The wild Brauer-Manin obstruction on K3 surfaces

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CHAPTER 1

The Brauer–Manin obstruction and the refined Swan conductor

The Brauer–Manin obstruction was introduced by Manin [Man71] in the 1970s to explain the failure of the local-global principle.

Let V be a variety over a number field k. For any place $\nu \in \Omega_k$ we have a natural inclusion $V(k) \hookrightarrow V(k_\nu)$. In particular, we have that

$$V(k) \neq \emptyset \Rightarrow V(k_\nu) \neq \emptyset \text{ for all } \nu \in \Omega_k.$$

We say that V/k satisfies the **local-global** principle if

$$V(k_\nu) \neq \emptyset \text{ for all } \nu \in \Omega_k \Rightarrow V(k) \neq \emptyset.$$

The Hasse-Minkowski Theorem gives a family of varieties for which it is enough to check the local solubility to determine if an equation admits a solution over k.

Theorem 1.0.1 (**Hasse-Minkowski**). *Let k be a number field and $V \subseteq \mathbb{P}^n_k$ be a hypersurface defined by a single homogeneous equation of degree 2. Then $V(k) \neq \emptyset$ if and only if $V(k_\nu) \neq \emptyset$ for every place ν of k.*

Proof. See [Ser73, Theorem IV.8]. □

It is well known that if we increase the degree of the polynomial or the number of equations defining V, then we might have a failure of the local-global principle. Here we mention some examples in this direction.

Example 1.0.2.

1. Lind and Reichardt showed independently that the system of equations

$$\begin{cases} u^2 - 17w^2 = 2z^2 \\ uw = v^2 \end{cases}$$

admits a non-trivial solution over $\mathbb{Q}_\nu$ for all places $\nu \in \Omega_\mathbb{Q}$ but does not admit a non-trivial rational solution.

2. Selmer showed that the following homogeneous equation of degree 3

$$3X^3 + 4Y^3 + 5Z^3 = 0$$

admits a non-trivial solution over $\mathbb{Q}_\nu$ for all places $\nu \in \Omega_\mathbb{Q}$ but does not admit a non-trivial rational solution.

3. Iskovskikh showed that equation

$$y^2 + z^2 = (3 - x^2)(x^2 - 2).$$

defining a smooth affine surface over $\mathbb{Q}$, admits a non-trivial solution over $\mathbb{Q}_\nu$ for all places $\nu \in \Omega_\mathbb{Q}$ but does not admit a non-trivial rational solution.

In all the examples the failure of the local-global principle can be explained by the **Brauer–Manin obstruction**.

Manin's idea was to use the Brauer group of a variety to construct a closed subset

$$\left(\prod_{\nu \in \Omega_k} V(k_\nu)\right)^{\mathrm{Br}} \subseteq \prod_{\nu \in \Omega_k} V(k_\nu)$$

that contains the k-points of V.

A crucial role in the construction of the Brauer–Manin set is played by the so-called **evaluation maps** attached to elements $\mathcal{A}$ in $\mathrm{Br}(V)$. We will see that for any element $\mathcal{A} \in \mathrm{Br}(V)$ and for every place $\nu \in \Omega_k$ we can define a map, the evaluation map

$$\mathrm{ev}_\mathcal{A} \colon V(k_\nu) \to \mathbb{Q}/\mathbb{Z}.$$

To detect which values these maps can take is one of the crucial steps in the study of the Brauer–Manin obstruction.

In this thesis we focus attention on the behaviour of the evaluation map at finite places of good reduction.

Definition 1.0.3. *We say that a prime $\mathfrak{p}$ of k is of **good reduction** for V if there exists a smooth proper $\mathcal{O}_\mathfrak{p}$-scheme $\mathcal{X}$ whose base change to $k_\mathfrak{p}$ is isomorphic to the base change of V to $k_\mathfrak{p}$.*

In Section 1.1.3 we will see that if $\mathfrak{p}$ is a finite place of good reduction for V with residue characteristic p and $\mathcal{A} \in \mathrm{Br}(V)$ has order prime to p, then the behaviour of the evaluation map can be studied via the so-called **residue map**, that links the prime to p part of the Brauer group of the base change of V to $k_{\mathfrak{p}}$ to the cohomology of the special fiber of a model. This is no longer possible when $\mathfrak{p}$ is a finite place of good reduction of residue characteristic p and $\mathcal{A} \in \mathrm{Br}(V)\{p\}$, i.e. $\mathcal{A}$ is a p-power element in the Brauer group of V. However, in [BN23] Bright and Newton use the refined Swan conductor to control the behaviour of the evaluation map also in this case.

In the first part of this chapter we introduce the notion of weak approximation and Brauer–Manin obstruction to weak approximation on a variety. In the second part of the chapter we introduce the notion of refined Swan conductor [Kat89]. Following Bright and Newton's recent work [BN23] we explain how this notion can be used to study the Brauer–Manin obstruction to weak approximation. Some original results are presented in this chapter: in particular, Lemma 1.2.24 and Corollary 1.3.9 are part of [Pag23].

1.1 The Brauer–Manin obstruction

Let k be a number field and V a proper k-variety. We want to study the set of k-points on V via the diagonal embedding

$$V(k) \hookrightarrow \prod_{\nu \in \Omega_k} V(k_\nu).$$

For every place $\nu \in \Omega_k$, the ν-adic topology on k_ν allows us to equip $V(k_\nu)$ with a topology, called the analytic topology, see [Con12, Proposition 3.1].

From now on we will think about $\prod_{\nu \in \Omega_k} V(k_\nu)$ as a topological space with respect to the product topology induced by the analytic topology on each component $V(k_\nu)$.

Definition 1.1.1. *We say that V satisfies **weak approximation** if the image of the diagonal map*

$$V(k) \hookrightarrow \prod_{\nu \in \Omega_k} V(k_\nu)$$

is dense.

For proper k-varieties it is possible to get the product $\prod_{\nu \in \Omega_k} V(k_\nu)$ as the set of $\mathbf{A}_k$-points on V, where $\mathbf{A}_k$ is the **ring of adèles** of the number field k.

Definition 1.1.2. *We define the **ring of adèles** of k as the restricted product of the completions of k with respect to the subgroups given by the ring of integers $\mathcal{O}_\nu \subseteq k_\nu$, i.e. :*

$$\mathbf{A}_k := \prod_{\nu \in \Omega_k} (k_\nu, \mathcal{O}_\nu) = \left\{ (x_\nu) \in \prod k_\nu \ : \ x_\nu \in \mathcal{O}_\nu \text{ for almost all } \nu \in \Omega_k \right\}.$$

The following proposition is well known; we include the proof because we are unable to find it in the literature.

Proposition 1.1.3. *Let R be an integral domain and let k be its function field. Suppose that V is a scheme of finite presentation over k. Then there exist a dense open subscheme $U \subseteq \mathrm{Spec}(R)$ and a scheme $\mathcal{V}$ of finite presentation over U such that $\mathcal{V}_k \simeq V$.*

Proof. Let $\{V_i\}_{i\in I} \subseteq V$ be a finite affine covering of V. Then, for every $i \in I$ the description of V_i involves only finitely many polynomials over k. Let R' be the localisation of R obtained by adjoining the inverse of all the denominators that appear either in the polynomials describing V_i or in the gluing morphisms between the opens V_i and V_j. Then, we can define $\mathcal{V}_i$ as the R'-scheme defined by the same equations of V_i and obtain $\mathcal{V}$ by gluing the $\mathcal{V}_i$'s together. □

Given a k-scheme of finite type V, we can always find a finite set of finite places $S \subseteq \Omega_k$ such that there exists an $\mathcal{O}_{k,S}$ -scheme of finite type $\mathcal{V}_S$ whose base change to k is isomorphic to V (i.e. a model of V over $\mathcal{O}_{k,S}$). Then, the set of adelic points $V(\mathbf{A}_k)$ can be described as

$$V(\mathbf{A}_k) = \prod_{\nu\in\Omega_k} (V(k_\nu), \mathcal{V}_S(\mathcal{O}_\nu)), \quad \text{[Con12, Theorem 3.6]}$$

In particular, if V is proper

$$V(\mathbf{A}_k) = \prod_{\nu\in\Omega_k} V(k_\nu).$$

1.1.1 Obstructions arising from functors

Let

$$\mathrm{F}\colon \mathrm{Sch}_k^{opp} \to \mathrm{Set}$$

be a functor. In this section we are going to show how it is possible, starting from F, to build subsets $V(\mathbf{A}_k)^{\mathrm{F}}$ of $V(\mathbf{A}_k)$ that contains $V(k)$.

Let R be a k-algebra, then every $x \in V(R)$ induces a map $\mathrm{F}(x)\colon \mathrm{F}(V) \to \mathrm{F}(R)$. For every $\mathcal{A} \in \mathrm{F}(V)$, we can define the map

$$\begin{aligned} \mathrm{ev}_{\mathcal{A}}\colon V(R) &\to \mathrm{F}(R) \\ x &\mapsto \mathrm{F}(x)(\mathcal{A}). \end{aligned}$$

In particular, we have the following commutative diagram:

$$\begin{array}{ccc} V(k) & \hookrightarrow & V(\mathbf{A}_k) \\ \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} \\ \mathrm{F}(k) & \longrightarrow & \mathrm{F}(\mathbf{A}_k) \end{array} \tag{1.1}$$

We define

$$V(\mathbf{A}_k)^{\mathcal{A}} := \{x \in V(\mathbf{A}_k) \ : \ \mathrm{ev}_{\mathcal{A}}(x) \in \mathrm{im}(\mathrm{F}(k) \to \mathrm{F}(\mathbf{A}_k))\}.$$

By the commutativity of the diagram above $V(k) \subseteq V(\mathbf{A}_k)^{\mathcal{A}}$. We define

$$V(\mathbf{A}_k)^{\mathrm{F}} := \bigcap_{\mathcal{A} \in \mathrm{F}(V)} V(\mathbf{A}_k)^{\mathcal{A}}.$$

Definition 1.1.4. *If $V(\mathbf{A}_k)^{\mathrm{F}}$ is a proper closed subset of $V(\mathbf{A}_k)$ we say that there is an* ***F-obstruction to weak approximation*** *on V.*

In particular, we see that in order to understand the set $V(\mathbf{A}_k)^{\mathrm{F}}$ we need to be able to understand both $\mathrm{F}(V)$ and for elements in $\mathrm{F}(V)$ the corresponding evaluation map.

In this thesis we focus the attention on the Brauer-Manin obstruction, more about obstructions attached to other functors (in particular about descent obstructions) can be found in [Poo17a, Section 8.4, 8.5, and 8.6].

1.1.2 The Brauer group and the Brauer–Manin obstruction

The Brauer–Manin obstruction is defined as the obstruction associated to the functor $\mathrm{F} := \mathrm{Br}(-)$, where $\mathrm{Br}(-) = \mathrm{H}^2_{\text{ét}}(-, \mathbb{G}_m)$ is the cohomological Brauer group (for an introduction to étale cohomology see [Mil80]).

In this section we will explain how it is possible to describe both the Brauer group of V and the evaluation map attached to an element $\mathcal{A} \in \mathrm{Br}(V)$.

1.1.2.1 The Brauer group of a variety

For a smooth, proper and geometrically integral variety V over a number field k there is an exact sequence [Č19, (6.4.4)]

$$0 \to \mathrm{Br}(V) \to \mathrm{Br}(k(V)) \xrightarrow{(\mathrm{r}_v)} \bigoplus_{v \in V^{(1)}} \mathrm{H}^1_{\text{ét}}(k(v), \mathbb{Q}/\mathbb{Z}) \tag{1.2}$$

where $V^{(1)}$ is the set of points $v \in V$ such that the local ring $\mathcal{O}_{V,v}$ is one dimensional. Since by assumption V is smooth, $\mathcal{O}_{V,v}$ is a regular noetherian ring of dimension 1 and hence a discrete valuation ring. Let $k(v)$ be the residue field of $\mathcal{O}_{V,v}$, then we have a map

$$\mathrm{r}_v \colon \mathrm{Br}(k(V)) \to \mathrm{H}^1(k(v), \mathbb{Q}/\mathbb{Z})$$

see [Poo17b, Section 6.8]. In particular, the Brauer group of the variety can be described in terms of the cohomology of the fields $k(V)$ and $k(v)$ for $v \in V^{(1)}$.

Given a field K, according to the definition given above, the Brauer group of K is defined as

$$\mathrm{Br}(K) := \mathrm{H}^2(\mathrm{Spec}(K)_{\text{ét}}, \mathbb{G}_m).$$

It is well known that for fields étale cohomology coincide with Galois cohomology, see [Mil80, Theorem 1.9, Chapter 1]. In particular, since Galois cohomology groups are torsion [CTS21, Corollary 1.3.6] we get that $\mathrm{Br}(V)$ is a torsion abelian group.

For fields it is possible to give an explicit construction of the Brauer group in terms of equivalence classes of central simple algebras. A finite-dimensional algebra $\mathcal{A}$ over K is called simple if it has no (two sided) ideals other than 0 and itself; it is called central if $Z(\mathcal{A}) = K$. Two central simple K-algebras $\mathcal{A}$ and $\mathcal{A}'$ are called Brauer equivalent if

$$\mathcal{A} \otimes_K M_n(K) \simeq \mathcal{A}' \otimes_K M_{n'}(K), \quad \text{for some } n, n' > 0.$$

Theorem 1.1.5. *The set of the equivalence classes of finite dimensional central simple K-algebras, with the group structure given by the tensor product, is isomorphic to* $\mathrm{Br}(K)$.

Proof. [GS17, Section 4.4]. □

1.1.2.2 Quaternion algebras

In the examples presented in this thesis we are mainly working with elements in the Brauer group represented by quaternion algebras.

Definition 1.1.6. *Assume the characteristic of K to be different from 2, then given $a, b \in K^\times$, we define the* ***quaternion algebra*** (a, b) *as the 4-dimensional K-algebra with basis $1, i, j, ij$ with*

$$i^2 = a, \; j^2 = b, \; ij = -ji.$$

It is possible to prove that quaternion algebras are central simple algebras and that operations with quaternion algebras inside the Brauer group are quite simple. Indeed, if we pick $a, b, c \in K^\times$ we have an isomorphism

$$(a, b) \otimes_k (a, c) \xrightarrow{\sim} (a, bc) \otimes_K M_2(K).$$

Given $a, b \in K^\times$, with abuse of notation we will denote by (a, b) also the class of the quaternion algebra in $\mathrm{Br}(K)$.

As mentioned before, the Brauer group of a variety is a torsion abelian group, hence it is enough to study for every n the exact sequence on the n-torsion induced by equation (1.2)

$$0 \to \mathrm{Br}(V)[n] \to \mathrm{Br}(k(V))[n] \xrightarrow{(\mathrm{r}_v)} \bigoplus_{v \in V^{(1)}} \mathrm{H}^1_{\text{ét}}(k(v), \mathbb{Q}/\mathbb{Z}))[n]. \tag{1.3}$$

From the short exact sequence $0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{n} \mathbb{Q}/\mathbb{Z} \to 0$ we get an isomorphism

$$\mathrm{H}^1_{\text{ét}}(k(v), \mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{H}^1_{\text{ét}}(k(v), \mathbb{Q}/\mathbb{Z})[n].$$

We have an explicit description of the residue map r_v on quaternion algebras in $\mathrm{Br}(k(V))$: for an element $(a,b) \in \mathrm{Br}(k(V))$ and a point $v \in V^{(1)}$ we have

$$\mathrm{r}_v(a,b) = \left[(-1)^{\mathrm{val}_v(a)\mathrm{val}_v(b)} \frac{a^{\mathrm{val}_v(b)}}{b^{\mathrm{val}_v(a)}}\right] \in \frac{k(v)^\times}{k(v)^{\times 2}} \simeq \mathrm{H}^1(k(v), \mathbb{Q}/\mathbb{Z})[2] \qquad (1.4)$$

where val_v is the discrete valuation on $k(V)$ coming from the discrete valuation ring $\mathcal{O}_{V,v}$. The description of the residue map given above follows from the definition of the tame symbols in Milnor K-theory together with the compatibility of the residue map r_v with the tame symbols given by the Galois symbols, see [GS17, Proposition 7.5.1]. The last isomorphism follows from the isomorphism between μ_2 and $\mathbb{Z}/2\mathbb{Z}$ together with the one coming from the Kummer sequence between $k(v)^\times/k(v)^{\times 2}$ and $\mathrm{H}^1_{\text{ét}}(k(\nu), \mu_2)$.

1.1.2.3 The evaluation map

Let $\nu \in \Omega_k$ be a finite place. Using local class field theory one can prove that there exists an isomorphism, called invariant map,

$$\mathrm{inv}_\nu \colon \mathrm{Br}(k_\nu) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}. \qquad (1.5)$$

A proof can be found in Section 1 and 2, Chapter III of [Mil20]. Moreover, if $\nu \in \Omega_k$ is an archimedean place, then k_ν is either isomorphic to $\mathbb{R}$ or to $\mathbb{C}$, for which

$$\mathrm{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathrm{Br}(\mathbb{C}) = \{0\}, \quad \text{[CTS21, Section 1.2.1]}.$$

Hence, for archimedean places we can define

$$\mathrm{inv}_\nu \colon \mathrm{Br}(k_\nu) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

as the map sending the non-trivial element of $\mathrm{Br}(k_\nu)$ to the class of $1/2$ in $\mathbb{Q}/\mathbb{Z}$. Finally, from global class field theory [Mil20, Chapter VII, VIII] we have the following short exact sequence:

$$0 \to \mathrm{Br}(k) \to \bigoplus_{\nu \in \Omega_k} \mathrm{Br}(k_\nu) \xrightarrow{\sum_\nu \mathrm{inv}_\nu} \mathbb{Q}/\mathbb{Z} \to 0.$$

Since it is possible to show that

$$\mathrm{Br}(\mathbf{A}_k) \simeq \bigoplus_{\nu \in \Omega_k} \mathrm{Br}(k_\nu), \quad \text{[Č19, Theorem 2.13]}$$

diagram (1.1) becomes

$$\begin{array}{ccccccccc} & & V(k) & \longrightarrow & V(\mathbf{A}_k) & & & & \\ & & \big\downarrow \mathrm{ev}_{\mathcal{A}} & & \big\downarrow \mathrm{ev}_{\mathcal{A}} & & & & \\ 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(\mathbf{A}_k) \simeq \bigoplus_{\nu \in \Omega_k} \mathrm{Br}(k_\nu) & \xrightarrow{\sum \mathrm{inv}_\nu} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0. \end{array} \qquad (1.6)$$

Putting everything together, we get that for every $\mathcal{A} \in \mathrm{Br}(V)$

$$V(\mathbf{A}_k)^{\mathcal{A}} = \left\{ (x_\nu) \in V(\mathbf{A}_k) \text{ such that } \sum_{\nu \in \Omega} \mathrm{inv}_\nu(\mathrm{ev}_{\mathcal{A}}(x_\nu)) = 0 \right\}.$$

Remark 1.1.7. From now on, with abuse of notation, we will denote by $\mathrm{ev}_{\mathcal{A}}$ also the composition of the evaluation map $\mathrm{ev}_{\mathcal{A}} \colon V(k_\nu) \to \mathrm{Br}(k_\nu)$ with the invariant map $\mathrm{inv}_\nu \colon \mathrm{Br}(k_\nu) \hookrightarrow \mathbb{Q}/\mathbb{Z}$.

The following result allows us to use the Brauer–Manin set to study weak approximation on V.

Proposition 1.1.8. *For any $\mathcal{A} \in \mathrm{Br}(V)$, the set $V(\mathbf{A}_k)^{\mathcal{A}}$ is open and closed in $V(\mathbf{A}_k)$.*

Proof. See [Poo17a, Corollary 8.2.11]. □

The set $V(\mathbf{A}_k)^{\mathrm{Br}}$ is therefore closed in $V(\mathbf{A}_k)$ and we have the following chain of inclusions

$$V(k) \subseteq \overline{V(k)} \subseteq V(\mathbf{A}_k)^{\mathrm{Br}} \subseteq V(\mathbf{A}_k).$$

Definition 1.1.9. *We say that there is a* ***Brauer–Manin obstruction to weak approximation*** *on V if the Brauer–Manin set $V(\mathbf{A}_k)^{\mathrm{Br}}$ is a proper subset of $V(\mathbf{A}_k)$.*

Assume $V(\mathbf{A}_k) \neq \emptyset$. Let $\omega \in \Omega_k$ be such that there exists $\mathcal{A} \in \mathrm{Br}(V)$ with

$$\mathrm{ev}_{\mathcal{A}} \colon V(k_\omega) \to \mathbb{Q}/\mathbb{Z}$$

non-constant. Then, we can take two points $(x_\nu), (y_\nu) \in V(\mathbf{A}_k)$ such that $x_\nu = y_\nu$ for all $\nu \neq \omega$ and $\mathrm{ev}_{\mathcal{A}}(x_\omega) \neq \mathrm{ev}_{\mathcal{A}}(y_\omega)$. These two points are such that

$$\sum_{\nu \in \Omega_k} \mathrm{ev}_{\mathcal{A}}(x_\nu) \neq \sum_{\nu \in \Omega_k} \mathrm{ev}_{\mathcal{A}}(y_\nu).$$

Hence, at least one of the two points does not belong to the Brauer–Manin set $V(\mathbf{A}_k)^{\mathrm{Br}}$ and thus there is a Brauer–Manin obstruction to weak approximation on V. Based on this observation we give the following definition.

Definition 1.1.10. *We say that a place $\omega \in \Omega_k$* ***plays a role*** *in the Brauer–Manin obstruction to weak approximation on V if there exists $\mathcal{A} \in \mathrm{Br}(V)$ such that $\mathrm{ev}_{\mathcal{A}} \colon V(k_\omega) \to \mathbb{Q}/\mathbb{Z}$ is non-constant.*

1.1.3 The prime to p part

In order to study the role of a prime $\mathfrak{p}$ in Brauer–Manin obstruction, we need to understand, for a given $\mathcal{A} \in \mathrm{Br}(V)$, the behaviour of the corresponding evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon V(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}.$$

If we denote by res: $\mathrm{Br}(V) \to \mathrm{Br}(V_{\mathfrak{p}})$ the natural restriction map form $\mathrm{Br}(V)$ to $\mathrm{Br}(V_{\mathfrak{p}})$, then for every point $P \in V(k_{\mathfrak{p}})$ we have a corresponding point $P_{\mathfrak{p}} \in V_{\mathfrak{p}}(k_{\mathfrak{p}})$ and

$$\mathrm{ev}_{\mathrm{res}(\mathcal{A})}(P_{\mathfrak{p}}) = \mathrm{ev}_{\mathcal{A}}(P).$$

Hence, if the evaluation map attached to $\mathcal{A}$ is non-constant on $V(k_{\mathfrak{p}})$, then the element $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(V_{\mathfrak{p}})$ is such that the evaluation map on $V_{\mathfrak{p}}(k_{\mathfrak{p}})$ is also non-constant. By what we just said together with the fact that we are interested in primes of good reduction, we will work under the setting described below.

p-adic setting: Let p be a prime number and L a finite field extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, uniformiser π and residue field ℓ. Let X be a smooth and geometrically irreducible L-variety having good reduction (i.e. there exists a smooth proper $\mathcal{O}_L$-scheme $\mathcal{X}$ whose generic fiber is isomorphic to X). We assume furthermore the special fiber $Y := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\ell)$ to be geometrically irreducible,

$$\begin{array}{ccccc} X & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) & \longleftarrow & \mathrm{Spec}(\ell). \end{array} \tag{1.7}$$

It is well known that if n is a positive integer prime to p, then we have the following exact sequence

$$0 \to \mathrm{Br}(\mathcal{X})[n] \to \mathrm{Br}(X)[n] \xrightarrow{\partial} \mathrm{H}^1(Y, \mathbb{Z}/n\mathbb{Z}) \tag{1.8}$$

see [CTS21, Chapter 3, equation (3.17)].

Theorem 1.1.11. *The evaluation map attached to $\mathcal{A}$ is constant and trivial over all finite field extensions L'/L if and only if $\mathcal{A} \in \mathrm{Br}(\mathcal{X})$.*

Proof. One of the two implication is quite straightforward, in fact if we start by $\mathcal{A} \in \mathrm{Br}(\mathcal{X})$, then the evaluation map factors as

$$\begin{array}{ccc} \mathcal{X}(\mathcal{O}_L) & \longrightarrow & \mathrm{Br}(\mathcal{O}_L) \\ \| & & \downarrow \\ X(L) & \xrightarrow{\mathrm{ev}_{\mathcal{A}}} & \mathrm{Br}(L) \end{array} \tag{1.9}$$

but $\mathrm{Br}(\mathcal{O}_L) = \{0\}$.

The other implication was proved by Colliot-Thélène and Saito [CTS96] using zero-cycles on X. The group of zero-cycles $C_0(X)$ is defined as the group of formal sums

$$\sum_i n_i P_i \quad \text{with } n_i \in \mathbb{Z},\ P_i \text{ closed point of } X.$$

The evaluation map $\mathcal{A}$ on a zero-cycle $\sum_i n_i P_i$ is defined as

$$\sum_i n_i \mathrm{cores}_{L(P_i)/L}\left(\mathrm{ev}_{\mathrm{res}_{L(P_i)/L}(\mathcal{A})}(P_i)\right).$$

For the definition of the restriction and co-restriction maps we refer to [GS17, Section 3.3]. In particular, they proved that $\mathrm{ev}_{\mathcal{A}}$ is trivial on $C_0(X)$ if and only if $\mathcal{A} \in \mathrm{Br}(\mathcal{X})[n]$. In particular, if $\mathcal{A}$ has constant and trivial evaluation map on all the closed points of X, then it has also constant and trivial evaluation map on zero-cycles. □

In general, as explained in [Bri15, Section 5.1] the evaluation map attached to an element $\mathcal{A} \in \mathrm{Br}(X)[n]$ with n prime to p factors through the special fiber Y, i.e. if two points $P_1, P_2 \in X(L) = \mathcal{X}(\mathcal{O}_L)$ have the same image in $Y(\ell)$ then the evaluation map on them coincide. Finally, the residue map appearing in (1.8) controls whether the evaluation map attached to an element $\mathcal{A} \in \mathrm{Br}(X)[n]$ is constant: $\mathcal{A}$ has constant evaluation map if and only if $\partial(\mathcal{A}) \in \mathrm{H}^1(\ell, \mathbb{Z}/n\mathbb{Z})$.

From the Purity Theorem [Č19, Theorem 1.2] we know that $\mathrm{Br}(\mathcal{X})$ is equal to $\mathrm{Br}(X) \cap \mathrm{Br}(\mathcal{O}_{\mathcal{X},Y})$, with the intersection taking place inside $\mathrm{Br}(k(X))$. Hence, in order to understand whether $\mathcal{A}$ has trivial evaluation map it is enough to understand if the image of $\mathcal{A}$ in $\mathrm{Br}(k(X))$ lies in $\mathrm{Br}(\mathcal{O}_{\mathcal{X},Y})$. Let K^h be the field of fraction of the henselianisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},Y}$, if n is prime to p, then

$$\mathrm{Br}(K^h)[n] = \ker(\mathrm{Br}(K^h) \to \mathrm{Br}(K^h_{un}))[n], \quad \text{[CTS21, Proposition 1.4.5]}.$$

This can be used to construct the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Br}(\mathcal{O}_{\mathcal{X},Y})[n] & \longrightarrow & \mathrm{Br}(k(X))[n] & \xrightarrow{\partial} & \mathrm{H}^1(F, \mathbb{Z}/n\mathbb{Z}) \\ & \downarrow & & \downarrow & & \| \\ 0 \longrightarrow & \mathrm{Br}(\mathcal{O}^h_{\mathcal{X},Y})[n] & \longrightarrow & \mathrm{Br}(K^h)[n] & \xrightarrow{\partial} & \mathrm{H}^1(F, \mathbb{Z}/n\mathbb{Z}) \end{array} \tag{1.10}$$

where the rows form exact sequences, see [CTS21, Theorem 3.6.1]. In particular, from diagram (1.10) we get that it is enough to compute the residue of the image of $\mathcal{A}$ in $\mathrm{Br}(K^h)[n]$ in order to detect whether the element $\mathcal{A}$ has trivial evaluation map.

If we start with an element $\mathcal{A} \in \mathrm{Br}(X)[p^m]$ for some m, then it is no longer true that its image in $\mathrm{Br}(K^h)$ is split by an unramified extension. However, Bright and Newton [BN23] recently proved that the behaviour of the evaluation map attached to $\mathcal{A}$ on the $k_{\mathfrak{p}}$-points of X is still determined by the value of some maps on the image of $\mathcal{A}$ in $\mathrm{Br}(K^h)[p^m]$. In this case the description of $\mathrm{Br}(K^h)[p^m]$ becomes more involved. Bright and Newton use the concept of **refined Swan conductor**, introduced by Kato in [Kat89], to give a filtration on $\mathrm{Br}(K^h)$ that captures the behaviour of the evaluation maps.

1.2 The refined Swan conductor

Set-up: Let K be a henselian discrete valuation field of mixed characteristic, whose residue field is not necessarily perfect. We denote by:

- $\mathcal{O}_K \subseteq K$ the ring of integers of K with uniformiser π and maximal ideal $\mathfrak{m}$;
- F the residue field of characteristic p.

The aim of this section is to introduce the notion of the refined Swan conductor and gather some of the main results around it that will be crucial in the rest of the thesis. The main references for this section are [Kat89, BN23]. In this section, instead of working just with the Brauer group, we will look at the Galois cohomology group $\mathrm{H}^q(K)$ defined in the next section. We will see how these cohomology groups are related to the differential forms of the residue field F.

1.2.1 The group $\mathrm{H}^q(R)$

Let R be a ring over $\mathbb{Q}$ or a smooth ring over a field of characteristic $p > 0$. Let n be a non-negative integer and $r \in \mathbb{Z}$. If n is invertible on R, then we denote by $\mathbb{Z}/n\mathbb{Z}(r)$ the usual Tate twist of the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on $R_{\text{ét}}$. If R is smooth over a field of characteristic p and $n = p^s m$ with $(p, m) = 1$, we define

$$\mathbb{Z}/n\mathbb{Z}(r) := \mathbb{Z}/m\mathbb{Z}(r) \oplus W_s\Omega^r_{R,\log}[-r].$$

For the definition of $W_s\Omega^r_{R,\log}$ see Definition 1.2.7. For $n \neq 0$, we denote by

$$\mathrm{H}^q_n(R) := \mathrm{H}^q(R_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(q-1)) \quad \text{and} \quad \mathrm{H}^q(R) := \varinjlim_n \mathrm{H}^q_n(R).$$

Note that:

$$\mathrm{H}^2(K) = \varinjlim_n \mathrm{H}^2(K_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(1)) = \mathrm{Br}(K).$$

We have an exact triangle

$$\mathbb{Z}/n\mathbb{Z}(1) \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m,$$

where the prime to the characteristic case comes from the Kummer sequence, while the p-part in characteristic p follows from [Ill79, Proposition I.3.23.2]. For $a \in R^\times$, let $\{a\} \in \mathrm{H}^1(R_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(1))$ be the image of a under the connecting map $R^\times \to \mathrm{H}^1(R_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(1))$. For $a_1, \ldots, a_r \in R^\times$ and $\chi \in \mathrm{H}^q_n(R)$ we denote by

$$\{\chi, a_1, \ldots, a_r\} := \chi \cup \{a_1\} \cup \cdots \cup \{a_r\} \in \mathrm{H}^{q+r}_n(R).$$

The symbol $\{\chi, a_1, \ldots, a_r\}$ is anti-symmetric and $\{\chi, a_1, \ldots, a_r\} = 0$ if there exist i, j with $i \neq j$ such that $a_i + a_j = 1$ or 0.

In order to describe the groups $\mathrm{H}^q(R)$ in the positive characteristic case, we will use the **Cartier operator** on differential forms.

1.2.1.1 The Cartier operator

Let ℓ be a perfect field of characteristic $p > 0$ and R be an ℓ-algebra. Let

$$Z_R^q := \ker(d\colon \Omega_R^q \to \Omega_R^{q+1}) \quad \text{and} \quad B_R^q := \operatorname{im}(d\colon \Omega_R^{q-1} \to \Omega_R^q).$$

Lemma 1.2.1 (Inverse Cartier operator). *Assume R to be regular; then there exists a unique morphism of groups*

$$C_R^{-1}\colon \Omega_R^1 \to \Omega_R^1/B_R^1$$

satisfying

- $C_F^{-1}(da) = a^{p-1}da \mod B_F^1$ *for all* $a \in F$*;*
- $C_R^{-1}(\lambda\omega) = \lambda^p C_R^{-1}(\omega)$ *for all* $\lambda \in R$*;*
- $d \circ C_R^{-1} = 0$.

Moreover, C^{-1} induces an isomorphism from Ω_R^1 to Z_R^1/B_R^1.

Proof. See [BK05, Theorem 1.3.4]. □

Remark 1.2.2. The subgroup B_R^1 has a natural structure of R-module, which is given by $\alpha \cdot d\beta = \alpha^p d\beta = d(\alpha^p\beta)$. If we denote by ${}^p\Omega_R^1$ the R-module structure on Ω_R^1 given by $\alpha \cdot \omega = \alpha^p\omega$, then the condition $C_R^{-1}(\lambda\omega) = \lambda^p C_F^{-1}(\omega)$ is equivalent to asking that $C_R^{-1}\colon \Omega_R^1 \to {}^p\Omega_R^1/{}^pB_R^1$ is a morphism of R-modules.

We can extend the definition of C_R^{-1} to higher differential forms by setting

$$C_R^{-1}(\omega_1 \wedge \cdots \wedge \omega_q) := C_R^{-1}(\omega_1) \wedge \cdots \wedge C_R^{-1}(\omega_q).$$

Theorem 1.2.3. *Let R be regular; then the morphism*

$$C_R^{-1}\colon \Omega_R^q \to Z_R^q/B_R^q$$

is an isomorphism for all $q \geq 0$. We will denote by C_R its inverse, which is called the Cartier operator.

Proof. See [BK05, Theorem 1.3.4]. □

The following corollary gives a way to characterise exact differential forms in terms of the Cartier operator.

Corollary 1.2.4. *Let R be regular. A q-form $\omega \in \Omega_R^q$ is exact if and only if $d(\omega) = 0$ and $C_R(\omega) = 0$.*

The last object we need to define is the subgroup of logarithmic q-differential forms on R, which will play a crucial role in this thesis.

Definition 1.2.5. *The **logarithmic** q-differential forms on R, denoted by $\Omega_{R,\log}^q$, are defined as the kernel of the map*

$$C_R^{-1} - \mathrm{id}\colon \Omega_R^q \to \Omega_R^q/B_R^q.$$

The following result gives a way to write down logarithmic differential forms explicitly in the case in which R is a field.

Theorem 1.2.6. *Let F be a field, finitely generated over a perfect field ℓ. The logarithmic differential q-forms $\Omega^q_{F,\log}$ is the subgroup of Ω^q_F generated by elements of the form*

$$\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}, \ \text{with } y_i \in F^\times.$$

Proof. It follows from surjectivity in the Bloch–Gabber–Kato Theorem [GS17, Theorem 9.5.2]. □

Finally, in [Ill79], Illusie defines a projective system $\{W_m\Omega^q_R\}_{m\geq 0}$ equipped with the Frobenius and the Verschiebung maps

$$F\colon W_{m+1}\Omega^q_R \to W_m\Omega^q_R \quad \text{and} \quad V\colon W_m\Omega^q_R \to W_{m+1}\Omega^q_R$$

for every $m \geq 0$. This projective system has transition maps given by the projection maps $R_{m+1}\colon W_{m+1}\Omega^j_R \to W_m\Omega^j_R$.

In [Ill79, Page 569, Proposition 3.3] Illusie generalises the inverse Cartier operator to $W_s\Omega^q_R$:

$$C_R^{-1}\colon W_s\Omega^q_R \to W_s\Omega^q_R/dV^{s-1}\Omega^{q-1}_R$$

Definition 1.2.7. *Given two positive integers q, s, the logarithmic subgroup $W_s\Omega^q_{R,\log}$ of $W_s\Omega^q_R$ is defined as the kernel of*

$$C_R^{-1} - \mathrm{id}\colon W_s\Omega^q_R \to W_s\Omega^q_R/dV^{s-1}\Omega^{q-1}_R.$$

If R is smooth over a field of positive characteristic and $n = p^s$, we can identify $\mathrm{H}^q_n(R) = \mathrm{H}^1(R_{\text{ét}}, W_s\Omega^r_{R,\log})$ with the cokernel of

$$C_R^{-1} - 1\colon W_s\Omega^{q-1}_R \to W_s\Omega^{q-1}_R/dV^{s-1}\Omega^{q-2}_R,$$

see [Kat89, Section 1.3]. We denote by δ_s both the map from $W_s\Omega^{q-1}_R/dV^{s-1}\Omega^{q-2}_R$ to $\mathrm{H}^q_n(R)$ and its composition with the natural map $\mathrm{H}^q_n(R) \to \mathrm{H}^q(R)$. We then have a commutative diagram

$$\begin{array}{ccc} W_s\Omega^{q-1}_R & \xrightarrow{\quad V \quad} & W_{s+1}\Omega^{q-1}_R \\ & {\scriptstyle \delta_s}\searrow \quad \swarrow{\scriptstyle \delta_{s+1}} & \\ & \mathrm{H}^q(R) & \end{array}$$

and the following relation [Kat89, Section 1.3]

$$\delta_s(\omega \wedge d\log(a_1) \wedge \cdots \wedge d\log(a_r)) = \{\delta_s(\omega), \{a_1, \ldots, a_r\}\} \text{ in } \mathrm{H}^{q+r}_n(R). \tag{1.11}$$

1.2.1.2 Description of $\mathrm{H}^q_p(F[T])$

In this section we are following [Kur88]. We are interested in describing the groups $\mathrm{H}^{q+1}_p(F[T])$. We recall that these groups fit in the following exact sequence

$$\Omega^q_{F[T]} \xrightarrow{C^{-1}-\mathrm{id}} \Omega^q_{F[T]}/B^q_{F[T]} \xrightarrow{\delta_1} \mathrm{H}^{q+1}_p(F[T]) \to 0. \tag{1.12}$$

Lemma 1.2.8. *Every element $\omega \in \Omega^q_{F[T]}$ can be written (in a unique way) as*

$$\sum_{i=0}^{n} \omega_i T^i + \sum_{j=1}^{m} \eta_j T^j \wedge d\log T$$

with $\omega_i \in \Omega^q_F$ and $\eta_j \in \Omega^{q-1}_F$.

Proof. Consider the following diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}[T] \\ \downarrow & & \downarrow{\scriptstyle \pi_1} \\ F & \xrightarrow{\pi_2} & \mathbb{Z}[T] \otimes_{\mathbb{Z}} F = F[T]. \end{array}$$

We know by the Künneth formula [Sta, 01V1] that

$$\Omega^q_{F[T]} = \bigoplus_{i+j=q} \pi_1^* \Omega^i_{\mathbb{Z}[T]} \otimes_{F[T]} \pi_2^* \Omega^j_F$$

where π_1^* and π_2^* are just the base change to $F[T]$ maps. Since $\Omega^i_{\mathbb{Z}[T]} = 0$ if $i \geq 2$, we get that

$$\Omega^q_{F[T]} = \left(F[T] \otimes_F \Omega^q_F\right) \oplus \left(F[T] \cdot dT \otimes_F \Omega^{q-1}_F\right)$$

which proves the result. □

We define increasing filtrations on $\mathrm{H}^{q+1}_p(F[T])$ and $\mathrm{H}^{q+1}_p(F[T, T^{-1}])$ such that fil_n is the group generated by elements of the form

$$\delta_1 \left[T^k \omega_1 + T^m \omega_2 \wedge d\log T\right]$$

where $\omega_1 \in \Omega^q_F$ and $\omega_2 \in \Omega^{q-1}_F$ and $k, m \leq n$. For $n \geq 0$ we define a morphism

$$\begin{aligned} \rho_n \colon \Omega^q_F \oplus \Omega^{q-1}_F &\to \mathrm{H}^{q+1}_p(F[T]) \\ (\omega_1, \omega_2) &\mapsto \delta_1 \left[T^n \omega_1 + T^n \omega_2 \wedge d\log T\right]. \end{aligned}$$

Similarly, we define the same morphisms with image in $\mathrm{H}^{q+1}_p(F[T, T^{-1}])$. It follows from the definitions and the description given in Lemma 1.2.8 that for $n \geq 1$, the morphism ρ_n induces a surjective map on

$$\mathrm{gr}_n := \frac{\mathrm{fil}_n \mathrm{H}^{q+1}_p(F[T])}{\mathrm{fil}_{n-1} \mathrm{H}^{q+1}_p(F[T])} = \frac{\mathrm{fil}_n \mathrm{H}^{q+1}_p(F[T, T^{-1}])}{\mathrm{fil}_{n-1} \mathrm{H}^{q+1}_p(F[T, T^{-1}])}.$$

From now on, for $n \geq 1$, we denote by ρ_n also the composition of ρ_n with the quotient map to gr_n.

Lemma 1.2.9. *Let $n \geq 1$ be prime to p. Then the map ρ_n induces an isomorphism*

$$\Omega^q_F \simeq \mathrm{gr}_n$$

Proof. By construction gr_n is generated by the elements of the form

$$\delta_1 \left[T^n \omega_1 + T^n \omega_2 \wedge d\log T\right].$$

If n is prime to p, then we have

$$T^n \omega_2 \wedge d\log T = d\left(\omega_2 \frac{T^n}{n}\right) - d(\omega_2)\frac{T^n}{n}.$$

Since δ_1 is zero on exact forms, we get that

$$\rho_n(\omega_1, \omega_2) = \delta_1 \left[T^n \left(\omega_1 - \frac{d\omega_2}{n}\right)\right] = \rho_n\left(\omega_1 - \frac{d\omega_2}{n}, 0\right).$$

This proves that the restriction of ρ_n to Ω^q_F is still surjective.

We are left to show injectivity. Let $\omega \in \Omega^q_F$ be such that $\rho_n(\omega, 0) = 0$. Then, there exist $\omega_0, \ldots \omega_{n_1} \in \Omega^q_F$ and $\eta_1, \ldots, \eta_{n-1} \in \Omega^{q-1}_F$ such that

$$\delta_1 \left[\omega T^n\right] + \delta_1 \left[\sum_{i=0}^{n-1} T^i \omega_i + \sum_{j=1}^{n-1} \eta_j T^j \wedge d\log T\right] = 0 \text{ in } \mathrm{H}^q_p(F[T]).$$

Hence, by equation (1.12) we get the following equality in $\Omega^{q-1}_{F[T]}/B^{q-1}_{F[T]}$

$$\omega T^n + \sum_{i=0}^{n-1} T^i \omega_i + \sum_{j=1}^{n-1} \eta_j T^j \wedge d\log T = (C^{-1} - \mathrm{id})\left(\sum_{i\geq 0} \alpha_i T^i + \sum_{j\geq 1} \beta_j T^j \wedge d\log T\right).$$

The right hand side of the identity can be re-written (using the properties of the inverse Cartier operator) as

$$\sum_{i=0}^{n} C^{-1}(\alpha_i) T^{pi} + \sum_{j=1}^{n} C^{-1}(\beta_j) T^{pj} \wedge d\log T - \sum_{i=0}^{n} \alpha_i T^i - \sum_{j=1}^{n} \beta_j T^j \wedge d\log T.$$

In particular,

$$\omega T^n = -\alpha_n T^n \text{ modulo } B^{q-1}_{F[T]} \quad \text{and} \quad C^{-1}(\alpha_n T^n) = C^{-1}(\alpha_n) T^{pn} = 0.$$

Hence, since C^{-1} is an isomorphism, $\alpha_n T^n = 0$ and $\omega T^n \in B^{q-1}_{F[T]}$. Hence,

$$\omega T^n = d\left(\sum_{i\geq 0} \gamma_i T^i + \sum_{j\geq 1} \theta_j T^{j-1} \wedge dT\right)$$

which implies that

$$d(\gamma_n) T^n = \omega T^n \quad \text{and} \quad (n\gamma_n - d\theta_n) T^{n-1} \wedge dT = 0.$$

Therefore $n\gamma_n = d\theta_n$ and $\omega = d(\gamma_n) = 0$. ☐

Remark 1.2.10. In particular, the isomorphism induced by ρ_1 factors as follows

$$\Omega_F^q \to \mathrm{fil}_1 \mathrm{H}_p^{q+1}(F[T]) \to \mathrm{gr}_1.$$

This implies that the map

$$\begin{aligned} \Omega_F^q &\to \mathrm{H}_p^{q+1}(F[T]) \\ \alpha &\mapsto \delta_1[T\alpha] \end{aligned}$$

is injective.

Lemma 1.2.11. *Let $n \geq 1$ be such that $n = pn_1$. Then the map ρ_n induces an isomorphism*

$$\Omega_F^q / Z_F^q \oplus \Omega_F^{q-1}/Z_F^{q-1} \simeq \mathrm{gr}_n$$

Proof. Let $(\omega_1, \omega_2) \in Z_F^q \oplus Z_F^{q-1}$, then there exists $(\theta_1, \theta_2) \in \Omega_F^q \oplus \Omega_F^{q-1}$ such that

$$(C^{-1}(\theta_1), C^{-1}(\theta_2)) = (\omega_1, \omega_2), \text{ in } Z_F^q/B_F^q \oplus Z_F^{q-1}/B_F^{q-1}$$

Hence, using the properties of the inverse Cartier operator

$$T^n\omega_1 + T^n\omega_2 \wedge d\log T = C^{-1}(\theta_1 T_1^n + \theta_2 T_1^n \wedge d\log T).$$

We know from equation (1.12) that the image via δ_1 of

$$(C^{-1} - \mathrm{id})\,[\theta_1 T^{n_1} + \theta_2 T^{n_1} \wedge d\log T]$$

is zero. Therefore, $\delta_1\,[T^n\omega_1 + T^n\omega_2 \wedge d\log T]$ is equal to

$$\delta_1\,[\theta_1 T^{n_1} + \theta_2 T^{n_1} \wedge d\log T]\,.$$

The latter is zero in gr_n, since $n_1 \leq n - 1$.

We are left to show that the kernel of ρ_n is contained in $Z_F^q \oplus Z_F^{q-1}$. Assume that $(\omega_1, \omega_2) \in \Omega_F^{q-1} \oplus \Omega_F^{q-2}$ is such that

$$\rho_n(\omega_1, \omega_2) = \delta_1\,[T^n\omega_1 + T^n\omega_2 \wedge d\log T] = 0.$$

An argument analogous to the one used in the previous lemma gives that there exist $\alpha_{n_1,p} \in \Omega_F^q$ and $\beta_{n_1,p} \in \Omega_F^{q-1}$ such that

$$\omega_1 = C^{-1}(\alpha_{n_1,p}) \quad \text{and} \quad \omega_2 = C^{-1}(\beta_{n_1,p}), \quad \mathrm{mod}\ B_F^{q-1} \oplus B_F^{q-2}.$$

The result now follows from the fact that the image of the inverse Cartier operator lies in the group of closed forms modulo exact forms. □

Let $g_1, g_2, h : F[T] \to F[T_1, T_2]$ be the F-morphisms defined by

$$g_1(T) = T_1, \quad g_2(T) = T_2, \quad \text{and} \quad h(T) = T_1 + T_2.$$

We denote in the same way the induced maps from $H_p^*(F[T])$ to $H_p^*(F[T_1, T_2])$.

Lemma 1.2.12. *Let $\omega \in \mathrm{H}^q_p(F[T])$ and assume that ω is such that*

(1) $g_1(\omega) + g_2(\omega) = h(\omega)$;

(2) the element $\{\omega, T\}$ in $\mathrm{H}^{q+1}_p(F(T))$ is equal to the image of an element τ in $\mathrm{H}^{q+1}_p(F[T])$ such that $g_1(\tau) + g_2(\tau) = h(\tau)$.

Then, there exists $\alpha \in \Omega^{q-1}_F$ such that

$$\omega = \delta_1(T\alpha).$$

Proof. A proof, that uses the previous two lemmas, can be found in [Kur88, Lemma 3.3.1]. □

1.2.2 Definition of Swan conductor

We recall that we work in the following setting: K is a henselian discrete valuation field of mixed characteristic, whose residue field is not necessarily perfect. We denote by:

- $\mathcal{O}_K \subseteq K$ the ring of integers of K with uniformiser π and maximal ideal $\mathfrak{m}$;
- F the residue field of characteristic p.

Let A be a ring over $\mathcal{O}_K$, $R := A/\mathfrak{m}A$, and i, j the inclusions of the special and generic fibers into $\mathrm{Spec}(A)$:

$$\mathrm{Spec}(A \otimes_{\mathcal{O}_K} K) \xrightarrow{\ j\ } \mathrm{Spec}(A) \xleftarrow{\ i\ } \mathrm{Spec}(R).$$

We define

$$V^q_n(A) := \mathrm{H}^q\left(R_{\text{ét}}, i^*\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1)\right)$$

and $V^q(A) := \varinjlim_n V^q_n(A)$. In particular, $\mathrm{H}^q(K) = V^q(\mathcal{O}_K)$. The natural map in $D^b(A_{\text{ét}})$

$$\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1) \to i_*i^*\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1)$$

induces a natural map $\mathrm{H}^q_n(A \otimes_{\mathcal{O}_K} K) \to V^q_n(A)$ for all n, q. Gabber [Gab94] proved that this map is an isomorphism if $(A, \mathfrak{m}A)$ is henselian. In this case, using the Kummer map $(A \otimes_{\mathcal{O}_K} K)^\times \to \mathrm{H}^1(A \otimes_{\mathcal{O}_K} K, \mathbb{Z}/n\mathbb{Z}(1))$ and the cup product, we define a product

$$\begin{aligned} V^q_n(A) \times ((A \otimes_{\mathcal{O}_K} K)^\times)^{\oplus r} &\to V^{q+r}_n(A) \\ (\chi, a_1, \dots, a_r) &\mapsto \{\chi, a_1, \dots, a_r\}. \end{aligned}$$

For a general A, the isomorphism

$$V^q_n(A) \simeq V^q_n(A^{(h)}) \tag{1.13}$$

allows to extend the product above. This isomorphism follows from the following remark.

Remark 1.2.13. One can check that the sheaves $i^*\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1)$, with i and j such that

$$\mathrm{Spec}(A\otimes_{\mathcal{O}_K}K)\xrightarrow{\ j\ }\mathrm{Spec}(A)\xleftarrow{\ i\ }\mathrm{Spec}(R).$$

and $(i^{(h)})^*\mathrm{R}(j^{(h)})_*\mathbb{Z}/n\mathbb{Z}(q-1)$ with $i^{(h)}$ and $j^{(h)}$ such that

$$\mathrm{Spec}(A^{(h)}\otimes_{\mathcal{O}_K}K)\xrightarrow{\ j^{(h)}\ }\mathrm{Spec}(A^{(h)})\xleftarrow{\ i^{(h)}\ }\mathrm{Spec}(R).$$

have the same stalks, and hence coincide on $R_{\text{ét}}$.

From now on we will identify $V_n^q(A)$ and $V_n^q(A^{(h)})$. In particular, note that if $A=\mathcal{O}_K[T]$, then all the polynomials of the form $1+\pi^n p(T)$ are invertible in $A^{(h)}$, see [Sta, 0EM7] for an overview on henselianisation of (not necessarily local) rings.

Definition 1.2.14. *The increasing filtration* $\{\mathrm{fil}_n H^q(K)\}_{n\geq 0}$ *on* $\mathrm{H}^q(K)$ *is defined by*

$$\chi\in\mathrm{fil}_n\mathrm{H}^q(K)\Leftrightarrow\{\chi,1+\pi^{n+1}T\}=0\ \textit{in}\ V^{q+1}(\mathcal{O}_K[T]).$$

We say that an element χ *in* $\mathrm{H}^q(K)$ *has* ***Swan conductor*** n, *if* $\chi\in\mathrm{fil}_n\mathrm{H}^q(K)$ *and* $\chi\notin\mathrm{fil}_{n-1}\mathrm{H}^q(K)$.

The following lemma implies that it is possible to define the Swan conductor of every element of $\mathrm{H}^q(K)$.

Lemma 1.2.15. *We have*

$$\mathrm{H}^q(K)=\bigcup_{n\geq 0}\mathrm{fil}_n\mathrm{H}^q(K).$$

Proof. This lemma is [Kat89, Lemma 2.2], we are adding some details in the proof. Let $A:=(\mathcal{O}_K[T])^h$ be the henselisation of $\mathcal{O}_K[T]$ with respect to the ideal $\mathfrak{m}\mathcal{O}_K[T]$. Then, by equation (1.13) $V_n^q(A)\simeq V_n^q(\mathcal{O}_K[T])$. Let $\chi\in\mathrm{H}^q(K)$, then there exists $s\geq 1$ such that $\chi\in\mathrm{H}^q_s(K)$. If $n\geq 0$ is such that $1+\pi^{n+1}T\in(A^\times)^s$, then

$$\{\chi,1+\pi^{n+1}T\}=\{\chi,a^s\}=\{s\cdot\chi,a\}=0$$

which implies that $\chi\in\mathrm{fil}_n\mathrm{H}^q(K)$. The result now follows from [Kat89, Lemma 2.4], in which Kato proves that for any $s\geq 1$ there exists some $n\geq 0$ such that the set $1+\pi^{n+1}A$ is contained in $(A^\times)^s$ □

Remark 1.2.16. Kato defines the Swan conductor also for henselian discrete valuation fields of equal characteristic. In particular, it is possible to relate the Swan conductor filtration to the filtration on $\mathrm{H}^q_p(F[T])$ defined in Section 1.2.1.2. If we take $K=F\left((T)\right)$ then the map $F[T]\to F\left((T)\right)$ sending T to T^{-1} allows us to pull-back the Swan conductor filtration on $\mathrm{H}^q_p(F[T])$; from [Kat89, Theorem 3.2(1)] we see that the filtration that we get in this way on $\mathrm{H}^q_p(F[T])$ coincides with the one defined in Section 1.2.1.2.

1.2.2.1 Elements with Swan conductor equal to zero

In [Kat89] and [BN23] some maps $\lambda_\pi\colon \mathrm{H}^q_n(R) \oplus \mathrm{H}^{q-1}_n(R) \to V^q_n(A)$ are defined under additional assumptions on the $\mathcal{O}_K$-algebra A. We give an overview of these maps that, since they are compatible with each other, we will always call λ_π.

- In [Kat89, Section 1.4] Kato defines for every n an injective map

  $$\lambda_\pi\colon \mathrm{H}^q_n(F) \oplus \mathrm{H}^{q-1}_n(F) \to \mathrm{H}^q_n(K).$$

  This collection of maps induces an injective map

  $$\lambda_\pi\colon \mathrm{H}^q(F) \oplus \mathrm{H}^{q-1}(F) \to \mathrm{H}^q(K).$$

- In [Kat89, Section 1.9] Kato extends the definition of λ_π from $\mathrm{H}^q_p(F) \oplus \mathrm{H}^{q-1}_p(F)$ to $\mathrm{H}^q_p(K)$ to any smooth $\mathcal{O}_K$-algebra A. In particular, he defines a map

  $$\lambda_\pi\colon \mathrm{H}^q_p(R) \oplus \mathrm{H}^{q-1}_p(R) \to V^q_p(A).$$

- In [BN23, Section 2.2] Bright and Newton generalise the previous map by defining

  $$\lambda_\pi\colon \mathrm{H}^q_{p^r}(R) \oplus \mathrm{H}^{q-1}_{p^r}(R) \to V^q_{p^r}(A)$$

  for any $r \geq 1$.

It is proven in [Kat89, Proposition 6.1] that the image of

$$\lambda_\pi\colon \mathrm{H}^q_n(F) \oplus \mathrm{H}^{q-1}_n(F) \to \mathrm{H}^q_n(K)$$

coincides with $\mathrm{fil}_0\mathrm{H}^q_n(K)$.

We state here a technical lemma proven by Bright and Newton that we will use several times in this thesis.

Lemma 1.2.17. *Let K^h be the fraction field of the henselisation of A with respect to the ideal generated by π. Let $r \geq 1$ and $q \geq 2$. Let χ be an element of $\mathrm{fil}_0\mathrm{H}^q_{p^r}(K^h)$ and write $\chi = \lambda_\pi(\alpha, \beta)$ with $(\alpha, \beta) \in \mathrm{H}^q_{p^r}(F) \oplus \mathrm{H}^{q-1}_{p^r}(F)$. If χ lies in the image of $V^q_{p^r}(A)$, then (α, β) lies in the image of $\mathrm{H}^q_{p^r}(R) \oplus \mathrm{H}^{q-1}_{p^r}(R)$.*

Proof. [BN23, Lemma 3.5] □

Following [Kat89] and [BN23] we sometimes use λ_π also to denote the composition

$$W_r\Omega^q_R \oplus W_r\Omega^{q-1}_R \xrightarrow{\delta_r} \mathrm{H}^q_{p^r}(R) \oplus \mathrm{H}^{q-1}_{p^r}(R) \xrightarrow{\lambda_\pi} V^q_{p^r}(A).$$

1.2.3 Construction of the refined Swan conductor

The aim of this section to define the refined Swan conductor of an element χ in $\mathrm{fil}_n\mathrm{H}^q(K)$. The key result we need is the following theorem, [Kat89, Theorem 5.1]

Theorem 1.2.18. *Let $\chi \in \mathrm{fil}_n\mathrm{H}^q(K)$, with $n \geq 1$; then there exists a unique pair (α, β) in $\Omega^q_F \oplus \Omega^{q-1}_F$ such that*

$$\{\chi, 1+\pi^n T\} = \lambda_\pi(T\alpha, T\beta) \quad \text{in } V_p^{q+1}(\mathcal{O}_K[T]). \tag{1.14}$$

Remark 1.2.19. The pair (α, β) in the theorem above depends on the choice of a uniformiser π in $\mathcal{O}_K$. However, we will mainly be interested in the vanishing of such a pair, and this is independent on the choice of the uniformiser. Let π' be another uniformiser and $a = \pi/\pi' \in \mathcal{O}_K^\times$, then we can define the map

$$\begin{aligned} m_a \colon \mathcal{O}_K &\to \mathcal{O}_K \\ x &\mapsto ax. \end{aligned}$$

We denote by m_a also the induced automorphism on $\mathcal{O}_K[T]$. Then [BN23, Lemma 2.13] tells us that

$$\lambda_{\pi'}(\bar{a}T(\alpha + \beta \wedge d\log \bar{a}), \bar{a}T\beta) = m_a^* \lambda_\pi(T\alpha, T\beta).$$

with $\bar{a}$ reduction of a in F and m_a^* automorphism induced by m_a on $\mathrm{H}^q(K)$.

The refined Swan conductor of an element $\mathcal{A} \in \mathrm{fil}_n\mathrm{H}^q(K)$ is defined as the element $(\alpha, \beta) \in \Omega^q_F \oplus \Omega^{q-1}_F$ such that

$$\{\chi, 1+\pi^n T\} = \lambda_\pi(T\alpha, T\beta).$$

Note that: we have that $\chi \in \mathrm{fil}_{n-1}\mathrm{H}^q(K)$ if and only if $\{\chi, 1+\pi^n T\} = 0$ and since λ_π is injective, this happens if and only if $(T\alpha, T\beta) = 0$. By Remark 1.2.10 this is equivalent to $(\alpha, \beta) = (0, 0)$. Hence, for every $n \geq 1$ we get an injective map, called **refined Swan conductor**

$$\mathrm{rsw}_{n,\pi} \colon \frac{\mathrm{fil}_n\mathrm{H}^q(K)}{\mathrm{fil}_{n-1}\mathrm{H}^q(K)} \hookrightarrow \Omega^q_F \oplus \Omega^{q-1}_F.$$

The rest of this section is devoted to the proof of Theorem 1.2.18. We start with the following lemma, [Kat89, Lemma 5.3].

Lemma 1.2.20. *Assume $\chi \in \mathrm{fil}_n\mathrm{H}^q(K)$ for $n \geq 1$. Then:*

(1) $\{\chi, 1+\pi^n T\} \in V^{q+1}(\mathcal{O}_K[T])$ is annihilated by p;

(2) $\{\chi, 1+\pi^n T_1, 1+\pi T_2\} = 0$ in $V^{q+2}(\mathcal{O}_K[T_1, T_1^{-1}, T_2])$;

(3) $\{\chi, 1+\pi^n(T_1+T_2)\} = \{\chi, 1+\pi^n T_1\} + \{\chi, 1+\pi^n T_2\}$ in $V^{q+1}(\mathcal{O}_K[T_1, T_2])$;

(4) $\{\chi, 1+\pi^n T, T\} = -\{\chi, 1+\pi^n T, -\pi^n\}$ in $V^{q+2}(\mathcal{O}_K[T, T^{-1}])$.

Proof. Note that $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$ implies, by functoriality of $V^q(-)$, that

$$\{\chi, 1 + \pi^{n+1} b\} = 0 \text{ in } V^q(B) \tag{1.15}$$

for all $\mathcal{O}_K$-algebras B and all $b \in B$.

1. We have

$$p \cdot \{\chi, 1 + \pi^n T\} = \{\chi, (1 + \pi^n T)^p\} = \{\chi, 1 + \pi^{n+1} q(T)\} = 0$$

with $q(T) = \sum_{s=1}^{p} \binom{p}{s} \frac{1}{\pi^{n+1}} (\pi^n T)^s \in \mathcal{O}_K[T]$. The last equality follows from equation (1.15).

2. Let M be the fraction field of the henselisation of the local ring $\mathcal{O}_K[T_1]_{(\pi)}$. By [Kat89, (1.8.1)] we know that the for any $s \geq 0$ the map

$$V^q_{p^s}(\mathcal{O}_K[T_1, T_2]) \to V^q_{p^s}(\mathcal{O}_M[T_2])$$

is injective. Hence, it is enough to prove the equality in $V^{q+2}(\mathcal{O}_M[T_2])$. Note that, for every $a \in \mathcal{O}_M$

$$\{\chi, 1 + \pi^n T + \pi^{n+1} a\} = \{\chi, 1 + \pi^n T\} \text{ in } V^q(\mathcal{O}_M). \tag{1.16}$$

In fact, $1 + \pi^n T \in \mathcal{O}_M^\times$ and

$$\{\chi, 1 + \pi^n T + \pi^{n+1} a\} - \{\chi, 1 + \pi^n T\} = \left\{\chi, 1 + \pi^{n+1} \frac{a}{1 + \pi^n T}\right\} = 0.$$

Using equation (1.16) we see that

$$\begin{aligned}
&\{\chi, 1 + \pi^n T_1, 1 + \pi T_2\} \\
&= \{\chi, 1 + \pi^n T_1(1 + \pi T_2), 1 + \pi T_2\} \\
&\overset{\star}{=} -\{\chi, 1 + \pi^n T_1(1 + \pi T_2), -\pi^n T_1\} \\
&\overset{\star}{=} -\{\chi, 1 + \pi^n T_1(1 + \pi T_2), -\pi^n T_1\} - \{\chi, 1 + \pi^n T_1, -\pi^n T_1\} \\
&= -\left\{\chi, 1 + \pi^{n+1} \frac{T_1 T_2}{1 + \pi^n T_1}, -\pi^n T_1\right\} = 0.
\end{aligned}$$

where in $\overset{\star}{=}$ we are using that $\{\chi, a, b\} = 0$ if $a + b = 1$.

3. We have

$$\begin{aligned}
&\{\chi, 1 + \pi^n T_1\} + \{\chi, 1 + \pi^n T_2\} - \{\chi, 1 + \pi^n (T_1 + T_2)\} \\
&= \left\{\chi, \frac{(1 + \pi^n T_1)(1 + \pi^n T_2)}{1 + \pi^n (T_1 + T_2)}\right\} = \left\{\chi, 1 + \pi^{2n} \frac{T_1 T_2}{1 + \pi^n (T_1 + T_2)}\right\} = 0.
\end{aligned}$$

4. This follows immediately from $\{\chi, a, b\} = 0$ if $a + b = 1$.

□

Proof of Theorem 1.2.18. Let $\chi \in \mathrm{fil}_n\mathrm{H}^q(K)$. Then, by Lemma 1.2.20(1) we know that

$$\theta := \{\chi, 1+\pi^n T\} \text{ lies in } V_p^{q+1}(\mathcal{O}_K[T]).$$

From Lemma 1.2.20(2) we have $\{\theta, 1+\pi T_2\} = 0$, hence θ satisfies the assumption of Lemma 1.2.17 with $A = \mathcal{O}_K[T]$ and $R = F[T]$. Hence, there exists (ω, σ) in $\mathrm{H}_p^{q+1}(F[T]) \oplus \mathrm{H}_p^q(F[T])$ such that $\lambda_\pi(\omega,\sigma) = \theta$. We are left to show that ω and σ satisfy the conditions of Lemma 1.2.12. For any $a \in \mathcal{O}_{K(T)}^\times$ we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^q_{p^r}(F(T)) \oplus \mathrm{H}^{q-1}_{p^r}(F(T)) & \xrightarrow{(\cup(-1)^{q-1}\{\bar{a}\}, \cup(-1)^q\{\bar{a}\})} & \mathrm{H}^{q+1}_{p^r}(F(T)) \oplus \mathrm{H}^q_{p^r}(F(T)) \\ \downarrow{\scriptstyle \lambda_\pi} & & \downarrow{\scriptstyle \lambda_\pi} \\ \mathrm{H}^q_{p^r}(K(T)) & \xrightarrow{(\cup\{a\})} & \mathrm{H}^{q+1}_{p^r}(K(T)) \end{array}$$

see the proof of [BN23, Lemma 2.12]. Thus, we get

$$\lambda_\pi(\omega \cup (-1)^{q-1}\{T\}, \sigma \cup (-1)^q\{T\}) = \{\theta, T\}.$$

As $\{\theta, T\} \in V_p^{q+2}(\mathcal{O}_K[T])$, from Lemma 1.2.17 we get that $\{\omega, T\}$ and $\{\sigma, T\}$ lie in $\mathrm{H}_p^{q+2}(F[T])$ and $\mathrm{H}_p^{q+1}(F[T])$ respectively. We denote by G_1, G_2 and H the maps from $\mathcal{O}_K[T]$ to $\mathcal{O}_K[T_1, T_2]$ defined by

$$G_1(T) = T_1, \quad G_2(T) = T_2, \quad \text{and} \quad H(T) = T_1 + T_2$$

and the corresponding maps from $V_p^\bullet(\mathcal{O}_K[T])$ to $V_p^\bullet(\mathcal{O}_K[T_1, T_2])$. Since λ_π is injective, in order to show that ω and σ satisfy the assumption of Lemma 1.2.12 it is enough to show that $G_1 + G_2 = H$ on θ and $\{\theta, T\}$. This is an immediate consequence of Lemma 1.2.20(3) and 1.2.20(4). □

We end this section by giving the definition of the residue map ∂ from $\mathrm{Br}(K)$ to $\mathrm{H}^1(F)$, which also uses the map λ_π (cf. [BN23, Section 2.5], [Kat89, Section 7.5]).

Definition 1.2.21. *The **residue map***

$$\partial\colon \mathrm{fil}_0\mathrm{Br}(K) \to \mathrm{H}^1(F)$$

is defined as the projection on the second component of the inverse of the isomorphism λ_π from $\mathrm{H}^2(F) \oplus \mathrm{H}^1(F)$ to $\mathrm{fil}_0\mathrm{Br}(K)$.

1.2.3.1 Image of the refined Swan conductor

We are now ready to state and prove some properties about the image of the refined Swan conductor. We denote by $e := \mathrm{ord}_K(p)$ the absolute ramification index of K and by $e' := ep(p-1)^{-1}$. We start with the following lemma.

Lemma 1.2.22. *Let χ be an element in $\mathrm{fil}_n\mathrm{H}^q(K)$ with*

$$\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta) \in \Omega_F^2 \oplus \Omega_F^1.$$

Then $d\alpha = 0$ and $d\beta = (-1)^q n\alpha$.

Proof. See [BN23, Lemma 2.17]. □

Remark 1.2.23. We get that:

(1) if $p \mid n$, then $d\alpha = 0$ and $d\beta = 0$, meaning that $(\alpha, \beta) \in Z_F^q \oplus Z_F^{q-1}$;

(2) if $p \nmid n$, then $\alpha = \bar{n}^{-1} d\beta$, meaning that the composition

$$\mathrm{fil}_n \mathrm{H}^q(K) \xrightarrow{\mathrm{rsw}_{n,\pi}} \Omega_F^q \oplus \Omega_F^{q-1} \xrightarrow{\mathrm{pr}_2} \Omega_F^{q-1}$$

has also kernel equal to $\mathrm{fil}_{n-1}\mathrm{H}^q(K)$.

In [BN23, Lemma 2.19] Bright and Newton are able to link the refined Swan conductor of an element $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$ with the refined Swan conductor of $p \cdot \chi$, whenever $n \geq e'$. More precisely, assume that $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta)$ and let $\bar{u}$ be the reduction modulo π of $p \cdot \pi^{-e}$, then $p \cdot \chi \in \mathrm{fil}_{n-e}\mathrm{H}^q(K)$ and

$$\mathrm{rsw}_{n-e,\pi}(p \cdot \chi) = \begin{cases} (\bar{u}\alpha, \bar{u}\beta) \text{ if } n > e'; \\ (\bar{u}\alpha + C(\alpha), \bar{u}\beta + C(\beta)) \text{ if } n = e'. \end{cases} \tag{1.17}$$

In the following lemma we prove a result analogous to the one proven by Bright and Newton for elements $\chi \in \mathrm{fil}_{np}\mathrm{H}^q(K)$, when $np < e'$.

Lemma 1.2.24. *Let $\chi \in \mathrm{fil}_{np}\mathrm{H}^q(K)$, with $np < e'$. Then $p \cdot \chi \in \mathrm{fil}_n \mathrm{Br}(K)$ and if $\mathrm{rsw}_{np,\pi}(\mathcal{A}) = (\alpha, \beta)$, then $d\alpha = 0$, $d\beta = 0$ and*

$$\mathrm{rsw}_{n,\pi}(p \cdot \chi) = (C(\alpha), C(\beta)).$$

Proof. From Remark 1.2.23(1) we know that $(\alpha, \beta) \in Z_F^q \oplus Z_F^{q-1}$, since clearly $p \mid np$. The condition $e' > np$ implies $\frac{p}{p-1} \cdot (e - np + n) > 0$, which implies

$$e - np + n > 0. \tag{1.18}$$

Let $u \in \mathcal{O}_K^\times$ be such that $p = u \cdot \pi^e$.

$$\{p \cdot \chi, 1 + \pi^{n+1}T\} = \{\chi, (1 + \pi^{n+1}T)^p\} = \{\chi, 1 + \pi^{np+1}b(T)\}$$

where

$$b(T) = \frac{(1 + \pi^{n+1}T)^p - 1}{\pi^{np+1}}.$$

We can rewrite $b(T)$ as

$$\sum_{k=1}^{p-1} \pi^{e+(n+1)k-(np+1)} a_k T^k + \pi^{p(n+1)-(np+1)} T^p.$$

Note that, for every $1 \leq k \leq p$, we have that from equation (1.18)

$$e + (n+1)k - np - 1 = (e - np) + (n+1)k - 1 \geq 0.$$

Therefore, $b(T) \in \mathcal{O}_K[T]$. Now, since by assumption $\chi \in \mathrm{fil}_{np}\mathrm{H}^q(K)$, we have that $\{\chi, 1+\pi^{np+1}b(T)\} = 0$ for all $b(T) \in \mathcal{O}_K[T]$, thus $p \cdot \chi \in \mathrm{fil}_n\mathrm{H}^q(K)$. In a similar way,

$$\{p \cdot \chi, 1+\pi^n T\} = \{\mathcal{A}, (1+\pi^n T)^p\} = \{\chi, 1+\pi^{np}c(T)\}$$

where

$$c(T) = \frac{(1+\pi^n T)^p - 1}{\pi^{np}}.$$

We can rewrite $c(T)$ as

$$\sum_{k=1}^{p-1} \pi^{e+nk-np} a_k T^k + T^p.$$

In this case, again equation 1.18 together with $1 \le k \le p-1$, implies $e+nk-np > 0$. Therefore, $c(T) \in \mathcal{O}_K[T]$ and its reduction modulo π is equal to T^p. It follows from [Kat89, (6.3.1)] that

$$\{\chi, 1+\pi^{np}c(T)\} = \lambda_\pi(\bar{c}(T)\alpha, \bar{c}(T)\beta) = \lambda_\pi(T^p\alpha, T^p\beta) = \lambda_\pi(TC(\alpha), TC(\beta))$$

where the last equality follows from [BN23, Lemma 2.18(2)]. □

Remark 1.2.25. For every non-negative integer d, we denote by $\mathrm{fil}_n\mathrm{H}^q_d(K)$ the intersection of $\mathrm{fil}_n\mathrm{H}^q(K)$ with $\mathrm{H}^q_d(K)$. Kato proves that for every non-negative integer d prime to p, $\mathrm{H}^q_d(K) = \mathrm{fil}_0\mathrm{H}^q_d(K)$, see [Kat89, Proposition 6.1]. We will now prove some useful properties of the refined Swan conductor on p-power order elements.[1]

(1) The filtration $\mathrm{fil}_n\mathrm{H}^q_{p^m}(K)$ is finite.

Assume first e' to be an integer, i.e. $(p-1) \mid e$. For $m = 1$ it is proven in [Kat89, Proposition 4.1] that $\mathrm{H}^q_p(K) = \mathrm{fil}_{e'}\mathrm{H}^q_p(K)$. This is equivalent to saying that for all $n > e'$ and $\chi \in \mathrm{fil}_n\mathrm{H}^q_p(K)$, we have $\mathrm{rsw}_{n,\pi}(\chi) = (0,0)$. Assume that $m > 1$, $\chi \in \mathrm{fil}_n\mathrm{H}^q_{p^m}(K)$ with $n > e' + (m-1)e$ and $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta)$. From equation (1.17) we know that $\mathrm{rsw}_{n-e,\pi}(p \cdot \chi) = (\bar{u}\alpha, \bar{u}\beta)$, hence working on induction on m we get that $\mathrm{rsw}_{n,\pi}(\chi) = (0,0)$. Note that, this result was essentially already proved in [Ier22, Proposition 17].

If e' is not an integer, then we take a primitive p-root of unity ζ and we consider the field extension $K(\zeta)/K$ with ramification index $e_{K(\zeta)/K}$. We denote by e_K and $e_{K(\zeta)}$ the absolute ramification indexes of K and $K(\zeta)$ respectively and by e'_K and $e'_{K(\zeta)}$ the products $e_K \cdot p \cdot (p-1)^{-1}$ and $e_{K(\zeta)} \cdot p \cdot (p-1)^{-1}$ respectively. Let $n > e'_K + (m-1) \cdot e_K$ and $\chi \in \mathrm{fil}_n\mathrm{H}^q_{p^m}(K)$, then

$$e_{K(\zeta)/K} \cdot n > e'_{K(\zeta)} + (m-1)e_{K(\zeta)}.$$

Since $(p-1) \mid e'_{K(\zeta)}$, we get (from what we said above) that

$$\mathrm{rsw}_{e_{K(\zeta)/K}\cdot n,\pi}(\mathrm{res}(\chi)) = (0,0)$$

[1] Parts (b) and (c) are already mentioned by Kato in [Kat89, Sections 4 and 5], however in Chapter 4 we are going to use parts (b) and (c) to prove [Kat89, Lemma 4.3] for which no proof is provided in literature.

where res is the natural map from $\mathrm{H}^q_{p^m}(K) \to \mathrm{H}^q_{p^m}(K(\zeta))$. It follows from [BN23, Lemma 2.16] that

$$\mathrm{rsw}_{e_{K(\zeta)/K}\cdot n,\pi}(\mathrm{res}(\chi)) = (\bar{a}^{-n}(\alpha + \beta \wedge d\log \bar{a}), \bar{a}^{-n} e_{K(\zeta)/K}\beta)$$

with $\bar{a}$ invertible in the residue field of $K(\zeta)$ (cf. Section 4.1). Hence, since $p \nmid e_{K(\zeta)/K}$ we can conclude that $\mathrm{rsw}_{n,\pi}(\chi) = (0,0)$.

(2) Let $\chi \in \mathrm{fil}_{np}\mathrm{H}^q_p(K)$ with $np < e'$ and $\mathrm{rsw}_{np,\pi}(\chi) = (\alpha, \beta)$. We know from Lemma 1.2.24 that $(\alpha, \beta) \in Z^q_F \oplus Z^{q-1}_F$ and since χ has order p

$$(C(\alpha), C(\beta)) = \mathrm{rsw}_{n,\pi}(p \cdot \chi) = (0,0)$$

Equivalently, from Corollary 1.2.4 we get that for $np < e'$, the refined Swan conductor on the p-torsion takes image in $B^q_F \oplus B^{q-1}_F$.

(3) Assume e' to be an integer, $\chi \in \mathrm{fil}_{e'}\mathrm{H}^q_p(K)$ and $\mathrm{rsw}_{e',\pi}(\chi) = (\alpha, \beta)$. From equation (1.17) we get that

$$-\bar{u}\alpha = C(\alpha) \text{ and } -\bar{u}\beta = C(\beta).$$

Let ζ be a primitive p-root of unity and $c = (\zeta - 1)^p \pi^{-e'}$, then $c = (c_1)^p$, with $c_1 = (\zeta - 1)\pi^{-e/(p-1)}$. By the properties of the Cartier operator

$$C(\bar{c}\alpha) = \bar{c}_1 C(\alpha) = -\bar{u}\bar{c}_1\alpha.$$

Since $(\zeta - 1)^{p-1} \equiv -p \mod \pi^{e+1}$ [2], $(\zeta - 1)^{p-1} = -p + \pi^{e+1}v$ for some $v \in \mathcal{O}_K$. Hence

$$c = \frac{(\zeta-1)^p}{\pi^{e'}} = \frac{\zeta - 1}{\pi^{e/(p-1)}} \cdot \frac{(\zeta-1)^{p-1}}{\pi^e} = c_1 \cdot \left(\frac{-p + \pi^{e+1}v}{\pi^e} \right) = c_1 \cdot (-u + \pi v).$$

Thus $\bar{c} = -\bar{u}\bar{c}_1$ and hence

$$\mathrm{mult}_{\bar{c}} \circ \mathrm{rsw}_{e',\pi}(\mathrm{H}^q_p(K)) \subseteq \Omega^q_{F,\log} \oplus \Omega^{q-1}_{F,\log}$$

where $\mathrm{mult}_{\bar{c}}$ is the map from Ω^q_F to Ω^q_F sending a q-form ω to $\bar{c} \cdot \omega$.

1.3 Refined Swan conductor and Brauer–Manin obstruction

We are ready to state some of the main results in [BN23].

We are working in the same setting as Section 1.1.3: p is a prime number and L a finite field extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, uniformiser π and

[2] In fact, $1 = ((\zeta - 1) + 1)^p = (\zeta - 1)^p + p(\zeta - 1)^{p-1} + \binom{p}{2}(\zeta - 1)^{p-2} + \cdots + p \cdot (\zeta - 1) + 1$. Since $\zeta \neq 1$ we get $(\zeta - 1)^{p-1} + p(\zeta - 1)^{p-2} + \binom{p}{2}(\zeta - 1)^{p-3} + \cdots + p = 0$. Now the congruence follows from the fact that $\mathrm{val}((\zeta - 1) \cdot p) \geq 1 + e$.

residue field ℓ. Let X be a smooth and geometrically irreducible L-variety having good reduction (i.e. there exists a smooth $\mathcal{O}_L$-scheme $\mathcal{X}$ whose generic fiber is isomorphic to X). We assume furthermore the special fiber, $Y := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\ell)$ to be geometrically irreducible,

$$\begin{array}{ccccc} X & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) & \longleftarrow & \mathrm{Spec}(\ell). \end{array}$$

Let K^h be the field of fractions of the henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},Y}$. Bright and Newton [BN23] define the filtration $\{\mathrm{fil}_n\mathrm{Br}(X)\}_{n\geq 0}$ on $\mathrm{Br}(X)$ as the pull-back via the natural map $\mathrm{Br}(X) \to \mathrm{Br}(K^h)$ of the filtration $\{\mathrm{fil}_n\mathrm{Br}(K^h)\}_{n\geq 0}$ on $\mathrm{Br}(K^h)$ defined in Section 1.2.2. Therefore it is possible to extend the definition of the residue map and the refined Swan conductor to elements in $\mathrm{Br}(X)$ simply as the residue map and refined Swan conductor of the image of $\mathcal{A}$ in $\mathrm{Br}(K^h)$. A priori these maps take values in $\mathrm{H}^1(F)$ and $\Omega^2_F \oplus \Omega^1_F$ respectively. However, Bright and Newton prove that the residue map of an element in $\mathrm{fil}_0\mathrm{Br}(X)$ takes values in $\mathrm{H}^1(Y, \mathbb{Q}/\mathbb{Z}) \subseteq \mathrm{H}^1(F)$ (see [BN23, Proposition 3.1(1)]) and that the refined Swan conductor on $\mathrm{fil}_n\mathrm{Br}(X)$ has image in $\mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \subseteq \Omega^2_F \oplus \Omega^1_F$ (a proof following [Kat89, Theorem 7.1] can be found in [BN23, Theorem B]).

The aim of this section is to transfer the results we got in Section 1.2.3.1 on the refined Swan conductor on $\mathrm{Br}(K^h)$ to the refined Swan conductor on $\mathrm{Br}(X)$. In [BN23] Bright and Newton define the following filtration, called the **Evaluation filtration**, on the Brauer group of X. Let L'/L be a finite field extension with ring of integer $\mathcal{O}_{L'}$ and uniformiser π'; for $r \geq 1$ and $P \in \mathcal{X}(\mathcal{O}_{L'})$, let $B(P, r)$ be the set of points $Q \in \mathcal{X}(\mathcal{O}_{L'})$ such that Q has the same image as P in $\mathcal{X}(O_{L'}/(\pi')^r)$. Then:

$$\begin{aligned} \mathrm{Ev}_n\mathrm{Br}X &:= \{\mathcal{B} \in \mathrm{Br}(X) \mid \forall L'/L \text{ finite}, \forall P \in \mathcal{X}(\mathcal{O}_{L'}) \\ &\qquad \mathrm{ev}_{\mathcal{B}} \text{ is constant on } B(P, e_{L'/L}(n+1))\}, \qquad (n \geq 0) \\ \mathrm{Ev}_{-1}\mathrm{Br}X &:= \{\mathcal{B} \in \mathrm{Br}(X) \mid \forall L'/L \text{ finite}, \mathrm{ev}_{\mathcal{B}} \text{ is constant on } \mathcal{X}(\mathcal{O}_{L'})\} \\ \mathrm{Ev}_{-2}\mathrm{Br}X &:= \{\mathcal{B} \in \mathrm{Br}(X) \mid \forall L'/L \text{ finite}, \mathrm{ev}_{\mathcal{B}} \text{ is zero on } \mathcal{X}(\mathcal{O}_{L'})\} \end{aligned}$$

For every positive integer m we denote by $\mathrm{Ev}_n\mathrm{Br}(X)[m]$ the restriction of $\mathrm{Ev}_n\mathrm{Br}(X)$ to $\mathrm{Br}(X)[m]$, i.e. $\mathrm{Ev}_n\mathrm{Br}(X)[m] := \mathrm{Ev}_n\mathrm{Br}(X) \cap \mathrm{Br}(X)[m]$. If $(m, p) = 1$ Colliot-Thélène and Skorobogatov [CTS13] and Bright [Bri15] proved that $\mathrm{Ev}_0\mathrm{Br}(X)[m] = \mathrm{Br}(X)[m]$. Moreover, the residue map ∂_m

$$0 \to \mathrm{Br}(\mathcal{X})[m] \to \mathrm{Br}(X)[m] \xrightarrow{\partial_m} \mathrm{H}^1(Y, \mathbb{Z}/m\mathbb{Z}) \tag{1.19}$$

is such that

$$\begin{aligned} \mathrm{Ev}_{-1}\mathrm{Br}(X)[m] &= \{\mathcal{A} \in \mathrm{Br}(X)[m] \mid \partial_m(\mathcal{A}) \in H^1(\ell, \mathbb{Z}/m\mathbb{Z})\} \\ \mathrm{Ev}_{-2}\mathrm{Br}(X)[m] &= \{\mathcal{A} \in \mathrm{Br}(X)[m] \mid \partial_m(\mathcal{A}) = 0\}. \end{aligned}$$

Cf. the end of Section 1.1.3.

In order to give a description of the the Evaluation filtration also on the p-power torsion part of $\mathrm{Br}(X)$, Bright and Newton use the filtration $\{\mathrm{fil}_n\mathrm{Br}(X)\}_{n\geq 0}$ on $\mathrm{Br}(X)$. The interaction between the two filtrations is described in the theorem that follows.

Theorem 1.3.1 (Theorem A, Bright and Newton)**.** *We have the following description of the Evaluation filtration*

(1) $\mathrm{fil}_0\mathrm{Br}(X)$ *coincides with* $\mathrm{Ev}_0\mathrm{Br}(X)$*;*

(2) $\mathrm{Ev}_{-1}\mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{Br}(X) \mid \partial(\mathcal{A}) \in H^1(\ell, \mathbb{Q}/\mathbb{Z})\}$*;*

(3) $\mathrm{Ev}_{-2}\mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{Br}(X) \mid \partial(\mathcal{A}) = 0\}$*;*

(4) For every $n \geq 1$

$$\mathrm{Ev}_n\mathrm{Br}(X) = \left\{\mathcal{A} \in \mathrm{fil}_{n+1}\mathrm{Br}(X) \mid \mathrm{rsw}_{n+1,\pi}(\mathcal{A}) \in \mathrm{H}^0(Y, \Omega^2_Y) \oplus 0\right\}.$$

Proof. This is a reformulation of [BN23, Theorem A]. □

It is clear at this point that in order to understand the evaluation filtration on the Brauer group of X we need to understand the residue map ∂ and the refined Swan conductor maps $\mathrm{rsw}_{n,\pi}$.

1.3.1 Cartier operator on varieties and image of the refined Swan conductor

It is possible to generalise the definition of inverse Cartier operator and Cartier operator to the sheaf of q-forms on a smooth and proper variety Y defined over a perfect field ℓ of positive characteristic. Following Illusie [Ill79], we will denote by F_Y the absolute Frobenius endomorphism of Y and by $Y^{(p)}$ the base change of Y via the absolute Frobenius σ_ℓ of the base field ℓ, namely

$$\begin{array}{ccccc} Y & \xrightarrow{F_{Y/\ell}} & Y^{(p)} & \xrightarrow{W} & Y \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(\ell) & \xrightarrow{\sigma_\ell} & \mathrm{Spec}(\ell) \end{array}$$

where $W \circ F_{Y/\ell} = F_Y$; we call $F_{Y/\ell}$ the relative Frobenius of Y over ℓ. Furthermore, we denote by $\Omega^\bullet_{Y/\ell}$ the De Rham complex of Y/ℓ.

Remark 1.3.2. We are following [Ill79, Section 0.2]. In his paper, Illusie works more generally with S-schemes, where the base S is a scheme of positive characteristic. In this thesis, we are only interested in varieties over perfect fields. In this case, the absolute De Rham complex $\Omega^\bullet_Y$ coincides with the relative De Rham complex $\Omega^\bullet_{Y/\ell}$ (since ℓ being perfect implies $\Omega^q_\ell = 0$).

We define for every $q \geq 0$

$$Z_Y^q := \ker(d\colon \Omega_Y^q \to \Omega_Y^{q+1}) \quad \text{and} \quad B_Y^q := \operatorname{im}(d\colon \Omega_Y^{q-1} \to \Omega_Y^q).$$

For every $q \geq 0$, the differential $d\colon \Omega_Y^q \to \Omega_Y^{q+1}$ is $\mathcal{O}_{Y^{(p)}}$-linear, hence the sheaves

$$(F_{Y/\ell})_* Z_Y^q \quad \text{and} \quad (F_{Y/\ell})_* B_Y^q$$

are $\mathcal{O}_{Y^{(p)}}$-modules and the abelian sheaf $\mathcal{H}^q((F_{Y/\ell})_*\Omega_Y^\bullet)$ is also a sheaf of $\mathcal{O}_{Y^{(p)}}$-modules.

Definition 1.3.3 (Inverse Cartier operator)**.** *For every $q \geq 0$ there is a morphism of $\mathcal{O}_Y$-modules, called the* ***inverse Cartier operator***

$$C_Y^{-1}\colon \Omega_Y^q \to W_*\mathcal{H}^i((F_{Y/\ell})_*\Omega_Y^\bullet).$$

Remark 1.3.4. Since $\Omega_{Y^{(p)}}^q = W^*\Omega_Y^q$, by adjunction we get a morphism of $\mathcal{O}_{Y^{(p)}}$-modules

$$C_{Y/\ell}^{-1}\colon \Omega_{Y^{(p)}}^q \to \mathcal{H}^q((F_{Y/\ell})_*\Omega_Y^\bullet).$$

Theorem 1.3.5. *If Y is a smooth variety over ℓ, then $C_{Y/\ell}^{-1}$ is an isomorphism.*

Proof. See [Ill79, Theorem 0.2.1.9]. □

From now on we are assuming that Y is a smooth and proper variety over ℓ. In this case, we denote by $C_{Y/\ell}$ the inverse of $C_{Y/\ell}^{-1}$. We are ready to define the sheaf of logarithmic forms on Y.

Definition 1.3.6 (Logarithmic forms)**.** *For every non-negative integer q, we denote by*

$$\Omega_{Y,\log}^q := \ker(W^* - C_{Y/\ell}\colon Z_Y^q \to \Omega_{Y^{(p)}}^q).$$

The sheaf $\Omega_{Y,\log}^q$ is called the sheaf of logarithmic *q-forms on Y.*

The following theorem is the analogue of Theorem 1.2.6 for fields of positive characteristic.

Theorem 1.3.7. *The sheaf $\Omega_{Y,\log}^q$ is the subsheaf of Ω_Y^q generated étale-locally by the logarithmic differentials, i.e. the sections of the form*

$$\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \text{ with } y_i \in \mathcal{O}_Y^*.$$

Proof. See [Ill79, Theorem 0.2.4.2]. □

The Cartier operator induces an exact sequence

$$0 \to \mathrm{H}^0(Y, B_Y^q) \to \mathrm{H}^0(Y, Z_Y^q) \xrightarrow{C_Y} \mathrm{H}^0(Y, \Omega_Y^q)$$

The following lemma describes the interaction between global differential forms on Y and their image in Ω_F^q.

Lemma 1.3.8. *Let* $\omega \in \mathrm{H}^0(Y, \Omega^q_Y)$*, then:*

(1) $\omega \in \mathrm{H}^0(Y, \Omega^q_{Y,\log})$ *if and only if the image of* ω *in* Ω^q_F *lies in* $\Omega^q_{F,\log}$*;*

(2) $\omega \in \mathrm{H}^0(Y, B^q_Y)$ *if and only if the image of* ω *in* Ω^q_F *lies in* B^q_F*.*

Proof. For all non-negative integers q, we have natural inclusions of $\mathrm{H}^0(Y, \Omega^q_Y)$ in Ω^q_F. These inclusions are compatible with the differential maps and with the Cartier operator. The proof of the first part of the lemma follows from the definition of logarithmic forms, while the second part is an immediate consequence of Corollary 1.2.4. ☐

Corollary 1.3.9 (On the image of $\mathrm{rsw}_{n,\pi}$)**.** *Let* $u = p\pi^{-e} \in \mathcal{O}_L^\times$*; then one of the following cases occurs:*

(1) *If* $p \nmid n$*, then*
$$\mathrm{pr}_2 \circ \mathrm{rsw}_{n,\pi} \colon \mathrm{fil}_n\mathrm{Br}(X) \to \mathrm{H}^0(Y, \Omega^1_Y)$$
has kernel equal to $\mathrm{fil}_{n-1}\mathrm{Br}(X)$*.*

(2) *If* $p \mid n$ *and* $n < e'$ *we write* $n = mp$*, then* (α, β) *lies in* $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ *and the following diagram*
$$\begin{array}{ccc} \mathrm{fil}_n\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\ \big\downarrow{\scriptstyle (-)^{\otimes p}} & & \big\downarrow{\scriptstyle C} \\ \mathrm{fil}_m\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{m,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y). \end{array}$$
commutes.

(3) *If* $n = e'$*, then* (α, β) *lies in* $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ *and the following diagram*
$$\begin{array}{ccc} \mathrm{fil}_{e'}\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\ \big\downarrow{\scriptstyle (-)^{\otimes p}} & & \big\downarrow{\scriptstyle \mathrm{mult}_{\bar{u}} + C} \\ \mathrm{fil}_{e'-e}\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n-e,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y). \end{array}$$
commutes. Moreover, let $\bar{c} \in \ell^\times$ *be the reduction of* $c = (\zeta - 1)^p\pi^{-e'} \in \mathcal{O}_L^\times$*, then* $\bar{c}$ *is such that*
$$\mathrm{mult}_{\bar{c}}\left(\mathrm{rsw}_{e',\pi}\mathrm{fil}_{e'}\mathrm{Br}(X)[p]\right) \subseteq \mathrm{H}^0(Y, \Omega^2_{Y,\log}) \oplus \mathrm{H}^0(Y, \Omega^1_{Y,\log}).$$

(4) *If* $p \mid n$ *and* $n > e'$*, then* (α, β) *lies in* $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ *and the following diagram*
$$\begin{array}{ccc} \mathrm{fil}_n\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\ \big\downarrow{\scriptstyle (-)^{\otimes p}} & & \big\downarrow{\scriptstyle \mathrm{mult}_{\bar{u}}} \\ \mathrm{fil}_{n-e}\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n-e,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y). \end{array}$$
commutes.

Proof. All these properties are a consequence of Section 1.2.3.1. More precisely: 1. follows from Remark 1.2.23(2); 2. follows from Lemma 1.2.24; 3. and 4. are a direct consequence of [BN23, Lemma 2.19]. □