2-torsion element $\mathcal{A} \in \mathrm{Br}(X)[2]$ is if $\mathcal{A}$ comes from a logarithmic 2-form through ρ_0.

4.5 Kummer K3 surfaces over 2-adic fields

In this section we are going to treat Kummer K3 surfaces. As already mentioned in Section 2.1.3 those surfaces arise as the resolution of singularities of the quotient of an abelian surface by its involution map. The details about the construction for fields of characteristic different from 2 can be found in [BŎ1, Section 10.5].

Let L be a 2-adic field. The recent papers [LS23], [Mat23] allow us to know whether the Kummer K3 surface attached to an abelian variety A/L with good reduction is still a K3 surface with good reduction (this was already known for K3 surfaces over p-adic fields with $p \neq 2$). In [SZ12] Skorobogatov and Zarhin link the transcendental part of the Brauer group of a Kummer K3 surface to the one of the corresponding abelian variety (cf. Section 4.5.1). All these results open up the possibility of building examples of K3 surfaces with good reduction at the prime 2 and for which we are able to study the Brauer group. In this section, we are going to show that for every pair of elliptic curves E_1, E_2 over $\mathbb{Q}$ with good ordinary reduction at $p = 2$ and full 2-torsion defined over $\mathbb{Q}_2$, the 2-torsion elements in the Brauer group of the corresponding Kummer K3 surface X do not play a role in the Brauer–Manin obstruction to weak approximation. In particular, this shows that the field extension in Theorem 3.1.6 is needed. We will then use these computations to exhibit an example of a K3 surface over $\mathbb{Q}_2$ with good ordinary reduction and such that $\mathrm{Br}(X) = \mathrm{Ev}_{-1}\mathrm{Br}(X)$, showing that the inverse of Theorem 3.1.1 does not hold in general.

4.5.1 Kummer K3 surfaces and their Brauer group: generalities

Let A be an abelian surface over a field k of characteristic different from 2 and $V = \mathrm{Kum}(A)$ the corresponding Kummer surface, Skorobogatov and Zarhin [SZ12] prove that there is a well-defined map

$$\pi^* \colon \mathrm{Br}(V) \to \mathrm{Br}(A)$$

that induces an injection of $\mathrm{Br}(V)/\mathrm{Br}_1(V)$ into $\mathrm{Br}(A)/\mathrm{Br}_1(A)$. They also prove that this injection is an isomorphism on the p-torsion for all odd primes, see [SZ12, Theorem 2.4]. We say that an element $\mathcal{A} \in \mathrm{Br}(A)$ **descends** to $\mathrm{Br}(V)$ if there exists $\mathcal{C} \in \mathrm{Br}(V)$ such that $\pi^*(\mathcal{C}) = \mathcal{A}$.

Lemma 4.5.1. *Let $V = \mathrm{Kum}(A)$, $\mathcal{C} \in \mathrm{Br}(V)$ and $\mathcal{B} := \pi^*(\mathcal{C}) \in \mathrm{Br}(A)$. Let $\mathfrak{p}$ be a prime in $\mathcal{O}_k$; if the image of $\mathcal{C}$ in $\mathrm{Br}(V_{\mathfrak{p}})$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(V_{\mathfrak{p}})$ then the image of $\mathcal{B}$ in $\mathrm{Br}(A_{\mathfrak{p}})$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(A_{\mathfrak{p}})$.*

Proof. The result follows from the fact that any finite field extension $M/k_{\mathfrak{p}}$ and $P \in A(M)$ we have

$$\mathrm{ev}_{\mathcal{B}}(P) = \mathrm{ev}_{\pi^*(\mathcal{C})}(P) = \mathrm{ev}_{\mathcal{C}}(\pi(P)).$$

□

Moreover, Skorobogatov and Zarhin [SZ12] show that given two elliptic curves E_1 and E_2 with Weierstrass equations

$$E_1 : v_1^2 = u_1 \cdot (u_1 - \gamma_{1,1}) \cdot (u_1 - \gamma_{1,2}), \quad E_2 : v_2^2 = u_2 \cdot (u_2 - \gamma_{2,1}) \cdot (u_2 - \gamma_{2,2})$$

the quotient $\mathrm{Br}(E_1 \times E_2)[2]/\mathrm{Br}_1(E_1 \times E_2)[2]$ is generated by the classes of the four Azumaya algebras

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = ((u_1 - \epsilon_1)(u_1 - \gamma_{1,2}), (u_2 - \epsilon_2)(u_2 - \gamma_{2,2})) \text{ with } \epsilon_i \in \{0, \gamma_{i,1}\}.$$

Finally, if M is the matrix

$$M = \begin{pmatrix} 1 & \gamma_{1,1}\cdot\gamma_{1,2} & \gamma_{2,1}\cdot\gamma_{2,2} & -\gamma_{1,1}\cdot\gamma_{2,1} \\ \gamma_{1,1}\cdot\gamma_{1,2} & 1 & \gamma_{1,1}\cdot\gamma_{2,1} & \gamma_{2,1}\cdot(\gamma_{2,1}-\gamma_{2,2}) \\ \gamma_{2,1}\cdot\gamma_{2,2} & \gamma_{1,1}\cdot\gamma_{2,1} & 1 & \gamma_{1,1}\cdot(\gamma_{1,1}-\gamma_{1,2}) \\ -\gamma_{1,1}\cdot\gamma_{2,1} & \gamma_{2,1}\cdot(\gamma_{2,1}-\gamma_{2,2}) & \gamma_{1,1}\cdot(\gamma_{1,1}-\gamma_{1,2}) & 1 \end{pmatrix}$$

then by [SZ12, Lemma 3.6]:

1. $\mathcal{A}_{\gamma_{1,1},\gamma_{2,1}}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the first row of M are all squares;
2. $\mathcal{A}_{\gamma_{1,1},0}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the second row of M are all squares;
3. $\mathcal{A}_{0,\gamma_{2,1}}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the third row of M are all squares;
4. $\mathcal{A}_{0,0}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the last row of M are all squares.

4.5.2 Product of elliptic curves with good reduction at 2 and full 2-torsion

In order to use the results summarised in the previous section we need to analyse what the 2-torsion points of an elliptic curve with good ordinary reduction at 2 look like. Let $E/\mathbb{Q}$ be the elliptic curve defined by the minimal Weierstrass equation

$$y^2 + xy + \delta y = x^3 + ax^2 + bx + c \tag{4.8}$$

with $\delta \in \{0, 1\}$ and $a, b, c \in \mathbb{Z}$ such that E has good reduction at 2. Assume furthermore that the 2-torsion of E is defined over $\mathbb{Q}_2$, i.e. $E(\mathbb{Q}_2)[2] = E(\bar{\mathbb{Q}}_2)[2]$. Let $\alpha_i, \beta_i \in \mathbb{Q}_2$ be such that $E(\mathbb{Q}_2)[2] = \{\mathcal{O}, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\}$, with $\mathcal{O}$ the point at infinity of E.

Lemma 4.5.2. *Assume that $\beta_1, \beta_2, \beta_3$ are ordered as*

$$\mathrm{ord}_2(\beta_1) \le \mathrm{ord}_2(\beta_2) \le \mathrm{ord}_2(\beta_3).$$

Then $\mathrm{ord}_2(\alpha_1) = -2$ *and* $\alpha_2, \alpha_3 \in \mathbb{Z}_2$.

Proof. The 2-torsion points on E can be computed through the 2-division polynomial of E, which is $\psi_2(x, y) = 2y + x + \delta$. In particular, $\alpha_i = -2\beta_i - \delta$ with β_i solution of

$$\Phi(y) := y^2 + (-2y - \delta)y + \delta y - ((-2y - \delta)^3 + a(-2y - \delta)^2 + b(-2y - \delta) + c).$$

The polynomial $\Phi(y)$ can be rewritten as

$$\Phi(y) = 8y^3 - (1 - 12\delta + 4a)y^2 - (-6\delta^2 + 4a\delta - 2b)y + \delta^3 - a\delta^2 + b\delta - c. \quad (4.9)$$

Looking at the coefficients of $\Phi(y)$ we get that

$$\begin{cases} \mathrm{ord}_2(\beta_1 + \beta_2 + \beta_3) = \mathrm{ord}_2(1 - 12\delta + 4a) - \mathrm{ord}_2(8) \\ \mathrm{ord}_2(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3) = \mathrm{ord}_2(-6\delta^2 + 4a\delta - 2b) - \mathrm{ord}_2(8) \\ \mathrm{ord}_2(\beta_1\beta_2\beta_3) = \mathrm{ord}_2(\delta^3 - a\delta^2 + b\delta - c) - \mathrm{ord}_2(8). \end{cases}$$

From the first equation, we get $\mathrm{ord}_2(\beta_1) \leq -3$ that combined with [Sil86, Theorem VIII.7.1] tells us that $\mathrm{ord}_2(\beta_1) = -3$. From the third equation, we get $\mathrm{ord}_2(\beta_2) + \mathrm{ord}_2(\beta_3) \geq 0$. Hence, if $\mathrm{ord}_2(\beta_2) = \mathrm{ord}_2(\beta_3)$ then β_2 and β_3 have both non-negative 2-adic valuation; otherwise, if $\mathrm{ord}_2(\beta_2) < \mathrm{ord}_2(\beta_3)$ then, from the second equation, we get $\mathrm{ord}_2(\beta_2) \geq 1$ which implies again that both β_2 and β_3 have non-negative 2-adic valuation. The result now follows from the fact that $\alpha_i = -2\beta_i - \delta$, with $\delta \in \{0, 1\}$. ☐

Lemma 4.5.3. *The change of variables given by*

$$\begin{cases} u = 4x - 4\alpha_1 \\ v = 4(2y + x + \delta) \end{cases} \quad (4.10)$$

induces an isomorphism between E and the elliptic curve given by the equation

$$v^2 = u(u - \gamma_1)(u - \gamma_2) \quad (4.11)$$

where $\gamma_1 = 4 \cdot (\alpha_2 - \alpha_1)$ and $\gamma_2 = 4 \cdot (\alpha_3 - \alpha_1)$.

Proof. The change of variables

$$\begin{cases} u_1 = 4x \\ v_1 = 4(2y + x + \delta) \end{cases}$$

sends the elliptic curve given by the equation

$$v_1^2 = u_1^3 + (4a + 1)u_1^2 + (16b + 8\delta)u_1 + 16c + 16\delta^2 \quad (4.12)$$

to the elliptic curve given by equation (4.8). Moreover, the 2-division polynomial of E is given by $2y + x + \delta$. Hence the non-trivial 2-torsion points on the elliptic curve given by equation (4.12) are sent to non-trivial 2-torsion points on E. It is therefore enough to consider the extra translation $u = u_1 - 4\alpha_1$ and $v = v_1$ to get the desired equation. ☐

Let E_1 and E_2 be two elliptic curves with equations of the form (4.8). We denote by $(\delta_i, a_i, b_i, c_i)$ the parameters that determine the equation attached to E_i, by $(\alpha_{i,j}, \beta_{i,j})$, $j \in \{1, 2, 3\}$ the non-trivial 2-torsion points of E_i and by A the abelian surface given by the product of E_1 with E_2. We denote by $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2] \rangle$ the subgroup of $\mathrm{Br}(A)[2]$ generated by $\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$ and $\mathrm{Br}_1(A)[2]$, where $\mathrm{Br}_1(A)[2]$ is the algebraic Brauer group of A.

Lemma 4.5.4. *Assume that ϵ_1 and ϵ_2 are as in Section 4.5.1; then the class of the quaternion algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ lies in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2]\rangle$ if and only if at least one among ϵ_1 and ϵ_2 is different from 0.*

Proof. We fix $\pi = 2$ as a uniformiser and $\xi = -1$ as a primitive 2-root of unity. We start by assuming that at least one among ϵ_1 and ϵ_2 is different from 0. By the symmetry of the statement, we can assume without loss of generality that $\epsilon_1 \neq 0$. Then

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = ((u_1 - \gamma_{1,1}) \cdot (u_1 - \gamma_{1,2})\, ,\, (u_2 - \epsilon_2) \cdot (u_2 - \gamma_{2,2})) = (u_1, f_{\epsilon_2}(u_2))$$

where $f_{\epsilon_2}(u_2) = (u_2 - \epsilon_2) \cdot (u_2 - \gamma_{2,2})$. The quaternion algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ corresponds via the change of variables of Lemma 4.5.3 to

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = (4 \cdot (x_1 - \alpha_{1,1})\, ,\, f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})) = (x_1 - \alpha_{1,1}\, ,\, f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})).$$

We define

$$g_{\epsilon_2}(x_2) := \begin{cases} (x_2 - \alpha_{2,2}) \cdot (x_2 - \alpha_{2,3}) & \text{if } \epsilon_2 = \gamma_{2,1}; \\ (x_2 - \alpha_{2,1}) \cdot (x_2 - \alpha_{2,3}) & \text{if } \epsilon_2 = 0. \end{cases}$$

Then, $16 \cdot g_{\epsilon_2}(x_2) = f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})$. Thus we can rewrite $\mathcal{A}_{\epsilon_1,\epsilon_2}$ as

$$(-\alpha_{1,1}\, ,\, g_{\epsilon_2}(x_2)) \otimes (1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, ,\, g_{\epsilon_2}(x_2)).$$

Since $(-\alpha_{1,1}\, ,\, g_{\epsilon_2}(x_2))$ lies in $\mathrm{Br}_1(A)[2]$, we are left to show that the class of the quaternion algebra $(1 + (-\alpha_{1,1})^{-1} \cdot x_1\, ,\, g_{\epsilon_2}(x_2))$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$. By Lemma 4.5.2 we know that $\mathrm{ord}_2(\alpha_{1,1}^{-1}) = 2$ and therefore by Proposition 4.2.2

$$(1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, ,\, g_{\epsilon_2}(x_2)) \in \mathrm{fil}_0\mathrm{Br}(A)[2].$$

By [BN23, Theorem C] in order to establish whether $(1 + (-\alpha_{1,1})^{-1} \cdot x_1\, ,\, g_{\epsilon_2}(x_2))$ belongs to $\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$ we need to compute $\partial(1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, ,\, g_{\epsilon_2}(x_2))$. We have that $g_{\epsilon_2}(x_2) \not\equiv 0 \mod 2$ and from Proposition 4.2.1.(1) together with Proposition 4.2.2 we get

$$\lambda_\pi\left(\bar{x}_1 \cdot d\log(\bar{g}_{\epsilon_2}(\bar{x}_2)), 0\right) = \Big(1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, ,\, g_{\epsilon_2}(x_2)\Big)$$

since $1 + (-\alpha_{1,1}^{-1}) \cdot x_1 = 1 + 4 \cdot (s^{-1} \cdot x_1)$ with $s = -4 \cdot \alpha_{1,1} \in \mathbb{Z}_2^\times$ and hence $s^{-1} \cdot x_1$ is a lift to characteristic 0 of $\bar{x}_1$. Therefore, by definition of the residue map ∂, we get

$$\partial((1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, ,\, g_{\epsilon_2}(x_2))) = 0$$

which by Theorem 1.3.1 implies that

$$(1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, ,\, g_{\epsilon_2}(x_2)) \in \mathrm{Ev}_{-2}\mathrm{Br}(A)[2] \subseteq \mathrm{Ev}_{-1}\mathrm{Br}(A)[2].$$

In order to end the proof we are left to show that $\mathcal{A}_{0,0} \notin \langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2]\rangle$. The change of variables of Lemma 4.5.3 sends the class of the quaternion algebra

$$\mathcal{A}_{0,0} = (u_1 \cdot (u_1 - \gamma_{1,2})\, ,\, u_2 \cdot (u_2 - \gamma_{2,2})) = (u_1 - \gamma_{1,1}\, ,\, u_2 - \gamma_{2,1})$$

to the class of the quaternion algebra

$$(4\cdot(x_1-\alpha_{1,2})\,,\,4\cdot(x_2-\alpha_{2,2}))=(x_1+2\beta_{1,2}+\delta\,,\,x_2+2\beta_{2,2}+\delta).$$

From Proposition 4.2.1(d) the latter is such that

$$\rho_0\left(\frac{d(\bar{x}_1+\delta)}{\bar{x}_1+\delta}\wedge\frac{d(\bar{x}_2+\delta)}{\bar{x}_2+\delta}\right)=\left[\left\{x_1+2\beta_{2,1}+\delta,x_2+2\beta_{2,2}+\delta\right\}\right]\in\mathrm{gr}^0.$$

In fact, $x_1+2\beta_{2,1}+\delta, x_2+2\beta_{2,2}+\delta$ and $x_2+2\beta_{2,2}+\delta$ are lifts to characteristic 0 of $\bar{x}_1+\delta$ and $\bar{x}_2+\delta$ respectively. Note that, $\frac{d(\bar{x}_1+\delta)}{\bar{x}_1+\delta}\wedge\frac{d(\bar{x}_2+\delta)}{\bar{x}_2+\delta}$ comes from a global 2-form on the special fibre Y of A and hence it is non-zero in its function field. Finally, using Proposition 4.2.2 we get that

$$\mathrm{rsw}_{2,\pi}((x_1+2\beta_{2,1}+\delta,x_2+2\beta_{2,2}+\delta))=\left(\frac{d(\bar{x}_1+\delta)}{\bar{x}_1+\delta}\wedge\frac{d(\bar{x}_2+\delta)}{\bar{x}_2+\delta},0\right)\neq(0,0)$$

and hence $(x_1+2\beta_{1,2}+\delta,x_2+2\beta_{2,2}+\delta)\notin\mathrm{fil}_1\mathrm{Br}(A)[2]\supseteq\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$. Moreover, as a consequence of Corollary 4.1.5 we get that $\mathcal{A}_{0,0}\notin\langle\mathrm{Ev}_{-1}\mathrm{Br}(A)[2],\mathrm{Br}_1(A)[2]\rangle$. In fact, otherwise, there would be an element $\mathcal{A}_1\in\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$ such that $\mathcal{A}_{0,0}\otimes\mathcal{A}_1\in\mathrm{Br}_1(A)[2]$, but $\mathcal{A}_{0,0}\otimes\mathcal{A}_1$ has the same refined Swan conductor as $\mathcal{A}_{0,0}$. □

Remark 4.5.5. We will later use a slightly stronger statement of the theorem above. Let $L/\mathbb{Q}_2$ be any field extension and res the natural map from $\mathrm{Br}(A)[2]$ to $\mathrm{Br}(A_L)[2]$. Then, $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2})\in\langle\mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2],\mathrm{Br}_1(A_L)[2]\rangle$ if and only if at least one among ϵ_1 and ϵ_2 is different from 0. We clearly have that $\mathcal{A}_{\epsilon_1,\epsilon_2}$ in $\langle\mathrm{Ev}_{-1}\mathrm{Br}(A)[2],\mathrm{Br}_1(A)[2]\rangle$ implies $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2})$ in $\langle\mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2],\mathrm{Br}_1(A_L)[2]\rangle$. Moreover, we have proven that the first component of $\mathrm{rsw}_{2,\pi}(\mathcal{A}_{0,0})$ is different from 0, and hence using Lemma 4.1.2 we get that $\mathrm{rsw}_{e_{L'/L}2,\pi}(\mathrm{res}(\mathcal{A}_{0,0}))\neq(0,0)$ and therefore in particular $\mathrm{res}(\mathcal{A}_{0,0})\notin\langle\mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2],\mathrm{Br}_1(A_L)[2]\rangle$

4.5.3 No Brauer–Manin obstruction from 2-torsion elements in Kum(A)

In this section, we show how, from the results of the previous section, we can deduce information on the 2-torsion elements in the Brauer group of the corresponding Kummer surface $V=\mathrm{Kum}(A_{\mathbb{Q}})$. We denote by A and X the base change of the abelian surface $A_{\mathbb{Q}}$ and the corresponding Kummer surface X to $\mathbb{Q}_2$. By Section 4.5.1 we know that $\mathcal{A}_{0,0}$ descends to X if and only if

$$\left[-\gamma_{1,1}\cdot\gamma_{2,1},\gamma_{2,1}(\gamma_{2,1}-\gamma_{2,2}),\gamma_{1,1}(\gamma_{1,1}-\gamma_{1,2}),1\right]\in(\mathbb{Q}_2^{\times 2})^4.$$

By construction, $\gamma_{1,1}=4\cdot(\alpha_{1,2}-\alpha_{1,1})=8\beta_{1,1}-8\beta_{1,2}$ and therefore

$$\gamma_{1,1}\equiv 8\beta_{1,1}\equiv 1-12\delta_1+4a_1\equiv 1-4(3\delta_1-a_1)\mod 8.$$

In fact, from Lemma 4.5.2 and more precisely from equation (4.9) we know that $8\beta_{1,1}+8\beta_{1,2}+8\beta_{1,3}=1-12\delta_1+4a_1$ and both $\mathrm{ord}_2(\beta_{1,2})$ and $\mathrm{ord}_2(\beta_{1,3})$ are non-negative. Similarly,

$$\gamma_{2,1}\equiv 8\beta_{2,1}\equiv 1-12\delta_2+4a_2\equiv 1-4(3\delta_2-a_2)\mod 8.$$

In particular, both $\gamma_{1,1}$ and $\gamma_{2,1}$ are either 1 or 5 modulo 8; hence $-\gamma_{1,1} \cdot \gamma_{2,1}$ is either -1 or 3 and therefore it is never a square. Summing up: we have shown $\mathcal{A}_{0,0}$ never descends to $\mathrm{Br}(X)$. We are ready to prove the main theorem of this section.

Theorem 4.5.6. *Let* $X = \mathrm{Kum}(A)$*, where* $A = E_1 \times E_2$ *is as in Section 4.5.1; then* $\mathrm{Br}(X)[2] = \mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$.

Proof. We recall that if $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)[2]$, we denote by $\mathcal{C}_{\epsilon_1,\epsilon_2}$ the corresponding element in $\mathrm{Br}(X)[2]$, i.e. $\mathcal{C}_{\epsilon_1,\epsilon_2}$ is such that $\pi^*(\mathcal{C}_{\epsilon_1,\epsilon_2}) = \mathcal{A}_{\epsilon_1,\epsilon_2}$. We need to prove that if $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)$, then $\mathcal{C}_{\epsilon_1,\epsilon_2}$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(X)$. Since we have already shown at the beginning of this section that $\mathcal{A}_{0,0}$ never descends to $\mathrm{Br}(X)$ we are left to show it for $(\epsilon_1, \epsilon_2) \neq (0,0)$.

Let $L/\mathbb{Q}_2$ be such that all elements appearing in the matrix M of Section 4.5.1 are squares, i.e. the injective map

$$\pi^* \colon \frac{\mathrm{Br}(X)[2]}{\mathrm{Br}_1(X)[2]} \hookrightarrow \frac{\mathrm{Br}(A)[2]}{\mathrm{Br}_1(A)[2]}$$

is an isomorphism.

With abuse of notation, we denote by res both

$$\mathrm{res}\colon \mathrm{Br}(A) \to \mathrm{Br}(A_L) \quad \text{and} \quad \mathrm{res}\colon \mathrm{Br}(X) \to \mathrm{Br}(X_L).$$

We denote by $(\mathcal{C}_{\epsilon_1,\epsilon_2})_L$ the pre-image of $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2}) \in \mathrm{Br}(A_L)[2]$.

From [LS23, Theorem 2] we know that the reduction of X is an ordinary K3 surface. Let e be the ramification index of $L/\mathbb{Q}_2$ and π_L the an uniformiser of $\mathcal{O}_L$; then we have $\mathrm{fil}_n\mathrm{Br}(X_L)[p] = \mathrm{fil}_0\mathrm{Br}(X_L)[p] = \mathrm{Ev}_{-1}\mathrm{Br}(X_L)[p]$ if $n < e' := 2e$ and $\mathrm{fil}_{e'}\mathrm{Br}(X)[p] = \mathrm{Br}(X_L)[p]$, see Remark 4.3.2. Hence, using Corollary 1.3.9(3) we have an injection

$$\mathrm{mult}_{\overline{c}} \circ \mathrm{rsw}_{e',\pi_L} : \frac{\mathrm{Br}(X_L)[2]}{\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]} \hookrightarrow \mathrm{H}^0(Y_\ell, \Omega^2_{Y_\ell,\log})$$

where ℓ is the residue field of L.

Since X (and hence all its base change) has good ordinary reduction, we know that $\mathrm{H}^0(\bar{Y}, \Omega^2_{\bar{Y},\log}) \otimes_{\mathbb{F}_2} \bar{\ell}$ is a one dimensional $\bar{\ell}$-vector space (cf. equation (3.2)) and hence $\mathrm{Br}(X_L)[2]/\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]$ is a vector space of dimension at most 1 over $\mathbb{F}_2$.

From Lemma 4.5.4 we know that $\mathcal{A}_{0,0} \notin \mathrm{fil}_1\mathrm{Br}(A)[2]$. Applying Lemma 4.1.2 we get that $\mathrm{res}(\mathcal{A}_{0,0}) \notin \mathrm{fil}_e\mathrm{Br}(A_L)[2]$ and by Lemma 4.5.1 $(\mathcal{C}_{0,0})_L \notin \mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]$. Therefore

$$\langle [(\mathcal{C}_{0,0})_L] \rangle = \frac{\mathrm{Br}(X_L)[2]}{\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]}.$$

Assume that there exists $(\epsilon_1, \epsilon_2) \neq (0,0)$ such that $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)$ and the corresponding element $\mathcal{C}_{\epsilon_1,\epsilon_2}$ does not lie in $\mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$. By Lemma 4.1.2 $\mathrm{res}(\mathcal{C}_{\epsilon_1,\epsilon_2})$ does not lie in $\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]$ and therefore, since the quotient $\mathrm{Br}(X_L)[2]/\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]$ is a 1-dimensional $\mathbb{F}_2$-vector space, we have that also the product $\mathrm{res}(\mathcal{C}_{\epsilon_1,\epsilon_2}) \otimes (\mathcal{C}_{0,0})_L$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$. This implies that also the corresponding element in $\mathrm{Br}(A_L)[2]$, $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2} \otimes \mathcal{A}_{0,0})$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(A_L)$. However, since by Remark 4.5.5 $\mathrm{res}(A_{\epsilon_1,\epsilon_2})$ lies in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2], \mathrm{Br}_1(X_L)[2]\rangle$, we get that also $\mathrm{res}(\mathcal{A}_{0,0})$ has to lie in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2], \mathrm{Br}_1(A_L)[2]\rangle$ which gives us the desired contradiction. □

Finally, we give an example of a K3 surface over $\mathbb{Q}$ with good ordinary reduction at 2 and such that $\mathrm{Br}(X) = \mathrm{Ev}_{-1}\mathrm{Br}(X)$. The existence of such an example shows that the converse of Theorem 3.1.1 is not true, i.e. it is not enough to have that $p-1 \mid e$ in order to find an element in $\mathrm{Br}(X)$ that does not lie in $\mathrm{Ev}_{-1}\mathrm{Br}(X)$.

Example 4.5.7. Let $A = E \times E$, where E is the elliptic curve given by the minimal Weierstrass equation

$$y^2 + xy + y = x^3 - 7 \cdot x + 5.$$

Then, with the same notation as in the previous sections $\beta_1 = -11/8$, $\beta_2 = -1$ and $\beta_3 = 1$. Hence

$$\alpha_1 = 7/4, \quad \alpha_2 = 1, \quad \alpha_3 = -3 \quad \text{and} \quad \gamma_1 = -3, \quad \gamma_2 = -21.$$

The matrix M is of the form

$$\begin{pmatrix} 1 & 3\cdot 21 & 3\cdot 21 & -9 \\ 3\cdot 21 & 1 & 9 & -3\cdot 18 \\ 3\cdot 21 & 9 & 1 & -3\cdot 18 \\ -9 & -3\cdot 18 & -3\cdot 18 & 1 \end{pmatrix}.$$

In particular, all the rows of M have at least one term which does not lie in $\mathbb{Q}_2^{\times 2}$. Moreover, using [SZ12, Proposition 3.7] we can compute the dimension as an $\mathbb{F}_2$-vector space of the quotient of $\mathrm{Br}(X)[2]$ by $\mathrm{Br}(\mathbb{Q}_2)[2]$ and in this case particular example:

$$\dim_{\mathbb{F}_2}\left(\frac{\mathrm{Br}(X)[2]}{\mathrm{Br}(\mathbb{Q}_2)[2]}\right) = 0.$$

We want to show that $\mathrm{Br}(X)\{2\} = \mathrm{Br}(\mathbb{Q}_2)\{2\}$. We work by induction on n; let $\mathcal{A}$ be in $\mathrm{Br}(X)[2^n]$, then

$$\mathcal{A}^{\otimes 2^{n-1}} \in \mathrm{Br}(X)[2] = \mathrm{Br}(\mathbb{Q}_2)[2].$$

In particular, given $P \in X(\mathbb{Q}_2)$, we have that

$$(\mathcal{A} \otimes \mathrm{ev}_{\mathcal{A}}(P))^{\otimes 2^{n-1}} = \mathcal{A}^{\otimes 2^{n-1}} \otimes \mathrm{ev}_{\mathcal{A}^{\otimes 2^{n-1}}}(P) = 0$$

hence $\mathcal{A} \otimes \mathrm{ev}_{\mathcal{A}}(P) \in \mathrm{Br}(X)[2^{n-1}]$, and by induction hypothesis that $\mathcal{A} \in \mathrm{Br}(\mathbb{Q}_2)[2^n]$.

4.5.4 Examples of Brauer–Manin obstruction

We continue this section by giving new examples of primes of good reduction that plays a role in the Brauer–Manin obstruction to weak approximation in the case $p = 3$ and $p = 5$.

Example 4.5.8. Let $L = \mathbb{Q}_3(\zeta)$ with ζ primitive 3-root of unity. Let π be a uniformiser of $\mathcal{O}_L$; then $e = e(L/\mathbb{Q}_3) = 2$ and the residue field ℓ is equal to $\mathbb{F}_3$.

We define X to be the Kummer K3 surface over L attached to the abelian surface $A = E \times E$, with E the elliptic curve over L defined by the Weierstrass equation

$$y^2 = x^3 + 4 \cdot x^2 + 3 \cdot x + 1.$$

The elliptic curve E (and hence A and X) has good ordinary reduction at the prime $\mathfrak{p} = (\pi)$. We will denote by $\{x, y, z\}$ and $\{u, v, w\}$ the variables corresponding to the embedding of respectively the first and the second copy of E in $\mathbb{P}^2_L$. We define the cyclic algebra

$$\mathcal{A} := \left(\frac{v-u}{w}, \frac{y-x}{z}\right)_\zeta \in \operatorname{Br}(L(A))[3].$$

Claim 1: $\mathcal{A}$ belongs to $\operatorname{Br}(A)[3]$.

Proof. First of all, notice that from $y^2z = x^3 + 4x^2z + 3xz^2 + z^3$ we get

$$z(y-x)(y+x) = (x+z)^3 \quad \text{and} \quad z(y^2 - 4x^2 - 3xz - z^2) = x^3.$$

Then:

- if $z = 0$, then $x = 0$ and therefore $y^2 - 4x^2 - 3xz - z^2 \neq 0$ and $x \neq y$ and from the equation above $\mathcal{A}$ is equivalent to

$$\left(\frac{v-u}{(v^2 - 4u^2 - 3uw - w^2)^{-1}}, \frac{y-x}{(y^2 - 4x^2 - 3xz - z^2)^{-1}}\right)_\zeta;$$

- if $x = y$, then $x \neq -y$ and $z \neq 0$ (since $z = 0$ implies $x \neq y$) and from the equation above $\mathcal{A}$ is equivalent to

$$\left(\frac{w}{v+u}, \frac{z}{x+y}\right)_\zeta.$$

Thus, we see that along all the divisors over which $\mathcal{A}$ is not well defined we are able to find an equivalent Azumaya algebra which on those divisors is well defined. Hence, $\mathcal{A}$ defines an element in $\operatorname{Br}(A)[3]$. □

Claim 2: $\mathcal{A}$ and does not lie in $\operatorname{Ev}_{-1}\operatorname{Br}(A)[3]$.

Proof. The regular global 1-form on the reduction of E modulo $\mathfrak{p}$ is given by the (local) formula

$$\frac{dx}{2y} = -\frac{1}{2} \cdot \frac{dx \cdot (x-y)}{y(y-x)} = -\frac{1}{2} \cdot \frac{dx \cdot \frac{x}{y} - dx}{y-x} = \frac{d(y-x)}{y-x}$$

where the last equality follows from the fact that on the special fibre $\frac{dx}{y} = \frac{dy}{x}$ and since we are in characteristic 3, $\frac{1}{2} = -1$. Hence, if we denote by Y the reduction modulo $\mathfrak{p}$ of A, we have that the global 2-form on Y is given by

$$\omega = \frac{d\left(\frac{v-u}{w}\right)}{\left(\frac{v-u}{w}\right)} \wedge \frac{d\left(\frac{y-x}{z}\right)}{\left(\frac{y-x}{z}\right)}.$$

Finally, $\rho_0(\omega, 0) = \left[\left\{\frac{v-u}{w}, \frac{y-x}{z}\right\}\right]$ and hence again by Proposition 4.2.2 we get that

$$\mathrm{rsw}_{3,\pi}(\mathcal{A}) \neq (\bar{c}^{-1} \cdot \omega, 0)$$

and therefore $\mathcal{A} \notin \mathrm{fil}_2\mathrm{Br}(A)[3] \supseteq \mathrm{Ev}_{-1}\mathrm{Br}(A)[3]$. □

Claim 3: The cyclic algebra $\mathcal{A}$ in $\mathrm{Br}(A)[3]$ is not algebraic, i.e. $\mathcal{A} \notin \mathrm{Br}_1(A)[3]$.

Proof. If follows directly from Corollary 4.1.5. □

Finally, from [SZ12, Theorem 2.4] the map

$$\pi^* \colon \frac{\mathrm{Br}(X)[3]}{\mathrm{Br}_1(X)[3]} \hookrightarrow \frac{\mathrm{Br}(A)[3]}{\mathrm{Br}_1(A)[3]}$$

is an isomorphism. Let $\mathcal{B} \in \mathrm{Br}(X)[3]$ be such that $\pi^*(\mathcal{B}) = \mathcal{A} \in \mathrm{Br}(A)[3]$. Then, from Lemma 4.5.1 we get that $\mathcal{B} \notin \mathrm{Ev}_{-1}\mathrm{Br}(X)[3]$, namely (since for K3 surfaces $\mathrm{Ev}_0\mathrm{Br}(X) = \mathrm{Ev}_{-1}\mathrm{Br}(X)$, see Section 4.3) the corresponding evaluation map on $X(L)$ is non-constant.

Before proceeding with the next example we need a lemma that shows how the evaluation map behaves under base change without the assumption of good reduction for X.

Lemma 4.5.9. *Let X be a variety over a p-adic field L, not necessarily having good reduction, and let $\mathcal{A} \in \mathrm{Br}(X)\{p\}$ be such that $\mathrm{ev}_{\mathcal{A}} \colon X(L) \to \mathrm{Br}(L)$ is non-constant. Then for every field extension L'/L with degree co-prime to p we have that $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(X_{L'})$ has also a non-constant evaluation map $\mathrm{ev}_{\mathrm{res}(\mathcal{A})} \colon X_{L'}(L') \to \mathrm{Br}(L')$.*

Proof. Let $P, Q \in X(L)$ be such that

$$\mathrm{ev}_{\mathcal{A}}(P) \neq \mathrm{ev}_{\mathcal{A}}(Q).$$

Denote by P' and Q' the base change of P and Q to L', i.e. we have the following commutative diagrams

$$\begin{array}{ccc} X_{L'} & \xrightarrow{\quad\psi\quad} & X_0 \\ \downarrow & & \downarrow \\ \operatorname{Spec}(L') & \xrightarrow{\varphi} & \operatorname{Spec}(L) \end{array} \qquad \begin{array}{ccc} \operatorname{Spec}(L') & \xrightarrow{P'} & X_{L'} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \operatorname{Spec}(L) & \xrightarrow{P} & X \end{array} \qquad \begin{array}{ccc} \operatorname{Spec}(L') & \xrightarrow{Q'} & X_{L'} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \operatorname{Spec}(L) & \xrightarrow{Q} & X. \end{array}$$

Then

$$\mathrm{ev}_{\mathrm{res}(\mathcal{A})}(P') = \mathrm{Br}(P')(\mathrm{Br}(\psi_L)(\mathcal{A}_0)) = \mathrm{Br}(\varphi_L)\mathrm{Br}(P)(\mathcal{A}) = \mathrm{Br}(\varphi_L)(\mathrm{ev}_{\mathcal{A}}(P)).$$

Finally since L'/L has degree co-prime to p the map $\mathrm{Br}(\varphi)\colon \mathrm{Br}(L) \to \mathrm{Br}(L')$ is injective on elements of p-order; hence

$$\mathrm{ev}_{\mathrm{res}(A)}(P') \neq \mathrm{ev}_{\mathrm{res}(A)}(Q').$$

□

Example 4.5.10. Let X be the diagonal quartic surface over $\mathbb{Q}_5$ defined by the equation:

$$5x^4 - 4y^4 = z^4 + w^4.$$

Skorobogatov and Ieronymou prove [IS15, Theorem 1.1], [IS15, Proposition 5.12] that there exists an element $\mathcal{A} \in \mathrm{Br}(X)[5]$ with surjective evaluation map. Let $L = \mathbb{Q}_5(\sqrt[4]{5})$, $e(L/\mathbb{Q}_5) = 4$ and $\alpha \in L$ be such that $\alpha^4 = 5$. Then the change of variables:

$$(x, y, z, w) \mapsto \left(\frac{x_1}{\alpha}, y_1, z_1, w_1\right)$$

sends X_L to the diagonal quartic $\tilde{X}/L$ given by the equation

$$x_1^4 - 4y_1^4 = z_1^4 + w_1^4.$$

The surface $\tilde{X}$ has good ordinary reduction over L. Finally, by Lemma 4.5.9 we know that $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(X_{L'}) = \mathrm{Br}(\tilde{X})$ has non-constant evaluation map.

Note that, at this point we have not write the algebra $\mathcal{A}$ as a cyclic algebra, and hence we are not able to show explicitly the link with the global logarithmic 2-forms on the special fibre and hence compute the refined Swan conductor of $\mathcal{A}$, even if from Remark 4.3.2 we know that $\mathcal{A} \in \mathrm{fil}_p\mathrm{Br}(X)[5]$.

4.6 Family of examples

We end this thesis by giving an example of a family of K3 surfaces.

Let $\alpha \in \bar{\mathbb{Q}}$ be such that $\alpha^2 \in \mathbb{Z}$ and let V_α be the K3 surface over $k := \mathbb{Q}(\alpha)$ defined by the equation

$$x^3y + y^3z + z^3w - w^4 + \alpha^2 \cdot xyzw - 2 \cdot \alpha^{-1} \cdot xzw^2 = 0. \tag{4.13}$$

Lemma 4.6.1. *The class of the quaternion algebra*

$$\mathcal{A} := \left(\frac{z^2 + \alpha^2 \cdot xy}{z^2}, -\frac{z}{x} \right) \in \mathrm{Br}(k(V_\alpha))$$

lies in $\mathrm{Br}(V_\alpha)[2]$.

Proof. Let $f := z^2 + \alpha^2 xy$ and C_x,C_z,C_f be the closed subsets of V_α defined by the equations $x = 0$, $z = 0$ and $f = 0$ respectively. The quaternion algebra $\mathcal{A}$ defines an element in $\mathrm{Br}(U)$, where $U := V_\alpha \setminus (C_x \cup C_z \cup C_f)$. The purity theorem for the Brauer group [CTS21, Theorem 3.7.2], assures us of the existence of the exact sequence

$$0 \to \mathrm{Br}(V_\alpha)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus \partial_D} \bigoplus_D \mathrm{H}^1(k(D), \mathbb{Z}/2) \tag{4.14}$$

where D ranges over the irreducible divisors of V_α with support in $X \setminus U$ and $k(D)$ denotes the residue field at the generic point of D.

In order to use the exact sequence (4.14) we need to understand what the prime divisors of V_α with support in $V_\alpha \setminus U = C_x \cup C_z \cup C_f$ look like. Using MAGMA [BCP97] it is possible to check the following:

- C_x has one irreducible component D_1 defined by the equations $\{x = 0, y^3 z + z^3 w + w^4 = 0\}$;
- C_z has one irreducible component D_2, defined by the equations $\{z = 0, x^3 y - w^4 = 0\}$;
- C_f has one irreducible component D_3, defined by the equations $\{\alpha^2 xy + z^2 = 0, x^3 z^2 + y^2 z^3 + 2\alpha x^2 z w^2 + \alpha^2 x w^4 = 0, \alpha^2 y^3 z - x^2 z^2 - 2\alpha x z w^2 - \alpha^2 w^4 = 0\}$.

Therefore, we can rewrite (4.14) in the following way:

$$0 \to \mathrm{Br}(V_\alpha)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus \partial_{D_i}} \bigoplus_{i=1}^{3} H^1(k(D_i), \mathbb{Z}/2). \tag{4.15}$$

Moreover, we have an explicit description of the residue map on quaternion algebras: for an element $(a, b) \in \mathrm{Br}(U)[2]$ we have

$$\partial_{D_i}(a, b) = \left[(-1)^{\nu_i(a)\nu_i(b)} \frac{a^{\nu_i(b)}}{b^{\nu_i(a)}} \right] \in \frac{k(D_i)^\times}{k(D_i)^{\times 2}} \simeq H^1(k(D_i), \mathbb{Z}/2) \tag{4.16}$$

where ν_i is the valuation associated to the prime divisor D_i. This follows from the definition of the tame symbols in Milnor K-theory together with the compatibility of the residue map with the tame symbols given by the Galois symbols (see [GS17], Proposition 7.5.1).

We can proceed with the computation of the residue maps ∂_{D_i} for $i = 1, \dots, 3$:

1. $\nu_1(x) = 1$ and $\nu_1(f) = \nu_1(z) = 0$. Hence,

$$\partial_{D_1}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[\left(\frac{f}{z^2}\right)^{-1}\right] = 1 \in \frac{k(D_1)^\times}{k(D_1)^{\times 2}}$$

where the last equality follows from the fact that $x = 0$ on D_1, thus $f\,|_{D_1} = z^2$.

2. $\nu_2(z) = 1$ and $\nu_2(f) = \nu_2(x) = 0$. Hence,

$$\partial_{D_2}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[\left(\frac{f}{z^2}\right)\left(-\frac{z}{x}\right)^2\right] = \left[\frac{f}{x^2}\right] = 1 \in \frac{k(D_2)^\times}{k(D_2)^{\times 2}}$$

where the last equality follows from the fact that $z = 0$ and $x^3y = w^4$ on D_2, thus $f\,|_{D_2} = \alpha^2 xy$ and $\frac{\alpha^2 y}{x} = \alpha^2\left(\frac{w}{x}\right)^4 = \left(\alpha\frac{w^2}{x^2}\right)^2$.

3. $\nu_3(f) = 1$ and $\nu_3(x) = \nu_3(z) = 0$. Hence,

$$\partial_{D_3}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[-\frac{z}{x}\right] = \left[\left(\frac{\alpha^3 x}{z^3}\left(w^2 + \frac{xz}{\alpha}\right)\right)^2\right] = 1 \in \frac{k(D_3)^\times}{k(D_3)^{\times 2}}$$

where the last equality follows from the following equalities on D_3:

- $y^3z = w^4 + \frac{2}{\alpha}xzw^2 + \frac{x^2z^2}{\alpha^2} = \left(w^2 + \frac{xz}{\alpha}\right)^2$;
- $xy = -\frac{1}{\alpha^2}z^2$ implies that $y^3z = (xy)^3\frac{z}{x}\frac{1}{x^2} = -\frac{z}{x}\left(\frac{z^2}{\alpha^2}\right)^3\frac{1}{x^2} = \frac{-z}{x}\left(\frac{z^3}{\alpha^3 x}\right)^2$

Therefore, $\partial_{D_i}(\mathcal{A}) = 0$ for all $i \in \{1, 2, 3\}$, hence $\mathcal{A} \in \mathrm{Br}(V_\alpha)$. □

Let $\mathfrak{p}$ be a prime above 2 and $\mathcal{O}_\mathfrak{p}$ be the valuation ring of $k_\mathfrak{p}$. We have that $2 \cdot \alpha^{-1} \in \mathcal{O}_\mathfrak{p}$ if and only if $\alpha^2 \not\equiv 0 \mod 8$; we can define $\mathcal{X}_\alpha$ to be the $\mathcal{O}_\mathfrak{p}$-scheme defined by equation (4.13). If $\alpha^2 \not\equiv 0 \mod 8$, then $\mathcal{X}_\alpha$ is smooth and hence V_α has good reduction at $\mathfrak{p}$, we denote by X_α the base change of V_α to $k_\mathfrak{p}$.

Theorem 4.6.2. *Assume that $\alpha^2 \not\equiv 0 \mod 8$. Then, $\mathcal{X}_\alpha$ has good ordinary reduction if and only if $\alpha^2 \equiv 1 \mod 2$. The evaluation map attached to $\mathcal{A}$*

$$\mathrm{ev}_\mathcal{A} \colon X_\alpha(k_\mathfrak{p}) \to \mathbb{Q}/\mathbb{Z}$$

is non-constant if and only if

$$\alpha^2 \not\equiv 0 \mod 4.$$

Proof. Recall that we know that for K3 surfaces the evaluation map attached to $\mathcal{A}$ is non-constant if and only if $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X_\alpha)$ (cf.Section 4.3).

- If $\alpha^2 \equiv 1 \mod 2$, then the special fibre Y_α is defined by the equation

$$x^3y + y^3z + z^3w + w^4 + xyzw.$$

From Lemma 4.3.1 we get that Y_α is an ordinary K3 surface. From Lemma 4.3.3 we know that a generator (as $k(\mathfrak{p})$-vector space) of $\mathrm{H}^0(Y_\alpha, \Omega^2_{Y_\alpha})$ is given by the global 2-form ω that can be written (locally) as

$$\frac{d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)}{\frac{z^3+xyz}{x^3}} = \frac{x^2}{z^2+xy} d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)\frac{x}{z} =$$
$$\left(\frac{x^2}{z^2+xy}\right) \cdot d\left(\frac{z^2+xy}{x^2}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right)$$

where the last equality follows from $d\left(\frac{z^2+xy}{x^2}\right) \wedge d\left(\frac{z}{x}\right) = d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)$. Hence, we can write ω as $\frac{df}{f} \wedge \frac{dg}{g}$, with $f = \frac{z^2+xy}{x^2}$, $g = \frac{z}{x}$. Finally, we see that, by Proposition 4.2.1

$$[\mathcal{A}] = \left[\left\{\frac{z^2+\alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right\}\right] = \left[\left\{\frac{z^2+\alpha^2 \cdot xy}{x^2}, -\frac{z}{x}\right\}\right] = \rho_0(\omega, 0)$$

since $\frac{z^2+\alpha^2 \cdot xy}{x^2}$ and $-\frac{z}{x}$ are lifts to characteristic 0 of f and g, respectively. Hence, using Proposition 4.2.2 we get that $\mathrm{rsw}_{e',\pi}(\mathcal{A}) \neq (0,0)$ and $\mathcal{A} \notin \mathrm{fil}_{e'-1}\mathrm{Br}(X_\alpha) \supseteq \mathrm{fil}_0\mathrm{Br}(X_\alpha)$.

- If $\alpha^2 \equiv 2 \mod 4$, then the special fibre Y_α is defined by the equation

$$x^3y + y^3z + z^3w + w^4$$

From Lemma 4.3.1 we get that Y_α is a non-ordinary K3 surface over $k(\mathfrak{p})$. From Lemma 4.3.3 we know that $\mathrm{H}^0(Y_\alpha, \Omega^2_{Y_\alpha})$ is generated (as a $k(\mathfrak{p})$-vector space) by the 2-form ω that can be written (locally) as

$$\frac{d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)}{\frac{z^3}{x^3}} = \left(\frac{x^2}{z^2}\right) \cdot d\left(\frac{y}{x}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right) = d\left(\frac{xy}{z^2}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right)$$

where the last equality follow from the fact that, since we are working over a field of characteristic 2, $\left(\frac{x}{z}\right)^2 d\left(\frac{y}{x}\right) = d\left(\frac{xy}{z^2}\right)$. Hence, in this case we can write ω as $d\left(f \cdot \frac{dg}{g}\right)$, with $f = \frac{xy}{z^2}$, $g = \frac{z}{x}$. Since $\alpha^2 \equiv 2 \mod 4$, the prime ideal (2) is ramified in the field extension $\mathbb{Q}_2(\alpha)/\mathbb{Q}_2$ and $\pi = \alpha$ is a uniformiser. From Proposition 4.2.1 we get that

$$\left[\left\{\frac{z^2+\alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right\}\right] = \left[\left\{1+\alpha^2\frac{xy}{z^2}, -\frac{z}{x}\right\}\right] = \rho_2\left(f \cdot \frac{dg}{g}\right)$$

since $\frac{xy}{z^2}$ and $-\frac{z}{x}$ are two lifts to characteristic 0 of f and g, respectively. Hence, by Proposition 4.2.2 $\mathrm{rsw}_{2,\pi}(\mathcal{A}) \neq (0,0)$ and hence $\mathcal{A} \notin \mathrm{fil}_1\mathrm{Br}(X_\alpha)$ and thus $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X_\alpha) \subseteq \mathrm{fil}_1\mathrm{Br}(X_\alpha)$.

- If $\alpha^2 \equiv 0 \mod 4$, then the special fibre Y_α is defined by the equation

$$x^3y + y^3z + z^3w + w^4 + xzw^2.$$

Again, from Lemma 4.3.1 we get that Y_α is a non-ordinary K3 surface over $k(\mathfrak{p})$. If $\mathbb{Q}_2(\alpha)/\mathbb{Q}_2$ is unramified then we are done by Theorem 3.2.1. If the field extension is ramified and π is a uniformiser, then since $\alpha^2 \equiv 0 \mod 4$ and $\alpha^2 \not\equiv 0 \mod 8$, we have that $\alpha^2 = \pi^4\beta$ with $\beta \in \mathcal{O}_{\mathfrak{p}}^\times$. Hence, if we look at the corresponding element in $k^2(K^h)$ via the isomorphism of equation (4.3), we have that

$$\mathcal{A} \mapsto \left\{ \frac{z^2 + \alpha^2 \cdot xy}{z^2}, -\frac{z}{x} \right\} = \left\{ 1 + \pi^4 \cdot \frac{\beta \cdot xy}{z^2}, -\frac{z}{x} \right\} \in U^4k^2(K^h).$$

Hence, using again Proposition 4.2.2, $\mathcal{A} \in \mathrm{fil}_0\mathrm{Br}(X_\alpha)[2]$.

□

Remark 4.6.3. The case $\alpha^2 \equiv 2 \mod 4$ proves that the bound appearing in Theorem 3.2.1 is optimal. In fact, we are able to find examples of K3 surface V over a quadratic field extensions of $\mathbb{Q}$ such that there is a prime above 2 whose ramification index is $e(\mathfrak{p}/2) = 2$ and that plays a role in the Brauer–Manin obstruction to weak approximation.

Finally, note that when $\alpha^2 \equiv 0 \mod 4$ and $e(\mathfrak{p}/2) = 1$, we know from Theorem 3.2.1 that there is an equality $\mathrm{Ev}_{-1}\mathrm{Br}(X_\alpha) = \mathrm{Br}(X_\alpha)$. However, if $e(\mathfrak{p}/2) = 2$, we just showed that the element $\mathcal{A}$ of the previous theorem lies in $\mathrm{Ev}_{-1}\mathrm{Br}(X_\alpha)[2]$ and not that $\mathrm{Br}(X_\alpha) = \mathrm{Ev}_{-1}\mathrm{Br}(X_\alpha)$.