The wild Brauer-Manin obstruction on K3 surfaces

Pagano, M.

CHAPTER 3

The role of primes of good reduction in the Brauer–Manin obstruction

This chapter is based on [Pag23], an article in which we continue to investigate around the following question.

Question 3.0.1. Assume $\mathrm{Pic}(\bar{V})$ to be torsion-free and finitely generated. Which primes can play a role in the Brauer–Manin obstruction to weak approximation on X?

We recall that [BN23, Theorem C] proves that, up to a base change to a finite field extension k'/k, we can always find a prime $\mathfrak{p}'$ of good reduction that plays a role in the Brauer–Manin obstruction to weak approximation on $V_{k'}$.

The main focus of this chapter is on the field extension k'/k appearing in Bright and Newton's result, with a particular emphasis on K3 surfaces. We are going to analyse how the reduction type and the absolute ramification index are involved in the possibility for a prime of good reduction to play a role in the Brauer–Manin obstruction to weak approximation. Let $\mathfrak{p}$ be a prime of good reduction for V, with model $\mathcal{V} \to \mathrm{Spec}(\mathcal{O}_k)$ such that the special fibre $\mathcal{V}(\mathfrak{p})$ has no global 1-form, i.e. $\mathrm{H}^0(\mathcal{V}(\mathfrak{p}), \Omega^1_{\mathcal{V}(\mathfrak{p})}) = 0$. Then, Bright and Newton prove that if the ramification index $e_{\mathfrak{p}}$ is smaller than $(p-1)$, the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V. Hence, already in Bright and Newton paper a link is inferred between the role of a prime $\mathfrak{p}$ of good reduction in the Brauer–Manin obstruction to weak approximation and the ramification index $e_{\mathfrak{p}}$.

In the previous chapter we have shown that there exists a variety V with torsion free and finitely generated geometric Picard group that satisfies the hypothesis of Theorem C of Bright and Newton and for which the field extension k'/k is not needed. In this example V is a K3 surface over the rational numbers and the prime of good ordinary reduction is $\mathfrak{p} = 2$. In particular, we have $e_{\mathfrak{p}} = 1 = 2-1 = p-1$. Hence, one might think that for a prime $\mathfrak{p}$ of good (ordinary) reduction it is enough to satisfy $e_{\mathfrak{p}} \geq (p-1)$, in order to play a role in the Brauer–Manin obstruction to weak approximation. In this chapter we are going to show that this is not the case.

Moreover, inspired by the example of the previous chapter, in Section 3.1.2 we improve the result of Theorem C. In particular, we show that for K3 surfaces it is always possible to find (over a finite field extension k'/k) an element whose order is exactly p and whose evaluation map is non-constant.

On the other hand, in the second part of this chapter we analyse the case of primes of good non-ordinary reduction for K3 surfaces. In particular, we will find also in this case that there is a relation between the possibility for a prime $\mathfrak{p}$ to play a role in the Brauer–Manin obstruction to weak approximation and the ramification index $e_{\mathfrak{p}}$.

3.1 Ordinary good reduction

Let V be a smooth, proper and geometrically integral k-variety and $\mathcal{V}$ an $\mathcal{O}_k$-model of V. In this section we are proving the following theorem.

Theorem 3.1.1. *Let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Assume that the special fibre $\mathcal{V}(\mathfrak{p})$ has no non-trivial global 1-forms, $\mathrm{H}^1(\overline{\mathcal{V}(\mathfrak{p})}, \mathbb{Z}/p\mathbb{Z}) = 0$ and $(p-1) \nmid e_{\mathfrak{p}}$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Let X be the base change to $k_{\mathfrak{p}}$ of V, $\mathcal{X}$ the base change of $\mathcal{V}$ to $\mathcal{O}_{\mathfrak{p}}$ and Y the special fibre $\mathcal{V}(\mathfrak{p})$. As already explained in Section 1.1.3 if we are able to show that for all elements $\mathcal{A} \in \mathrm{Br}(X)$ the evaluation map $\mathrm{ev}_{\mathcal{A}} \colon X(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant, then we get that the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.

We start by giving the definition of ordinary variety.

Definition 3.1.2. *Let Y be a smooth, proper and geometrically integral variety over a perfect field ℓ of positive characteristic. We say that Y is **ordinary** if $\mathrm{H}^i(Y, B^q_Y) = 0$ for every i, q.*

As pointed out in [BK86, Example 7.4] if Y is an abelian variety, then we get back the usual definition of ordinary, namely Y is ordinary if and only if there is an isomorphism between $Y(\bar{\ell})[p]$ and $(\mathbb{Z}/p\mathbb{Z})^{\dim(Y)}$. In general, being ordinary can be defined in many different equivalent ways, see for example [BK86, Proposition 7.3]. In particular, it is related to the slopes of the Newton polygon defined by the action of the Frobenius on the crystalline cohomology groups $\mathrm{H}^q_{\mathrm{cris}}(Y/W)$. Roughly speaking, the role played by the Dieudonné module for abelian varieties is replaced by the F-crystals $\mathrm{H}^q(Y/W)$, with $q \geq 0$. However, since for the aim of this section we can avoid introducing crystalline cohomology and the de Rham-Witt complex we prefer not to add too many technicalities and use the definition of ordinary variety needing to introduce the least number of new concepts. We will see in Section 3.2 how working with the de Rham–Witt complex can be used to get information about the de Rham cohomology of Y and consequently about the Evaluation filtration.

3.1.1 Proof of Theorem 3.1.1

Throughout this section, we will assume that X is such that its special fibre Y is smooth, **ordinary** and such that both $\mathrm{H}^0(Y, \Omega^1_Y)$ and $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z})$ are trivial. In this case, the Cartier operator gives a bijection between the global closed q-forms and the global q-forms on Y. In fact, for every q the short exact sequence $0 \to B^q_Y \to Z^q_Y \xrightarrow{C} \Omega^q_Y \to 0$ induces a long exact sequence in cohomology

$$0 \to \mathrm{H}^0(Y, B^q_Y) \to \mathrm{H}^0(Y, Z^q_Y) \xrightarrow{C} \mathrm{H}^0(Y, \Omega^2_Y) \to \mathrm{H}^1(Y, B^q_Y) \to \dots.$$

The ordinary condition assures the vanishing of $\mathrm{H}^0(Y, B^q_Y)$ and $\mathrm{H}^1(Y, B^q_Y)$ and hence the bijectivity of the Cartier operator $C\colon \mathrm{H}^0(Y, Z^q_Y) \to \mathrm{H}^0(Y, \Omega^q_Y)$.

Lemma 3.1.3. *Let n be an integer such that $0 < n < e'$; then for every $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ we have*

$$\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0$$

Proof. The assumption $\mathrm{H}^0(Y, \Omega^1_Y) = 0$ together with Lemma 1.3.9(1) assure us that if $p \nmid n$, then $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0$. Moreover, as already pointed out in Section 1.2.3.1 if $\mathcal{A} \in \mathrm{Br}(X)$ has order coprime to p, then $\mathcal{A} \in \mathrm{fil}_0\mathrm{Br}(X)$. Therefore we are reduced to the case $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)[p^r]$, with $r \geq 1$ and $p \mid n$. We will prove the lemma by induction on r. If $r = 1$ and $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$, by Corollary 1.3.9(2)

$$(0,0) = \mathrm{rsw}_{n/p,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha), C(\beta)).$$

By Lemma 1.3.8(2) it follows that $(\alpha, \beta) \in \mathrm{H}^0(Y, B^2_Y) \oplus \mathrm{H}^0(Y, B^1_Y)$, and we are done since by the ordinary assumption there are no non-trivial global exact 1 and 2-forms. Assume the result to be true for $r-1$ and let $\mathcal{A} \in \mathrm{Br}(X)[p^r]$, then again by Corollary 1.3.9(2) together with the induction hypothesis we have that

$$(0,0) = \mathrm{rsw}_{n/p,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha), C(\beta))$$

and once again we get that $(\alpha, \beta) \in \mathrm{H}^0(Y, B^2_Y) \oplus \mathrm{H}^0(Y, B^1_Y)$ and therefore both α and β are zero. $\square$

Lemma 3.1.4. *Assume that* $(p-1) \nmid e$*; then for every* $n \geq 1$ *and for every* $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ *we have*

$$\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0.$$

Proof. Let $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)[p^r]$. By the previous lemma, we already know that the result is true if $n < e'$ and we know that the result is true if $p \nmid n$, see 1.3.9(1). Again, we work by induction on r. For $r = 1$, we know from Remark 1.2.25(1) that if $n > e'$ then $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0, 0)$. Hence, we can assume $n \leq e'$. Since by assumption e' is not an integer $n \leq e'$ is equivalent to $n < e'$, we can therefore conclude by applying the previous lemma. Let $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)[p^r]$ and $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$. By assumption we know that if $n \geq e'$ then $n > e'$. From Corollary 1.3.9(3)

$$\mathrm{rsw}_{n-e}(\mathcal{A}^{\otimes p}) = (\bar{u}\alpha, \bar{u}\beta)$$

with $\bar{u} \in \ell^\times$ and the result follows from the induction hypothesis. □

We get therefore the following theorem.

Theorem 3.1.5. *Assume that* Y *is an ordinary variety with no global* 1*-forms and that* $(p-1) \nmid e$*. Then* $\mathrm{Br}(X) = \mathrm{fil}_0\mathrm{Br}(X)$.

The assumption $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$ implies that $\mathrm{Ev}_{-1}\mathrm{Br}(X)\{p\} = \mathrm{fil}_0\mathrm{Br}(X)\{p\}$, see [BN23, Lemma 11.3]. Hence, combining this result with Theorem 3.1.5 we get the proof of Theorem 3.1.1.

3.1.2 On the existence of a Brauer–Manin obstruction over a field extension

In this section we prove a slightly stronger version of [BN23, Theorem C] for K3 surfaces having good ordinary reduction. We show that it is always possible (after a finite field extension) to find an element of order **exactly** p with non-constant evaluation map.

Theorem 3.1.6. *Let* V *be* K3 *surface over a number field* k *and* $\mathfrak{p}$ *be a prime of good ordinary reduction for* V*. Then there exists a finite field extension* k'/k *and an element* $\mathcal{A} \in \mathrm{Br}(V_{k'})[p]$ *that obstructs weak approximation on* $V_{k'}$.

For the proof we change setting and we work over the p-adic field $L := k_\mathfrak{p}$ and we denote by $X, \mathcal{X}$ the base change of $V, \mathcal{V}$ to $k_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ respectively. Moreover, we denote by Y the base change of $\mathcal{V}$ at $\ell := k(\mathfrak{p})$.

Without loss of generality we can assume that L contains a primitive p-root of unity. We fix, for every q, an isomorphism on $X_{\text{ét}}$ between $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}(q)$. Define

$$\mathcal{M}_1^q := i^*\mathrm{R}^q j_* \left(\mathbb{Z}/p\mathbb{Z}(1)\right)$$

where i and j denote the inclusion of the generic and special fibre (X and Y, respectively) in $\mathcal{X}$, cf. Section 1.3.

Remark 3.1.7. The main reference for this section is [BK86]. In [BK86] Bloch and Kato link a "twisted" version of the sheaves $\mathcal{M}_1^q$ with the sheaf of logarithmic forms on Y. In particular, they work with the sheaves $M_1^q := i^* R^q j_* \left(\mathbb{Z}/p\mathbb{Z}(q)\right)$. Since for every q we fixed an isomorphism between $\mathbb{Z}/p\mathbb{Z}(1)$ and $\mathbb{Z}/p\mathbb{Z}(q)$, the same results apply to the sheaves $\mathcal{M}_1^q$.

The sheaves $\mathcal{M}_1^q$ build the spectral sequence of vanishing cycles [BK86, 0.2]:

$$E_2^{s,t} := \mathrm{H}^s(Y, \mathcal{M}_1^t) \Rightarrow \mathrm{H}^{s+t}\left(X, \mathbb{Z}/p\mathbb{Z}(1)\right).$$

By comparing the spectral sequence of vanishing cycles for X and for K^h (which we recall to be defined as the fraction field of the henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},Y}$) we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^2\left(X, \mathbb{Z}/p\mathbb{Z}(1)\right) \arrow[r, "f"] \arrow[d] & \mathrm{H}^0(Y, \mathcal{M}_1^2) \arrow[d, "g"] \\
\mathrm{Br}(K^h)[p] \arrow[r, "\mathrm{res}"] & \mathrm{H}^0(K^h, \mathrm{Br}(K^h_{nr})[p]).
\end{tikzcd}$$

In [BN23, Lemma 3.4] it is proven that the map g is injective. The map f is defined as the composition of the projection $\mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}(1)) \twoheadrightarrow E_\infty^{0,2}$ with inclusion map $E_\infty^{0,2} \hookrightarrow E_2^{0,2}$. Furthermore, the injectivity of g allows us to define a map $f_1 \colon \mathrm{Br}(X)[p] \to \mathrm{H}^0(Y, \mathcal{M}_1^2)$ such that the diagram

$$\begin{tikzcd}
\mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}(1)) \arrow[r, "f"] \arrow[d] & \mathrm{H}^0(Y, \mathcal{M}_1^2) \\
\mathrm{Br}(X)[p] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

commutes, where the vertical arrow comes from the Kummer exact sequence [CTS21, Section 1.3.4]. In fact, every element in $\mathrm{H}^2\left(X, \mathbb{Z}/p\mathbb{Z}(1)\right)$ coming from an element $\delta \in \mathrm{Pic}(X)$ is sent to 0 by f, since its image in $\mathrm{Br}(K^h)[p]$ comes from an element in $\mathrm{Pic}(K^h)$, which is trivial by Hilbert's theorem 90 [CTS21, Theorem 1.3.2].

By [Kat89, Proposition 6.1] we also know that $\ker(\mathrm{res}) = \mathrm{fil}_0\mathrm{Br}(K^h)[p]$, hence we have the diagram

$$\begin{tikzcd}
\mathrm{fil}_0\mathrm{Br}(X)[p] \arrow[r] \arrow[d] & \mathrm{Br}(X)[p] \arrow[r, "f_1"] \arrow[d] & \mathrm{H}^0(Y, \mathcal{M}_1^2) \arrow[d, "g"] \\
\mathrm{fil}_0\mathrm{Br}(K^h)[p] \arrow[r] & \mathrm{Br}(K^h)[p] \arrow[r, "\mathrm{res}"] & \mathrm{H}^0(K^h, \mathrm{Br}(K^h_{nr})[p]).
\end{tikzcd} \tag{3.1}$$

Note that we can build a diagram analogous to (3.1) for every finite field extension L'/L.

We are left to prove the existence (over a field extension L' of L) of an element $\mathcal{A}$ in $\mathrm{Br}(X_{L'})[p]$ such that $f_{L'}(\mathcal{A}) \neq 0$. In order to do this, we analyse the spectral sequence of vanishing cycles over an algebraic closure $\bar{L}$ of L. Let Λ be the integral closure of $\mathcal{O}_L$ in $\bar{L}$ and $\bar{\ell}$ the residue field of Λ. Let $\bar{X}$, $\bar{\mathcal{X}}$ and $\bar{Y}$ be the base change of $X, \mathcal{X}$ and Y to $\bar{L}, \Lambda$ and $\bar{\ell}$ respectively. We define $\bar{\mathcal{M}}_1^q := \bar{i}^* \mathrm{R}^q \bar{j}_* \left(\mathbb{Z}/p\mathbb{Z}(1)\right)$, where $\bar{i}$ and $\bar{j}$ denote the inclusion of the generic and special fibre ($\bar{X}$ and $\bar{Y}$, respectively) in $\bar{\mathcal{X}}$. These sheaves build a spectral sequence of vanishing cycles:

$$\bar{E}_2^{s,t} := \mathrm{H}^s(\bar{Y}, \bar{\mathcal{M}}_1^t) \Rightarrow \mathrm{H}^{s+t}\left(\bar{X}, \mathbb{Z}/p\mathbb{Z}(1)\right).$$

We are interested in the map $\bar{f}_1 \colon \mathrm{Br}(\bar{X})[p] \to \bar{E}_\infty^{0,2}$, defined in the same way as the map f_1.

Combining [BK86, Theorem 8.1] with the short exact sequence (8.0.1) and Proposition 7.3 of [BK86] we get that if Y is ordinary, then

$$\mathrm{H}^n(\bar{Y}, \bar{\mathcal{M}}_1^q) \simeq \mathrm{H}^n(\bar{Y}, \Omega^q_{\bar{Y},\log}) \text{ and } \mathrm{H}^n(\bar{Y}, \Omega^q_{\bar{Y},\log}) \otimes_{\mathbb{F}_p} \bar{\ell} \xrightarrow{\sim} \mathrm{H}^n(\bar{Y}, \Omega^q_{\bar{Y}}). \tag{3.2}$$

for every q, n. From equation (3.2) $\bar{E}_2^{2,1} = \mathrm{H}^2(\bar{Y}, \bar{\mathcal{M}}_1^1) \simeq \mathrm{H}^2(\bar{Y}, \Omega^1_{\bar{Y},\log}) = 0$, where the last equality follows from the fact that for K3 surfaces $\mathrm{H}^2(\bar{Y}, \Omega^1_{\bar{Y}}) = 0$. Moreover, since for K3 surfaces $\mathrm{H}^3(\bar{Y}, \Omega^0_{\bar{Y}}) = 0$, we get from equation (3.2) that also $\bar{E}_2^{3,0}$ vanishes. Thus,

$$\bar{E}_\infty^{0,2} = \ker(\bar{E}_2^{0,2} \to \bar{E}_2^{2,1}) = \bar{E}_2^{0,2}$$

and hence (by construction) the map $\bar{f}_1 \colon \mathrm{Br}(\bar{X})[p] \to \mathrm{H}^0(\bar{Y}, \bar{\mathcal{M}}_1^2)$ is surjective. Finally, using again equation (3.2) we get that $\mathrm{H}^0(\bar{Y}, \bar{\mathcal{M}}_1^2)$ is not trivial.

Proof of Theorem 3.1.6. Let V be a K3 surface over a number field k that contains a primitive p-root of unity ζ and let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Let L be the p-adic field $k_{\mathfrak{p}}$ and X be the base change of V to L. Let $\mathcal{A} \in \mathrm{Br}(\bar{X})[p]$ be such that $\bar{f}_1(\mathcal{A}) \neq 0$, then there is a finite field extension L'/L such that $\mathcal{A}$ is defined over L' (i.e. $\mathcal{A} \in \mathrm{Br}(X_{L'})[p]$) and $f_{1,L'}(\mathcal{A}) \neq 0$. Namely, by what we said above $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X_{L'})[p]$.

Since $\mathrm{Br}(\bar{V})[p] \simeq \mathrm{Br}(V \times_k \overline{k_{\mathfrak{p}}})[p]$ there exists $\mathcal{A} \in \mathrm{Br}(\bar{V})$ such that $\bar{f}_1(\mathcal{A}) \neq 0$. Hence, if k'/k is a finite field extension such that $\mathcal{A}$ is defined over k', then we know from what we have showed above that there is a prime $\mathfrak{p}'$ above $\mathfrak{p}$ such that $\mathrm{ev}_{\mathcal{A}}$ is non-constant on $X_{k'}(k'_{\mathfrak{p}'})$. □

3.2 Non-ordinary good reduction

In this section we analyse also what happens, in the case of K3 surfaces, for primes of good non-ordinary reduction.

Theorem 3.2.1. *Let V be a K3 surface and $\mathfrak{p}$ be a prime of good non-ordinary reduction for V with $e_{\mathfrak{p}} \leq (p-1)$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Like in the previous section, let X be the base change to $k_{\mathfrak{p}}$ of V, then we denote by $\mathcal{X}$ the base change of $\mathcal{V}$ to $\mathcal{O}_{\mathfrak{p}}$ and by Y the special fibre $\mathcal{V}(\mathfrak{p})$. As already explained before it is enough to show that for all element $\mathcal{A} \in \mathrm{Br}(X)$ the evaluation map $\mathrm{ev}_{\mathcal{A}} \colon X(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant.

3.2.1 De Rham–Witt complex and logarithmic forms

The aim of this section is to briefly introduce the de Rham–Witt complex $W\Omega_Y^\bullet$ of a scheme Y over a perfect field ℓ of characteristic p and see how the ordinary condition can be read from it. We will not go into the details of the de Rham–Witt complex nor of crystalline cohomology, we will just use them to study the global sections the sheaf of logarithmic 2-forms on Y.

In [Ill79], Illusie defines the de Rham–Witt complex of Y:

$$W\Omega_Y^\bullet\colon \quad W\Omega_Y^0 = W\mathcal{O}_Y \xrightarrow{d} W\Omega_Y^1 \xrightarrow{d} \dots \xrightarrow{d} W\Omega_Y^q \xrightarrow{d} \dots .$$

To define the de Rham–Witt complex, Illusie begins by building a projective system $\{W_m\Omega_Y^q\}_{m\geq 0}$ equipped with the Frobenius and the Verschiebung maps

$$F\colon W_{m+1}\Omega_Y^q \to W_m\Omega_Y^q \quad \text{and} \quad V\colon W_m\Omega_Y^q \to W_{m+1}\Omega_Y^q$$

for every $m \geq 0$. This projective system has transition maps given by the projection maps $R\colon W_{m+1}\Omega_Y^j \to W_m\Omega_Y^j$. The sheaves $W\Omega_Y^q$ are defined as the inverse limit of the projective system $\{W_m\Omega_Y^q\}_{m\geq 0}$, which is the initial object in a suitable category, see [Ill79, Chapter I]. We will state here some properties of the de Rham–Witt complex, without going into the details.

a. For every non-negative i, j, let $\mathrm{H}^i(Y, W\Omega_Y^j)$ be the corresponding de Rham–Witt cohomology group. The de Rham–Witt cohomology groups are finitely generated W-modules modulo torsion, where $W = W(\ell)$ is the ring of Witt vectors of ℓ.

b. The de Rham–Witt cohomology groups are strictly related to the crystalline cohomology groups of Y (see [BO78] for the definition of the crystalline site and crystalline cohomology). More precisely, there is a spectral sequence, called the the slope spectral sequence:

$$E_1^{i,j} := \mathrm{H}^j(Y, W\Omega_Y^i) \Longrightarrow H^{i+j}_{\text{cris}}(Y/W)$$

and the following degeneracy results [Ill79]:

- the slope spectral sequence always degenerates modulo torsion at E_1;
- the slope spectral sequence degenerates at E_1 if and only if $\mathrm{H}^j(Y, W\Omega_Y^i)$ is a finitely generated W-module for all i, j.

Let σ be the lift of the Frobenius of ℓ to W. The complex $W\Omega_Y^\bullet$ is endowed with a natural σ-semilinear endomorphism $F\colon W\Omega_Y^\bullet \to W\Omega_Y^\bullet$, induced by the Frobenius $F\colon W_{m+1}\Omega_Y^q \to W_m\Omega_Y^q$. Ordinary varieties (see Definition 3.1.2) can also be defined in terms of the action of F on the cohomology groups $\mathrm{H}^j(Y, W\Omega_Y^i)$.

Lemma 3.2.2. *A proper smooth variety Y is ordinary if and only if*

$$F\colon \mathrm{H}^j(Y, W\Omega_Y^i) \to \mathrm{H}^j(Y, W\Omega_Y^i)$$

is bijective for all i, j.

Proof. See [BK86, Section 7]. □

In order to understand how being ordinary is related to logarithmic forms on Y, we first need to introduce the logarithmic Hodge–Witt sheaf.

Definition 3.2.3. *Given two positive integers i, m, the* ***logarithmic Hodge–Witt sheaf*** $W_m\Omega^i_{X,\log}$ *is defined as*

$$W_m\Omega^i_{X,\log} := \operatorname{im}(s\colon (\mathcal{O}_X^\times)^{\otimes i} \to W_m\Omega^i_X),$$

where s is defined by

$$s(x_1 \otimes \cdots \otimes x_i) := d\log \underline{x}_1 \wedge \cdots \wedge d\log \underline{x}_i$$

with $\underline{x} = (x, 0, \dots, 0) \in W_m\mathcal{O}_X$ the Teichmüller representative of $x \in \mathcal{O}_X$. The system $\{W_m\Omega^i_{Y,\log}\}_{m\geq 0}$ is a sub-system of $\{W_m\Omega^i_Y\}_{m\geq 0}$. Moreover, for every i, q

$$\mathrm{H}^i(Y, W\Omega^q_{Y,\log}) := \varprojlim_m \mathrm{H}^i(Y, W_m\Omega^q_{Y,\log})$$

where the limit is take over the projection maps $R\colon W_{m+1}\Omega^j_{Y,\log} \to W_m\Omega^j_{Y,\log}$.

3.2.2 Mittag-Leffler systems

Assume Y to be smooth and proper over ℓ. A crucial fact that we will use several time in the next section is that for every i, j the inverse systems of abelian groups $\{\mathrm{H}^i(Y, W_m\Omega^j_Y)\}_{m\geq 0}$ and $\{\mathrm{H}^i(Y, W_m\Omega^j_{Y,\log})\}_{m\geq 0}$ both satisfy the Mittag-Leffler (ML) condition [IR83, Corollary 3.5, page 194], [Ill79, proof of Proposition 2.1, page 607] and

$$\mathrm{H}^i(Y, W\Omega^j_Y) = \varprojlim_m \mathrm{H}^i(Y, W_m\Omega^j_Y) \quad \text{[Ill79, Proposition 2.1].}$$

The aim of this section is to briefly recap what Mittag-Leffler systems (ML) are and prove some auxiliary lemmas that we will need afterwards. The main reference for this section is [Sta, 02MY].

Let $\mathcal{C}$ be an abelian category. An inverse system consists of a pair (A_i, φ_{ji}), where for every $i \in \mathbb{N}$, $A_i \in \mathrm{Ob}(\mathcal{C})$ and for each $j > i$ there is a map $\varphi_{ji}\colon A_j \to A_i$ such that $\varphi_{ji} \circ \varphi_{kj} = \varphi_{ki}$ for all $k > j > i$. We will often omit the transition maps from the notation. We will denote by $\varprojlim_i A_i$ the inverse limit of the system (A_i); this limit always exists and can be described as follow

$$\varprojlim_i A_i = \left\{ (a_i) \in \prod_i A_i \mid \varphi_{i+1,i}(a_{i+1}) = a_i,\ i = 0, 1, 2, \dots \right\}.$$

See [Sta, 02MY] for more details.

Definition 3.2.4. *We say that (A_i) satisfies the* ***Mittag-Leffler (ML)*** *condition if for every i there is a $c = c(i) \geq i$ such that*

$$\operatorname{Im}(A_k \xrightarrow{\varphi_{ki}} A_i) = \operatorname{Im}(A_c \xrightarrow{\varphi_{ci}} A_i)$$

for all $k \geq c$.

Lemma 3.2.5. *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups.

(1) *If (B_i) is (ML), then also (C_i) is ML.*

(2) *If (A_i) is (ML), then*

$$0 \to \varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i \to 0$$

is exact.

Proof. This is part (2) and (3) of [Sta, 02N1]. □

Lemma 3.2.6. *Let (A_i, φ_{ji}) and (B_i, ψ_{ji}) be two inverse systems of abelian groups. Assume that there are maps of inverse systems $(\pi_i)\colon (A_i) \to (B_i)$ and $(\lambda_i)\colon (B_i) \to (A_{i-1})$ such that $\varphi_{i(i-1)} = \lambda_i \circ \pi_i$ and $\psi_{i(i-1)} = \pi_{i-1} \circ \lambda_i$. Then both (π_i) and (λ_i) induce an isomorphism between $\varprojlim_i A_i$ and $\varprojlim_i B_i$. Moreover, (A_i) is (ML) if and only if (B_i) is (ML).*

Proof. Given an inverse system (A_i) and a positive integer n, we denote by $(A[n]_i)$ the inverse system given by

$$A[n]_i = A_{i-n} \text{ and } \varphi[n]_{(i+1)i} = \varphi_{(i-n+1)(i-n)}$$

whenever $i > n$ and the trivial group otherwise.

The inverse limits of the systems $A[n]_i$ and $A[m]_i$ are isomorphic. In fact, without loss of generality, we can assume $m > n$. The transition maps $\varphi_{(i-n)(i-m)}$ from $A[n]_i$ to $A_i[m]$ induce a map between inverse systems

$$(\varphi[n, m]_i)\colon (A[n]_i) \to (A[m]_i).$$

This map induces an isomorphism $\varphi[n, m]$ between $\varprojlim_i A[n]_i$ and $\varprojlim_i A[m]_i$ sending an element $(a_{i-n}) \in \varprojlim_i A[n]_i$ to $(\varphi_{i-n,i-m}(a_i)) = (a_{i-m}) \in \varprojlim_i A[m]_i$.

Similarly, for every n, m with $m > n$ we can build maps

$$\psi[n, m]\colon \varprojlim_i B[n]_i \to \varprojlim_i B[m]_i$$

$$\lambda[n]\colon \varprojlim_i B[n]_i \to \varprojlim_i A[n+1]_i$$

$$\pi[n]\colon \varprojlim_i A[n]_i \to \varprojlim B[n]_i$$

that fit in following commutative diagram

$$\begin{tikzcd}
\varprojlim_i A[1]_i \arrow[r, "\varphi[1,2]"] & \varprojlim_i A[2]_i \arrow[r, "\varphi[2,3]"] & \varprojlim_i A[3]_i \\
\varprojlim_i B_i \arrow[u, "\lambda[0]"] \arrow[r, "\psi[0,1]"'] & \varprojlim_i B[1]_i \arrow[u, "\lambda[1]"] \arrow[r, "\psi[1,2]"'] \arrow[from=1-1, "\pi[1]"] & \varprojlim_i B[2]_i . \arrow[u, "\lambda[2]"] \arrow[from=1-2, "\pi[2]"]
\end{tikzcd}$$

As the horizontal maps are isomorphism, the maps (π_i) and $(\lambda_{(i+1)i})$ induce the desired isomorphisms.

Assume now that (A_i) is (ML). By definition for every i, there is a $c = c(i)$ such that for all $k \geq c$

$$\mathrm{Im}(A_k \xrightarrow{\varphi_{ki}} A_i) = \mathrm{Im}(A_c \xrightarrow{\varphi_{ci}} A_i).$$

Assume that $k \geq c+1$. Since (B_i) is an inverse system, $\mathrm{Im}(\psi_{ki}) \subseteq \mathrm{Im}(\psi_{(c+1)i})$ always.

At the same time, from the following commutative diagram

$$\begin{tikzcd}
A_k \arrow[r, "\varphi_{kc}"] \arrow[d, "\pi_k"] & A_c \arrow[r, "\varphi_{ci}"] & A_i \arrow[d, "\pi_i"] \\
B_k \arrow[r, "\psi_{k(c+1)}"'] & B_{c+1} \arrow[u, "\lambda_{c+1}"] \arrow[r, "\psi_{(c+1)i}"'] & B_i
\end{tikzcd}$$

we get that for every $b \in B_{c+1}$

$$\psi_{(c+1)i}(b) = \pi_i \circ \varphi_{ci} \circ \lambda_{c+1}(b).$$

Using that the system (A_i) is (ML) we get that there is an element a in A_k such that $\varphi_{ki}(a) = \varphi_{ci} \circ \lambda_{c+1}(b)$. Putting everything together we get that

$$\psi_{ki}(\pi_k(a)) = \psi_{(c+1)i} \circ \psi_{k(c+1)}(\pi_k(a)) = \pi_i \circ \varphi_{ki}(a) = \pi_i \circ \varphi_{(c)i} \circ \lambda_{c+1}(b) = \psi_{(c+1)i}(b).$$

Hence, for all $k \geq c+1$ every element in the image of $\psi_{(c+1)i}$ lies also in the image of ψ_{ki}, namely (B_i) is (ML). Finally, (B_i) being (ML) implies that (A_i) is (ML) as well just follows from replacing (B_i) with $(B[1]_i)$ and switching the roles of (A_i) and (B_i) in what we just proved. $\square$

Lemma 3.2.7. *Let*

$$0 \to (A_{1,i}) \to (A_{2,i}) \to \dots$$

be a long exact sequence of inverse systems of abelian groups. Assume that for every n, the inverse system $(A_{n,i})$ is (ML). Then we have an induced long exact sequence of abelian groups

$$0 \to \varprojlim_i A_{1,i} \to \varprojlim_i A_{2,i} \to \dots$$

Proof. For every n, we define $B_{n,i} := \mathrm{Im}(A_{n-1,i} \to A_{n,i}) = \mathrm{Ker}(A_{n,i} \to A_{n+1,i})$. For every n we get short exact sequences

$$0 \to (B_{n,i}) \to (A_{n,i}) \to (B_{n+1,i}) \to 0.$$

From Lemma 3.2.5(1) together with the assumption on the $(A_{n,i})$'s we get that, for every n, $(B_{n,i})$ is (ML). Using Lemma 3.2.5(2) we get, for every n, a short exact sequence of abelian groups

$$0 \to \varprojlim_i B_{n,i} \to \varprojlim_i A_{n,i} \to \varprojlim_i B_{n+1,i} \to 0.$$

The result now follows from putting all these short exact sequences together. $\square$

3.2.3 Global logarithmic 2-forms

We are now ready to go back to the inverse systems given by the cohomology groups associated to $\{W_m\Omega^q_Y\}_{m\geq 0}$ and $\{W_m\Omega^q_{Y,\log}\}_{m\geq 0}$

Lemma 3.2.8. *For every positive integer i there is a long exact sequence in cohomology*

$$\begin{tikzcd}
\mathrm{H}^0(Y, W\Omega^i_{Y,\log}) \arrow[r, hook] & \mathrm{H}^0(Y, W\Omega^i_Y) \arrow[r, "R-F"] & \mathrm{H}^0(Y, W\Omega^i_Y) \arrow[dll, out=-30, in=150] \\
\mathrm{H}^1(Y, W\Omega^i_{Y,\log}) \arrow[r] & \mathrm{H}^1(Y, W\Omega^i_Y) \arrow[r, "R-F"] & \mathrm{H}^1(Y, W\Omega^i_Y) \arrow[dll, out=-30, in=150] \\
\mathrm{H}^2(Y, W\Omega^i_{Y,\log}) \arrow[r] & \ldots. &
\end{tikzcd}$$

Proof. Following Illusie [Ill79, page 567] we define for every non-negative m

$$\mathrm{Fil}^m W_{m+1}\Omega^i_Y := \ker(R\colon W_{m+1}\Omega^i_Y \to W_m\Omega^i_Y).$$

It is proven in [Ill79, Proposition 3.2, page 568] that $\mathrm{Fil}^m W_{m+1}\Omega^i_Y = V^m\Omega^i_Y + dV^m\Omega^{i-1}_Y$ and hence

$$F(\mathrm{Fil}^m W_{m+1}\Omega^i_Y) = dV^{m-1}\Omega^{i-1}_Y$$

where the last equality follows from the fact that $FV = p$ and $FdV = d$, see [Ill79, Chapter I, page 541].[1] Therefore, the map $F\colon W_{m+1}\Omega^i_Y \to W_m\Omega^i_Y$ induces a map

$$F'\colon W_m\Omega^i_Y \to W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y.$$

In more details: the map R_{m+1} induces an isomorphism between $W_m\Omega^i_Y$ and $W_{m+1}\Omega^i_Y/\ker(R_{m+1})$. We can therefore look at the map induced by F from

[1] In fact, $FV = VF = p$, implies $FV^m = pV^{m-1} = V^{m-1}p$ and $p \cdot \Omega^i_Y = 0$.

$W_{m+1}\Omega^i_Y/\ker(R_{m+1})$ to $W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y$. We denote by π_m the quotient map from $W_m\Omega^i_Y$ to $W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y$ and by $\tilde{\pi}_{m+1}$ the projection from $W_{m+1}\Omega^i_Y$ to $W_{m+1}\Omega^i_Y/\ker(R_{m+1})$. We have the following commutative diagram

$$\begin{tikzcd}
 & W_{m+1}\Omega^i_Y \arrow[dl, "R_{m+1}"'] \arrow[d, "\tilde{\pi}_{m+1}"] \arrow[dr, "F"] & \\
W_m\Omega^i_Y \arrow[r, "\bar{R}^{-1}_{m+1}"] \arrow[dr, "F'"'] & W_{m+1}\Omega^i_Y/\ker(R_{m+1}) \arrow[d, "\bar{F}"] & W_m\Omega^i_Y \arrow[dl, "\pi_m"] \\
 & W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y &
\end{tikzcd}$$

Note that, in particular

$$(\pi_m - F') \circ R_{m+1} = \pi_m \circ (R_{m+1} - F). \tag{3.3}$$

By [CTSS83, Lemme 2] for every non-negative integer m there is a short exact sequence

$$0 \to W_m\Omega^i_{Y,\log} \to W_m\Omega^i_Y \xrightarrow{\pi_m - F'} W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y \to 0.$$

These sequences (as m varies) induce long exact sequences of inverse systems of abelian groups

$$\begin{tikzcd}
\dots \arrow[r] & \{\mathrm{H}^j(Y, W_m\Omega^i_Y)\}_{m\geq 0} \arrow[r, "\pi - F'"] & \{\mathrm{H}^j(Y, W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y)\}_{m\geq 0} \arrow[dll, out=-10, in=170] \\
\{\mathrm{H}^{j+1}(Y, W_m\Omega^i_{Y,\log})\}_{m\geq 0} \arrow[rr] & & \dots
\end{tikzcd}$$

The projection map $R_m \colon W_m\Omega^i_Y \to W_{m-1}\Omega^i_Y$ factors as

$$W_m\Omega^i_Y \xrightarrow{\pi_m} W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y \xrightarrow{\lambda_m} W_{m-1}\Omega^i_Y,$$

see [Ill79, Proposition 3.2, page 568]. Hence, by Lemma 3.2.6 also the inverse system given by $\{\mathrm{H}^j(Y, W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y)\}_{m\geq 0}$ satisfies the Mittag–Leffler condition and π_m induces an isomorphism π

$$\mathrm{H}^j(Y, W\Omega^i_Y) \simeq \varprojlim_m \mathrm{H}^j(Y, W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y).$$

Finally, in order to conclude the proof it is enough to apply Lemma 3.2.7 and to notice that equation (3.3) implies that the isomorphism R induced by the transition maps of the system $\{\mathrm{H}^j(Y, W_m\Omega^i_Y)\}_{m\geq 0}$

$$R\colon \varprojlim_m \mathrm{H}^j(Y, W_{m+1}\Omega^i_Y) \xrightarrow{\sim} \varprojlim_m \mathrm{H}^j(Y, W_m\Omega^i_Y)$$

is such that $(\pi - F') \circ R = \pi \circ (R - F)$. $\square$

The following lemma gives a way to relate the $\mathbb{Z}_p$-module $\mathrm{H}^j(Y, W\Omega^i_{Y,\log})$ to the $\mathbb{Z}/p^m\mathbb{Z}$-modules $\mathrm{H}^j(Y, W_m\Omega^i_{Y,\log})$.

Lemma 3.2.9. *For every positive integer n we have a long exact sequence in cohomology*

$$\begin{tikzcd}
\mathrm{H}^0(Y, W\Omega^i_{Y,\log}) \arrow[r, hook, "p^m"] & \mathrm{H}^0(Y, W\Omega^i_{Y,\log}) \arrow[r] & \mathrm{H}^0(Y, W_m\Omega^i_{Y,\log}) \arrow[dll, rounded corners] \\
\mathrm{H}^1(Y, W\Omega^i_{Y,\log}) \arrow[r, "p^m"] & \mathrm{H}^1(Y, W\Omega^i_{Y,\log}) \arrow[r] & \mathrm{H}^1(Y, W_m\Omega^i_{Y,\log}) \arrow[dll, rounded corners] \\
\mathrm{H}^2(Y, W\Omega^i_{Y,\log}) \arrow[r, "p^m"] & \dots &
\end{tikzcd}$$

Proof. In [CTSS83, Lemme 3] the authors show that there is, for every positive m, a short exact sequence of sheaves

$$0 \to W_n\Omega^i_{\log} \xrightarrow{p^m} W_{n+m}\Omega^i_{\log} \to W_m\Omega^i_{\log} \to 0$$

Again, the result follows from taking the long exact sequence of inverse systems of abelian groups in cohomology, then using that the systems involved satisfy the Mittag–Leffler condition and hence applying Lemma 3.2.7. ☐

Proposition 3.2.10. *Let Y be a smooth, proper variety over a a perfect field ℓ of positive characteristic. Assume that $\mathrm{H}^0(Y, W\Omega^q_Y) = 0$ and $\mathrm{H}^1(Y, W\Omega^q_Y)$ is torsion-free. Then for all $m \geq 1$*

$$\mathrm{H}^0(Y, W_m\Omega^q_{Y,\log}) = 0.$$

Proof. The long exact sequence in cohomology of Lemma 3.2.8 together with the vanishing of $\mathrm{H}^0(Y, W\Omega^q_Y)$ imply that

$$\mathrm{H}^0(Y, W\Omega^q_{Y,\log}) = 0 \quad \text{and} \quad \mathrm{H}^1(Y, W\Omega^q_{Y,\log}) \subseteq \mathrm{H}^1(Y, W\Omega^q_Y).$$

In particular, $\mathrm{H}^1(Y, W\Omega^q_{Y,\log})$ is torsion-free as well. Finally, we can use the long exact sequence of Lemma 3.2.9 to get that

$$\mathrm{H}^0(Y, W_m\Omega^q_{Y,\log}) = \ker(\mathrm{H}^1(Y, W\Omega^q_{Y,\log}) \xrightarrow{p^m} \mathrm{H}^1(Y, W\Omega^q_{Y,\log}))$$

which proves the result. ☐

The idea is to use the lemma we just proved to show that for non-ordinary K3 surfaces over perfect fields there are no non-trivial global logarithmic 2-forms. We will see at the end of the section how a proof of this fact follows immediately from the following lemma and some results proven by Illusie [Ill79].

Lemma 3.2.11. *Assume that* $\mathrm{H}^0(Y, W\Omega^2_Y)$ *is trivial and* $\mathrm{H}^1(Y, W\Omega^2_Y)$ *is torsion free. Then*

$$\mathrm{H}^0(Y, \Omega^2_{Y,\log}) = 0.$$

If $\mathrm{H}^0(Y, \Omega^2_Y)$ *is a* 1*-dimensional* ℓ*-vector space, then the Cartier operator is zero on the space of global* 2*-forms.*

Proof. The first part is a direct consequence of Proposition 3.2.10 with $q = 2$.

If $\mathrm{H}^0(Y, \Omega^2_Y)$ is a 1-dimensional ℓ-vector space, for every $\omega \in \mathrm{H}^0(Y, \Omega^2_Y)$ there exists $\lambda \in \ell$ such that

$$C(\omega) = \lambda \cdot \omega.$$

Assume that $\lambda \neq 0$, then there exists a finite field extension ℓ'/ℓ that contains an element λ_0 such that $\lambda = \lambda_0^{p-1}$. In particular, we get

$$C(\lambda_0^p \cdot \omega) = \lambda_0 \cdot C(\omega) = (\lambda_0 \lambda) \cdot \omega = \lambda_0^p \cdot \omega.$$

This implies, by definition of logarithmic forms, that $\lambda_0^p \cdot \omega \in \mathrm{H}^0(Y_{\ell'}, \Omega^2_{Y_{\ell'},\log})$, which gives the desired contradiction (in fact the first part of the Lemma applies also to the base change of Y to ℓ'). $\square$

We go back to the general setting of Section 1.3 and we work under the extra assumption that the special fibre Y has good non-ordinary reduction and no global 1-forms.

Lemma 3.2.12. *Assume that the special fibre* Y *of the* $\mathcal{O}_L$*-model* $\mathcal{X}$ *is such that there are no non-trivial global logarithmic* 2*-forms and* $\mathrm{H}^0(Y, \Omega^2_Y)$ *is a* 1*-dimensional* ℓ*-vector space. Then, if the absolute ramification index* $e \leq p-1$*, we have* $\mathrm{Br}(X) = \mathrm{fil}_0\mathrm{Br}(X)$*.*

Proof. If $e < p-1$ the result is proven by Bright and Newton, [BN23, Lemma 11.2] without any assumption on the global section of the sheaves $\Omega^2_{Y,\log}$ and Ω^2_Y. If $e = p-1$, let $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ with $\mathrm{rsw}_{n,\pi} = (\alpha, \beta)$. In order to prove the Lemma it is enough to show that $(\alpha, \beta) = (0, 0)$.

- If $n < e'$ then $p \nmid n$ and hence by Corollary 1.3.9(1) and the assumption that $\mathrm{H}^0(Y, \Omega^1_Y) = 0$ we get $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0, 0)$.
- If $n > e'$ then $p \nmid n$ or $p \nmid n-e$ and hence either $\mathrm{rsw}_{n,\pi}(\mathcal{A})$ or $\mathrm{rsw}_{n-e,\pi}(\mathcal{A}^{\otimes p})$ is zero and therefore by Corollary 1.3.9(2) $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0, 0)$.
- If $n = e'$, then by Corollary 1.3.9(3) we have

  $$\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha) - \bar{u}\alpha, C(\beta) - \bar{u}\beta).$$

  However, by assumption on Y there are no non-trivial global 1-forms, hence $\beta = 0$. Moreover, by Lemma 3.2.11 $C(\alpha) = 0$. Hence,

  $$\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = (-\bar{u}\alpha, 0).$$

  Finally, $e' - e < e'$ and hence $\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = 0$, which implies that $\alpha = 0$ and thus also in this case $\mathrm{rsw}_{e',\pi}(\mathcal{A}) = (0, 0)$.

□

In [Ill79, Chapter 7] Illusie shows that if Y is a non-ordinary K3 surface, then Y satisfies the assumptions of Lemma 3.2.11. Moreover, as already mentioned at the end of Section 4.3 from [Ier22, Proposition 2.3] we know that $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X)$ if and only if $\mathrm{ev}_{\mathcal{A}}$ is non-constant on $X(L)$. This shows how the ordinary assumption was needed in the example that we gave in Chapter 2, cf. Theorem 2.1.10.