The wild Brauer-Manin obstruction on K3 surfaces

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Introduction

The focus of this thesis is on the study of rational points on varieties. This kind of problem stems from the desire to be able to describe the rational solutions of a polynomial equation, i.e. given a polynomial $f(x_0, \ldots, x_n) \in \mathbb{Q}[x_0, \ldots, x_n]$, what can we say about

$$\mathrm{Z}(f)(\mathbb{Q}) := \{(\alpha_0, \ldots, \alpha_n) \in \mathbb{Q}^{n+1} \mid f(\alpha_0, \ldots, \alpha_n) = 0\}?$$

A first way to study this set is by looking at the equation over the real numbers. In fact, if there is no $(\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$ such that $f(\alpha_0, \ldots, \alpha_n) = 0$ then $\mathrm{Z}(f)(\mathbb{Q})$ is also the empty set. The real numbers are a complete field, which makes the study of the zeros of functions defined over it much more accessible. However, $\mathbb{R}$ is just one of the possible completions of $\mathbb{Q}$, the one with respect to the Euclidean metric. The p-adic metrics give for every prime p a field, $\mathbb{Q}_p$, which is also a complete field that contains $\mathbb{Q}$. Putting all of these together gives a natural inclusion

$$\mathrm{Z}(f)(\mathbb{Q}) \hookrightarrow \prod_{p \text{ prime}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R}).$$

The sets $\mathrm{Z}(f)(\mathbb{Q}_p)$ and $\mathrm{Z}(f)(\mathbb{R})$ are subsets of $\mathbb{Q}_p^{n+1}$ and $\mathbb{R}^{n+1}$, hence they inherit a topology coming from the topology on $\mathbb{Q}_p$ and $\mathbb{R}$ respectively. We are particularly interested in understanding the closure of the set $\mathrm{Z}(f)(\mathbb{Q})$ in the product $\prod_{p \text{ prime}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R})$.

In the language of algebraic geometry, the polynomial f defines a variety and the set $\mathrm{Z}(f)(\mathbb{Q})$ can be recovered as the set of morphisms from $\mathrm{Spec}(\mathbb{Q})$ to the variety (similarly for $\mathbb{Q}_p$ and $\mathbb{R}$). In particular, we work in the following setting:

$$\begin{array}{ccc} \mathbb{Q} & \rightsquigarrow & k \text{ number field;} \\ \mathrm{Z}(f) & \rightsquigarrow & V \text{ proper, smooth and geometrically integral } k\text{-variety;} \\ \prod_p \mathbb{Q}_p \times \mathbb{R} & \rightsquigarrow & \mathbf{A}_k, \text{ the ring of adèles of } k. \end{array}$$

As we will see in Chapter 1, the ring of adèles encodes the information about all completions of k with respect to the places ν of the number field k. Moreover, if V is a proper variety, then we will see that the $\mathbf{A}_k$-points on V coincide with $\prod_{\nu\in\Omega_k} V(k_\nu)$.

Definition 1. *We say that V satisfies* ***weak approximation*** *if the image of $V(k)$ in $V(\mathbf{A}_k)$ is dense.*

In 1970 Manin [Man71] introduced the use of the Brauer group of a variety V, defined in Chapter 1, to study the image of $V(k)$ in $V(\mathbf{A}_k)$. In particular, he used the Brauer group to build an intermediate set $V(\mathbf{A}_k)^{\mathrm{Br}}$ such that

$$V(k) \subseteq V(\mathbf{A}_k)^{\mathrm{Br}} \subseteq V(\mathbf{A}_k).$$

Using class field theory it is possible to prove that for every place $\nu \in \Omega_k$ and for every element $\mathcal{A} \in \mathrm{Br}(V)$, there is a map, called the evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon V(k_\nu) \to \mathbb{Q}/\mathbb{Z}$$

such that

$$V(\mathbf{A}_k)^{\mathrm{Br}} := \left\{ (x_\nu) \in X(\mathbf{A}_k) \mid \forall \mathcal{A} \in \mathrm{Br}(V), \quad \sum_{\nu\in\Omega_k} \mathrm{ev}_{\mathcal{A}}(x_\nu) = 0 \right\} \tag{1}$$

is a **closed** subset of $V(\mathbf{A}_k)$ that contains the set of k-points $V(k)$.

Definition 2. *We say that there is a* ***Brauer–Manin obstruction*** *to weak approximation on V if $V(\mathbf{A}_k)^{\mathrm{Br}}$ is a proper subset of $V(\mathbf{A}_k)$.*

As we will see in more detail in Section 1.1.2.3 from equation (1) we get that if for a place ν there is an element $\mathcal{A} \in \mathrm{Br}(V)$ such that $\mathrm{ev}_{\mathcal{A}} \colon V(k_\nu) \to \mathbb{Q}/\mathbb{Z}$ is non-constant, then $V(\mathbf{A}_k)^{\mathrm{Br}} \subsetneq V(\mathbf{A}_k)$. In this case we say that the place ν **plays a role** in the Brauer–Manin obstruction to weak approximation on V.

Let $\bar{k}$ be an algebraic closure of k and $\bar{V}$ be the base change of V to $\bar{k}$, i.e. $\bar{V} := V \times_k \bar{k}$. The results of this thesis are inspired by the following question:

Question 3. Assume $\mathrm{Pic}(\bar{V})$ to be torsion-free and finitely generated. Which places can play a role in the Brauer–Manin obstruction to weak approximation on V?

This question is a reformulation of a question that was originally asked by Swinnerton–Dyer; he asked whether under the assumption of Question 3 the only places that can play a role in the Brauer–Manin obstruction to weak approximation are the archimedean ones and the ones of bad reduction for the variety. The necessity of having such a condition on the geometric Picard group $\mathrm{Pic}(\bar{V})$ is a consequence of work of Harari [Har00], see Section 2.1.1.

It turns out that under the assumption of Question 3 if a prime of good reduction plays a role, then the corresponding element $\mathcal{A}$ in the Brauer group cannot be algebraic, i.e. it cannot lie in the kernel of $\mathrm{Br}(V) \to \mathrm{Br}(\bar{V})$[1]. In [CTS13] Colliot-Thélène and Skorobogatov showed that, if the transcendental Brauer group is finite, then the only places that can play a role are the archimedean places, the places of bad reduction and the places whose residue characteristic divides the order of the transcendental Brauer group, see [CTS13]. Using this result, they give several examples of varieties for which the answer to Swinnerton-Dyer's question is positive.

For curves and surfaces with negative Kodaira dimension we have all the elements in the Brauer group are algebraic, i.e. $\mathrm{Br}(V) = \mathrm{Br}_1(V)$. Hence, K3 surfaces are one of the first example of varieties where the transcendental Brauer group is potentially non-trivial. However, this is not always the case: for example in [ISZ11] the authors show that, under certain conditions, the whole Brauer group of a diagonal quartic surface over $\mathbb{Q}$ is algebraic. The first example of a transcendental element in the Brauer group of a K3 surface defined over a number field was given by Wittenberg in [Wit04]. In particular, Wittenberg constructed a 2-torsion transcendental element that obstructs weak approximation on the surface. Other examples of 2-torsion transcendental elements that obstruct weak approximation can be found in [HVAV11] and [Ier10]. In all these articles, the obstruction to weak approximation comes from the fact that the transcendental quaternion algebra has non-constant evaluation at the place at infinity. With a construction similar to the one used in [HVAV11], Hassett and Várilly-Alvarado [HVA13] have also built an example of a 2-torsion element on a K3 surface that obstructs the Hasse principle.

Furthermore, there are examples of transcendental elements of order 3 on K3 surfaces that obstruct weak approximation (for example, see [Pre13], [New16] and [BVA20]). In all these cases, the evaluation map at the place at infinity has to be trivial, since $\mathrm{Br}(\mathbb{R})$ does not contain elements of order 3, and the obstruction to weak approximation comes from the evaluation map at the prime 3, which in every example is a prime of bad reduction for the K3 surface taken into account. Therefore, none of the examples mentioned above can be used to give a negative answer to the question formulated by Swinnerton-Dyer.

After Colliot-Thélène and Skorobogatov's work the main remaining difficulty was to control the evaluation map of a p-power element in the Brauer group on the $V(k_{\mathfrak{p}})$-points, where $\mathfrak{p}$ is a prime of good reduction with residue characteristic p. In 2020 Bright and Newton [BN23] developed new techniques that allow one to control also the behaviour of the evaluation map in this case. Roughly speaking, they use work of Kato [Kat89] to introduce a new filtration on the Brauer group which they prove to be strongly related to the behaviour of the evaluation map attached to elements in $\mathrm{Br}(V)$. The first chapter of this thesis is devoted to introducing these new techniques and extending some results of Bright and Newton.

Bright and Newton also prove that if we start with a variety V with non-trivial $\mathrm{H}^0(V, \Omega^2_V)$, then for primes $\mathfrak{p}$ having good ordinary reduction we can always find a finite field extension k'/k and a prime above $\mathfrak{p}$ playing a role in the Brauer–Manin obstruction to weak approximation: see [BN23, Theorem C] or Chapter 2 for the

[1] Elements in the Brauer group that are not algebraic are called transcendental.

precise statement.

Hence, if we take a variety V such that:

- the geometric Picard group $\mathrm{Pic}(\bar{V})$ is torsion-free and finitely generated,
- $\mathrm{H}^0(V, \Omega^2_V)$ is non-trivial,
- there is a prime $\mathfrak{p}$ of good ordinary reduction,

then up to a base change to a finite field extension k'/k we can **always** find a prime of good (ordinary) reduction that plays a role in the Brauer–Manin obstruction to weak approximation on $V_{k'}$. Since K3 surfaces satisfy all the properties listed above, Bright and Newton's result leads to a negative answer to the question asked by Swinnerton-Dyer. In Chapter 3 we will improve this theorem for K3 surfaces.

Theorem 4. *Let V be a K3 surface over a number field k and $\mathfrak{p}$ be a prime of good ordinary reduction for V. Then there exist a finite field extension k'/k and an element $\mathcal{A} \in \mathrm{Br}(V_{k'})[p]$ that obstructs weak approximation on $V_{k'}$.*

In particular, in Bright and Newton's result the element $\mathcal{A} \in \mathrm{Br}(V_{k'})$ giving an obstruction to weak approximation has order a p-power, where p is the residue characteristic of the prime $\mathfrak{p}$ having good ordinary reduction. We prove that for K3 surfaces, you can always find an element of order exactly p that gives an obstruction to weak approximation.

However, neither result says anything about how large the field extension k'/k can be. In particular, they do not say whether it is possible to find already over $\mathbb{Q}$ a Brauer-Manin obstruction arising from a prime of good reduction. The aim of Chapter 2 is to build a K3 surface over the rational numbers for which a prime of good reduction is involved in the Brauer–Manin obstruction to weak approximation. This is the first example of a K3 surface (defined over $\mathbb{Q}$) for which a prime of good reduction plays a role in the Brauer–Manin obstruction to weak approximation.

Theorem 5. *Let $V \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the projective K3 surface defined by the equation*

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \tag{2}$$

The class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right) \in \mathrm{Br}\,\mathbb{Q}(V)$$

defines an element in $\mathrm{Br}(V)$. The evaluation map $\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{Q}_2) \to \mathrm{Br}(\mathbb{Q}_2)$ is non-constant. Moreover, $V(\mathbb{Q})$ is not dense in $V(\mathbb{Q}_2)$.

This theorem proves that the field extension appearing in Theorem 4 is not always needed.

At this point some natural questions arise:

Question 6.

1. When is the field extension k'/k appearing in Bright and Newton's result needed?
2. Is the ordinary condition necessary?

Answering these two questions is the aim of Chapter 3. In particular, we prove the following results.

Theorem 7. *Let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Assume that the special fibre at $\mathfrak{p}$, $\mathcal{V}(\mathfrak{p})$ has no non-trivial global 1-forms, $\mathrm{H}^1(\overline{\mathcal{V}(\mathfrak{p})}, \mathbb{Z}/p\mathbb{Z}) = 0$ and $(p-1) \nmid e_{\mathfrak{p}}$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Theorem 8. *Let V be a K3 surface and $\mathfrak{p}$ be a prime of good non-ordinary reduction for V with $e_{\mathfrak{p}} \leq (p-1)$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Finally, in the last chapter of this thesis we will show that the conditions in Theorem 7 and Theorem 8 are optimal. Part of the chapter is devoted to the proof of the following theorem.

Theorem 9. *Let A be the abelian surface given by the product of two elliptic curves E_1 and E_2 defined over $\mathbb{Q}$. Assume that both E_1 and E_2 have good ordinary reduction at the prime 2 and full two torsion defined over $\mathbb{Q}_2$. Let $V = \mathrm{Kum}(A)$ be the corresponding Kummer K3 surface. Then, V has good ordinary reduction at 2 and every element $\mathcal{A} \in \mathrm{Br}(V)[2]$ has constant evaluation map $\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{Q}_2) \to \mathbb{Q}/\mathbb{Z}$.*

In particular, this proves that the field extension appearing in Theorem 4 is in general needed. Moreover, starting from this theorem we will build an example for which 2 is a prime of good ordinary reduction but does not play a role in the Brauer–Manin obstruction, even though the ramification index (which is 1) is divisible by $p-1$. This shows that the condition $(p-1) \nmid e_{\mathfrak{p}}$ in Theorem 7 is sufficient but not necessary. Moreover, we will also produce an example of a K3 surface for which there is a prime of good non-ordinary reduction $\mathfrak{p}$ that plays a role in the Brauer–Manin obstruction to weak approximation and for which $e_{\mathfrak{p}} = p$. The latter example shows that the bound appearing in Theorem 8 is optimal.

Notation and conventions

We include here some notation and conventions that we will follow throughout this thesis. Most of the notation will be introduced once again in the thesis when needed, however we include them here as well for convenience.

Varieties over number fields

We denote by k any number field and $\mathcal{O}_k$ its ring of integers. Let Ω_k be the set of places of k, then for any place $\nu \in \Omega_k$ we denote by k_ν the completion of k at ν. We call places corresponding to archimedean valuation archimedean, while places corresponding to non-archimedean valuation will be called finite. For $\nu \in \Omega_k$ archimedean, we define $\mathcal{O}_\nu := k_\nu$. If ν is a finite place we will often denote it by $\mathfrak{p}$, where $\mathfrak{p}$ is the corresponding prime ideal in $\mathcal{O}_k$ and we denote by $\mathcal{O}_\mathfrak{p}$ (sometimes by $\mathcal{O}_\nu$) its ring of integers and by $k(\mathfrak{p})$ (sometimes by $k(\nu)$) its residue field. For any finite set of places $S \subset \Omega_k$ we define $\mathcal{O}_{k,S}$ as the intersection in k of the local rings $\mathcal{O}_\nu$ for $\nu \in S$.

Finally, we denote by V any proper, smooth and geometrically integral variety over k. We refer to [Liu02] for an introduction to schemes and their properties. Let $\mathcal{V}$ be an $\mathcal{O}_{k,S}$-scheme of finite type such that there exists an isomorphism between $\mathcal{V} \times_{\mathcal{O}_{k,S}} \mathrm{Spec}(k)$ and V (i.e. $\mathcal{V}$ is an $\mathcal{O}_{k,S}$ model for the scheme V); for any finite place $\mathfrak{p}$ not in S, we denote by $\mathcal{V}(\mathfrak{p})$ the special fibre at $\mathfrak{p}$, i.e. the base change of $\mathcal{V}$ to $\mathrm{Spec}(k(\mathfrak{p}))$, by $\mathcal{V}_\mathfrak{p}$ the base change of $\mathcal{V}$ to $\mathcal{O}_\mathfrak{p}$ and by $V_\mathfrak{p}$ the base change of V to $k_\mathfrak{p}$.

Varieties over p-adic fields

Let p be a prime number. We denote by L any p-adic field (i.e. a finite field extension of the field of p-adic numbers $\mathbb{Q}_p$), with ring of integers $\mathcal{O}_L$ and residue field ℓ, which is a finite field of positive characteristic. We denote by X any smooth and geometrically integral variety over L and by $\mathcal{X}$ any smooth model over $\mathcal{O}_L$ with geometrically integral fibre, i.e. $\mathcal{X}$ is such that there exists an isomorphism between $\mathcal{X} \times_{\mathcal{O}_L} \mathrm{Spec}(L)$ and X. Moreover, we denote by Y the special fibre of $\mathcal{X}$, which is the base change of $\mathcal{X}$ to ℓ and is a smooth, proper and geometrically integral variety over the finite field ℓ, since both being smooth and proper are stable under base change.

Remark 10. In particular, for any finite place $\mathfrak{p}$ of k, the completion of k at $\mathfrak{p}$, $k_\mathfrak{p}$, is a p-adic field with residue field $k(\mathfrak{p})$ of positive characteristic p. Hence, the base change $V_\mathfrak{p}$ of V to $k_\mathfrak{p}$ is a variety over a p-adic field and the special fibre at $\mathfrak{p}$, $\mathcal{V}(\mathfrak{p})$ is a variety over the finite field $k(\mathfrak{p})$, which is the residue field of the p-adic field $k_\mathfrak{p}$.