The wild Brauer-Manin obstruction on K3 surfaces

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CHAPTER 2

An example of Brauer–Manin obstruction coming from a prime of good reduction

2.1 Swinnerton-Dyer's question

This chapter is based on [Pag22]. The main result of this chapter gives an answer, over the rational numbers, to the following question, asked by Swinnerton-Dyer to Colliot-Thélène and Skorobogatov [CTS13, Question 1].

Question 2.1.1 (Swinnerton-Dyer)**.** Let k be a number field and let S be a finite set of places of k containing the archimedean places. Let $\mathcal{V}$ be a smooth projective $\mathcal{O}_{k,S}$-scheme with geometrically integral fibres, and let V/k be the generic fibre. Assume that $\mathrm{Pic}(\overline{V})$ is finitely generated and torsion-free. Swinnerton-Dyer asks if there is an open and closed $Z \subseteq \prod_{\nu\in S} V(k_\nu)$ such that

$$V(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{\nu\notin S} V(k_\nu).$$

If $V(\mathbf{A}_k)^{\mathrm{Br}}$ is non-empty and can be described as in Swinnerton-Dyer's question, then for any $\omega \notin S$, and for any $\mathcal{A} \in \mathrm{Br}(V)$ the corresponding evaluation map $\mathrm{ev}_{\mathcal{A}}\colon V(k_\omega) \to \mathbb{Q}/\mathbb{Z}$ has to be constant. In fact, for any $(x_\nu) \in Z\times\prod_{\nu\notin S\cup\{\omega\}} V(k_\nu)$ we have that, for any $y_\omega, \tilde{y}_\omega \in V(k_\omega)$

$$\sum_{\nu\in\Omega_k\setminus\{\omega\}} \mathrm{ev}_{\mathcal{A}}(x_\nu) + \mathrm{ev}_{\mathcal{A}}(y_\omega) = \sum_{\nu\in\Omega_k\setminus\{\omega\}} \mathrm{ev}_{\mathcal{A}}(x_\nu) + \mathrm{ev}_{\mathcal{A}}(\tilde{y}_\omega) = 0.$$

This is because, from the description of the Brauer–Manin set:

$$(x_\nu)_{\nu\neq\omega} \cup (y_\omega),\ (x_\nu)_{\nu\neq\omega} \cup (\tilde{y}_\omega) \in V(\mathbf{A}_k)^{\mathrm{Br}}$$

That is, all the places $\omega \notin S$ do not play a role in the Brauer–Manin obstruction to weak approximation on X (cf. Definition 1.1.10).

We can therefore reformulate Swinnerton-Dyer's question in the following way: is it true that under the assumption of the question asked above the only places that can play a role in the Brauer–Manin obstruction to weak approximation are the places of bad reduction and the archimedean places?

2.1.1 Torsion in the geometric Picard group

The aim of this section is to briefly explain why the assumption on the torsion of the geometric Picard group is needed. This is a consequence of work of Harari [Har00] and Skorobogatov [Sko01]. In their work they use abelian descent obstruction, which arise from torsors (for an introduction to torsors we refer to [Sko01, Chapter 2]).

Let V be a smooth, proper and geometrically integral variety over a number field k. In 1989 Minchev [Min89] has shown that the existence of an étale morphism of geometrically integral varieties implies the failure of weak approximation on V (he proved the failure of strong approximation, which for proper varieties coincide with weak approximation). In particular, if $\mathrm{Pic}(\bar{V})$ has torsion, then we immediately get that the étale fundamental group (cf. [Mil80, Chapter I, Section 5]) $\pi^1_{\text{ét}}(X)$ is non-trivial and, by a result of Harari [Har00, proof of Lemma 5.2(1)], there exists a geometrically integral étale morphism. However, if we want to link the obstruction arising from this morphism to the Brauer–Manin obstruction, we need the covering to be abelian, which is something we can guarantee up to enlarging the base field k.

The argument in this section is divided into three parts: in the first part, it is shown how the obstruction coming from an abelian Galois covering is linked to the Brauer-Manin obstruction; in the second, it is shown how the presence of torsion in the geometric Picard group implies the existence of a non-trivial abelian Galois covering defined over a field extension of the base field k; in the third, it is shown how the existence of a non-trivial Galois covering implies the presence of an obstruction to weak approximation.

Following Colliot-Thélène and Sansuc [CTS87], in [Sko01, Section 6.1] Skorobogatov proves how it possible to describe the algebraic Brauer–Manin obstruction using torsors of multiplicative type. Let G be a k-group of multiplicative type with dual group M, then there exists a natural map

$$\text{type}\colon \mathrm{H}^1(V,G) \to \mathrm{Hom}_{\Gamma_k}(M, \mathrm{Pic}(\bar{V})). \tag{2.1}$$

Any element $\gamma \in M = \mathrm{Hom}(G, \mathbb{G}_m)$ induces a map $\gamma^*\colon \mathrm{H}^1(\bar{V},G) \to \mathrm{H}^1(\bar{V},\mathbb{G}_m)$, and the type of $f \in \mathrm{H}^1(V,G)$ is defined as the map sending γ to $\gamma^*(f)$.

Given a torsor $f\colon W \to V$ under G, we define

$$V(\mathbf{A}_k)^f := \bigcup_{\sigma \in \mathrm{H}^1(k,G)} f^\sigma(W^\sigma(\mathbf{A}_k)).$$

For $\sigma \in \mathrm{H}^1(k, G)$, $f^\sigma\colon W^\sigma \to V$ is defined as the twist of the torsor f, see [Sko01, Section 2.2, Example 2]. In general, f^σ is a G^σ-torsor, where G^σ is the Galois twist of G by σ, [Sko01, Section 2.1] and if G is abelian, then $G^\sigma = G$.

Remark 2.1.2. The torsor f under G represents an element of $\mathrm{H}^1(V, G)$; the set $V(\mathbf{A}_k)^f$ coincide with the set defined in Chapter 1, Section 1.1.1 with respect to the functor $\mathrm{F} := \mathrm{H}^1(-, G)$, see [Sko01, Section 5.3].

Skorobogatov proved the following theorem.

Theorem 2.1.3. *Let M be a Γ_k-module of finite type, S its dual group of multiplicative type and $\lambda \in \mathrm{Hom}_{\Gamma_k}(M, \mathrm{Pic}(\bar{V}))$. Then*

(a) there are only finitely many isomorphism classes of torsors $f\colon W \to V$ of type λ such that $W(\mathbf{A}_k) \neq \emptyset$;

(b) there exists a subgroup $\mathrm{Br}_\lambda(V) \subseteq \mathrm{Br}_1(V)$ such that

$$V(\mathbf{A}_k)^{\mathrm{Br}_\lambda} = \bigcup_{\mathrm{type}(W,f)=\lambda} f(W(\mathbf{A}_k)).$$

Proof. A proof can be found in [Sko01, Section 6.1]. In particular, note that since V is proper and smooth, we always have $\bar{k}[V]^* = \bar{k}^*$. □

We want to show that, up to enlarging the base field k, there is a type λ for which the corresponding Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$ is not of the shape appearing in Swinnerton-Dyer's question.

Assume that there is torsion in $\mathrm{Pic}(\bar{V})$, then up to enlarging the base field, we can assume that there is torsion already defined over the base field k. By Kummer theory there exists n such that $f \in \mathrm{H}^1(V, \mu_n)$ does not become trivial in $\mathrm{H}^1(\bar{V}, \mu_n)$. Moreover, again up to (possibly) enlarging the base field, we can assume that it contains a primitive n-root of unity and hence to have an isomorphism

$$\mathrm{H}^1(V, \mu_n) \simeq \mathrm{H}^1(V, \mathbb{Z}/n\mathbb{Z}) \simeq \pi_1(V, \mathbb{Z}/n\mathbb{Z}).$$

We denote by $\rho\colon W \to V$ the Galois covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to the μ_n-torsor $f\colon W \to V$. In [Har00] Harari proves the following result for Galois coverings.

Lemma 2.1.4. *Let $W \xrightarrow{\rho} V$ be a geometrically non-trivial Galois covering, with V geometrically connected of positive dimension. Then there exists a finite field extension k'/k such that, for almost all places ν which are totally split for k'/k, there exists a k_ν point x_ν of V such that $x_\nu \in V(k_\nu)$ but $x_\nu \notin \rho(W(k_\nu))$.*

Proof. This is a reformulation of [Har00, Lemma 2.3] with $g = \mathrm{id}$. □

We know that there exists at least one geometrically non-trivial μ_n-torsor $f\colon W \to V$. We denote by λ its type. Then, there exists $f_1, \dots, f_r$ torsors under μ_n of type λ such that

$$V(\mathbf{A}_k)^{\mathrm{Br}_\lambda} = \bigcup_{i=1}^{r} f_i(W_i(\mathbf{A}_k)).$$

Let S be any finite set of places. Then from Lemma 2.1.4 we know that for any $i \in \{1, \dots, r\}$ there exists a finite field extension k_i/k and infinitely many places ν_i that are totally split for k_i/k such that there is a point $(x_{\nu_i}) \in V(k_{\nu_i})$ but $(x_{\nu_i}) \notin f_i(W_i(k_{\nu_i}))$. If we pick the places ν_i all different from each other and not in S, then any point $(x_\nu) \in V(\mathbf{A}_k)$ such that

$$x_{\nu_i} \notin f_i(W_i(k_{\nu_i})) \text{ for } i = 1, \dots, r$$

does not belong to $V(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$.

However, if there exists a finite set of places S such that

$$V(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{\nu \notin S} V(k_\nu) \subseteq V(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$$

then for every $z \in Z$ and every $(x_\nu) \in \prod_{\nu \notin S} V(k_\nu)$, the point $z \cup (x_\nu)$ has to lie in $V(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$.

2.1.2 Previous works

In this section we give an overview of the main results that have been proved in the attempt to answer the question asked by Swinnerton-Dyer.

The question appears for the first time in [CTS13], paper in which Colliot-Thélène and Skorobogatov prove some results around it. In particular, they work under the following setting: L is a p-adic field and $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_L)$ is a smooth proper model with geometrically integral fibres of a variety X defined over L. In [CTS13] Colliot-Thélène and Skorobogatov prove the following result [CTS21, Proposition 10.4.3].

Proposition 2.1.5. *Let q be a prime, $q \neq \mathrm{char}(\ell)$. Assume that the closed geometric fibre Y of $\mathcal{X}$ has no connected unramified cyclic covering of degree q. Then $\mathrm{Br}(X)\{q\}$ is generated by the images of $\mathrm{Br}(\mathcal{X})\{q\}$ and $\mathrm{Br}(L)\{q\}$.*

The condition on the closed geometric fibre is equivalent to the vanishing of the étale cohomology group $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/q\mathbb{Z})$. This condition holds if there is no q-torsion in the geometric Picard group of the generic fibre, see [BN23, Lemma 11.4].

In [CTS13] Colliot-Thélène and Skorobovatov prove also that if the Picard group is torsion free and finitely generated, then

$$\mathrm{Br}_1(X) = \ker\left(\mathrm{Br}(X) \to \mathrm{Br}(X_{un})\right), \quad \text{[CTS21, Proposition 10.4.2]}$$

where X_{un} is the base change of X to the maximal unramified extension L_{un} of L. Moreover, they prove that the map

$$\mathrm{Br}(L) \oplus \ker\big(\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_{un})\big) \to \ker\big(\mathrm{Br}(X) \to \mathrm{Br}(X_{un})\big)$$

is surjective. Combining these two results with diagram (1.9) we get that every element $\mathcal{A} \in \mathrm{Br}_1(X)$ has constant evaluation map. Putting everything together they get the following theorem, for varieties over number fields.

Theorem 2.1.6. *Let k be a number field. Let S be a finite set of places of k containing the archimedean places, and let $\mathcal{O}_{k,S}$ be the ring of S-integers of k. Let $\mathcal{V}$ over $\mathcal{O}_S$ be a smooth and proper scheme with geometrically integral generic fibre V. Assume that*

1. $\mathrm{Pic}(\bar{V})$ *is torsion free and finitely generated;*
2. *the transcendental Brauer group* $\mathrm{Br}(V)/\mathrm{Br}_1(V)$ *is a finite abelian group of order invertible in $\mathcal{O}_{k,S}$.*

Then $V(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{\nu \notin S} V(k_\nu)$, where $Z \subseteq \prod_{\nu \in S} V(k_\nu)$ is an open and closed subset.

Proof. This is [CTS21, Theorem 13.3.15]. Note that in their result Colliot-Thélène and Skorobogatov split the assumption on the geometric Picard group in two assumptions: the vanishing of $\mathrm{H}^1(V, \mathcal{O}_V)$ and the torsion freeness of the geometric Neron-Severi group $\mathrm{NS}(\bar{V})$. □

This theorem shows that under the finiteness assumption on the transcendental Brauer group, up to enlarging the set S, the question asked by Swinnerton-Dyer has a positive answer. Some years later Bright and Newton [BN23] were able to prove another result in the same direction, but without any assumption on the transcendental Brauer group, using the notion of refined Swan conductor.

Theorem 2.1.7. *Let k be a number field. Assume V to be a smooth, proper and geometrically integral k-variety. Assume* $\mathrm{Pic}(\bar{V})$ *to be torsion free and finitely generated. Let S be the following set of places:*

1. *Archimedean places;*
2. *places of bad reduction for V;*
3. *places $\mathfrak{p}$ satisfying $e_{\mathfrak{p}} \geq p-1$, with p residue characteristic of $\mathfrak{p}$;*
4. *places $\mathfrak{p}$ with $\mathrm{H}^0(\mathcal{V}(\mathfrak{p}), \Omega^1_{\mathcal{V}(\mathfrak{p})})$ non-trivial, for any smooth integral model $\mathcal{V} \to \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}})$, where $\mathcal{V}(\mathfrak{p})$ is the special fibre at $\mathfrak{p}$.*

Then the set S is finite and there exists $Z \subseteq \prod_{\nu \in S} V(k_\nu)$ open and closed such that $V(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{\nu \notin S} V(k_\nu)$.

Proof. [BN23, Theorem D]. □

On the other hand, in [BN23] Bright and Newton prove also a result showing that after a finite field extension primes of good **ordinary** reduction always play a role in the Brauer–Manin obstruction to weak approximation (for the definition of good ordinary reduction see Chapter 3, Definition 3.1.2).

Theorem 2.1.8. *Let V be a smooth, proper and geometrically integral variety over a number field k such that $\mathrm{H}^0(V, \Omega^2_V) \neq 0$. Let $\mathfrak{p}$ be a finite place of k at which V has good ordinary reduction, with residue characteristic p. Then there exist a finite extension k'/k, a place $\mathfrak{p}'$ of k' over $\mathfrak{p}$ and an element $\mathcal{A} \in \mathrm{Br}(V_{k'})\{p\}$ such that the evaluation map $\mathrm{ev}_{\mathcal{A}} \colon V(k'_{\mathfrak{p}'}) \to \mathbb{Q}/\mathbb{Z}$ is non-constant.*

In particular, this proves that in general the question asked by Swinnerton-Dyer has a negative answer.

The rest of this chapter is devoted to build the first example of a variety satisfying the assumption of Swinnerton-Dyer's question for which a prime of good reduction plays a role in the Brauer–Manin obstruction to weak approximation. This example consist of a K3 surface over the rational numbers, proving that the question asked by Swinnerton-Dyer has a negative answer already over $\mathbb{Q}$.

Before proceeding with the main result of this chapter, we give a short overview of K3 surfaces and explain why they form the natural playground in trying to build counterexample to Swinnerton-Dyer's question.

2.1.3 K3 surfaces

Let k be any field, we will work just with algebraic K3 surfaces. For a smooth variety, we write ω_X for the canonical sheaf of X.

Definition 2.1.9. *An algebraic K3 surface is a smooth projective 2-dimensional variety over a field k such that $\omega_X \simeq \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.*

In this thesis the examples will always be of these two kinds (see [Huy16, Section 1.1] for more details):

1. X smooth quartic surface in $\mathbb{P}^3_k$;

2. $X = \mathrm{Kum}(A)$, where A is an abelian surface over a field of characteristic different from 2 and X is obtain by resolving singularities from the quotient A/ι, with $\iota \colon A \to A$ involution given by $x \mapsto -x$.

The de Rham cohomology of K3 surfaces over algebraically closed fields can be summarised in the Hodge diamond, which looks like this

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

where $h^{i,j} = \dim_k \mathrm{H}^i(X, \Omega^j_X)$.

As mentioned before, K3 surfaces are the natural playground to build examples of varieties that satisfy both the assumption in Swinnerton-Dyer's question and in Bright and Newton Theorem 2.1.8. The Picard group is finitely generated and torsion free, see [Huy16, Section 1.2]. If we start with a K3 surface over a number field k then, since the canonical bundle is trivial by definition, we get that $\mathrm{H}^0(X, \Omega^2_X)$ has dimension 1 as a k-vector space. Moreover, Bogomolov and Zarin [BZ09] prove that there exists a finite field extension k'/k and a density 1 set of finite places Σ of k' such that $X_{k'}$ has ordinary good reduction at every place $\nu \in \Sigma$.

We are ready to state the main theorem of this chapter and of [Pag22].

Theorem 2.1.10. *Let $V \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the projective K3 surface defined by the equation*

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \tag{2.2}$$

The class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right) \in \operatorname{Br} \mathbb{Q}(V)$$

defines an element in $\operatorname{Br}(V)$*. The evaluation map* $\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{Q}_2) \to \operatorname{Br}(\mathbb{Q}_2)$ *is non-constant, and therefore gives an obstruction to weak approximation on* V*. Finally,* $V(\mathbb{Q})$ *is not dense in* $V(\mathbb{Q}_2)$*, with respect to the analytic topology.*

2.2 Proof of Theorem 2.1.10

In the first part of the proof we will show that the element $\mathcal{A} \in \operatorname{Br}(k(V))$ lies in $\operatorname{Br}(V)$. Next, we will exhibit two points $P_1, P_2 \in V(\mathbb{Q}_2)$ such that

$$\mathrm{ev}_{\mathcal{A}}(P_1) \neq \mathrm{ev}_{\mathcal{A}}(P_2).$$

Finally, we will prove that, for every place ν different from 2, the evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{Q}_\nu) \to \mathbb{Q}/\mathbb{Z}$$

is constant.

Proof of Theorem 2.1.10. Let $f := z^3 + w^2x + xyz$ and C_x,C_z,C_f be the closed subsets of V defined by the equations $x = 0$, $z = 0$ and $f = 0$ respectively. The quaternion algebra $\mathcal{A}$ defines an element in $\mathrm{Br}(U)$, where $U := V \setminus (C_x \cup C_z \cup C_f)$. The purity theorem for the Brauer group [CTS21, Theorem 3.7.2], assures us the existence of the exact sequence

$$0 \to \mathrm{Br}(V)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus \partial_D} \bigoplus_D \mathrm{H}^1(k(D), \mathbb{Z}/2) \tag{2.3}$$

where D ranges over the irreducible divisors of X with support in $X \setminus U$ and $k(D)$ denotes the residue field at the generic point of D.

In order to use the exact sequence (4.14) we need to understand what the prime divisors of V with support in $X \setminus U = C_x \cup C_z \cup C_f$ look like. It is possible to check the following:

- C_x has as irreducible components D_1 and D_2, defined by the equations $\{x = 0, z = 0\}$ and $\{x = 0, y^3 + z^2w = 0\}$ respectively;
- C_z has as irreducible components D_1 and D_3, where D_3 is defined by the equations $\{z = 0, x^2y + w^3 = 0\}$;
- C_f has as irreducible components D_1, D_4 and D_5, where D_4 and D_5 are defined by the equations $\{z^3 + xw^2 = 0, y = 0\}$ and $\{y^3z - x^2z^2 + y^2w^2 = 0, x^3 + y^2z = 0, xyz + z^3 + xw^2 = 0\}$ respectively.

Therefore, we can rewrite (4.14) in the following way:

$$0 \to \mathrm{Br}(V)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus \partial_{D_i}} \bigoplus_{i=1}^{5} H^1(k(D_i), \mathbb{Z}/2). \tag{2.4}$$

Moreover, we have an explicit description of the residue map on quaternion algebras: for an element $(a, b) \in \mathrm{Br}(U)[2]$ we have

$$\partial_{D_i}(a, b) = \left[(-1)^{\nu_i(a)\nu_i(b)} \frac{a^{\nu_i(b)}}{b^{\nu_i(a)}}\right] \in \frac{k(D_i)^\times}{k(D_i)^{\times 2}} \simeq H^1(k(D_i), \mathbb{Z}/2) \tag{2.5}$$

where ν_i is the valuation associated to the prime divisor D_i. This is already explained in Chapter 1, (1.4).

We can proceed with the computation of the residue maps ∂_{D_i} for $i = 1, \ldots, 5$:

1. $\nu_1(f) = \nu_1(x) = \nu_1(z) = 1$. Hence,
$$\partial_{D_1}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[\left(-\frac{z}{x}\right)^2\right] = 1 \in \frac{k(D_1)^\times}{k(D_1)^{\times 2}}.$$
2. $\nu_2(x) = 1$ and $\nu_2(f) = \nu_2(z) = 0$. Hence,
$$\partial_{D_2}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[-\left(\frac{f}{x^3}\right)^{-1}\left(-\frac{z}{x}\right)^3\right] = \left[\frac{z^3}{f}\right] = 1 \in \frac{k(D_2)^\times}{k(D_2)^{\times 2}}$$

where the last equality follows from the fact that $x = 0$ on D_2, thus we have $f\,|_{D_2} = z^3$.

3. $\nu_3(x) = \nu_3(f) = 0$ and $\nu_3(z) = 1$. Hence,

$$\partial_{D_3}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[\left(\frac{f}{x^3}\right)\right] = \left[\left(\frac{w}{x}\right)^2\right] = 1 \in \frac{k(D_3)^\times}{k(D_3)^{\times 2}}$$

where the last equality follows from the fact that $z = 0$ on D_3, thus we have $f\,|_{D_3} = w^2 x$.

4. $\nu_4(x) = \nu_4(z) = 0$ and $\nu_4(f) = 1$. Hence,

$$\partial_{D_4}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[\left(-\frac{x}{z}\right)\right] = \left[\left(\frac{w}{z}\right)^2\right] = 1 \in \frac{k(D_4)^\times}{k(D_4)^{\times 2}}$$

where the last equality follows from the fact that $z^3 + w^2 x = 0$ on D_4, thus $-\frac{x}{z} = \left(\frac{w}{z}\right)^2$.

5. $\nu_5(x) = \nu_5(z) = 0$ and $\nu_5(f) = 1$. Hence,

$$\partial_{D_5}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[\left(-\frac{x}{z}\right)\right] = \left[\left(\frac{y}{x}\right)^2\right] = 1 \in \frac{k(D_5)^\times}{k(D_5)^{\times 2}}$$

where the last equality follows from the fact that $x^3 + y^2 z = 0$ on D_5, thus $-\frac{x}{z} = \left(\frac{y}{x}\right)^2$.

Therefore, $\partial_{D_i}(\mathcal{A}) = 0$ for all $i \in \{1, \ldots, 5\}$, hence $\mathcal{A} \in \mathrm{Br}(X)$.

We now show that the element $\mathcal{A}$ obstructs weak approximation on V. Let $\mathcal{V} \subseteq \mathbb{P}^3_{\mathbb{Z}}$ be the projective scheme defined by the equation

$$x^3 y + y^3 z + z^3 w + w^3 x + xyzw = 0. \tag{2.6}$$

$\mathcal{V}$ is a $\mathbb{Z}$-model for V and has good reduction at the prime 2.

Let $P_1 := (1 : 0 : 1 : 0) \in \mathcal{V}(\mathbb{Z}_2)$; then P_1 is such that $\mathrm{ev}_{\mathcal{A}}(P_1) = (1, -1)$. Therefore $\mathrm{ev}_{\mathcal{A}}(P_1)$ is the trivial class in $\mathrm{Br}(\mathbb{Q}_2)$. Moreover, Hensel's lemma assures us of the existence of a solution $P_2 = (1 : 2 : 1 : d) \in \mathcal{V}(\mathbb{Z}_2)$ whose reduction modulo 8 is $(1 : 2 : 1 : 2)$. Hence,

$$\mathrm{ev}_{\mathcal{A}}(P_2) = (f(P_2), -1) \quad \text{with} \quad f(P_2) \equiv 7 \pmod 8.$$

Therefore, we get that $\mathrm{ev}_{\mathcal{A}}(P_2)$ defines a non-trivial element in the Brauer group of $\mathbb{Q}_2$ [Ser73, Theorem 3.1]. The existence of such points implies that there is a Brauer–Manin obstruction to weak approximation arising from $\mathcal{A}$.

In order to conclude the proof of the theorem we investigate the behaviour of the evaluation map at the other primes and at infinity. For every prime p let $\mathcal{V}_p$ be the base change of $\mathcal{V}$ to $\mathbb{Z}_p$. We distinguish the following cases.

Case $p \notin \{3, 5, 17, \infty\}$. In this case, $\mathcal{V}$ has good reduction at p. Therefore, we can use [CTS21, Proposition 10.4.3] to conclude that the evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon \mathcal{V}(\mathbb{Z}_p) \to \mathbb{Q}/\mathbb{Z}$$

is constant. Moreover, $P = (1 : 0 : 1 : 0) \in \mathcal{X}(\mathbb{Z}_p)$ and

$$\mathrm{ev}_{\mathcal{A}}(P) = (1, -1)$$

which is trivial in $\mathrm{Br}(\mathbb{Q}_p)$; hence the evaluation map is trivial on the whole $\mathcal{V}(\mathbb{Z}_p)$, which coincides with $V(\mathbb{Q}_p)$.

Case $p \in \{3, 5, 17\}$. Under this assumption, $\mathcal{V}_p/\mathbb{Z}_p$ is not smooth. In these three cases, we want to show that the evaluation map is trivial on $\mathcal{V}(\mathbb{Z}_p)$ by showing that it factors through $\mathrm{Br}(\mathbb{Z}_p)$.

The special fibre $\mathcal{V}(p) := \mathcal{V}_p \times_{\mathbb{Z}_p} \mathrm{Spec}(\mathbb{F}_p)$ is a non-smooth $\mathbb{F}_p$-scheme. However, $\mathcal{V}(p)$ is an irreducible $\mathbb{F}_p$-scheme, with just isolated singularities. The $\mathbb{Z}_p$-points of $\mathcal{V}_p$ are all smooth. In fact, $\mathcal{V}(p)$ contains just one singular point defined over $\mathbb{F}_p$ that does not even lift to a $\mathbb{Z}/p^2\mathbb{Z}$-point. Let $\mathcal{U}$ be the smooth locus of $\mathcal{V}_p$; because of what we have just said we have

$$V(\mathbb{Q}_p) = \mathcal{V}(\mathbb{Z}_p) = \mathcal{U}(\mathbb{Z}_p).$$

Let U be the base change of $\mathcal{U}$ to $\mathrm{Spec}(\mathbb{Q}_p)$. The purity theorem on $\mathcal{U}$ [CTS21, Theorem 3.7.1] gives us the exact sequence

$$\mathrm{Br}(\mathcal{U})[2] \to \mathrm{Br}(U)[2] \xrightarrow{\partial_{D_p}} H^1(k(D_p), \mathbb{Z}/2\mathbb{Z})$$

where D_p is the divisor associated to the special fibre (D_p is the smooth locus of $\mathcal{V}(p)$). We just need to show that $\partial_{D_p}(\mathcal{A}) = 0$. Let ν_p be the valuation corresponding to the prime divisor D_p; then

$$\nu_p\left(\frac{f}{x^3}\right) = 0 \quad \text{and} \quad \nu_p\left(-\frac{z}{x}\right) = 0.$$

Indeed, the point $(1 : 0 : 1 : 0) \in \mathcal{V}(p)(\mathbb{F}_p)$ is smooth, hence it lies in D_p. Moreover

$$\frac{f}{x^3}(1 : 0 : 1 : 0) = 1 \quad \text{and} \quad -\frac{z}{x}(1 : 0 : 1 : 0) = -1.$$

Therefore both $\frac{f}{x^3}$ and $-\frac{z}{x}$ do not vanish on D_p, which implies that $\partial_{D_p}(\mathcal{A}) = 0$. Therefore, $\mathcal{A}$ lies in $\mathrm{Br}(\mathcal{U}) \subseteq \mathrm{Br}(V_p)$ and the evaluation map factors as

$$\begin{array}{ccc} \mathcal{U}(\mathbb{Z}_p) & \xrightarrow{\quad |\mathcal{A}| \quad} & \mathrm{Br}(\mathbb{Q}_p). \\ & \searrow \quad \nearrow & \\ & \mathrm{Br}(\mathbb{Z}_p) & \end{array}$$

Since $\mathrm{Br}(\mathbb{Z}_p)$ is trivial, the evaluation map has to be constant and trivial.

Case $p = \infty$. The evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{R}) \to \mathbb{Q}/\mathbb{Z}$$

is constant and equal to 0.

We will show that it is constant on the dense open subset

$$W := \{P \in X(\mathbb{R}) : x(P), z(P), f(P) \neq 0\} \subseteq V(\mathbb{R}).$$

Indeed, from the continuity of the evaluation map it follows that it has to be constant also on the whole of $V(\mathbb{R})$. Let $P = (\alpha : \beta : \gamma : \delta) \in W$, thus $\gamma \neq 0$. First, assume that $-\frac{\gamma}{\alpha} > 0$. Then

$$\mathrm{ev}_{\mathcal{A}}(P) = \left(\frac{f(P)}{x(P)^3}, -\frac{z(P)}{x(P)} \right) = \left(\frac{f(P)}{\alpha^3}, -\frac{\gamma}{\alpha} \right)$$

is trivial in $\mathrm{Br}(\mathbb{R})$. Now, suppose that $-\frac{\gamma}{\alpha} < 0$. Without loss of generality, we can assume that both α and γ are positive. We want to show that in this case $f(P)$ has to be positive:

- if $\delta = 0$, then $P \in V(\mathbb{R})$ implies that $\beta(\alpha^3 + \beta^2\gamma) = 0$. Therefore $\beta = 0$, since $\alpha^3 + \beta^2\gamma \geq \alpha^3 > 0$. Hence, $f(P) = \gamma^3 > 0$;
- if $\delta \neq 0$ then $P \in V(\mathbb{R})$ implies

  $$f(P) = -\frac{\beta}{\delta}(\alpha^3 + \beta^2\gamma).$$

  Hence, since $\alpha^3 + \beta^2\gamma > 0$,

  $$f(P) > 0 \quad \text{if and only if} \quad -\frac{\beta}{\delta} > 0.$$

  Equivalently, β, δ do not have the same sign. Hence, we just need to show that there is no point $P \in W$ with α, γ positive and β, δ with the same sign. First, we observe that β, δ can not be both positive, since otherwise

  $$\alpha^3\beta + \beta^3\gamma + \gamma^3\delta + \delta^3\alpha + \alpha\beta\gamma\delta > 0.$$

  On the other hand β, δ cannot also be both negative. Indeed, we have that $P \in V(\mathbb{R})$ if and only if

  $$\alpha^3(-\beta) + (-\beta)^3\gamma + \gamma^3(-\delta) + (-\delta)^3\alpha = \alpha(-\beta)\gamma(-\delta).$$

  Without loss of generality we may assume that $\alpha \geq \max\{-\beta, \gamma, -\delta\}$; but if $\alpha, -\beta, \gamma, -\delta$ are all positive, then

  $$\alpha^3(-\beta) + (-\beta)^3\gamma + \gamma^3(-\delta) + (-\delta)^3\alpha > \alpha(-\beta)\gamma(-\delta).$$

  Hence, $(\alpha : \beta : \gamma : \delta) \notin V(\mathbb{R})$. □

2.2.1 Remark on the quaternion algebra $\mathcal{A}$

By what we said before Theorem 2.1.6, since 2 is a prime of good reduction and $\mathcal{A}$ is obstructing weak approximation, $\mathcal{A}$ is not an algebraic element. In this section we give a rough idea of the strategy behind the construction of the quaternion algebra $\mathcal{A}$, more details will be provided in Chapter 4.

Let X and $\mathcal{X}$ be the base change of V and $\mathcal{V}$ to $\mathbb{Q}_2$ and $\mathbb{Z}_2$ respectively. Let Y be the special fibre of $\mathcal{X}$,

$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & \mathcal{X} & \xleftarrow{\ i\ } & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Q}_2) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}_2) & \longleftarrow & \mathrm{Spec}(\mathbb{F}_2). \end{array} \tag{2.7}$$

Let π be a uniformiser in $\mathbb{Z}_2$. By Corollary 1.3.9(3) we know that there is a non-zero constant $\bar{c}$ in the residue field (that in this case is 1, since the residue field is $\mathbb{F}_2$) such that

$$\mathrm{mult}_{\bar{c}}\left(\mathrm{rsw}_{2,\pi}(\mathrm{Br}(X)[2])\right) \subseteq \mathrm{H}^0(Y, \Omega^2_{Y,\log}) \oplus \mathrm{H}^0(Y, \Omega^1_{Y,\log})$$

Moreover, the special fibre of a K3 surface having good reduction is still a K3 surface, see for example [BN23, Remark 11.5]. Hence, $\mathrm{H}^0(Y, \Omega^2_Y)$ is a one dimensional $\mathbb{F}_2$-vector space. We will prove in Chapter 4, using the results of Section 4.3, that 2 is a prime of good ordinary reduction for X and the only non-trivial two form ω is logarithmic. Finally, in Chapter 4 we will developed the techniques needed to compute the refined Swan conductor on p-torsion elements in the Brauer group and show that $\mathcal{A} \in \mathrm{fil}_2\mathrm{Br}(V)[2]$ and $\mathrm{rsw}_{2,\pi}(\mathcal{A}) = (\omega, 0) \neq (0,0)$. In particular, this implies that $\mathcal{A} \notin \mathrm{Ev}_{-1}\mathrm{Br}(X)[p]$. See Section 4.4 for the details.

2.3 A family of K3 surfaces with the same property

In this section we will show that the first part of Theorem 2.1.10 can be easily generalised to a family of K3 surfaces that share some properties with our K3 surface V.

Let a, b, c, d, e be odd integers, $\underline{\alpha} = (a, b, c, d, e)$ and $V_{\underline{\alpha}}$ be the K3 surface in $\mathbb{P}^3_{\mathbb{Q}}$ associated to the equation

$$a \cdot x^3 y + b \cdot y^3 z + c \cdot z^3 w + d \cdot w^3 x + e \cdot xyzw = 0. \tag{2.8}$$

Let $\mathcal{V}_{\underline{\alpha}}$ be the projective scheme over $\mathbb{Z}$ defined by the polynomial equation (2.8). Then $\mathcal{V}_{\underline{\alpha}}$ is a $\mathbb{Z}$-model of $V_{\underline{\alpha}}$. Moreover, since a, b, c, d, e are all odd integers, all these varieties have the same reduction, which we will denote by Y, modulo the prime 2. A natural question that arises at this point is if also for all these K3 surfaces there exists an element $\mathcal{A} \in \mathrm{Br}(V_{\underline{\alpha}})[2]$ such that the corresponding evaluation map on $V_{\underline{\alpha}}(\mathbb{Q}_2)$ is non-constant, i.e. that obstruct weak approximation. The following theorem gives a partial answer to the question.

Theorem 2.3.1. *Assume that* $\Delta := abcd \in \mathbb{Q}^{\times 2}$. *Then, the class of the quaternion algebra*

$$\mathcal{A} = \left(d \cdot \frac{c \cdot z^3 + d \cdot w^2 x + e \cdot xyz}{x^3}, -(cd) \cdot \frac{z}{x} \right) \in \mathrm{Br}(\mathbb{Q}(V_{\underline{\alpha}}))$$

defines an element in $\mathrm{Br}(V_{\underline{\alpha}})$. *The evaluation map* $\mathrm{ev}_{\mathcal{A}} \colon V_{\underline{\alpha}}(\mathbb{Q}_2) \to \mathbb{Q}/\mathbb{Z}$ *is non-constant, and therefore gives an obstruction to weak approximation on* V.

Proof. The proof is very similar to the first part of the proof of Theorem 2.1.10. We denote by f the polynomial $c \cdot z^3 + d \cdot w^2 x + e \cdot xyz$. Also in this case, let C_x, C_z, C_f be the closed subsets of $V_{\underline{\alpha}}$ defined by the equations $x = 0, z = 0$ and $f = 0$ respectively. Let U be the open subset of $V_{\underline{\alpha}}$ defined as the complement of $C_x \cup C_z \cup C_f$. Clearly, $\mathcal{A} \in \mathrm{Br}(U)$. Moreover,

- C_x consists of two irreducible components, $D_1 = \{x, z\}$ and $D_2 = \{x, b \cdot y^3 + c \cdot z^2 w\}$.
- C_z consists of two irreducible components, D_1 and $D_3 = \{z, a \cdot x^2 y + d \cdot w^3\}$.
- C_f consists of three irreducible components, D_1, $D_4 = \{y, c \cdot z^3 + d \cdot w^2 x\}$ and $D_5 = \{f, a \cdot x^3 + b \cdot y^2 z, be \cdot y^3 z - ac \cdot x^2 z^2 + bd \cdot y^2 w^2\}$.

In order to show that the quaternion algebra $\mathcal{A}$ lies in the Brauer group of $V_{\underline{\alpha}}$ we will use the exact sequence (4.15) coming from the purity theorem and the explicit description of the residue map given in equation (4.16). We will denote by ν_i the valuation associated to the prime divisor D_i.

1. $\nu_1(f) = \nu_1(x) = \nu_1(z) = 1$, and so $\nu_1\left(\frac{f}{x^3}\right) = -2$ and $\nu_1\left(-\frac{z}{x}\right) = 0$. Hence

$$\begin{aligned} \partial_{D_1}(\mathcal{A}) &= \left[(-1)^0 \left(d \cdot \frac{f}{x^3} \right)^0 \left(-\frac{1}{cd} \cdot \frac{x}{z} \right)^{-2} \right] \\ &= \left[\left(\frac{1}{cd} \cdot \frac{z}{x} \right)^2 \right] = 1 \in \frac{k(D_1)^\times}{k(D_1)^{\times 2}}. \end{aligned}$$

2. $\nu_2(f) = \nu_2(z) = 0$ and $\nu_2(x) = 1$, and so $\nu_2\left(\frac{f}{x^3}\right) = -3$ and $\nu_2\left(-\frac{z}{x}\right) = -1$. Hence

$$\begin{aligned} \partial_{D_2}(\mathcal{A}) &= \left[(-1)^3 \left(d \cdot \frac{f}{x^3} \right)^{-1} \left(-\frac{1}{cd} \cdot \frac{x}{z} \right)^{-3} \right] \\ &= \left[\left(\frac{(cd)^3}{d} \frac{x^3}{c \cdot z^3} \cdot \frac{z^3}{x^3} \right) \right] = 1 \in \frac{k(D_2)^\times}{k(D_2)^{\times 2}} \end{aligned}$$

where the second equality follows from the fact that $f\,|_{D_2} = c \cdot z^3$.

3. $\nu_3(f) = \nu_3(x) = 0$ and $\nu_3(z) = 1$, and so $\nu_3\left(\frac{f}{x^3}\right) = 0$ and $\nu_3\left(-\frac{z}{x}\right) = 1$. Hence
$$\partial_{D_3}(\mathcal{A}) = \left[(-1)^0\left(d\cdot\frac{f}{x^3}\right)^1\left(-\frac{1}{cd}\cdot\frac{x}{z}\right)^0\right] = \left[d\cdot\frac{f}{x^3}\right] = 1 \in \frac{k(D_3)^\times}{k(D_3)^{\times 2}}$$
where the last equality follows from the fact that $f\mid_{D_3} = d\cdot w^2 x$.

4. $\nu_4(f) = 1$ and $\nu_4(x) = \nu_4(z) = 0$, and so $\nu_4\left(\frac{f}{x^3}\right) = 1$ and $\nu_4\left(-\frac{z}{x}\right) = 0$. Hence
$$\partial_{D_4}(\mathcal{A}) = \left[(-1)^0\left(d\cdot\frac{f}{x^3}\right)^0\left(-\frac{1}{cd}\cdot\frac{x}{z}\right)^1\right] = \left[\frac{1}{cd}\cdot\frac{c}{d}\right] = 1 \in \frac{k(D_4)^\times}{k(D_4)^{\times 2}}$$
where the second equality follows from the fact that $-\frac{z}{x} = \frac{d}{c}\left(\frac{w}{z}\right)^2$ on D_4.

5. $\nu_5(f) = 1$ and $\nu_5(x) = \nu_5(z) = 0$, and so $\nu_5\left(\frac{f}{x^3}\right) = 1$ and $\nu_5\left(-\frac{z}{x}\right) = 0$. Hence
$$\partial_{D_5}(\mathcal{A}) = \left[(-1)^0\left(d\cdot\frac{f}{x^3}\right)^0\left(-\frac{1}{cd}\cdot\frac{x}{z}\right)^1\right] = \left[\frac{b}{acd}\right] = 1 \in \frac{k(D_5)^\times}{k(D_5)^{\times 2}}$$
where the last equality follows from the fact that $-\frac{z}{x} = \frac{a}{b}\left(\frac{x}{y}\right)^2$ on D_5 and the assumption that $abcd$ is a square in $\mathbb{Q}$.

The above computations together with the purity theorem show indeed that $\mathcal{A}$ lies in $\mathrm{Br}(V_{\underline{\alpha}})$. Finally, we need to show that the evaluation map on the $\mathbb{Q}_2$-points of $V_{\underline{\alpha}}$ is non-constant. Let
$$P_1 := (1:0:1:0) \in V_{\underline{\alpha}}(\mathbb{Q}).$$
Then, P_1 is such that $\mathrm{ev}_{\mathcal{A}}(P_1) = (dc, -dc)$, which is trivial in $\mathrm{Br}(\mathbb{Q}_2)$. Furthermore, let
$$P_2 := \left(cd : y : 1 : -2\cdot\frac{acde}{2c+cd}\right) \in V_{\underline{\alpha}}(\mathbb{Q}_2)$$
be such that the reduction modulo 8 of y is equal to $2\cdot de$. Then
$$f(P_2) \equiv c + d\cdot 4\cdot(cd) + e\cdot(cd)\cdot 2\cdot ed \equiv 7\cdot c \mod 8$$
and therefore evaluation map at P_2 is
$$\mathrm{ev}_{\mathcal{A}}(P_2) = \left(d\cdot\frac{f(P_2)}{(cd)^3}, -cd\frac{1}{cd}\right) = (g(P_2), -1) \quad \text{with} \quad g(P_2) \equiv 7 \mod 8.$$
Thus, $\mathrm{ev}_{\mathcal{A}}(P_2)$ defines a non-trivial element in $\mathrm{Br}(\mathbb{Q}_2)$. Hence, the element $\mathcal{A} \in \mathrm{Br}(V_{\underline{\alpha}})$ gives an obstruction to weak approximation on $V_{\underline{\alpha}}$. □

A natural question that arises at this point is what happens if $\Delta := abcd$ is not a square in $\mathbb{Q}$. Note that, in this case we can repeat the same computations that we did in the proof of Theorem 2.3.1. That is, for every divisor $D \neq D_5$ we get $\partial_D(\mathcal{A}) = 1$, while for D_5 we have

$$\partial_{D_5}(\mathcal{A}) = [\Delta] \in \frac{k(D_5)^\times}{k(D_5)^{\times 2}}.$$

Hence, in this case, $\mathcal{A}$ defines an element in the Brauer group of the base change of $V_{\underline{\alpha}}$ to $\mathbb{Q}(\sqrt{\Delta})$. With an argument similar to the one sketched in Section 2.2.1, also in this case we expect to be able to find two points P_1, P_2 defined over $\mathbb{Q}_2(\sqrt{\Delta})$ such that $\mathrm{ev}_{\mathcal{A}}(P_1) \neq \mathrm{ev}_{\mathcal{A}}(P_2)$.

2.3.1 Final considerations

The results of this chapter arise from the wish to use Theorem 2.1.8 in order to produce example of varieties in which primes of good reduction play a role in the Brauer–Manin obstruction to weak approximation.

In our example,

$$V = \mathrm{Proj}\left(\frac{\mathbb{Q}[x,y,z,w]}{x^3y + y^3z + z^3w + w^3x + xyzw}\right) \subseteq \mathbb{P}^3_{\mathbb{Q}}$$

is a smooth projective variety defined over the number field $\mathbb{Q}$.

Since V is a K3 surface, the hypotheses of Theorem 2.1.8 are satisfied. In this example, we were able to construct an element $\mathcal{A}$ that satisfies Theorem 2.1.8 which is already defined over the rational numbers and the corresponding evaluation map is non-constant on the 2-adic points. Moreover, $\mathcal{A}$ does not just lie in the 2-primary part of the Brauer group of X, it has order exactly 2.

The next two chapters are devoted to the investigation on whether one can hope to extend this strategy to a more general setting.