The wild Brauer-Manin obstruction on K3 surfaces

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THE WILD BRAUER–MANIN OBSTRUCTION ON K3 SURFACES

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Introduction

The focus of this thesis is on the study of rational points on varieties. This kind of problem stems from the desire to be able to describe the rational solutions of a polynomial equation, i.e. given a polynomial $f(x_0, \dots, x_n) \in \mathbb{Q}[x_0, \dots, x_n]$, what can we say about

$$\mathrm{Z}(f)(\mathbb{Q}) := \{(\alpha_0, \dots, \alpha_n) \in \mathbb{Q}^{n+1} \mid f(\alpha_0, \dots, \alpha_n) = 0\}?$$

A first way to study this set is by looking at the equation over the real numbers. In fact, if there is no $(\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1}$ such that $f(\alpha_0, \dots, \alpha_n) = 0$ then $\mathrm{Z}(f)(\mathbb{Q})$ is also the empty set. The real numbers are a complete field, which makes the study of the zeros of functions defined over it much more accessible. However, $\mathbb{R}$ is just one of the possible completions of $\mathbb{Q}$, the one with respect to the Euclidean metric. The p-adic metrics give for every prime p a field, $\mathbb{Q}_p$, which is also a complete field that contains $\mathbb{Q}$. Putting all of these together gives a natural inclusion

$$\mathrm{Z}(f)(\mathbb{Q}) \hookrightarrow \prod_{p \text{ prime}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R}).$$

The sets $\mathrm{Z}(f)(\mathbb{Q}_p)$ and $\mathrm{Z}(f)(\mathbb{R})$ are subsets of $\mathbb{Q}_p^{n+1}$ and $\mathbb{R}^{n+1}$, hence they inherit a topology coming from the topology on $\mathbb{Q}_p$ and $\mathbb{R}$ respectively. We are particularly interested in understanding the closure of the set $\mathrm{Z}(f)(\mathbb{Q})$ in the product $\prod_{p \text{ prime}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R})$.

In the language of algebraic geometry, the polynomial f defines a variety and the set $\mathrm{Z}(f)(\mathbb{Q})$ can be recovered as the set of morphisms from $\mathrm{Spec}(\mathbb{Q})$ to the variety (similarly for $\mathbb{Q}_p$ and $\mathbb{R}$). In particular, we work in the following setting:

$\mathbb{Q}$	$\rightsquigarrow$	k number field;
$\mathrm{Z}(f)$	$\rightsquigarrow$	V proper, smooth and geometrically integral k-variety;
$\prod_p \mathbb{Q}_p \times \mathbb{R}$	$\rightsquigarrow$	$\mathbf{A}_k$, the ring of adèles of k.

As we will see in Chapter 1, the ring of adèles encodes the information about all completions of k with respect to the places ν of the number field k. Moreover, if V is a proper variety, then we will see that the $\mathbf{A}_k$-points on V coincide with $\prod_{\nu\in\Omega_k} V(k_\nu)$.

Definition 1. *We say that V satisfies **weak approximation** if the image of $V(k)$ in $V(\mathbf{A}_k)$ is dense.*

In 1970 Manin [Man71] introduced the use of the Brauer group of a variety V, defined in Chapter 1, to study the image of $V(k)$ in $V(\mathbf{A}_k)$. In particular, he used the Brauer group to build an intermediate set $V(\mathbf{A}_k)^{\mathrm{Br}}$ such that

$$V(k) \subseteq V(\mathbf{A}_k)^{\mathrm{Br}} \subseteq V(\mathbf{A}_k).$$

Using class field theory it is possible to prove that for every place $\nu \in \Omega_k$ and for every element $\mathcal{A} \in \mathrm{Br}(V)$, there is a map, called the evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon V(k_\nu) \to \mathbb{Q}/\mathbb{Z}$$

such that

$$V(\mathbf{A}_k)^{\mathrm{Br}} := \left\{ (x_\nu) \in X(\mathbf{A}_k) \mid \forall \mathcal{A} \in \mathrm{Br}(V), \quad \sum_{\nu\in\Omega_k} \mathrm{ev}_{\mathcal{A}}(x_\nu) = 0 \right\} \tag{1}$$

is a **closed** subset of $V(\mathbf{A}_k)$ that contains the set of k-points $V(k)$.

Definition 2. *We say that there is a **Brauer–Manin obstruction** to weak approximation on V if $V(\mathbf{A}_k)^{\mathrm{Br}}$ is a proper subset of $V(\mathbf{A}_k)$.*

As we will see in more detail in Section 1.1.2.3 from equation (1) we get that if for a place ν there is an element $\mathcal{A} \in \mathrm{Br}(V)$ such that $\mathrm{ev}_{\mathcal{A}} \colon V(k_\nu) \to \mathbb{Q}/\mathbb{Z}$ is non-constant, then $V(\mathbf{A}_k)^{\mathrm{Br}} \subsetneq V(\mathbf{A}_k)$. In this case we say that the place ν **plays a role** in the Brauer–Manin obstruction to weak approximation on V.

Let $\bar{k}$ be an algebraic closure of k and $\bar{V}$ be the base change of V to $\bar{k}$, i.e. $\bar{V} := V \times_k \bar{k}$. The results of this thesis are inspired by the following question:

Question 3. Assume $\mathrm{Pic}(\bar{V})$ to be torsion-free and finitely generated. Which places can play a role in the Brauer–Manin obstruction to weak approximation on V?

This question is a reformulation of a question that was originally asked by Swinnerton–Dyer; he asked whether under the assumption of Question 3 the only places that can play a role in the Brauer–Manin obstruction to weak approximation are the archimedean ones and the ones of bad reduction for the variety. The necessity of having such a condition on the geometric Picard group $\mathrm{Pic}(\bar{V})$ is a consequence of work of Harari [Har00], see Section 2.1.1.

It turns out that under the assumption of Question 3 if a prime of good reduction plays a role, then the corresponding element $\mathcal{A}$ in the Brauer group cannot be algebraic, i.e. it cannot lie in the kernel of $\mathrm{Br}(V) \to \mathrm{Br}(\bar{V})$[1]. In [CTS13] Colliot-Thélène and Skorobogatov showed that, if the transcendental Brauer group is finite, then the only places that can play a role are the archimedean places, the places of bad reduction and the places whose residue characteristic divides the order of the transcendental Brauer group, see [CTS13]. Using this result, they give several examples of varieties for which the answer to Swinnerton-Dyer's question is positive.

For curves and surfaces with negative Kodaira dimension we have all the elements in the Brauer group are algebraic, i.e. $\mathrm{Br}(V) = \mathrm{Br}_1(V)$. Hence, K3 surfaces are one of the first example of varieties where the transcendental Brauer group is potentially non-trivial. However, this is not always the case: for example in [ISZ11] the authors show that, under certain conditions, the whole Brauer group of a diagonal quartic surface over $\mathbb{Q}$ is algebraic. The first example of a transcendental element in the Brauer group of a K3 surface defined over a number field was given by Wittenberg in [Wit04]. In particular, Wittenberg constructed a 2-torsion transcendental element that obstructs weak approximation on the surface. Other examples of 2-torsion transcendental elements that obstruct weak approximation can be found in [HVAV11] and [Ier10]. In all these articles, the obstruction to weak approximation comes from the fact that the transcendental quaternion algebra has non-constant evaluation at the place at infinity. With a construction similar to the one used in [HVAV11], Hassett and Várilly-Alvarado [HVA13] have also built an example of a 2-torsion element on a K3 surface that obstructs the Hasse principle.

Furthermore, there are examples of transcendental elements of order 3 on K3 surfaces that obstruct weak approximation (for example, see [Pre13], [New16] and [BVA20]). In all these cases, the evaluation map at the place at infinity has to be trivial, since $\mathrm{Br}(\mathbb{R})$ does not contain elements of order 3, and the obstruction to weak approximation comes from the evaluation map at the prime 3, which in every example is a prime of bad reduction for the K3 surface taken into account. Therefore, none of the examples mentioned above can be used to give a negative answer to the question formulated by Swinnerton-Dyer.

After Colliot-Thélène and Skorobogatov's work the main remaining difficulty was to control the evaluation map of a p-power element in the Brauer group on the $V(k_\mathfrak{p})$-points, where $\mathfrak{p}$ is a prime of good reduction with residue characteristic p. In 2020 Bright and Newton [BN23] developed new techniques that allow one to control also the behaviour of the evaluation map in this case. Roughly speaking, they use work of Kato [Kat89] to introduce a new filtration on the Brauer group which they prove to be strongly related to the behaviour of the evaluation map attached to elements in $\mathrm{Br}(V)$. The first chapter of this thesis is devoted to introducing these new techniques and extending some results of Bright and Newton.

Bright and Newton also prove that if we start with a variety V with non-trivial $\mathrm{H}^0(V, \Omega^2_V)$, then for primes $\mathfrak{p}$ having good ordinary reduction we can always find a finite field extension k'/k and a prime above $\mathfrak{p}$ playing a role in the Brauer–Manin obstruction to weak approximation: see [BN23, Theorem C] or Chapter 2 for the

[1]Elements in the Brauer group that are not algebraic are called transcendental.

precise statement.

Hence, if we take a variety V such that:

- the geometric Picard group $\operatorname{Pic}(\bar{V})$ is torsion-free and finitely generated,
- $\mathrm{H}^0(V, \Omega_V^2)$ is non-trivial,
- there is a prime $\mathfrak{p}$ of good ordinary reduction,

then up to a base change to a finite field extension k'/k we can **always** find a prime of good (ordinary) reduction that plays a role in the Brauer–Manin obstruction to weak approximation on $V_{k'}$. Since K3 surfaces satisfy all the properties listed above, Bright and Newton's result leads to a negative answer to the question asked by Swinnerton-Dyer. In Chapter 3 we will improve this theorem for K3 surfaces.

Theorem 4. *Let V be a K3 surface over a number field k and $\mathfrak{p}$ be a prime of good ordinary reduction for V. Then there exist a finite field extension k'/k and an element $\mathcal{A} \in \operatorname{Br}(V_{k'})[p]$ that obstructs weak approximation on $V_{k'}$.*

In particular, in Bright and Newton's result the element $\mathcal{A} \in \operatorname{Br}(V_{k'})$ giving an obstruction to weak approximation has order a p-power, where p is the residue characteristic of the prime $\mathfrak{p}$ having good ordinary reduction. We prove that for K3 surfaces, you can always find an element of order exactly p that gives an obstruction to weak approximation.

However, neither result says anything about how large the field extension k'/k can be. In particular, they do not say whether it is possible to find already over $\mathbb{Q}$ a Brauer-Manin obstruction arising from a prime of good reduction. The aim of Chapter 2 is to build a K3 surface over the rational numbers for which a prime of good reduction is involved in the Brauer–Manin obstruction to weak approximation. This is the first example of a K3 surface (defined over $\mathbb{Q}$) for which a prime of good reduction plays a role in the Brauer–Manin obstruction to weak approximation.

Theorem 5. *Let $V \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the projective K3 surface defined by the equation*

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \tag{2}$$

The class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right) \in \operatorname{Br} \mathbb{Q}(V)$$

defines an element in $\operatorname{Br}(V)$. The evaluation map $\operatorname{ev}_{\mathcal{A}} \colon V(\mathbb{Q}_2) \to \operatorname{Br}(\mathbb{Q}_2)$ is non-constant. Moreover, $V(\mathbb{Q})$ is not dense in $V(\mathbb{Q}_2)$.

This theorem proves that the field extension appearing in Theorem 4 is not always needed.

At this point some natural questions arise:

Question 6.

1. When is the field extension k'/k appearing in Bright and Newton's result needed?

2. Is the ordinary condition necessary?

Answering these two questions is the aim of Chapter 3. In particular, we prove the following results.

Theorem 7. *Let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Assume that the special fibre at $\mathfrak{p}$, $\mathcal{V}(\mathfrak{p})$ has no non-trivial global 1-forms, $\mathrm{H}^1(\overline{\mathcal{V}(\mathfrak{p})}, \mathbb{Z}/p\mathbb{Z}) = 0$ and $(p-1) \nmid e_{\mathfrak{p}}$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Theorem 8. *Let V be a K3 surface and $\mathfrak{p}$ be a prime of good non-ordinary reduction for V with $e_{\mathfrak{p}} \leq (p-1)$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Finally, in the last chapter of this thesis we will show that the conditions in Theorem 7 and Theorem 8 are optimal. Part of the chapter is devoted to the proof of the following theorem.

Theorem 9. *Let A be the abelian surface given by the product of two elliptic curves E_1 and E_2 defined over $\mathbb{Q}$. Assume that both E_1 and E_2 have good ordinary reduction at the prime 2 and full two torsion defined over $\mathbb{Q}_2$. Let $V = \mathrm{Kum}(A)$ be the corresponding Kummer K3 surface. Then, V has good ordinary reduction at 2 and every element $\mathcal{A} \in \mathrm{Br}(V)[2]$ has constant evaluation map $\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{Q}_2) \to \mathbb{Q}/\mathbb{Z}$.*

In particular, this proves that the field extension appearing in Theorem 4 is in general needed. Moreover, starting from this theorem we will build an example for which 2 is a prime of good ordinary reduction but does not play a role in the Brauer–Manin obstruction, even though the ramification index (which is 1) is divisible by $p-1$. This shows that the condition $(p-1) \nmid e_{\mathfrak{p}}$ in Theorem 7 is sufficient but not necessary. Moreover, we will also produce an example of a K3 surface for which there is a prime of good non-ordinary reduction $\mathfrak{p}$ that plays a role in the Brauer–Manin obstruction to weak approximation and for which $e_{\mathfrak{p}} = p$. The latter example shows that the bound appearing in Theorem 8 is optimal.

Notation and conventions

We include here some notation and conventions that we will follow throughout this thesis. Most of the notation will be introduced once again in the thesis when needed, however we include them here as well for convenience.

Varieties over number fields

We denote by k any number field and $\mathcal{O}_k$ its ring of integers. Let Ω_k be the set of places of k, then for any place $\nu \in \Omega_k$ we denote by k_ν the completion of k at ν. We call places corresponding to archimedean valuation archimedean, while places corresponding to non-archimedean valuation will be called finite. For $\nu \in \Omega_k$ archimedean, we define $\mathcal{O}_\nu := k_\nu$. If ν is a finite place we will often denote it by $\mathfrak{p}$, where $\mathfrak{p}$ is the corresponding prime ideal in $\mathcal{O}_k$ and we denote by $\mathcal{O}_\mathfrak{p}$ (sometimes by $\mathcal{O}_\nu$) its ring of integers and by $k(\mathfrak{p})$ (sometimes by $k(\nu)$) its residue field. For any finite set of places $S \subset \Omega_k$ we define $\mathcal{O}_{k,S}$ as the intersection in k of the local rings $\mathcal{O}_\nu$ for $\nu \in S$.

Finally, we denote by V any proper, smooth and geometrically integral variety over k. We refer to [Liu02] for an introduction to schemes and their properties. Let $\mathcal{V}$ be an $\mathcal{O}_{k,S}$-scheme of finite type such that there exists an isomorphism between $\mathcal{V} \times_{\mathcal{O}_{k,S}} \operatorname{Spec}(k)$ and V (i.e. $\mathcal{V}$ is an $\mathcal{O}_{k,S}$ model for the scheme V); for any finite place $\mathfrak{p}$ not in S, we denote by $\mathcal{V}(\mathfrak{p})$ the special fibre at $\mathfrak{p}$, i.e. the base change of $\mathcal{V}$ to $\operatorname{Spec}(k(\mathfrak{p}))$, by $\mathcal{V}_\mathfrak{p}$ the base change of $\mathcal{V}$ to $\mathcal{O}_\mathfrak{p}$ and by $V_\mathfrak{p}$ the base change of V to $k_\mathfrak{p}$.

Varieties over p-adic fields

Let p be a prime number. We denote by L any p-adic field (i.e. a finite field extension of the field of p-adic numbers $\mathbb{Q}_p$), with ring of integers $\mathcal{O}_L$ and residue field ℓ, which is a finite field of positive characteristic. We denote by X any smooth and geometrically integral variety over L and by $\mathcal{X}$ any smooth model over $\mathcal{O}_L$ with geometrically integral fibre, i.e. $\mathcal{X}$ is such that there exists an isomorphism between $\mathcal{X} \times_{\mathcal{O}_L} \operatorname{Spec}(L)$ and X. Moreover, we denote by Y the special fibre of $\mathcal{X}$, which is the base change of $\mathcal{X}$ to ℓ and is a smooth, proper and geometrically integral variety over the finite field ℓ, since both being smooth and proper are stable under base change.

Remark 10. In particular, for any finite place $\mathfrak{p}$ of k, the completion of k at $\mathfrak{p}$, $k_\mathfrak{p}$, is a p-adic field with residue field $k(\mathfrak{p})$ of positive characteristic p. Hence, the base change $V_\mathfrak{p}$ of V to $k_\mathfrak{p}$ is a variety over a p-adic field and the special fibre at $\mathfrak{p}$, $\mathcal{V}(\mathfrak{p})$ is a variety over the finite field $k(\mathfrak{p})$, which is the residue field of the p-adic field $k_\mathfrak{p}$.

CHAPTER 1

The Brauer–Manin obstruction and the refined Swan conductor

The Brauer–Manin obstruction was introduced by Manin [Man71] in the 1970s to explain the failure of the local-global principle.

Let V be a variety over a number field k. For any place $\nu \in \Omega_k$ we have a natural inclusion $V(k) \hookrightarrow V(k_\nu)$. In particular, we have that

$$V(k) \neq \emptyset \Rightarrow V(k_\nu) \neq \emptyset \text{ for all } \nu \in \Omega_k.$$

We say that V/k satisfies the **local-global** principle if

$$V(k_\nu) \neq \emptyset \text{ for all } \nu \in \Omega_k \Rightarrow V(k) \neq \emptyset.$$

The Hasse-Minkowski Theorem gives a family of varieties for which it is enough to check the local solubility to determine if an equation admits a solution over k.

Theorem 1.0.1 (**Hasse-Minkowski**). *Let k be a number field and $V \subseteq \mathbb{P}^n_k$ be a hypersurface defined by a single homogeneous equation of degree 2. Then $V(k) \neq \emptyset$ if and only if $V(k_\nu) \neq \emptyset$ for every place ν of k.*

Proof. See [Ser73, Theorem IV.8]. □

It is well known that if we increase the degree of the polynomial or the number of equations defining V, then we might have a failure of the local-global principle. Here we mention some examples in this direction.

Example 1.0.2.

1. Lind and Reichardt showed independently that the system of equations

$$\begin{cases} u^2 - 17w^2 = 2z^2 \\ uw = v^2 \end{cases}$$

admits a non-trivial solution over $\mathbb{Q}_\nu$ for all places $\nu \in \Omega_\mathbb{Q}$ but does not admit a non-trivial rational solution.

2. Selmer showed that the following homogeneous equation of degree 3

$$3X^3 + 4Y^3 + 5Z^3 = 0$$

admits a non-trivial solution over $\mathbb{Q}_\nu$ for all places $\nu \in \Omega_\mathbb{Q}$ but does not admit a non-trivial rational solution.

3. Iskovskikh showed that equation

$$y^2 + z^2 = (3 - x^2)(x^2 - 2).$$

defining a smooth affine surface over $\mathbb{Q}$, admits a non-trivial solution over $\mathbb{Q}_\nu$ for all places $\nu \in \Omega_\mathbb{Q}$ but does not admit a non-trivial rational solution.

In all the examples the failure of the local-global principle can be explained by the **Brauer–Manin obstruction**.

Manin's idea was to use the Brauer group of a variety to construct a closed subset

$$\left(\prod_{\nu \in \Omega_k} V(k_\nu)\right)^{\mathrm{Br}} \subseteq \prod_{\nu \in \Omega_k} V(k_\nu)$$

that contains the k-points of V.

A crucial role in the construction of the Brauer–Manin set is played by the so-called **evaluation maps** attached to elements $\mathcal{A}$ in $\mathrm{Br}(V)$. We will see that for any element $\mathcal{A} \in \mathrm{Br}(V)$ and for every place $\nu \in \Omega_k$ we can define a map, the evaluation map

$$\mathrm{ev}_\mathcal{A} \colon V(k_\nu) \to \mathbb{Q}/\mathbb{Z}.$$

To detect which values these maps can take is one of the crucial steps in the study of the Brauer–Manin obstruction.

In this thesis we focus attention on the behaviour of the evaluation map at finite places of good reduction.

Definition 1.0.3. *We say that a prime $\mathfrak{p}$ of k is of* ***good reduction*** *for V if there exists a smooth proper $\mathcal{O}_\mathfrak{p}$-scheme $\mathcal{X}$ whose base change to $k_\mathfrak{p}$ is isomorphic to the base change of V to $k_\mathfrak{p}$.*

In Section 1.1.3 we will see that if $\mathfrak{p}$ is a finite place of good reduction for V with residue characteristic p and $\mathcal{A} \in \mathrm{Br}(V)$ has order prime to p, then the behaviour of the evaluation map can be studied via the so-called **residue map**, that links the prime to p part of the Brauer group of the base change of V to $k_\mathfrak{p}$ to the cohomology of the special fiber of a model. This is no longer possible when $\mathfrak{p}$ is a finite place of good reduction of residue characteristic p and $\mathcal{A} \in \mathrm{Br}(V)\{p\}$, i.e. $\mathcal{A}$ is a p-power element in the Brauer group of V. However, in [BN23] Bright and Newton use the refined Swan conductor to control the behaviour of the evaluation map also in this case.

In the first part of this chapter we introduce the notion of weak approximation and Brauer–Manin obstruction to weak approximation on a variety. In the second part of the chapter we introduce the notion of refined Swan conductor [Kat89]. Following Bright and Newton's recent work [BN23] we explain how this notion can be used to study the Brauer–Manin obstruction to weak approximation. Some original results are presented in this chapter: in particular, Lemma 1.2.24 and Corollary 1.3.9 are part of [Pag23].

1.1 The Brauer–Manin obstruction

Let k be a number field and V a proper k-variety. We want to study the set of k-points on V via the diagonal embedding

$$V(k) \hookrightarrow \prod_{\nu \in \Omega_k} V(k_\nu).$$

For every place $\nu \in \Omega_k$, the ν-adic topology on k_ν allows us to equip $V(k_\nu)$ with a topology, called the analytic topology, see [Con12, Proposition 3.1].

From now on we will think about $\prod_{\nu \in \Omega_k} V(k_\nu)$ as a topological space with respect to the product topology induced by the analytic topology on each component $V(k_\nu)$.

Definition 1.1.1. *We say that V satisfies* ***weak approximation*** *if the image of the diagonal map*

$$V(k) \hookrightarrow \prod_{\nu \in \Omega_k} V(k_\nu)$$

is dense.

For proper k-varieties it is possible to get the product $\prod_{\nu \in \Omega_k} V(k_\nu)$ as the set of $\mathbf{A}_k$-points on V, where $\mathbf{A}_k$ is the **ring of adèles** of the number field k.

Definition 1.1.2. *We define the* ***ring of adèles*** *of k as the restricted product of the completions of k with respect to the subgroups given by the ring of integers $\mathcal{O}_\nu \subseteq k_\nu$, i.e. :*

$$\mathbf{A}_k := \prod_{\nu \in \Omega_k} (k_\nu, \mathcal{O}_\nu) = \left\{ (x_\nu) \in \prod k_\nu \ : \ x_\nu \in \mathcal{O}_\nu \text{ for almost all } \nu \in \Omega_k \right\}.$$

The following proposition is well known; we include the proof because we are unable to find it in the literature.

Proposition 1.1.3. *Let R be an integral domain and let k be its function field. Suppose that V is a scheme of finite presentation over k. Then there exist a dense open subscheme $U \subseteq \mathrm{Spec}(R)$ and a scheme $\mathcal{V}$ of finite presentation over U such that $\mathcal{V}_k \simeq V$.*

Proof. Let $\{V_i\}_{i\in I} \subseteq V$ be a finite affine covering of V. Then, for every $i \in I$ the description of V_i involves only finitely many polynomials over k. Let R' be the localisation of R obtained by adjoining the inverse of all the denominators that appear either in the polynomials describing V_i or in the gluing morphisms between the opens V_i and V_j. Then, we can define $\mathcal{V}_i$ as the R'-scheme defined by the same equations of V_i and obtain $\mathcal{V}$ by gluing the $\mathcal{V}_i$'s together. □

Given a k-scheme of finite type V, we can always find a finite set of finite places $S \subseteq \Omega_k$ such that there exists an $\mathcal{O}_{k,S}$ -scheme of finite type $\mathcal{V}_S$ whose base change to k is isomorphic to V (i.e. a model of V over $\mathcal{O}_{k,S}$). Then, the set of adelic points $V(\mathbf{A}_k)$ can be described as

$$V(\mathbf{A}_k) = \prod_{\nu\in\Omega_k} (V(k_\nu), \mathcal{V}_S(\mathcal{O}_\nu)), \quad \text{[Con12, Theorem 3.6]}$$

In particular, if V is proper

$$V(\mathbf{A}_k) = \prod_{\nu\in\Omega_k} V(k_\nu).$$

1.1.1 Obstructions arising from functors

Let

$$\mathrm{F}\colon \mathrm{Sch}_k^{opp} \to \mathrm{Set}$$

be a functor. In this section we are going to show how it is possible, starting from F, to build subsets $V(\mathbf{A}_k)^{\mathrm{F}}$ of $V(\mathbf{A}_k)$ that contains $V(k)$.

Let R be a k-algebra, then every $x \in V(R)$ induces a map $\mathrm{F}(x)\colon \mathrm{F}(V) \to \mathrm{F}(R)$. For every $\mathcal{A} \in \mathrm{F}(V)$, we can define the map

$$\begin{aligned} \mathrm{ev}_{\mathcal{A}}\colon V(R) &\to \mathrm{F}(R) \\ x &\mapsto \mathrm{F}(x)(\mathcal{A}). \end{aligned}$$

In particular, we have the following commutative diagram:

$$\begin{array}{ccc} V(k) & \hookrightarrow & V(\mathbf{A}_k) \\ \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} \\ \mathrm{F}(k) & \longrightarrow & \mathrm{F}(\mathbf{A}_k) \end{array} \tag{1.1}$$

We define

$$V(\mathbf{A}_k)^{\mathcal{A}} := \{x \in V(\mathbf{A}_k) \ : \ \mathrm{ev}_{\mathcal{A}}(x) \in \mathrm{im}(\mathrm{F}(k) \to \mathrm{F}(\mathbf{A}_k))\}.$$

By the commutativity of the diagram above $V(k) \subseteq V(\mathbf{A}_k)^{\mathcal{A}}$. We define

$$V(\mathbf{A}_k)^{\mathrm{F}} := \bigcap_{\mathcal{A} \in \mathrm{F}(V)} V(\mathbf{A}_k)^{\mathcal{A}}.$$

Definition 1.1.4. *If $V(\mathbf{A}_k)^{\mathrm{F}}$ is a proper closed subset of $V(\mathbf{A}_k)$ we say that there is an* ***F-obstruction to weak approximation*** *on V.*

In particular, we see that in order to understand the set $V(\mathbf{A}_k)^{\mathrm{F}}$ we need to be able to understand both $\mathrm{F}(V)$ and for elements in $\mathrm{F}(V)$ the corresponding evaluation map.

In this thesis we focus the attention on the Brauer-Manin obstruction, more about obstructions attached to other functors (in particular about descent obstructions) can be found in [Poo17a, Section 8.4, 8.5, and 8.6].

1.1.2 The Brauer group and the Brauer–Manin obstruction

The Brauer–Manin obstruction is defined as the obstruction associated to the functor $\mathrm{F} := \mathrm{Br}(-)$, where $\mathrm{Br}(-) = \mathrm{H}^2_{\text{ét}}(-, \mathbb{G}_m)$ is the cohomological Brauer group (for an introduction to étale cohomology see [Mil80]).

In this section we will explain how it is possible to describe both the Brauer group of V and the evaluation map attached to an element $\mathcal{A} \in \mathrm{Br}(V)$.

1.1.2.1 The Brauer group of a variety

For a smooth, proper and geometrically integral variety V over a number field k there is an exact sequence [Č19, (6.4.4)]

$$0 \to \mathrm{Br}(V) \to \mathrm{Br}(k(V)) \xrightarrow{(\mathrm{r}_v)} \bigoplus_{v \in V^{(1)}} \mathrm{H}^1_{\text{ét}}(k(v), \mathbb{Q}/\mathbb{Z}) \tag{1.2}$$

where $V^{(1)}$ is the set of points $v \in V$ such that the local ring $\mathcal{O}_{V,v}$ is one dimensional. Since by assumption V is smooth, $\mathcal{O}_{V,v}$ is a regular noetherian ring of dimension 1 and hence a discrete valuation ring. Let $k(v)$ be the residue field of $\mathcal{O}_{V,v}$, then we have a map

$$\mathrm{r}_v \colon \mathrm{Br}(k(V)) \to \mathrm{H}^1(k(v), \mathbb{Q}/\mathbb{Z})$$

see [Poo17b, Section 6.8]. In particular, the Brauer group of the variety can be described in terms of the cohomology of the fields $k(V)$ and $k(v)$ for $v \in V^{(1)}$.

Given a field K, according to the definition given above, the Brauer group of K is defined as

$$\mathrm{Br}(K) := \mathrm{H}^2(\mathrm{Spec}(K)_{\text{ét}}, \mathbb{G}_m).$$

It is well known that for fields étale cohomology coincide with Galois cohomology, see [Mil80, Theorem 1.9, Chapter 1]. In particular, since Galois cohomology groups are torsion [CTS21, Corollary 1.3.6] we get that $\mathrm{Br}(V)$ is a torsion abelian group.

For fields it is possible to give an explicit construction of the Brauer group in terms of equivalence classes of central simple algebras. A finite-dimensional algebra $\mathcal{A}$ over K is called simple if it has no (two sided) ideals other than 0 and itself; it is called central if $Z(\mathcal{A}) = K$. Two central simple K-algebras $\mathcal{A}$ and $\mathcal{A}'$ are called Brauer equivalent if

$$\mathcal{A} \otimes_K M_n(K) \simeq \mathcal{A}' \otimes_K M_{n'}(K), \quad \text{for some } n, n' > 0.$$

Theorem 1.1.5. *The set of the equivalence classes of finite dimensional central simple K-algebras, with the group structure given by the tensor product, is isomorphic to* $\mathrm{Br}(K)$.

Proof. [GS17, Section 4.4]. □

1.1.2.2 Quaternion algebras

In the examples presented in this thesis we are mainly working with elements in the Brauer group represented by quaternion algebras.

Definition 1.1.6. *Assume the characteristic of K to be different from 2, then given $a, b \in K^\times$, we define the* ***quaternion algebra*** (a, b) *as the 4-dimensional K-algebra with basis $1, i, j, ij$ with*

$$i^2 = a, \ j^2 = b, \ ij = -ji.$$

It is possible to prove that quaternion algebras are central simple algebras and that operations with quaternion algebras inside the Brauer group are quite simple. Indeed, if we pick $a, b, c \in K^\times$ we have an isomorphism

$$(a, b) \otimes_k (a, c) \xrightarrow{\sim} (a, bc) \otimes_K M_2(K).$$

Given $a, b \in K^\times$, with abuse of notation we will denote by (a, b) also the class of the quaternion algebra in $\mathrm{Br}(K)$.

As mentioned before, the Brauer group of a variety is a torsion abelian group, hence it is enough to study for every n the exact sequence on the n-torsion induced by equation (1.2)

$$0 \to \mathrm{Br}(V)[n] \to \mathrm{Br}(k(V))[n] \xrightarrow{(\mathrm{r}_v)} \bigoplus_{v \in V^{(1)}} \mathrm{H}^1_{\text{ét}}(k(v), \mathbb{Q}/\mathbb{Z}))[n]. \tag{1.3}$$

From the short exact sequence $0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{n} \mathbb{Q}/\mathbb{Z} \to 0$ we get an isomorphism

$$\mathrm{H}^1_{\text{ét}}(k(v), \mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{H}^1_{\text{ét}}(k(v), \mathbb{Q}/\mathbb{Z})[n].$$

We have an explicit description of the residue map r_v on quaternion algebras in $\mathrm{Br}(k(V))$: for an element $(a,b) \in \mathrm{Br}(k(V))$ and a point $v \in V^{(1)}$ we have

$$\mathrm{r}_v(a,b) = \left[(-1)^{\mathrm{val}_v(a)\mathrm{val}_v(b)} \frac{a^{\mathrm{val}_v(b)}}{b^{\mathrm{val}_v(a)}}\right] \in \frac{k(v)^\times}{k(v)^{\times 2}} \simeq \mathrm{H}^1(k(v), \mathbb{Q}/\mathbb{Z})[2] \tag{1.4}$$

where val_v is the discrete valuation on $k(V)$ coming from the discrete valuation ring $\mathcal{O}_{V,v}$. The description of the residue map given above follows from the definition of the tame symbols in Milnor K-theory together with the compatibility of the residue map r_v with the tame symbols given by the Galois symbols, see [GS17, Proposition 7.5.1]. The last isomorphism follows from the isomorphism between μ_2 and $\mathbb{Z}/2\mathbb{Z}$ together with the one coming from the Kummer sequence between $k(v)^\times/k(v)^{\times 2}$ and $\mathrm{H}^1_{\text{ét}}(k(\nu), \mu_2)$.

1.1.2.3 The evaluation map

Let $\nu \in \Omega_k$ be a finite place. Using local class field theory one can prove that there exists an isomorphism, called invariant map,

$$\mathrm{inv}_\nu \colon \mathrm{Br}(k_\nu) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}. \tag{1.5}$$

A proof can be found in Section 1 and 2, Chapter III of [Mil20]. Moreover, if $\nu \in \Omega_k$ is an archimedean place, then k_ν is either isomorphic to $\mathbb{R}$ or to $\mathbb{C}$, for which

$$\mathrm{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathrm{Br}(\mathbb{C}) = \{0\}, \quad \text{[CTS21, Section 1.2.1]}.$$

Hence, for archimedean places we can define

$$\mathrm{inv}_\nu \colon \mathrm{Br}(k_\nu) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

as the map sending the non-trivial element of $\mathrm{Br}(k_\nu)$ to the class of $1/2$ in $\mathbb{Q}/\mathbb{Z}$. Finally, from global class field theory [Mil20, Chapter VII, VIII] we have the following short exact sequence:

$$0 \to \mathrm{Br}(k) \to \bigoplus_{\nu \in \Omega_k} \mathrm{Br}(k_\nu) \xrightarrow{\sum_\nu \mathrm{inv}_\nu} \mathbb{Q}/\mathbb{Z} \to 0.$$

Since it is possible to show that

$$\mathrm{Br}(\mathbf{A}_k) \simeq \bigoplus_{\nu \in \Omega_k} \mathrm{Br}(k_\nu), \quad \text{[Č19, Theorem 2.13]}$$

diagram (1.1) becomes

$$\begin{array}{ccccccccc} & & V(k) & \longrightarrow & V(\mathbf{A}_k) & & & & \\ & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & & & \\ 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(\mathbf{A}_k) \simeq \bigoplus_{\nu\in\Omega_k} \mathrm{Br}(k_\nu) & \xrightarrow{\sum \mathrm{inv}_\nu} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0. \end{array} \tag{1.6}$$

Putting everything together, we get that for every $\mathcal{A} \in \mathrm{Br}(V)$

$$V(\mathbf{A}_k)^{\mathcal{A}} = \left\{ (x_\nu) \in V(\mathbf{A}_k) \text{ such that } \sum_{\nu \in \Omega} \mathrm{inv}_\nu(\mathrm{ev}_{\mathcal{A}}(x_\nu)) = 0 \right\}.$$

Remark 1.1.7. From now on, with abuse of notation, we will denote by $\mathrm{ev}_{\mathcal{A}}$ also the composition of the evaluation map $\mathrm{ev}_{\mathcal{A}} \colon V(k_\nu) \to \mathrm{Br}(k_\nu)$ with the invariant map $\mathrm{inv}_\nu \colon \mathrm{Br}(k_\nu) \hookrightarrow \mathbb{Q}/\mathbb{Z}$.

The following result allows us to use the Brauer–Manin set to study weak approximation on V.

Proposition 1.1.8. *For any $\mathcal{A} \in \mathrm{Br}(V)$, the set $V(\mathbf{A}_k)^{\mathcal{A}}$ is open and closed in $V(\mathbf{A}_k)$.*

Proof. See [Poo17a, Corollary 8.2.11]. □

The set $V(\mathbf{A}_k)^{\mathrm{Br}}$ is therefore closed in $V(\mathbf{A}_k)$ and we have the following chain of inclusions

$$V(k) \subseteq \overline{V(k)} \subseteq V(\mathbf{A}_k)^{\mathrm{Br}} \subseteq V(\mathbf{A}_k).$$

Definition 1.1.9. *We say that there is a* ***Brauer–Manin obstruction to weak approximation*** *on V if the Brauer–Manin set $V(\mathbf{A}_k)^{\mathrm{Br}}$ is a proper subset of $V(\mathbf{A}_k)$.*

Assume $V(\mathbf{A}_k) \neq \emptyset$. Let $\omega \in \Omega_k$ be such that there exists $\mathcal{A} \in \mathrm{Br}(V)$ with

$$\mathrm{ev}_{\mathcal{A}} \colon V(k_\omega) \to \mathbb{Q}/\mathbb{Z}$$

non-constant. Then, we can take two points $(x_\nu), (y_\nu) \in V(\mathbf{A}_k)$ such that $x_\nu = y_\nu$ for all $\nu \neq \omega$ and $\mathrm{ev}_{\mathcal{A}}(x_\omega) \neq \mathrm{ev}_{\mathcal{A}}(y_\omega)$. These two points are such that

$$\sum_{\nu \in \Omega_k} \mathrm{ev}_{\mathcal{A}}(x_\nu) \neq \sum_{\nu \in \Omega_k} \mathrm{ev}_{\mathcal{A}}(y_\nu).$$

Hence, at least one of the two points does not belong to the Brauer–Manin set $V(\mathbf{A}_k)^{\mathrm{Br}}$ and thus there is a Brauer–Manin obstruction to weak approximation on V. Based on this observation we give the following definition.

Definition 1.1.10. *We say that a place $\omega \in \Omega_k$* ***plays a role*** *in the Brauer–Manin obstruction to weak approximation on V if there exists $\mathcal{A} \in \mathrm{Br}(V)$ such that $\mathrm{ev}_{\mathcal{A}} \colon V(k_\omega) \to \mathbb{Q}/\mathbb{Z}$ is non-constant.*

1.1.3 The prime to p part

In order to study the role of a prime $\mathfrak{p}$ in Brauer–Manin obstruction, we need to understand, for a given $\mathcal{A} \in \mathrm{Br}(V)$, the behaviour of the corresponding evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon V(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}.$$

If we denote by $\mathrm{res} \colon \mathrm{Br}(V) \to \mathrm{Br}(V_{\mathfrak{p}})$ the natural restriction map form $\mathrm{Br}(V)$ to $\mathrm{Br}(V_{\mathfrak{p}})$, then for every point $P \in V(k_{\mathfrak{p}})$ we have a corresponding point $P_{\mathfrak{p}} \in V_{\mathfrak{p}}(k_{\mathfrak{p}})$ and

$$\mathrm{ev}_{\mathrm{res}(\mathcal{A})}(P_{\mathfrak{p}}) = \mathrm{ev}_{\mathcal{A}}(P).$$

Hence, if the evaluation map attached to $\mathcal{A}$ is non-constant on $V(k_{\mathfrak{p}})$, then the element $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(V_{\mathfrak{p}})$ is such that the evaluation map on $V_{\mathfrak{p}}(k_{\mathfrak{p}})$ is also non-constant. By what we just said together with the fact that we are interested in primes of good reduction, we will work under the setting described below.

p-adic setting: Let p be a prime number and L a finite field extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, uniformiser π and residue field ℓ. Let X be a smooth and geometrically irreducible L-variety having good reduction (i.e. there exists a smooth proper $\mathcal{O}_L$-scheme $\mathcal{X}$ whose generic fiber is isomorphic to X). We assume furthermore the special fiber $Y := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\ell)$ to be geometrically irreducible,

$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & \mathcal{X} & \xleftarrow{\ i\ } & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) & \longleftarrow & \mathrm{Spec}(\ell). \end{array} \tag{1.7}$$

It is well known that if n is a positive integer prime to p, then we have the following exact sequence

$$0 \to \mathrm{Br}(\mathcal{X})[n] \to \mathrm{Br}(X)[n] \xrightarrow{\partial} \mathrm{H}^1(Y, \mathbb{Z}/n\mathbb{Z}) \tag{1.8}$$

see [CTS21, Chapter 3, equation (3.17)].

Theorem 1.1.11. *The evaluation map attached to $\mathcal{A}$ is constant and trivial over all finite field extensions L'/L if and only if $\mathcal{A} \in \mathrm{Br}(\mathcal{X})$.*

Proof. One of the two implication is quite straightforward, in fact if we start by $\mathcal{A} \in \mathrm{Br}(\mathcal{X})$, then the evaluation map factors as

$$\begin{array}{ccc} \mathcal{X}(\mathcal{O}_L) & \longrightarrow & \mathrm{Br}(\mathcal{O}_L) \\ \| & & \downarrow \\ X(L) & \xrightarrow{\mathrm{ev}_{\mathcal{A}}} & \mathrm{Br}(L) \end{array} \tag{1.9}$$

but $\mathrm{Br}(\mathcal{O}_L) = \{0\}$.

The other implication was proved by Colliot-Thélène and Saito [CTS96] using zero-cycles on X. The group of zero-cycles $C_0(X)$ is defined as the group of formal sums

$$\sum_i n_i P_i \quad \text{with } n_i \in \mathbb{Z},\ P_i \text{ closed point of } X.$$

The evaluation map $\mathcal{A}$ on a zero-cycle $\sum_i n_i P_i$ is defined as

$$\sum_i n_i \mathrm{cores}_{L(P_i)/L}\left(\mathrm{ev}_{\mathrm{res}_{L(P_i)/L}(\mathcal{A})}(P_i)\right).$$

For the definition of the restriction and co-restriction maps we refer to [GS17, Section 3.3]. In particular, they proved that $\mathrm{ev}_{\mathcal{A}}$ is trivial on $C_0(X)$ if and only if $\mathcal{A} \in \mathrm{Br}(\mathcal{X})[n]$. In particular, if $\mathcal{A}$ has constant and trivial evaluation map on all the closed points of X, then it has also constant and trivial evaluation map on zero-cycles. □

In general, as explained in [Bri15, Section 5.1] the evaluation map attached to an element $\mathcal{A} \in \mathrm{Br}(X)[n]$ with n prime to p factors through the special fiber Y, i.e. if two points $P_1, P_2 \in X(L) = \mathcal{X}(\mathcal{O}_L)$ have the same image in $Y(\ell)$ then the evaluation map on them coincide. Finally, the residue map appearing in (1.8) controls whether the evaluation map attached to an element $\mathcal{A} \in \mathrm{Br}(X)[n]$ is constant: $\mathcal{A}$ has constant evaluation map if and only if $\partial(\mathcal{A}) \in \mathrm{H}^1(\ell, \mathbb{Z}/n\mathbb{Z})$.

From the Purity Theorem [Č19, Theorem 1.2] we know that $\mathrm{Br}(\mathcal{X})$ is equal to $\mathrm{Br}(X) \cap \mathrm{Br}(\mathcal{O}_{\mathcal{X},Y})$, with the intersection taking place inside $\mathrm{Br}(k(X))$. Hence, in order to understand whether $\mathcal{A}$ has trivial evaluation map it is enough to understand if the image of $\mathcal{A}$ in $\mathrm{Br}(k(X))$ lies in $\mathrm{Br}(\mathcal{O}_{\mathcal{X},Y})$. Let K^h be the field of fraction of the henselianisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},Y}$, if n is prime to p, then

$$\mathrm{Br}(K^h)[n] = \ker(\mathrm{Br}(K^h) \to \mathrm{Br}(K^h_{un}))[n], \quad \text{[CTS21, Proposition 1.4.5]}.$$

This can be used to construct the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Br}(\mathcal{O}_{\mathcal{X},Y})[n] & \longrightarrow & \mathrm{Br}(k(X))[n] & \xrightarrow{\partial} & \mathrm{H}^1(F, \mathbb{Z}/n\mathbb{Z}) \\ & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \mathrm{Br}(\mathcal{O}^h_{\mathcal{X},Y})[n] & \longrightarrow & \mathrm{Br}(K^h)[n] & \xrightarrow{\partial} & \mathrm{H}^1(F, \mathbb{Z}/n\mathbb{Z}) \end{array} \tag{1.10}$$

where the rows form exact sequences, see [CTS21, Theorem 3.6.1]. In particular, from diagram (1.10) we get that it is enough to compute the residue of the image of $\mathcal{A}$ in $\mathrm{Br}(K^h)[n]$ in order to detect whether the element $\mathcal{A}$ has trivial evaluation map.

If we start with an element $\mathcal{A} \in \mathrm{Br}(X)[p^m]$ for some m, then it is no longer true that its image in $\mathrm{Br}(K^h)$ is split by an unramified extension. However, Bright and Newton [BN23] recently proved that the behaviour of the evaluation map attached to $\mathcal{A}$ on the $k_{\mathfrak{p}}$-points of X is still determined by the value of some maps on the image of $\mathcal{A}$ in $\mathrm{Br}(K^h)[p^m]$. In this case the description of $\mathrm{Br}(K^h)[p^m]$ becomes more involved. Bright and Newton use the concept of **refined Swan conductor**, introduced by Kato in [Kat89], to give a filtration on $\mathrm{Br}(K^h)$ that captures the behaviour of the evaluation maps.

1.2 The refined Swan conductor

Set-up: Let K be a henselian discrete valuation field of mixed characteristic, whose residue field is not necessarily perfect. We denote by:

- $\mathcal{O}_K \subseteq K$ the ring of integers of K with uniformiser π and maximal ideal $\mathfrak{m}$;
- F the residue field of characteristic p.

The aim of this section is to introduce the notion of the refined Swan conductor and gather some of the main results around it that will be crucial in the rest of the thesis. The main references for this section are [Kat89, BN23]. In this section, instead of working just with the Brauer group, we will look at the Galois cohomology group $\mathrm{H}^q(K)$ defined in the next section. We will see how these cohomology groups are related to the differential forms of the residue field F.

1.2.1 The group $\mathrm{H}^q(R)$

Let R be a ring over $\mathbb{Q}$ or a smooth ring over a field of characteristic $p > 0$. Let n be a non-negative integer and $r \in \mathbb{Z}$. If n is invertible on R, then we denote by $\mathbb{Z}/n\mathbb{Z}(r)$ the usual Tate twist of the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on $R_{\text{ét}}$. If R is smooth over a field of characteristic p and $n = p^s m$ with $(p, m) = 1$, we define

$$\mathbb{Z}/n\mathbb{Z}(r) := \mathbb{Z}/m\mathbb{Z}(r) \oplus W_s\Omega^r_{R,\log}[-r].$$

For the definition of $W_s\Omega^r_{R,\log}$ see Definition 1.2.7. For $n \neq 0$, we denote by

$$\mathrm{H}^q_n(R) := \mathrm{H}^q(R_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(q-1)) \quad \text{and} \quad \mathrm{H}^q(R) := \varinjlim_n \mathrm{H}^q_n(R).$$

Note that:

$$\mathrm{H}^2(K) = \varinjlim_n \mathrm{H}^2(K_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(1)) = \mathrm{Br}(K).$$

We have an exact triangle

$$\mathbb{Z}/n\mathbb{Z}(1) \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m,$$

where the prime to the characteristic case comes from the Kummer sequence, while the p-part in characteristic p follows from [Ill79, Proposition I.3.23.2]. For $a \in R^\times$, let $\{a\} \in \mathrm{H}^1(R_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(1))$ be the image of a under the connecting map $R^\times \to \mathrm{H}^1(R_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(1))$. For $a_1, \ldots, a_r \in R^\times$ and $\chi \in \mathrm{H}^q_n(R)$ we denote by

$$\{\chi, a_1, \ldots, a_r\} := \chi \cup \{a_1\} \cup \cdots \cup \{a_r\} \in \mathrm{H}^{q+r}_n(R).$$

The symbol $\{\chi, a_1, \ldots, a_r\}$ is anti-symmetric and $\{\chi, a_1, \ldots, a_r\} = 0$ if there exist i, j with $i \neq j$ such that $a_i + a_j = 1$ or 0.

In order to describe the groups $\mathrm{H}^q(R)$ in the positive characteristic case, we will use the **Cartier operator** on differential forms.

1.2.1.1 The Cartier operator

Let ℓ be a perfect field of characteristic $p > 0$ and R be an ℓ-algebra. Let

$$Z_R^q := \ker(d\colon \Omega_R^q \to \Omega_R^{q+1}) \quad \text{and} \quad B_R^q := \operatorname{im}(d\colon \Omega_R^{q-1} \to \Omega_R^q).$$

Lemma 1.2.1 (Inverse Cartier operator). *Assume R to be regular; then there exists a unique morphism of groups*

$$C_R^{-1}\colon \Omega_R^1 \to \Omega_R^1/B_R^1$$

satisfying

- $C_F^{-1}(da) = a^{p-1}da \mod B_F^1$ *for all* $a \in F$*;*
- $C_R^{-1}(\lambda\omega) = \lambda^p C_R^{-1}(\omega)$ *for all* $\lambda \in R$*;*
- $d \circ C_R^{-1} = 0$.

Moreover, C^{-1} induces an isomorphism from Ω_R^1 to Z_R^1/B_R^1.

Proof. See [BK05, Theorem 1.3.4]. □

Remark 1.2.2. The subgroup B_R^1 has a natural structure of R-module, which is given by $\alpha \cdot d\beta = \alpha^p d\beta = d(\alpha^p\beta)$. If we denote by ${}^p\Omega_R^1$ the R-module structure on Ω_R^1 given by $\alpha \cdot \omega = \alpha^p\omega$, then the condition $C_R^{-1}(\lambda\omega) = \lambda^p C_F^{-1}(\omega)$ is equivalent to asking that $C_R^{-1}\colon \Omega_R^1 \to {}^p\Omega_R^1/{}^pB_R^1$ is a morphism of R-modules.

We can extend the definition of C_R^{-1} to higher differential forms by setting

$$C_R^{-1}(\omega_1 \wedge \cdots \wedge \omega_q) := C_R^{-1}(\omega_1) \wedge \cdots \wedge C_R^{-1}(\omega_q).$$

Theorem 1.2.3. *Let R be regular; then the morphism*

$$C_R^{-1}\colon \Omega_R^q \to Z_R^q/B_R^q$$

is an isomorphism for all $q \geq 0$. We will denote by C_R its inverse, which is called the Cartier operator.

Proof. See [BK05, Theorem 1.3.4]. □

The following corollary gives a way to characterise exact differential forms in terms of the Cartier operator.

Corollary 1.2.4. *Let R be regular. A q-form $\omega \in \Omega_R^q$ is exact if and only if $d(\omega) = 0$ and $C_R(\omega) = 0$.*

The last object we need to define is the subgroup of logarithmic q-differential forms on R, which will play a crucial role in this thesis.

Definition 1.2.5. *The **logarithmic** q-differential forms on R, denoted by $\Omega_{R,\log}^q$, are defined as the kernel of the map*

$$C_R^{-1} - \mathrm{id}\colon \Omega_R^q \to \Omega_R^q/B_R^q.$$

The following result gives a way to write down logarithmic differential forms explicitly in the case in which R is a field.

Theorem 1.2.6. *Let F be a field, finitely generated over a perfect field ℓ. The logarithmic differential q-forms $\Omega^q_{F,\log}$ is the subgroup of Ω^q_F generated by elements of the form*

$$\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}, \quad \textit{with } y_i \in F^\times.$$

Proof. It follows from surjectivity in the Bloch–Gabber–Kato Theorem [GS17, Theorem 9.5.2]. □

Finally, in [Ill79], Illusie defines a projective system $\{W_m\Omega^q_R\}_{m\geq 0}$ equipped with the Frobenius and the Verschiebung maps

$$F\colon W_{m+1}\Omega^q_R \to W_m\Omega^q_R \quad \text{and} \quad V\colon W_m\Omega^q_R \to W_{m+1}\Omega^q_R$$

for every $m \geq 0$. This projective system has transition maps given by the projection maps $R_{m+1}\colon W_{m+1}\Omega^j_R \to W_m\Omega^j_R$.

In [Ill79, Page 569, Proposition 3.3] Illusie generalises the inverse Cartier operator to $W_s\Omega^q_R$:

$$C_R^{-1}\colon W_s\Omega^q_R \to W_s\Omega^q_R/dV^{s-1}\Omega^{q-1}_R$$

Definition 1.2.7. *Given two positive integers q, s, the logarithmic subgroup $W_s\Omega^q_{R,\log}$ of $W_s\Omega^q_R$ is defined as the kernel of*

$$C_R^{-1} - \mathrm{id}\colon W_s\Omega^q_R \to W_s\Omega^q_R/dV^{s-1}\Omega^{q-1}_R.$$

If R is smooth over a field of positive characteristic and $n = p^s$, we can identify $\mathrm{H}^q_n(R) = \mathrm{H}^1(R_{\text{ét}}, W_s\Omega^r_{R,\log})$ with the cokernel of

$$C_R^{-1} - 1\colon W_s\Omega^{q-1}_R \to W_s\Omega^{q-1}_R/dV^{s-1}\Omega^{q-2}_R,$$

see [Kat89, Section 1.3]. We denote by δ_s both the map from $W_s\Omega^{q-1}_R/dV^{s-1}\Omega^{q-2}_R$ to $\mathrm{H}^q_n(R)$ and its composition with the natural map $\mathrm{H}^q_n(R) \to \mathrm{H}^q(R)$. We then have a commutative diagram

$$\begin{array}{ccc} W_s\Omega^{q-1}_R & \xrightarrow{\quad V \quad} & W_{s+1}\Omega^{q-1}_R \\ & {\scriptstyle\delta_s}\searrow \quad \swarrow{\scriptstyle\delta_{s+1}} & \\ & \mathrm{H}^q(R) & \end{array}$$

and the following relation [Kat89, Section 1.3]

$$\delta_s(\omega \wedge d\log(a_1) \wedge \cdots \wedge d\log(a_r)) = \{\delta_s(\omega), \{a_1, \ldots, a_r\}\} \text{ in } \mathrm{H}^{q+r}_n(R). \tag{1.11}$$

1.2.1.2 Description of $\mathrm{H}_p^q(F[T])$

In this section we are following [Kur88]. We are interested in describing the groups $\mathrm{H}_p^{q+1}(F[T])$. We recall that these groups fit in the following exact sequence

$$\Omega^q_{F[T]} \xrightarrow{C^{-1}-\mathrm{id}} \Omega^q_{F[T]}/B^q_{F[T]} \xrightarrow{\delta_1} \mathrm{H}_p^{q+1}(F[T]) \to 0. \tag{1.12}$$

Lemma 1.2.8. *Every element* $\omega \in \Omega^q_{F[T]}$ *can be written (in a unique way) as*

$$\sum_{i=0}^{n} \omega_i T^i + \sum_{j=1}^{m} \eta_j T^j \wedge d\log T$$

with $\omega_i \in \Omega^q_F$ *and* $\eta_j \in \Omega^{q-1}_F$.

Proof. Consider the following diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}[T] \\ \downarrow & & \downarrow \pi_1 \\ F & \xrightarrow{\pi_2} & \mathbb{Z}[T] \otimes_{\mathbb{Z}} F = F[T]. \end{array}$$

We know by the Künneth formula [Sta, 01V1] that

$$\Omega^q_{F[T]} = \bigoplus_{i+j=q} \pi_1^* \Omega^i_{\mathbb{Z}[T]} \otimes_{F[T]} \pi_2^* \Omega^j_F$$

where π_1^* and π_2^* are just the base change to $F[T]$ maps. Since $\Omega^i_{\mathbb{Z}[T]} = 0$ if $i \geq 2$, we get that

$$\Omega^q_{F[T]} = \left(F[T] \otimes_F \Omega^q_F\right) \oplus \left(F[T] \cdot dT \otimes_F \Omega^{q-1}_F\right)$$

which proves the result. □

We define increasing filtrations on $\mathrm{H}_p^{q+1}(F[T])$ and $\mathrm{H}_p^{q+1}(F[T,T^{-1}])$ such that fil_n is the group generated by elements of the form

$$\delta_1\left[T^k \omega_1 + T^m \omega_2 \wedge d\log T\right]$$

where $\omega_1 \in \Omega^q_F$ and $\omega_2 \in \Omega^{q-1}_F$ and $k, m \leq n$. For $n \geq 0$ we define a morphism

$$\begin{aligned} \rho_n \colon \Omega^q_F \oplus \Omega^{q-1}_F &\to \mathrm{H}_p^{q+1}(F[T]) \\ (\omega_1, \omega_2) &\mapsto \delta_1\left[T^n \omega_1 + T^n \omega_2 \wedge d\log T\right]. \end{aligned}$$

Similarly, we define the same morphisms with image in $\mathrm{H}_p^{q+1}(F[T,T^{-1}])$. It follows from the definitions and the description given in Lemma 1.2.8 that for $n \geq 1$, the morphism ρ_n induces a surjective map on

$$\mathrm{gr}_n := \frac{\mathrm{fil}_n \mathrm{H}_p^{q+1}(F[T])}{\mathrm{fil}_{n-1} \mathrm{H}_p^{q+1}(F[T])} = \frac{\mathrm{fil}_n \mathrm{H}_p^{q+1}(F[T,T^{-1}])}{\mathrm{fil}_{n-1} \mathrm{H}_p^{q+1}(F[T,T^{-1}])}.$$

From now on, for $n \geq 1$, we denote by ρ_n also the composition of ρ_n with the quotient map to gr_n.

Lemma 1.2.9. *Let $n \geq 1$ be prime to p. Then the map ρ_n induces an isomorphism*

$$\Omega^q_F \simeq \mathrm{gr}_n$$

Proof. By construction gr_n is generated by the elements of the form

$$\delta_1 \left[T^n\omega_1 + T^n\omega_2 \wedge d\log T\right].$$

If n is prime to p, then we have

$$T^n\omega_2 \wedge d\log T = d\left(\omega_2 \frac{T^n}{n}\right) - d(\omega_2)\frac{T^n}{n}.$$

Since δ_1 is zero on exact forms, we get that

$$\rho_n(\omega_1, \omega_2) = \delta_1 \left[T^n\left(\omega_1 - \frac{d\omega_2}{n}\right)\right] = \rho_n\left(\omega_1 - \frac{d\omega_2}{n}, 0\right).$$

This proves that the restriction of ρ_n to Ω^q_F is still surjective.

We are left to show injectivity. Let $\omega \in \Omega^q_F$ be such that $\rho_n(\omega, 0) = 0$. Then, there exist $\omega_0, \ldots \omega_{n_1} \in \Omega^q_F$ and $\eta_1, \ldots, \eta_{n-1} \in \Omega^{q-1}_F$ such that

$$\delta_1 \left[\omega T^n\right] + \delta_1 \left[\sum_{i=0}^{n-1} T^i\omega_i + \sum_{j=1}^{n-1} \eta_j T^j \wedge d\log T\right] = 0 \text{ in } \mathrm{H}^q_p(F[T]).$$

Hence, by equation (1.12) we get the following equality in $\Omega^{q-1}_{F[T]}/B^{q-1}_{F[T]}$

$$\omega T^n + \sum_{i=0}^{n-1} T^i\omega_i + \sum_{j=1}^{n-1} \eta_j T^j \wedge d\log T = (C^{-1} - \mathrm{id})\left(\sum_{i\geq 0} \alpha_i T^i + \sum_{j\geq 1} \beta_j T^j \wedge d\log T\right).$$

The right hand side of the identity can be re-written (using the properties of the inverse Cartier operator) as

$$\sum_{i=0}^{n} C^{-1}(\alpha_i)T^{pi} + \sum_{j=1}^{n} C^{-1}(\beta_j)T^{pj} \wedge d\log T - \sum_{i=0}^{n} \alpha_i T^i - \sum_{j=1}^{n} \beta_j T^j \wedge d\log T.$$

In particular,

$$\omega T^n = -\alpha_n T^n \text{ modulo } B^{q-1}_{F[T]} \quad \text{and} \quad C^{-1}(\alpha_n T^n) = C^{-1}(\alpha_n)T^{pn} = 0.$$

Hence, since C^{-1} is an isomorphism, $\alpha_n T^n = 0$ and $\omega T^n \in B^{q-1}_{F[T]}$. Hence,

$$\omega T^n = d\left(\sum_{i\geq 0} \gamma_i T^i + \sum_{j\geq 1} \theta_j T^{j-1} \wedge dT\right)$$

which implies that

$$d(\gamma_n)T^n = \omega T^n \quad \text{and} \quad (n\gamma_n - d\theta_n)T^{n-1} \wedge dT = 0.$$

Therefore $n\gamma_n = d\theta_n$ and $\omega = d(\gamma_n) = 0$. □

Remark 1.2.10. In particular, the isomorphism induced by ρ_1 factors as follows

$$\Omega^q_F \to \mathrm{fil}_1 \mathrm{H}^{q+1}_p(F[T]) \to \mathrm{gr}_1.$$

This implies that the map

$$\begin{aligned} \Omega^q_F &\to \mathrm{H}^{q+1}_p(F[T]) \\ \alpha &\mapsto \delta_1[T\alpha] \end{aligned}$$

is injective.

Lemma 1.2.11. *Let $n \geq 1$ be such that $n = pn_1$. Then the map ρ_n induces an isomorphism*

$$\Omega^q_F/Z^q_F \oplus \Omega^{q-1}_F/Z^{q-1}_F \simeq \mathrm{gr}_n$$

Proof. Let $(\omega_1, \omega_2) \in Z^q_F \oplus Z^{q-1}_F$, then there exists $(\theta_1, \theta_2) \in \Omega^q_F \oplus \Omega^{q-1}_F$ such that

$$(C^{-1}(\theta_1), C^{-1}(\theta_2)) = (\omega_1, \omega_2), \text{ in } Z^q_F/B^q_F \oplus Z^{q-1}_F/B^{q-1}_F$$

Hence, using the properties of the inverse Cartier operator

$$T^n\omega_1 + T^n\omega_2 \wedge d\log T = C^{-1}(\theta_1 T^n_1 + \theta_2 T^n_1 \wedge d\log T).$$

We know from equation (1.12) that the image via δ_1 of

$$(C^{-1} - \mathrm{id})\,[\theta_1 T^{n_1} + \theta_2 T^{n_1} \wedge d\log T]$$

is zero. Therefore, $\delta_1\,[T^n\omega_1 + T^n\omega_2 \wedge d\log T]$ is equal to

$$\delta_1\,[\theta_1 T^{n_1} + \theta_2 T^{n_1} \wedge d\log T]\,.$$

The latter is zero in gr_n, since $n_1 \leq n-1$.

We are left to show that the kernel of ρ_n is contained in $Z^q_F \oplus Z^{q-1}_F$. Assume that $(\omega_1, \omega_2) \in \Omega^{q-1}_F \oplus \Omega^{q-2}_F$ is such that

$$\rho_n(\omega_1, \omega_2) = \delta_1\,[T^n\omega_1 + T^n\omega_2 \wedge d\log T] = 0.$$

An argument analogous to the one used in the previous lemma gives that there exist $\alpha_{n_1,p} \in \Omega^q_F$ and $\beta_{n_1,p} \in \Omega^{q-1}_F$ such that

$$\omega_1 = C^{-1}(\alpha_{n_1,p}) \quad \text{and} \quad \omega_2 = C^{-1}(\beta_{n_1,p}), \quad \text{mod } B^{q-1}_F \oplus B^{q-2}_F.$$

The result now follows from the fact that the image of the inverse Cartier operator lies in the group of closed forms modulo exact forms. □

Let $g_1, g_2, h : F[T] \to F[T_1, T_2]$ be the F-morphisms defined by

$$g_1(T) = T_1, \quad g_2(T) = T_2, \quad \text{and} \quad h(T) = T_1 + T_2.$$

We denote in the same way the induced maps from $H^*_p(F[T])$ to $H^*_p(F[T_1, T_2])$.

Lemma 1.2.12. *Let $\omega \in \mathrm{H}^q_p(F[T])$ and assume that ω is such that*

(1) $g_1(\omega) + g_2(\omega) = h(\omega)$;

(2) the element $\{\omega, T\}$ in $\mathrm{H}^{q+1}_p(F(T))$ is equal to the image of an element τ in $\mathrm{H}^{q+1}_p(F[T])$ such that $g_1(\tau) + g_2(\tau) = h(\tau)$.

Then, there exists $\alpha \in \Omega^{q-1}_F$ such that

$$\omega = \delta_1(T\alpha).$$

Proof. A proof, that uses the previous two lemmas, can be found in [Kur88, Lemma 3.3.1]. □

1.2.2 Definition of Swan conductor

We recall that we work in the following setting: K is a henselian discrete valuation field of mixed characteristic, whose residue field is not necessarily perfect. We denote by:

- $\mathcal{O}_K \subseteq K$ the ring of integers of K with uniformiser π and maximal ideal $\mathfrak{m}$;
- F the residue field of characteristic p.

Let A be a ring over $\mathcal{O}_K$, $R := A/\mathfrak{m}A$, and i, j the inclusions of the special and generic fibers into $\mathrm{Spec}(A)$:

$$\mathrm{Spec}(A \otimes_{\mathcal{O}_K} K) \xrightarrow{\ j\ } \mathrm{Spec}(A) \xleftarrow{\ i\ } \mathrm{Spec}(R).$$

We define

$$V^q_n(A) := \mathrm{H}^q\left(R_{\text{ét}}, i^*\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1)\right)$$

and $V^q(A) := \varinjlim_n V^q_n(A)$. In particular, $\mathrm{H}^q(K) = V^q(\mathcal{O}_K)$. The natural map in $D^b(A_{\text{ét}})$

$$\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1) \to i_*i^*\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1)$$

induces a natural map $\mathrm{H}^q_n(A \otimes_{\mathcal{O}_K} K) \to V^q_n(A)$ for all n, q. Gabber [Gab94] proved that this map is an isomorphism if $(A, \mathfrak{m}A)$ is henselian. In this case, using the Kummer map $(A \otimes_{\mathcal{O}_K} K)^\times \to \mathrm{H}^1(A \otimes_{\mathcal{O}_K} K, \mathbb{Z}/n\mathbb{Z}(1))$ and the cup product, we define a product

$$\begin{aligned} V^q_n(A) \times ((A \otimes_{\mathcal{O}_K} K)^\times)^{\oplus r} &\to V^{q+r}_n(A) \\ (\chi, a_1, \dots, a_r) &\mapsto \{\chi, a_1, \dots, a_r\}. \end{aligned}$$

For a general A, the isomorphism

$$V^q_n(A) \simeq V^q_n(A^{(h)}) \tag{1.13}$$

allows to extend the product above. This isomorphism follows from the following remark.

Remark 1.2.13. One can check that the sheaves $i^*\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1)$, with i and j such that

$$\mathrm{Spec}(A\otimes_{\mathcal{O}_K}K)\xrightarrow{\ j\ }\mathrm{Spec}(A)\xleftarrow{\ i\ }\mathrm{Spec}(R).$$

and $(i^{(h)})^*\mathrm{R}(j^{(h)})_*\mathbb{Z}/n\mathbb{Z}(q-1)$ with $i^{(h)}$ and $j^{(h)}$ such that

$$\mathrm{Spec}(A^{(h)}\otimes_{\mathcal{O}_K}K)\xrightarrow{\ j^{(h)}\ }\mathrm{Spec}(A^{(h)})\xleftarrow{\ i^{(h)}\ }\mathrm{Spec}(R).$$

have the same stalks, and hence coincide on $R_{\text{ét}}$.

From now on we will identify $V_n^q(A)$ and $V_n^q(A^{(h)})$. In particular, note that if $A=\mathcal{O}_K[T]$, then all the polynomials of the form $1+\pi^n p(T)$ are invertible in $A^{(h)}$, see [Sta, 0EM7] for an overview on henselianisation of (not necessarily local) rings.

Definition 1.2.14. *The increasing filtration* $\{\mathrm{fil}_n H^q(K)\}_{n\geq 0}$ *on* $\mathrm{H}^q(K)$ *is defined by*

$$\chi\in\mathrm{fil}_n\mathrm{H}^q(K)\Leftrightarrow\{\chi,1+\pi^{n+1}T\}=0\ \textit{in}\ V^{q+1}(\mathcal{O}_K[T]).$$

We say that an element χ *in* $\mathrm{H}^q(K)$ *has* ***Swan conductor*** n*, if* $\chi\in\mathrm{fil}_n\mathrm{H}^q(K)$ *and* $\chi\notin\mathrm{fil}_{n-1}\mathrm{H}^q(K)$*.*

The following lemma implies that it is possible to define the Swan conductor of every element of $\mathrm{H}^q(K)$.

Lemma 1.2.15. *We have*

$$\mathrm{H}^q(K)=\bigcup_{n\geq 0}\mathrm{fil}_n\mathrm{H}^q(K).$$

Proof. This lemma is [Kat89, Lemma 2.2], we are adding some details in the proof. Let $A:=(\mathcal{O}_K[T])^h$ be the henselisation of $\mathcal{O}_K[T]$ with respect to the ideal $\mathfrak{m}\mathcal{O}_K[T]$. Then, by equation (1.13) $V_n^q(A)\simeq V_n^q(\mathcal{O}_K[T])$. Let $\chi\in\mathrm{H}^q(K)$, then there exists $s\geq 1$ such that $\chi\in\mathrm{H}^q_s(K)$. If $n\geq 0$ is such that $1+\pi^{n+1}T\in(A^\times)^s$, then

$$\{\chi,1+\pi^{n+1}T\}=\{\chi,a^s\}=\{s\cdot\chi,a\}=0$$

which implies that $\chi\in\mathrm{fil}_n\mathrm{H}^q(K)$. The result now follows from [Kat89, Lemma 2.4], in which Kato proves that for any $s\geq 1$ there exists some $n\geq 0$ such that the set $1+\pi^{n+1}A$ is contained in $(A^\times)^s$ □

Remark 1.2.16. Kato defines the Swan conductor also for henselian discrete valuation fields of equal characteristic. In particular, it is possible to relate the Swan conductor filtration to the filtration on $\mathrm{H}^q_p(F[T])$ defined in Section 1.2.1.2. If we take $K=F\left((T)\right)$ then the map $F[T]\to F\left((T)\right)$ sending T to T^{-1} allows us to pull-back the Swan conductor filtration on $\mathrm{H}^q_p(F[T])$; from [Kat89, Theorem 3.2(1)] we see that the filtration that we get in this way on $\mathrm{H}^q_p(F[T])$ coincides with the one defined in Section 1.2.1.2.

1.2.2.1 Elements with Swan conductor equal to zero

In [Kat89] and [BN23] some maps $\lambda_\pi\colon \mathrm{H}^q_n(R) \oplus \mathrm{H}^{q-1}_n(R) \to V^q_n(A)$ are defined under additional assumptions on the $\mathcal{O}_K$-algebra A. We give an overview of these maps that, since they are compatible with each other, we will always call λ_π.

- In [Kat89, Section 1.4] Kato defines for every n an injective map

  $$\lambda_\pi\colon \mathrm{H}^q_n(F) \oplus \mathrm{H}^{q-1}_n(F) \to \mathrm{H}^q_n(K).$$

  This collection of maps induces an injective map

  $$\lambda_\pi\colon \mathrm{H}^q(F) \oplus \mathrm{H}^{q-1}(F) \to \mathrm{H}^q(K).$$

- In [Kat89, Section 1.9] Kato extends the definition of λ_π from $\mathrm{H}^q_p(F) \oplus \mathrm{H}^{q-1}_p(F)$ to $\mathrm{H}^q_p(K)$ to any smooth $\mathcal{O}_K$-algebra A. In particular, he defines a map

  $$\lambda_\pi\colon \mathrm{H}^q_p(R) \oplus \mathrm{H}^{q-1}_p(R) \to V^q_p(A).$$

- In [BN23, Section 2.2] Bright and Newton generalise the previous map by defining

  $$\lambda_\pi\colon \mathrm{H}^q_{p^r}(R) \oplus \mathrm{H}^{q-1}_{p^r}(R) \to V^q_{p^r}(A)$$

  for any $r \geq 1$.

It is proven in [Kat89, Proposition 6.1] that the image of

$$\lambda_\pi\colon \mathrm{H}^q_n(F) \oplus \mathrm{H}^{q-1}_n(F) \to \mathrm{H}^q_n(K)$$

coincides with $\mathrm{fil}_0\mathrm{H}^q_n(K)$.

We state here a technical lemma proven by Bright and Newton that we will use several times in this thesis.

Lemma 1.2.17. *Let K^h be the fraction field of the henselisation of A with respect to the ideal generated by π. Let $r \geq 1$ and $q \geq 2$. Let χ be an element of $\mathrm{fil}_0\mathrm{H}^q_{p^r}(K^h)$ and write $\chi = \lambda_\pi(\alpha, \beta)$ with $(\alpha, \beta) \in \mathrm{H}^q_{p^r}(F) \oplus \mathrm{H}^{q-1}_{p^r}(F)$. If χ lies in the image of $V^q_{p^r}(A)$, then (α, β) lies in the image of $\mathrm{H}^q_{p^r}(R) \oplus \mathrm{H}^{q-1}_{p^r}(R)$.*

Proof. [BN23, Lemma 3.5] □

Following [Kat89] and [BN23] we sometimes use λ_π also to denote the composition

$$W_r\Omega^q_R \oplus W_r\Omega^{q-1}_R \xrightarrow{\delta_r} \mathrm{H}^q_{p^r}(R) \oplus \mathrm{H}^{q-1}_{p^r}(R) \xrightarrow{\lambda_\pi} V^q_{p^r}(A).$$

1.2.3 Construction of the refined Swan conductor

The aim of this section to define the refined Swan conductor of an element χ in $\mathrm{fil}_n \mathrm{H}^q(K)$. The key result we need is the following theorem, [Kat89, Theorem 5.1]

Theorem 1.2.18. *Let $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$, with $n \geq 1$; then there exists a unique pair (α, β) in $\Omega^q_F \oplus \Omega^{q-1}_F$ such that*

$$\{\chi, 1 + \pi^n T\} = \lambda_\pi(T\alpha, T\beta) \quad in\ V^{q+1}_p(\mathcal{O}_K[T]). \tag{1.14}$$

Remark 1.2.19. The pair (α, β) in the theorem above depends on the choice of a uniformiser π in $\mathcal{O}_K$. However, we will mainly be interested in the vanishing of such a pair, and this is independent on the choice of the uniformiser. Let π' be another uniformiser and $a = \pi/\pi' \in \mathcal{O}^\times_K$, then we can define the map

$$\begin{aligned} m_a \colon \mathcal{O}_K &\to \mathcal{O}_K \\ x &\mapsto ax. \end{aligned}$$

We denote by m_a also the induced automorphism on $\mathcal{O}_K[T]$. Then [BN23, Lemma 2.13] tells us that

$$\lambda_{\pi'}(\bar{a}T(\alpha + \beta \wedge d\log \bar{a}), \bar{a}T\beta) = m^*_a \lambda_\pi(T\alpha, T\beta).$$

with $\bar{a}$ reduction of a in F and m^*_a automorphism induced by m_a on $\mathrm{H}^q(K)$.

The refined Swan conductor of an element $\mathcal{A} \in \mathrm{fil}_n \mathrm{H}^q(K)$ is defined as the element $(\alpha, \beta) \in \Omega^q_F \oplus \Omega^{q-1}_F$ such that

$$\{\chi, 1 + \pi^n T\} = \lambda_\pi(T\alpha, T\beta).$$

Note that: we have that $\chi \in \mathrm{fil}_{n-1} \mathrm{H}^q(K)$ if and only if $\{\chi, 1+\pi^n T\} = 0$ and since λ_π is injective, this happens if and only if $(T\alpha, T\beta) = 0$. By Remark 1.2.10 this is equivalent to $(\alpha, \beta) = (0, 0)$. Hence, for every $n \geq 1$ we get an injective map, called **refined Swan conductor**

$$\mathrm{rsw}_{n,\pi} \colon \frac{\mathrm{fil}_n \mathrm{H}^q(K)}{\mathrm{fil}_{n-1} \mathrm{H}^q(K)} \hookrightarrow \Omega^q_F \oplus \Omega^{q-1}_F.$$

The rest of this section is devoted to the proof of Theorem 1.2.18. We start with the following lemma, [Kat89, Lemma 5.3].

Lemma 1.2.20. *Assume $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$ for $n \geq 1$. Then:*

(1) $\{\chi, 1 + \pi^n T\} \in V^{q+1}(\mathcal{O}_K[T])$ *is annihilated by* p*;*

(2) $\{\chi, 1 + \pi^n T_1, 1 + \pi T_2\} = 0$ *in* $V^{q+2}(\mathcal{O}_K[T_1, T_1^{-1}, T_2])$*;*

(3) $\{\chi, 1 + \pi^n(T_1 + T_2)\} = \{\chi, 1 + \pi^n T_1\} + \{\chi, 1 + \pi^n T_2\}$ *in* $V^{q+1}(\mathcal{O}_K[T_1, T_2])$*;*

(4) $\{\chi, 1 + \pi^n T, T\} = -\{\chi, 1 + \pi^n T, -\pi^n\}$ *in* $V^{q+2}(\mathcal{O}_K[T, T^{-1}])$*.*

Proof. Note that $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$ implies, by functoriality of $V^q(-)$, that

$$\{\chi, 1 + \pi^{n+1} b\} = 0 \text{ in } V^q(B) \tag{1.15}$$

for all $\mathcal{O}_K$-algebras B and all $b \in B$.

1. We have

$$p \cdot \{\chi, 1 + \pi^n T\} = \{\chi, (1 + \pi^n T)^p\} = \{\chi, 1 + \pi^{n+1} q(T)\} = 0$$

   with $q(T) = \sum_{s=1}^{p} \binom{p}{s} \frac{1}{\pi^{n+1}} (\pi^n T)^s \in \mathcal{O}_K[T]$. The last equality follows from equation (1.15).

2. Let M be the fraction field of the henselisation of the local ring $\mathcal{O}_K[T_1]_{(\pi)}$. By [Kat89, (1.8.1)] we know that the for any $s \geq 0$ the map

$$V^q_{p^s}(\mathcal{O}_K[T_1, T_2]) \to V^q_{p^s}(\mathcal{O}_M[T_2])$$

   is injective. Hence, it is enough to prove the equality in $V^{q+2}(\mathcal{O}_M[T_2])$. Note that, for every $a \in \mathcal{O}_M$

$$\{\chi, 1 + \pi^n T + \pi^{n+1} a\} = \{\chi, 1 + \pi^n T\} \text{ in } V^q(\mathcal{O}_M). \tag{1.16}$$

   In fact, $1 + \pi^n T \in \mathcal{O}_M^\times$ and

$$\{\chi, 1 + \pi^n T + \pi^{n+1} a\} - \{\chi, 1 + \pi^n T\} = \left\{\chi, 1 + \pi^{n+1} \frac{a}{1 + \pi^n T}\right\} = 0.$$

   Using equation (1.16) we see that

$$\begin{aligned}
&\{\chi, 1 + \pi^n T_1, 1 + \pi T_2\} \\
&= \{\chi, 1 + \pi^n T_1(1 + \pi T_2), 1 + \pi T_2\} \\
&\overset{\star}{=} -\{\chi, 1 + \pi^n T_1(1 + \pi T_2), -\pi^n T_1\} \\
&\overset{\star}{=} -\{\chi, 1 + \pi^n T_1(1 + \pi T_2), -\pi^n T_1\} - \{\chi, 1 + \pi^n T_1, -\pi^n T_1\} \\
&= -\left\{\chi, 1 + \pi^{n+1} \frac{T_1 T_2}{1 + \pi^n T_1}, -\pi^n T_1\right\} = 0.
\end{aligned}$$

   where in $\overset{\star}{=}$ we are using that $\{\chi, a, b\} = 0$ if $a + b = 1$.

3. We have

$$\begin{aligned}
&\{\chi, 1 + \pi^n T_1\} + \{\chi, 1 + \pi^n T_2\} - \{\chi, 1 + \pi^n (T_1 + T_2)\} \\
&= \left\{\chi, \frac{(1 + \pi^n T_1)(1 + \pi^n T_2)}{1 + \pi^n (T_1 + T_2)}\right\} = \left\{\chi, 1 + \pi^{2n} \frac{T_1 T_2}{1 + \pi^n (T_1 + T_2)}\right\} = 0.
\end{aligned}$$

4. This follows immediately from $\{\chi, a, b\} = 0$ if $a + b = 1$.

□

Proof of Theorem 1.2.18. Let $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$. Then, by Lemma 1.2.20(1) we know that

$$\theta := \{\chi, 1+\pi^n T\} \text{ lies in } V_p^{q+1}(\mathcal{O}_K[T]).$$

From Lemma 1.2.20(2) we have $\{\theta, 1+\pi T_2\} = 0$, hence θ satisfies the assumption of Lemma 1.2.17 with $A = \mathcal{O}_K[T]$ and $R = F[T]$. Hence, there exists (ω, σ) in $\mathrm{H}_p^{q+1}(F[T]) \oplus \mathrm{H}_p^q(F[T])$ such that $\lambda_\pi(\omega, \sigma) = \theta$. We are left to show that ω and σ satisfy the conditions of Lemma 1.2.12. For any $a \in \mathcal{O}_{K(T)}^\times$ we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^q_{p^r}(F(T)) \oplus \mathrm{H}^{q-1}_{p^r}(F(T)) & \xrightarrow{(\cup(-1)^{q-1}\{\bar{a}\}, \cup(-1)^q\{\bar{a}\})} & \mathrm{H}^{q+1}_{p^r}(F(T)) \oplus \mathrm{H}^q_{p^r}(F(T)) \\ \downarrow \lambda_\pi & & \downarrow \lambda_\pi \\ \mathrm{H}^q_{p^r}(K(T)) & \xrightarrow{(\cup\{a\})} & \mathrm{H}^{q+1}_{p^r}(K(T)) \end{array}$$

see the proof of [BN23, Lemma 2.12]. Thus, we get

$$\lambda_\pi(\omega \cup (-1)^{q-1}\{T\}, \sigma \cup (-1)^q\{T\}) = \{\theta, T\}.$$

As $\{\theta, T\} \in V_p^{q+2}(\mathcal{O}_K[T])$, from Lemma 1.2.17 we get that $\{\omega, T\}$ and $\{\sigma, T\}$ lie in $\mathrm{H}_p^{q+2}(F[T])$ and $\mathrm{H}_p^{q+1}(F[T])$ respectively. We denote by G_1, G_2 and H the maps from $\mathcal{O}_K[T]$ to $\mathcal{O}_K[T_1, T_2]$ defined by

$$G_1(T) = T_1, \quad G_2(T) = T_2, \quad \text{and} \quad H(T) = T_1 + T_2$$

and the corresponding maps from $V_p^\bullet(\mathcal{O}_K[T])$ to $V_p^\bullet(\mathcal{O}_K[T_1, T_2])$. Since λ_π is injective, in order to show that ω and σ satisfy the assumption of Lemma 1.2.12 it is enough to show that $G_1 + G_2 = H$ on θ and $\{\theta, T\}$. This is an immediate consequence of Lemma 1.2.20(3) and 1.2.20(4). □

We end this section by giving the definition of the residue map ∂ from $\mathrm{Br}(K)$ to $\mathrm{H}^1(F)$, which also uses the map λ_π (cf. [BN23, Section 2.5], [Kat89, Section 7.5]).

Definition 1.2.21. *The* ***residue map***

$$\partial \colon \mathrm{fil}_0 \mathrm{Br}(K) \to \mathrm{H}^1(F)$$

is defined as the projection on the second component of the inverse of the isomorphism λ_π from $\mathrm{H}^2(F) \oplus \mathrm{H}^1(F)$ to $\mathrm{fil}_0 \mathrm{Br}(K)$.

1.2.3.1 Image of the refined Swan conductor

We are now ready to state and prove some properties about the image of the refined Swan conductor. We denote by $e := \mathrm{ord}_K(p)$ the absolute ramification index of K and by $e' := ep(p-1)^{-1}$. We start with the following lemma.

Lemma 1.2.22. *Let χ be an element in $\mathrm{fil}_n \mathrm{H}^q(K)$ with*

$$\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta) \in \Omega_F^2 \oplus \Omega_F^1.$$

Then $d\alpha = 0$ and $d\beta = (-1)^q n \alpha$.

Proof. See [BN23, Lemma 2.17]. □

Remark 1.2.23. We get that:

(1) if $p \mid n$, then $d\alpha = 0$ and $d\beta = 0$, meaning that $(\alpha, \beta) \in Z_F^q \oplus Z_F^{q-1}$;

(2) if $p \nmid n$, then $\alpha = \bar{n}^{-1} d\beta$, meaning that the composition

$$\mathrm{fil}_n \mathrm{H}^q(K) \xrightarrow{\mathrm{rsw}_{n,\pi}} \Omega_F^q \oplus \Omega_F^{q-1} \xrightarrow{\mathrm{pr}_2} \Omega_F^{q-1}$$

has also kernel equal to $\mathrm{fil}_{n-1}\mathrm{H}^q(K)$.

In [BN23, Lemma 2.19] Bright and Newton are able to link the refined Swan conductor of an element $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$ with the refined Swan conductor of $p \cdot \chi$, whenever $n \geq e'$. More precisely, assume that $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta)$ and let $\bar{u}$ be the reduction modulo π of $p \cdot \pi^{-e}$, then $p \cdot \chi \in \mathrm{fil}_{n-e}\mathrm{H}^q(K)$ and

$$\mathrm{rsw}_{n-e,\pi}(p \cdot \chi) = \begin{cases} (\bar{u}\alpha, \bar{u}\beta) \text{ if } n > e'; \\ (\bar{u}\alpha + C(\alpha), \bar{u}\beta + C(\beta)) \text{ if } n = e'. \end{cases} \tag{1.17}$$

In the following lemma we prove a result analogous to the one proven by Bright and Newton for elements $\chi \in \mathrm{fil}_{np}\mathrm{H}^q(K)$, when $np < e'$.

Lemma 1.2.24. *Let $\chi \in \mathrm{fil}_{np}\mathrm{H}^q(K)$, with $np < e'$. Then $p \cdot \chi \in \mathrm{fil}_n\mathrm{Br}(K)$ and if $\mathrm{rsw}_{np,\pi}(\mathcal{A}) = (\alpha, \beta)$, then $d\alpha = 0$, $d\beta = 0$ and*

$$\mathrm{rsw}_{n,\pi}(p \cdot \chi) = (C(\alpha), C(\beta)).$$

Proof. From Remark 1.2.23(1) we know that $(\alpha, \beta) \in Z_F^q \oplus Z_F^{q-1}$, since clearly $p \mid np$. The condition $e' > np$ implies $\frac{p}{p-1} \cdot (e - np + n) > 0$, which implies

$$e - np + n > 0. \tag{1.18}$$

Let $u \in \mathcal{O}_K^\times$ be such that $p = u \cdot \pi^e$.

$$\{p \cdot \chi, 1 + \pi^{n+1}T\} = \{\chi, (1 + \pi^{n+1}T)^p\} = \{\chi, 1 + \pi^{np+1}b(T)\}$$

where

$$b(T) = \frac{(1 + \pi^{n+1}T)^p - 1}{\pi^{np+1}}.$$

We can rewrite $b(T)$ as

$$\sum_{k=1}^{p-1} \pi^{e+(n+1)k-(np+1)} a_k T^k + \pi^{p(n+1)-(np+1)} T^p.$$

Note that, for every $1 \leq k \leq p$, we have that from equation (1.18)

$$e + (n+1)k - np - 1 = (e - np) + (n+1)k - 1 \geq 0.$$

Therefore, $b(T) \in \mathcal{O}_K[T]$. Now, since by assumption $\chi \in \mathrm{fil}_{np}\mathrm{H}^q(K)$, we have that $\{\chi, 1 + \pi^{np+1}b(T)\} = 0$ for all $b(T) \in \mathcal{O}_K[T]$, thus $p \cdot \chi \in \mathrm{fil}_n\mathrm{H}^q(K)$. In a similar way,

$$\{p \cdot \chi, 1 + \pi^n T\} = \{\mathcal{A}, (1 + \pi^n T)^p\} = \{\chi, 1 + \pi^{np}c(T)\}$$

where

$$c(T) = \frac{(1 + \pi^n T)^p - 1}{\pi^{np}}.$$

We can rewrite $c(T)$ as

$$\sum_{k=1}^{p-1} \pi^{e+nk-np} a_k T^k + T^p.$$

In this case, again equation 1.18 together with $1 \leq k \leq p-1$, implies $e+nk-np > 0$. Therefore, $c(T) \in \mathcal{O}_K[T]$ and its reduction modulo π is equal to T^p. It follows from [Kat89, (6.3.1)] that

$$\{\chi, 1 + \pi^{np}c(T)\} = \lambda_\pi(\bar{c}(T)\alpha, \bar{c}(T)\beta) = \lambda_\pi(T^p\alpha, T^p\beta) = \lambda_\pi(TC(\alpha), TC(\beta))$$

where the last equality follows from [BN23, Lemma 2.18(2)]. □

Remark 1.2.25. For every non-negative integer d, we denote by $\mathrm{fil}_n\mathrm{H}^q_d(K)$ the intersection of $\mathrm{fil}_n\mathrm{H}^q(K)$ with $\mathrm{H}^q_d(K)$. Kato proves that for every non-negative integer d prime to p, $\mathrm{H}^q_d(K) = \mathrm{fil}_0\mathrm{H}^q_d(K)$, see [Kat89, Proposition 6.1]. We will now prove some useful properties of the refined Swan conductor on p-power order elements.[1]

(1) The filtration $\mathrm{fil}_n\mathrm{H}^q_{p^m}(K)$ is finite.

Assume first e' to be an integer, i.e. $(p-1) \mid e$. For $m = 1$ it is proven in [Kat89, Proposition 4.1] that $\mathrm{H}^q_p(K) = \mathrm{fil}_{e'}\mathrm{H}^q_p(K)$. This is equivalent to saying that for all $n > e'$ and $\chi \in \mathrm{fil}_n\mathrm{H}^q_p(K)$, we have $\mathrm{rsw}_{n,\pi}(\chi) = (0,0)$. Assume that $m > 1$, $\chi \in \mathrm{fil}_n\mathrm{H}^q_{p^m}(K)$ with $n > e' + (m-1)e$ and $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta)$. From equation (1.17) we know that $\mathrm{rsw}_{n-e,\pi}(p \cdot \chi) = (\bar{u}\alpha, \bar{u}\beta)$, hence working on induction on m we get that $\mathrm{rsw}_{n,\pi}(\chi) = (0,0)$. Note that, this result was essentially already proved in [Ier22, Proposition 17].

If e' is not an integer, then we take a primitive p-root of unity ζ and we consider the field extension $K(\zeta)/K$ with ramification index $e_{K(\zeta)/K}$. We denote by e_K and $e_{K(\zeta)}$ the absolute ramification indexes of K and $K(\zeta)$ respectively and by e'_K and $e'_{K(\zeta)}$ the products $e_K \cdot p \cdot (p-1)^{-1}$ and $e_{K(\zeta)} \cdot p \cdot (p-1)^{-1}$ respectively. Let $n > e'_K + (m-1) \cdot e_K$ and $\chi \in \mathrm{fil}_n\mathrm{H}^q_{p^m}(K)$, then

$$e_{K(\zeta)/K} \cdot n > e'_{K(\zeta)} + (m-1)e_{K(\zeta)}.$$

Since $(p-1) \mid e'_{K(\zeta)}$, we get (from what we said above) that

$$\mathrm{rsw}_{e_{K(\zeta)/K} \cdot n, \pi}(\mathrm{res}(\chi)) = (0,0)$$

[1] Parts (b) and (c) are already mentioned by Kato in [Kat89, Sections 4 and 5], however in Chapter 4 we are going to use parts (b) and (c) to prove [Kat89, Lemma 4.3] for which no proof is provided in literature.

where res is the natural map from $\mathrm{H}^q_{p^m}(K) \to \mathrm{H}^q_{p^m}(K(\zeta))$. It follows from [BN23, Lemma 2.16] that

$$\mathrm{rsw}_{e_{K(\zeta)/K}\cdot n,\pi}(\mathrm{res}(\chi)) = (\bar{a}^{-n}(\alpha + \beta \wedge d\log \bar{a}), \bar{a}^{-n} e_{K(\zeta)/K}\beta)$$

with $\bar{a}$ invertible in the residue field of $K(\zeta)$ (cf. Section 4.1). Hence, since $p \nmid e_{K(\zeta)/K}$ we can conclude that $\mathrm{rsw}_{n,\pi}(\chi) = (0,0)$.

(2) Let $\chi \in \mathrm{fil}_{np}\mathrm{H}^q_p(K)$ with $np < e'$ and $\mathrm{rsw}_{np,\pi}(\chi) = (\alpha, \beta)$. We know from Lemma 1.2.24 that $(\alpha,\beta) \in Z^q_F \oplus Z^{q-1}_F$ and since χ has order p

$$(C(\alpha), C(\beta)) = \mathrm{rsw}_{n,\pi}(p \cdot \chi) = (0,0)$$

Equivalently, from Corollary 1.2.4 we get that for $np < e'$, the refined Swan conductor on the p-torsion takes image in $B^q_F \oplus B^{q-1}_F$.

(3) Assume e' to be an integer, $\chi \in \mathrm{fil}_{e'}\mathrm{H}^q_p(K)$ and $\mathrm{rsw}_{e',\pi}(\chi) = (\alpha, \beta)$. From equation (1.17) we get that

$$-\bar{u}\alpha = C(\alpha) \text{ and } -\bar{u}\beta = C(\beta).$$

Let ζ be a primitive p-root of unity and $c = (\zeta - 1)^p \pi^{-e'}$, then $c = (c_1)^p$, with $c_1 = (\zeta - 1)\pi^{-e/(p-1)}$. By the properties of the Cartier operator

$$C(\bar{c}\alpha) = \bar{c}_1 C(\alpha) = -\bar{u}\bar{c}_1\alpha.$$

Since $(\zeta - 1)^{p-1} \equiv -p \mod \pi^{e+1}$ [2], $(\zeta-1)^{p-1} = -p + \pi^{e+1}v$ for some $v \in \mathcal{O}_K$. Hence

$$c = \frac{(\zeta-1)^p}{\pi^{e'}} = \frac{\zeta - 1}{\pi^{e/(p-1)}} \cdot \frac{(\zeta-1)^{p-1}}{\pi^e} = c_1 \cdot \left(\frac{-p + \pi^{e+1}v}{\pi^e}\right) = c_1 \cdot (-u + \pi v).$$

Thus $\bar{c} = -\bar{u}\bar{c}_1$ and hence

$$\mathrm{mult}_{\bar{c}} \circ \mathrm{rsw}_{e',\pi}(\mathrm{H}^q_p(K)) \subseteq \Omega^q_{F,\log} \oplus \Omega^{q-1}_{F,\log}$$

where $\mathrm{mult}_{\bar{c}}$ is the map from Ω^q_F to Ω^q_F sending a q-form ω to $\bar{c} \cdot \omega$.

1.3 Refined Swan conductor and Brauer–Manin obstruction

We are ready to state some of the main results in [BN23].

We are working in the same setting as Section 1.1.3: p is a prime number and L a finite field extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, uniformiser π and

[2] In fact, $1 = ((\zeta-1)+1)^p = (\zeta-1)^p + p(\zeta-1)^{p-1} + \binom{p}{2}(\zeta-1)^{p-2} + \cdots + p\cdot(\zeta-1) + 1$. Since $\zeta \neq 1$ we get $(\zeta-1)^{p-1} + p(\zeta-1)^{p-2} + \binom{p}{2}(\zeta-1)^{p-3} + \cdots + p = 0$. Now the congruence follows from the fact that $\mathrm{val}((\zeta-1)\cdot p) \geq 1 + e$.

residue field ℓ. Let X be a smooth and geometrically irreducible L-variety having good reduction (i.e. there exists a smooth $\mathcal{O}_L$-scheme $\mathcal{X}$ whose generic fiber is isomorphic to X). We assume furthermore the special fiber, $Y := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\ell)$ to be geometrically irreducible,

$$\begin{array}{ccccc} X & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) & \longleftarrow & \mathrm{Spec}(\ell). \end{array}$$

Let K^h be the field of fractions of the henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},Y}$. Bright and Newton [BN23] define the filtration $\{\mathrm{fil}_n\mathrm{Br}(X)\}_{n\geq 0}$ on $\mathrm{Br}(X)$ as the pull-back via the natural map $\mathrm{Br}(X) \to \mathrm{Br}(K^h)$ of the filtration $\{\mathrm{fil}_n\mathrm{Br}(K^h)\}_{n\geq 0}$ on $\mathrm{Br}(K^h)$ defined in Section 1.2.2. Therefore it is possible to extend the definition of the residue map and the refined Swan conductor to elements in $\mathrm{Br}(X)$ simply as the residue map and refined Swan conductor of the image of $\mathcal{A}$ in $\mathrm{Br}(K^h)$. A priori these maps take values in $\mathrm{H}^1(F)$ and $\Omega^2_F \oplus \Omega^1_F$ respectively. However, Bright and Newton prove that the residue map of an element in $\mathrm{fil}_0\mathrm{Br}(X)$ takes values in $\mathrm{H}^1(Y, \mathbb{Q}/\mathbb{Z}) \subseteq \mathrm{H}^1(F)$ (see [BN23, Proposition 3.1(1)]) and that the refined Swan conductor on $\mathrm{fil}_n\mathrm{Br}(X)$ has image in $\mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \subseteq \Omega^2_F \oplus \Omega^1_F$ (a proof following [Kat89, Theorem 7.1] can be found in [BN23, Theorem B]).

The aim of this section is to transfer the results we got in Section 1.2.3.1 on the refined Swan conductor on $\mathrm{Br}(K^h)$ to the refined Swan conductor on $\mathrm{Br}(X)$. In [BN23] Bright and Newton define the following filtration, called the **Evaluation filtration**, on the Brauer group of X. Let L'/L be a finite field extension with ring of integer $\mathcal{O}_{L'}$ and uniformiser π'; for $r \geq 1$ and $P \in \mathcal{X}(\mathcal{O}_{L'})$, let $B(P, r)$ be the set of points $Q \in \mathcal{X}(\mathcal{O}_{L'})$ such that Q has the same image as P in $\mathcal{X}(O_{L'}/(\pi')^r)$. Then:

$$\begin{aligned} \mathrm{Ev}_n\mathrm{Br}X &:= \{\mathcal{B} \in \mathrm{Br}(X) \mid \forall L'/L \text{ finite}, \forall P \in \mathcal{X}(\mathcal{O}_{L'}) \\ &\qquad \mathrm{ev}_{\mathcal{B}} \text{ is constant on } B(P, e_{L'/L}(n+1))\}, \qquad (n \geq 0) \\ \mathrm{Ev}_{-1}\mathrm{Br}X &:= \{\mathcal{B} \in \mathrm{Br}(X) \mid \forall L'/L \text{ finite}, \mathrm{ev}_{\mathcal{B}} \text{ is constant on } \mathcal{X}(\mathcal{O}_{L'})\} \\ \mathrm{Ev}_{-2}\mathrm{Br}X &:= \{\mathcal{B} \in \mathrm{Br}(X) \mid \forall L'/L \text{ finite}, \mathrm{ev}_{\mathcal{B}} \text{ is zero on } \mathcal{X}(\mathcal{O}_{L'})\} \end{aligned}$$

For every positive integer m we denote by $\mathrm{Ev}_n\mathrm{Br}(X)[m]$ the restriction of $\mathrm{Ev}_n\mathrm{Br}(X)$ to $\mathrm{Br}(X)[m]$, i.e. $\mathrm{Ev}_n\mathrm{Br}(X)[m] := \mathrm{Ev}_n\mathrm{Br}(X) \cap \mathrm{Br}(X)[m]$. If $(m, p) = 1$ Colliot-Thélène and Skorobogatov [CTS13] and Bright [Bri15] proved that $\mathrm{Ev}_0\mathrm{Br}(X)[m] = \mathrm{Br}(X)[m]$. Moreover, the residue map ∂_m

$$0 \to \mathrm{Br}(\mathcal{X})[m] \to \mathrm{Br}(X)[m] \xrightarrow{\partial_m} \mathrm{H}^1(Y, \mathbb{Z}/m\mathbb{Z}) \tag{1.19}$$

is such that

$$\begin{aligned} \mathrm{Ev}_{-1}\mathrm{Br}(X)[m] &= \{\mathcal{A} \in \mathrm{Br}(X)[m] \mid \partial_m(\mathcal{A}) \in H^1(\ell, \mathbb{Z}/m\mathbb{Z})\} \\ \mathrm{Ev}_{-2}\mathrm{Br}(X)[m] &= \{\mathcal{A} \in \mathrm{Br}(X)[m] \mid \partial_m(\mathcal{A}) = 0\}. \end{aligned}$$

Cf. the end of Section 1.1.3.

In order to give a description of the the Evaluation filtration also on the p-power torsion part of $\mathrm{Br}(X)$, Bright and Newton use the filtration $\{\mathrm{fil}_n\mathrm{Br}(X)\}_{n\geq 0}$ on $\mathrm{Br}(X)$. The interaction between the two filtrations is described in the theorem that follows.

Theorem 1.3.1 (Theorem A, Bright and Newton)**.** *We have the following description of the Evaluation filtration*

(1) $\mathrm{fil}_0\mathrm{Br}(X)$ *coincides with* $\mathrm{Ev}_0\mathrm{Br}(X)$*;*

(2) $\mathrm{Ev}_{-1}\mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{Br}(X) \mid \partial(\mathcal{A}) \in H^1(\ell, \mathbb{Q}/\mathbb{Z})\}$*;*

(3) $\mathrm{Ev}_{-2}\mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{Br}(X) \mid \partial(\mathcal{A}) = 0\}$*;*

(4) For every $n \geq 1$

$$\mathrm{Ev}_n\mathrm{Br}(X) = \left\{\mathcal{A} \in \mathrm{fil}_{n+1}\mathrm{Br}(X) \mid \mathrm{rsw}_{n+1,\pi}(\mathcal{A}) \in \mathrm{H}^0(Y, \Omega^2_Y) \oplus 0\right\}.$$

Proof. This is a reformulation of [BN23, Theorem A]. □

It is clear at this point that in order to understand the evaluation filtration on the Brauer group of X we need to understand the residue map ∂ and the refined Swan conductor maps $\mathrm{rsw}_{n,\pi}$.

1.3.1 Cartier operator on varieties and image of the refined Swan conductor

It is possible to generalise the definition of inverse Cartier operator and Cartier operator to the sheaf of q-forms on a smooth and proper variety Y defined over a perfect field ℓ of positive characteristic. Following Illusie [Ill79], we will denote by F_Y the absolute Frobenius endomorphism of Y and by $Y^{(p)}$ the base change of Y via the absolute Frobenius σ_ℓ of the base field ℓ, namely

$$\begin{array}{ccccc} Y & \xrightarrow{F_{Y/\ell}} & Y^{(p)} & \xrightarrow{W} & Y \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(\ell) & \xrightarrow{\sigma_\ell} & \mathrm{Spec}(\ell) \end{array}$$

where $W \circ F_{Y/\ell} = F_Y$; we call $F_{Y/\ell}$ the relative Frobenius of Y over ℓ. Furthermore, we denote by $\Omega^\bullet_{Y/\ell}$ the De Rham complex of Y/ℓ.

Remark 1.3.2. We are following [Ill79, Section 0.2]. In his paper, Illusie works more generally with S-schemes, where the base S is a scheme of positive characteristic. In this thesis, we are only interested in varieties over perfect fields. In this case, the absolute De Rham complex $\Omega^\bullet_Y$ coincides with the relative De Rham complex $\Omega^\bullet_{Y/\ell}$ (since ℓ being perfect implies $\Omega^q_\ell = 0$).

We define for every $q \geq 0$

$$Z^q_Y := \ker(d\colon \Omega^q_Y \to \Omega^{q+1}_Y) \quad \text{and} \quad B^q_Y := \operatorname{im}(d\colon \Omega^{q-1}_Y \to \Omega^q_Y).$$

For every $q \geq 0$, the differential $d\colon \Omega^q_Y \to \Omega^{q+1}_Y$ is $\mathcal{O}_{Y^{(p)}}$-linear, hence the sheaves

$$(F_{Y/\ell})_* Z^q_Y \quad \text{and} \quad (F_{Y/\ell})_* B^q_Y$$

are $\mathcal{O}_{Y^{(p)}}$-modules and the abelian sheaf $\mathcal{H}^q((F_{Y/\ell})_*\Omega^\bullet_Y)$ is also a sheaf of $\mathcal{O}_{Y^{(p)}}$-modules.

Definition 1.3.3 (Inverse Cartier operator)**.** *For every $q \geq 0$ there is a morphism of $\mathcal{O}_Y$-modules, called the* ***inverse Cartier operator***

$$C^{-1}_Y\colon \Omega^q_Y \to W_*\mathcal{H}^i((F_{Y/\ell})_*\Omega^\bullet_Y).$$

Remark 1.3.4. Since $\Omega^q_{Y^{(p)}} = W^*\Omega^q_Y$, by adjunction we get a morphism of $\mathcal{O}_{Y^{(p)}}$-modules

$$C^{-1}_{Y/\ell}\colon \Omega^q_{Y^{(p)}} \to \mathcal{H}^q((F_{Y/\ell})_*\Omega^\bullet_Y).$$

Theorem 1.3.5. *If Y is a smooth variety over ℓ, then $C^{-1}_{Y/\ell}$ is an isomorphism.*

Proof. See [Ill79, Theorem 0.2.1.9]. □

From now on we are assuming that Y is a smooth and proper variety over ℓ. In this case, we denote by $C_{Y/\ell}$ the inverse of $C^{-1}_{Y/\ell}$. We are ready to define the sheaf of logarithmic forms on Y.

Definition 1.3.6 (Logarithmic forms)**.** *For every non-negative integer q, we denote by*

$$\Omega^q_{Y,\log} := \ker(W^* - C_{Y/\ell}\colon Z^q_Y \to \Omega^q_{Y^{(p)}}).$$

The sheaf $\Omega^q_{Y,\log}$ is called the sheaf of logarithmic *q-forms on Y.*

The following theorem is the analogue of Theorem 1.2.6 for fields of positive characteristic.

Theorem 1.3.7. *The sheaf $\Omega^q_{Y,\log}$ is the subsheaf of Ω^q_Y generated étale-locally by the logarithmic differentials, i.e. the sections of the form*

$$\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \ \textit{with } y_i \in \mathcal{O}^*_Y.$$

Proof. See [Ill79, Theorem 0.2.4.2]. □

The Cartier operator induces an exact sequence

$$0 \to \mathrm{H}^0(Y, B^q_Y) \to \mathrm{H}^0(Y, Z^q_Y) \xrightarrow{C_Y} \mathrm{H}^0(Y, \Omega^q_Y)$$

The following lemma describes the interaction between global differential forms on Y and their image in Ω^q_F.

Lemma 1.3.8. *Let $\omega \in \mathrm{H}^0(Y, \Omega^q_Y)$, then:*

(1) $\omega \in \mathrm{H}^0(Y, \Omega^q_{Y,\log})$ if and only if the image of ω in Ω^q_F lies in $\Omega^q_{F,\log}$;

(2) $\omega \in \mathrm{H}^0(Y, B^q_Y)$ if and only if the image of ω in Ω^q_F lies in B^q_F.

Proof. For all non-negative integers q, we have natural inclusions of $\mathrm{H}^0(Y, \Omega^q_Y)$ in Ω^q_F. These inclusions are compatible with the differential maps and with the Cartier operator. The proof of the first part of the lemma follows from the definition of logarithmic forms, while the second part is an immediate consequence of Corollary 1.2.4. □

Corollary 1.3.9 (On the image of $\mathrm{rsw}_{n,\pi}$)**.** *Let $u = p\pi^{-e} \in \mathcal{O}_L^\times$; then one of the following cases occurs:*

(1) If $p \nmid n$, then

$$\mathrm{pr}_2 \circ \mathrm{rsw}_{n,\pi} \colon \mathrm{fil}_n\mathrm{Br}(X) \to \mathrm{H}^0(Y, \Omega^1_Y)$$

has kernel equal to $\mathrm{fil}_{n-1}\mathrm{Br}(X)$.

(2) If $p \mid n$ and $n < e'$ we write $n = mp$, then (α, β) lies in $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ and the following diagram

$$\begin{array}{ccc}
\mathrm{fil}_n\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\
\downarrow{\scriptstyle (-)^{\otimes p}} & & \downarrow{\scriptstyle C} \\
\mathrm{fil}_m\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{m,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y).
\end{array}$$

commutes.

(3) If $n = e'$, then (α, β) lies in $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ and the following diagram

$$\begin{array}{ccc}
\mathrm{fil}_{e'}\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\
\downarrow{\scriptstyle (-)^{\otimes p}} & & \downarrow{\scriptstyle \mathrm{mult}_{\bar u} + C} \\
\mathrm{fil}_{e'-e}\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n-e,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y).
\end{array}$$

commutes. Moreover, let $\bar c \in \ell^\times$ be the reduction of $c = (\zeta - 1)^p \pi^{-e'} \in \mathcal{O}_L^\times$, then $\bar c$ is such that

$$\mathrm{mult}_{\bar c}\left(\mathrm{rsw}_{e',\pi}\mathrm{fil}_{e'}\mathrm{Br}(X)[p]\right) \subseteq \mathrm{H}^0(Y, \Omega^2_{Y,\log}) \oplus \mathrm{H}^0(Y, \Omega^1_{Y,\log}).$$

(4) If $p \mid n$ and $n > e'$, then (α, β) lies in $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ and the following diagram

$$\begin{array}{ccc}
\mathrm{fil}_n\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\
\downarrow{\scriptstyle (-)^{\otimes p}} & & \downarrow{\scriptstyle \mathrm{mult}_{\bar u}} \\
\mathrm{fil}_{n-e}\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n-e,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y).
\end{array}$$

commutes.

Proof. All these properties are a consequence of Section 1.2.3.1. More precisely: 1. follows from Remark 1.2.23(2); 2. follows from Lemma 1.2.24; 3. and 4. are a direct consequence of [BN23, Lemma 2.19]. □

CHAPTER 2

An example of Brauer–Manin obstruction coming from a prime of good reduction

2.1 Swinnerton-Dyer's question

This chapter is based on [Pag22]. The main result of this chapter gives an answer, over the rational numbers, to the following question, asked by Swinnerton-Dyer to Colliot-Thélène and Skorobogatov [CTS13, Question 1].

Question 2.1.1 (Swinnerton-Dyer)**.** Let k be a number field and let S be a finite set of places of k containing the archimedean places. Let $\mathcal{V}$ be a smooth projective $\mathcal{O}_{k,S}$-scheme with geometrically integral fibres, and let V/k be the generic fibre. Assume that $\operatorname{Pic}(\overline{V})$ is finitely generated and torsion-free. Swinnerton-Dyer asks if there is an open and closed $Z \subseteq \prod_{\nu\in S} V(k_\nu)$ such that

$$V(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{\nu\notin S} V(k_\nu).$$

If $V(\mathbf{A}_k)^{\mathrm{Br}}$ is non-empty and can be described as in Swinnerton-Dyer's question, then for any $\omega \notin S$, and for any $\mathcal{A} \in \operatorname{Br}(V)$ the corresponding evaluation map $\operatorname{ev}_{\mathcal{A}} \colon V(k_\omega) \to \mathbb{Q}/\mathbb{Z}$ has to be constant. In fact, for any $(x_\nu) \in Z\times\prod_{\nu\notin S\cup\{\omega\}} V(k_\nu)$ we have that, for any $y_\omega, \tilde{y}_\omega \in V(k_\omega)$

$$\sum_{\nu\in\Omega_k\setminus\{\omega\}} \operatorname{ev}_{\mathcal{A}}(x_\nu) + \operatorname{ev}_{\mathcal{A}}(y_\omega) = \sum_{\nu\in\Omega_k\setminus\{\omega\}} \operatorname{ev}_{\mathcal{A}}(x_\nu) + \operatorname{ev}_{\mathcal{A}}(\tilde{y}_\omega) = 0.$$

This is because, from the description of the Brauer–Manin set:

$$(x_\nu)_{\nu\neq\omega} \cup (y_\omega),\ (x_\nu)_{\nu\neq\omega} \cup (\tilde{y}_\omega) \in V(\mathbf{A}_k)^{\mathrm{Br}}$$

That is, all the places $\omega \notin S$ do not play a role in the Brauer–Manin obstruction to weak approximation on X (cf. Definition 1.1.10).

We can therefore reformulate Swinnerton-Dyer's question in the following way: is it true that under the assumption of the question asked above the only places that can play a role in the Brauer–Manin obstruction to weak approximation are the places of bad reduction and the archimedean places?

2.1.1 Torsion in the geometric Picard group

The aim of this section is to briefly explain why the assumption on the torsion of the geometric Picard group is needed. This is a consequence of work of Harari [Har00] and Skorobogatov [Sko01]. In their work they use abelian descent obstruction, which arise from torsors (for an introduction to torsors we refer to [Sko01, Chapter 2]).

Let V be a smooth, proper and geometrically integral variety over a number field k. In 1989 Minchev [Min89] has shown that the existence of an étale morphism of geometrically integral varieties implies the failure of weak approximation on V (he proved the failure of strong approximation, which for proper varieties coincide with weak approximation). In particular, if $\mathrm{Pic}(\bar{V})$ has torsion, then we immediately get that the étale fundamental group (cf. [Mil80, Chapter I, Section 5]) $\pi^1_{\text{ét}}(X)$ is non-trivial and, by a result of Harari [Har00, proof of Lemma 5.2(1)], there exists a geometrically integral étale morphism. However, if we want to link the obstruction arising from this morphism to the Brauer–Manin obstruction, we need the covering to be abelian, which is something we can guarantee up to enlarging the base field k.

The argument in this section is divided into three parts: in the first part, it is shown how the obstruction coming from an abelian Galois covering is linked to the Brauer-Manin obstruction; in the second, it is shown how the presence of torsion in the geometric Picard group implies the existence of a non-trivial abelian Galois covering defined over a field extension of the base field k; in the third, it is shown how the existence of a non-trivial Galois covering implies the presence of an obstruction to weak approximation.

Following Colliot-Thélène and Sansuc [CTS87], in [Sko01, Section 6.1] Skorobogatov proves how it possible to describe the algebraic Brauer–Manin obstruction using torsors of multiplicative type. Let G be a k-group of multiplicative type with dual group M, then there exists a natural map

$$\text{type}\colon \mathrm{H}^1(V,G) \to \mathrm{Hom}_{\Gamma_k}(M, \mathrm{Pic}(\bar{V})). \tag{2.1}$$

Any element $\gamma \in M = \mathrm{Hom}(G, \mathbb{G}_m)$ induces a map $\gamma^*\colon \mathrm{H}^1(\bar{V},G) \to \mathrm{H}^1(\bar{V},\mathbb{G}_m)$, and the type of $f \in \mathrm{H}^1(V,G)$ is defined as the map sending γ to $\gamma^*(f)$.

Given a torsor $f\colon W \to V$ under G, we define

$$V(\mathbf{A}_k)^f := \bigcup_{\sigma \in \mathrm{H}^1(k,G)} f^\sigma(W^\sigma(\mathbf{A}_k)).$$

For $\sigma \in \mathrm{H}^1(k, G)$, $f^\sigma \colon W^\sigma \to V$ is defined as the twist of the torsor f, see [Sko01, Section 2.2, Example 2]. In general, f^σ is a G^σ-torsor, where G^σ is the Galois twist of G by σ, [Sko01, Section 2.1] and if G is abelian, then $G^\sigma = G$.

Remark 2.1.2. The torsor f under G represents an element of $\mathrm{H}^1(V, G)$; the set $V(\mathbf{A}_k)^f$ coincide with the set defined in Chapter 1, Section 1.1.1 with respect to the functor $\mathrm{F} := \mathrm{H}^1(-, G)$, see [Sko01, Section 5.3].

Skorobogatov proved the following theorem.

Theorem 2.1.3. *Let M be a Γ_k-module of finite type, S its dual group of multiplicative type and $\lambda \in \mathrm{Hom}_{\Gamma_k}(M, \mathrm{Pic}(\bar{V}))$. Then*

(a) there are only finitely many isomorphism classes of torsors $f\colon W \to V$ of type λ such that $W(\mathbf{A}_k) \neq \emptyset$;

(b) there exists a subgroup $\mathrm{Br}_\lambda(V) \subseteq \mathrm{Br}_1(V)$ such that

$$V(\mathbf{A}_k)^{\mathrm{Br}_\lambda} = \bigcup_{\mathrm{type}(W,f)=\lambda} f(W(\mathbf{A}_k)).$$

Proof. A proof can be found in [Sko01, Section 6.1]. In particular, note that since V is proper and smooth, we always have $\bar{k}[V]^* = \bar{k}^*$. ☐

We want to show that, up to enlarging the base field k, there is a type λ for which the corresponding Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$ is not of the shape appearing in Swinnerton-Dyer's question.

Assume that there is torsion in $\mathrm{Pic}(\bar{V})$, then up to enlarging the base field, we can assume that there is torsion already defined over the base field k. By Kummer theory there exists n such that $f \in \mathrm{H}^1(V, \mu_n)$ does not become trivial in $\mathrm{H}^1(\bar{V}, \mu_n)$. Moreover, again up to (possibly) enlarging the base field, we can assume that it contains a primitive n-root of unity and hence to have an isomorphism

$$\mathrm{H}^1(V, \mu_n) \simeq \mathrm{H}^1(V, \mathbb{Z}/n\mathbb{Z}) \simeq \pi_1(V, \mathbb{Z}/n\mathbb{Z}).$$

We denote by $\rho\colon W \to V$ the Galois covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to the μ_n-torsor $f\colon W \to V$. In [Har00] Harari proves the following result for Galois coverings.

Lemma 2.1.4. *Let $W \xrightarrow{\rho} V$ be a geometrically non-trivial Galois covering, with V geometrically connected of positive dimension. Then there exists a finite field extension k'/k such that, for almost all places ν which are totally split for k'/k, there exists a k_ν point x_ν of V such that $x_\nu \in V(k_\nu)$ but $x_\nu \notin \rho(W(k_\nu))$.*

Proof. This is a reformulation of [Har00, Lemma 2.3] with $g = \mathrm{id}$. □

We know that there exists at least one geometrically non-trivial μ_n-torsor $f\colon W \to V$. We denote by λ its type. Then, there exists $f_1, \dots, f_r$ torsors under μ_n of type λ such that

$$V(\mathbf{A}_k)^{\mathrm{Br}_\lambda} = \bigcup_{i=1}^{r} f_i(W_i(\mathbf{A}_k)).$$

Let S be any finite set of places. Then from Lemma 2.1.4 we know that for any $i \in \{1, \dots, r\}$ there exists a finite field extension k_i/k and infinitely many places ν_i that are totally split for k_i/k such that there is a point $(x_{\nu_i}) \in V(k_{\nu_i})$ but $(x_{\nu_i}) \notin f_i(W_i(k_{\nu_i}))$. If we pick the places ν_i all different from each other and not in S, then any point $(x_\nu) \in V(\mathbf{A}_k)$ such that

$$x_{\nu_i} \notin f_i(W_i(k_{\nu_i})) \text{ for } i = 1, \dots, r$$

does not belong to $V(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$.

However, if there exists a finite set of places S such that

$$V(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{\nu \notin S} V(k_\nu) \subseteq V(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$$

then for every $z \in Z$ and every $(x_\nu) \in \prod_{\nu \notin S} V(k_\nu)$, the point $z \cup (x_\nu)$ has to lie in $V(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$.

2.1.2 Previous works

In this section we give an overview of the main results that have been proved in the attempt to answer the question asked by Swinnerton-Dyer.

The question appears for the first time in [CTS13], paper in which Colliot-Thélène and Skorobogatov prove some results around it. In particular, they work under the following setting: L is a p-adic field and $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_L)$ is a smooth proper model with geometrically integral fibres of a variety X defined over L. In [CTS13] Colliot-Thélène and Skorobogatov prove the following result [CTS21, Proposition 10.4.3].

Proposition 2.1.5. *Let q be a prime, $q \neq \mathrm{char}(\ell)$. Assume that the closed geometric fibre Y of $\mathcal{X}$ has no connected unramified cyclic covering of degree q. Then* $\mathrm{Br}(X)\{q\}$ *is generated by the images of* $\mathrm{Br}(\mathcal{X})\{q\}$ *and* $\mathrm{Br}(L)\{q\}$.

The condition on the closed geometric fibre is equivalent to the vanishing of the étale cohomology group $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/q\mathbb{Z})$. This condition holds if there is no q-torsion in the geometric Picard group of the generic fibre, see [BN23, Lemma 11.4].

In [CTS13] Colliot-Thélène and Skorobovatov prove also that if the Picard group is torsion free and finitely generated, then

$$\mathrm{Br}_1(X) = \ker\left(\mathrm{Br}(X) \to \mathrm{Br}(X_{un})\right), \qquad \text{[CTS21, Proposition 10.4.2]}$$

where X_{un} is the base change of X to the maximal unramified extension L_{un} of L. Moreover, they prove that the map

$$\mathrm{Br}(L) \oplus \ker\big(\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_{un})\big) \to \ker\big(\mathrm{Br}(X) \to \mathrm{Br}(X_{un})\big)$$

is surjective. Combining these two results with diagram (1.9) we get that every element $\mathcal{A} \in \mathrm{Br}_1(X)$ has constant evaluation map. Putting everything together they get the following theorem, for varieties over number fields.

Theorem 2.1.6. *Let k be a number field. Let S be a finite set of places of k containing the archimedean places, and let $\mathcal{O}_{k,S}$ be the ring of S-integers of k. Let $\mathcal{V}$ over $\mathcal{O}_S$ be a smooth and proper scheme with geometrically integral generic fibre V. Assume that*

1. $\mathrm{Pic}(\bar{V})$ *is torsion free and finitely generated;*
2. *the transcendental Brauer group* $\mathrm{Br}(V)/\mathrm{Br}_1(V)$ *is a finite abelian group of order invertible in $\mathcal{O}_{k,S}$.*

Then $V(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{\nu \notin S} V(k_\nu)$, where $Z \subseteq \prod_{\nu \in S} V(k_\nu)$ is an open and closed subset.

Proof. This is [CTS21, Theorem 13.3.15]. Note that in their result Colliot-Thélène and Skorobogatov split the assumption on the geometric Picard group in two assumptions: the vanishing of $\mathrm{H}^1(V, \mathcal{O}_V)$ and the torsion freeness of the geometric Neron-Severi group $\mathrm{NS}(\bar{V})$. □

This theorem shows that under the finiteness assumption on the transcendental Brauer group, up to enlarging the set S, the question asked by Swinnerton-Dyer has a positive answer. Some years later Bright and Newton [BN23] were able to prove another result in the same direction, but without any assumption on the transcendental Brauer group, using the notion of refined Swan conductor.

Theorem 2.1.7. *Let k be a number field. Assume V to be a smooth, proper and geometrically integral k-variety. Assume* $\mathrm{Pic}(\bar{V})$ *to be torsion free and finitely generated. Let S be the following set of places:*

1. *Archimedean places;*
2. *places of bad reduction for V;*
3. *places $\mathfrak{p}$ satisfying $e_{\mathfrak{p}} \geq p - 1$, with p residue characteristic of $\mathfrak{p}$;*
4. *places $\mathfrak{p}$ with $\mathrm{H}^0(\mathcal{V}(\mathfrak{p}), \Omega^1_{\mathcal{V}(\mathfrak{p})})$ non-trivial, for any smooth integral model $\mathcal{V} \to$* $\mathrm{Spec}(\mathcal{O}_{\mathfrak{p}})$*, where $\mathcal{V}(\mathfrak{p})$ is the special fibre at $\mathfrak{p}$.*

Then the set S is finite and there exists $Z \subseteq \prod_{\nu \in S} V(k_\nu)$ open and closed such that $V(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{\nu \notin S} V(k_\nu)$.

Proof. [BN23, Theorem D]. □

On the other hand, in [BN23] Bright and Newton prove also a result showing that after a finite field extension primes of good **ordinary** reduction always play a role in the Brauer–Manin obstruction to weak approximation (for the definition of good ordinary reduction see Chapter 3, Definition 3.1.2).

Theorem 2.1.8. *Let V be a smooth, proper and geometrically integral variety over a number field k such that $\mathrm{H}^0(V, \Omega^2_V) \neq 0$. Let $\mathfrak{p}$ be a finite place of k at which V has good ordinary reduction, with residue characteristic p. Then there exist a finite extension k'/k, a place $\mathfrak{p}'$ of k' over $\mathfrak{p}$ and an element $\mathcal{A} \in \mathrm{Br}(V_{k'})\{p\}$ such that the evaluation map $\mathrm{ev}_{\mathcal{A}} \colon V(k'_{\mathfrak{p}'}) \to \mathbb{Q}/\mathbb{Z}$ is non-constant.*

In particular, this proves that in general the question asked by Swinnerton-Dyer has a negative answer.

The rest of this chapter is devoted to build the first example of a variety satisfying the assumption of Swinnerton-Dyer's question for which a prime of good reduction plays a role in the Brauer–Manin obstruction to weak approximation. This example consist of a K3 surface over the rational numbers, proving that the question asked by Swinnerton-Dyer has a negative answer already over $\mathbb{Q}$.

Before proceeding with the main result of this chapter, we give a short overview of K3 surfaces and explain why they form the natural playground in trying to build counterexample to Swinnerton-Dyer's question.

2.1.3 K3 surfaces

Let k be any field, we will work just with algebraic K3 surfaces. For a smooth variety, we write ω_X for the canonical sheaf of X.

Definition 2.1.9. *An algebraic K3 surface is a smooth projective 2-dimensional variety over a field k such that $\omega_X \simeq \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.*

In this thesis the examples will always be of these two kinds (see [Huy16, Section 1.1] for more details):

1. X smooth quartic surface in $\mathbb{P}^3_k$;

2. $X = \mathrm{Kum}(A)$, where A is an abelian surface over a field of characteristic different from 2 and X is obtain by resolving singularities from the quotient A/ι, with $\iota \colon A \to A$ involution given by $x \mapsto -x$.

The de Rham cohomology of K3 surfaces over algebraically closed fields can be summarised in the Hodge diamond, which looks like this

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

where $h^{i,j} = \dim_k \mathrm{H}^i(X, \Omega^j_X)$.

As mentioned before, K3 surfaces are the natural playground to build examples of varieties that satisfy both the assumption in Swinnerton-Dyer's question and in Bright and Newton Theorem 2.1.8. The Picard group is finitely generated and torsion free, see [Huy16, Section 1.2]. If we start with a K3 surface over a number field k then, since the canonical bundle is trivial by definition, we get that $\mathrm{H}^0(X, \Omega^2_X)$ has dimension 1 as a k-vector space. Moreover, Bogomolov and Zarin [BZ09] prove that there exists a finite field extension k'/k and a density 1 set of finite places Σ of k' such that $X_{k'}$ has ordinary good reduction at every place $\nu \in \Sigma$.

We are ready to state the main theorem of this chapter and of [Pag22].

Theorem 2.1.10. *Let $V \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the projective K3 surface defined by the equation*

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \tag{2.2}$$

The class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right) \in \operatorname{Br} \mathbb{Q}(V)$$

defines an element in $\operatorname{Br}(V)$. *The evaluation map* $\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{Q}_2) \to \operatorname{Br}(\mathbb{Q}_2)$ *is non-constant, and therefore gives an obstruction to weak approximation on* V. *Finally,* $V(\mathbb{Q})$ *is not dense in* $V(\mathbb{Q}_2)$*, with respect to the analytic topology.*

2.2 Proof of Theorem 2.1.10

In the first part of the proof we will show that the element $\mathcal{A} \in \operatorname{Br}(k(V))$ lies in $\operatorname{Br}(V)$. Next, we will exhibit two points $P_1, P_2 \in V(\mathbb{Q}_2)$ such that

$$\mathrm{ev}_{\mathcal{A}}(P_1) \neq \mathrm{ev}_{\mathcal{A}}(P_2).$$

Finally, we will prove that, for every place ν different from 2, the evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{Q}_\nu) \to \mathbb{Q}/\mathbb{Z}$$

is constant.

Proof of Theorem 2.1.10. Let $f := z^3 + w^2x + xyz$ and C_x,C_z,C_f be the closed subsets of V defined by the equations $x = 0$, $z = 0$ and $f = 0$ respectively. The quaternion algebra $\mathcal{A}$ defines an element in $\mathrm{Br}(U)$, where $U := V \setminus (C_x \cup C_z \cup C_f)$. The purity theorem for the Brauer group [CTS21, Theorem 3.7.2], assures us the existence of the exact sequence

$$0 \to \mathrm{Br}(V)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus \partial_D} \bigoplus_D \mathrm{H}^1(k(D), \mathbb{Z}/2) \tag{2.3}$$

where D ranges over the irreducible divisors of X with support in $X \setminus U$ and $k(D)$ denotes the residue field at the generic point of D.

In order to use the exact sequence (4.14) we need to understand what the prime divisors of V with support in $X \setminus U = C_x \cup C_z \cup C_f$ look like. It is possible to check the following:

- C_x has as irreducible components D_1 and D_2, defined by the equations $\{x = 0, z = 0\}$ and $\{x = 0, y^3 + z^2w = 0\}$ respectively;
- C_z has as irreducible components D_1 and D_3, where D_3 is defined by the equations $\{z = 0, x^2y + w^3 = 0\}$;
- C_f has as irreducible components D_1, D_4 and D_5, where D_4 and D_5 are defined by the equations $\{z^3 + xw^2 = 0, y = 0\}$ and $\{y^3z - x^2z^2 + y^2w^2 = 0, x^3 + y^2z = 0, xyz + z^3 + xw^2 = 0\}$ respectively.

Therefore, we can rewrite (4.14) in the following way:

$$0 \to \mathrm{Br}(V)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus \partial_{D_i}} \bigoplus_{i=1}^{5} H^1(k(D_i), \mathbb{Z}/2). \tag{2.4}$$

Moreover, we have an explicit description of the residue map on quaternion algebras: for an element $(a, b) \in \mathrm{Br}(U)[2]$ we have

$$\partial_{D_i}(a, b) = \left[(-1)^{\nu_i(a)\nu_i(b)} \frac{a^{\nu_i(b)}}{b^{\nu_i(a)}} \right] \in \frac{k(D_i)^\times}{k(D_i)^{\times 2}} \simeq H^1(k(D_i), \mathbb{Z}/2) \tag{2.5}$$

where ν_i is the valuation associated to the prime divisor D_i. This is already explained in Chapter 1, (1.4).

We can proceed with the computation of the residue maps ∂_{D_i} for $i = 1, \ldots, 5$:

1. $\nu_1(f) = \nu_1(x) = \nu_1(z) = 1$. Hence,

$$\partial_{D_1}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[\left(-\frac{z}{x}\right)^2\right] = 1 \in \frac{k(D_1)^\times}{k(D_1)^{\times 2}}.$$

2. $\nu_2(x) = 1$ and $\nu_2(f) = \nu_2(z) = 0$. Hence,

$$\partial_{D_2}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[-\left(\frac{f}{x^3}\right)^{-1}\left(-\frac{z}{x}\right)^3\right] = \left[\frac{z^3}{f}\right] = 1 \in \frac{k(D_2)^\times}{k(D_2)^{\times 2}}$$

where the last equality follows from the fact that $x = 0$ on D_2, thus we have $f\mid_{D_2} = z^3$.

3. $\nu_3(x) = \nu_3(f) = 0$ and $\nu_3(z) = 1$. Hence,

$$\partial_{D_3}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[\left(\frac{f}{x^3}\right)\right] = \left[\left(\frac{w}{x}\right)^2\right] = 1 \in \frac{k(D_3)^\times}{k(D_3)^{\times 2}}$$

where the last equality follows from the fact that $z = 0$ on D_3, thus we have $f\mid_{D_3} = w^2 x$.

4. $\nu_4(x) = \nu_4(z) = 0$ and $\nu_4(f) = 1$. Hence,

$$\partial_{D_4}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[\left(-\frac{x}{z}\right)\right] = \left[\left(\frac{w}{z}\right)^2\right] = 1 \in \frac{k(D_4)^\times}{k(D_4)^{\times 2}}$$

where the last equality follows from the fact that $z^3 + w^2 x = 0$ on D_4, thus $-\frac{x}{z} = \left(\frac{w}{z}\right)^2$.

5. $\nu_5(x) = \nu_5(z) = 0$ and $\nu_5(f) = 1$. Hence,

$$\partial_{D_5}\left(\frac{f}{x^3}, -\frac{z}{x}\right) = \left[\left(-\frac{x}{z}\right)\right] = \left[\left(\frac{y}{x}\right)^2\right] = 1 \in \frac{k(D_5)^\times}{k(D_5)^{\times 2}}$$

where the last equality follows from the fact that $x^3 + y^2 z = 0$ on D_5, thus $-\frac{x}{z} = \left(\frac{y}{x}\right)^2$.

Therefore, $\partial_{D_i}(\mathcal{A}) = 0$ for all $i \in \{1, \dots, 5\}$, hence $\mathcal{A} \in \mathrm{Br}(X)$.

We now show that the element $\mathcal{A}$ obstructs weak approximation on V. Let $\mathcal{V} \subseteq \mathbb{P}^3_{\mathbb{Z}}$ be the projective scheme defined by the equation

$$x^3 y + y^3 z + z^3 w + w^3 x + xyzw = 0. \tag{2.6}$$

$\mathcal{V}$ is a $\mathbb{Z}$-model for V and has good reduction at the prime 2.

Let $P_1 := (1 : 0 : 1 : 0) \in \mathcal{V}(\mathbb{Z}_2)$; then P_1 is such that $\mathrm{ev}_{\mathcal{A}}(P_1) = (1, -1)$. Therefore $\mathrm{ev}_{\mathcal{A}}(P_1)$ is the trivial class in $\mathrm{Br}(\mathbb{Q}_2)$. Moreover, Hensel's lemma assures us of the existence of a solution $P_2 = (1 : 2 : 1 : d) \in \mathcal{V}(\mathbb{Z}_2)$ whose reduction modulo 8 is $(1 : 2 : 1 : 2)$. Hence,

$$\mathrm{ev}_{\mathcal{A}}(P_2) = (f(P_2), -1) \quad \text{with} \quad f(P_2) \equiv 7 \pmod 8.$$

Therefore, we get that $\mathrm{ev}_{\mathcal{A}}(P_2)$ defines a non-trivial element in the Brauer group of $\mathbb{Q}_2$ [Ser73, Theorem 3.1]. The existence of such points implies that there is a Brauer–Manin obstruction to weak approximation arising from $\mathcal{A}$.

In order to conclude the proof of the theorem we investigate the behaviour of the evaluation map at the other primes and at infinity. For every prime p let $\mathcal{V}_p$ be the base change of $\mathcal{V}$ to $\mathbb{Z}_p$. We distinguish the following cases.

Case $p \notin \{3, 5, 17, \infty\}$. In this case, $\mathcal{V}$ has good reduction at p. Therefore, we can use [CTS21, Proposition 10.4.3] to conclude that the evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon \mathcal{V}(\mathbb{Z}_p) \to \mathbb{Q}/\mathbb{Z}$$

is constant. Moreover, $P = (1 : 0 : 1 : 0) \in \mathcal{X}(\mathbb{Z}_p)$ and

$$\mathrm{ev}_{\mathcal{A}}(P) = (1, -1)$$

which is trivial in $\mathrm{Br}(\mathbb{Q}_p)$; hence the evaluation map is trivial on the whole $\mathcal{V}(\mathbb{Z}_p)$, which coincides with $V(\mathbb{Q}_p)$.

Case $p \in \{3, 5, 17\}$. Under this assumption, $\mathcal{V}_p/\mathbb{Z}_p$ is not smooth. In these three cases, we want to show that the evaluation map is trivial on $\mathcal{V}(\mathbb{Z}_p)$ by showing that it factors through $\mathrm{Br}(\mathbb{Z}_p)$.

The special fibre $\mathcal{V}(p) := \mathcal{V}_p \times_{\mathbb{Z}_p} \mathrm{Spec}(\mathbb{F}_p)$ is a non-smooth $\mathbb{F}_p$-scheme. However, $\mathcal{V}(p)$ is an irreducible $\mathbb{F}_p$-scheme, with just isolated singularities. The $\mathbb{Z}_p$-points of $\mathcal{V}_p$ are all smooth. In fact, $\mathcal{V}(p)$ contains just one singular point defined over $\mathbb{F}_p$ that does not even lift to a $\mathbb{Z}/p^2\mathbb{Z}$-point. Let $\mathcal{U}$ be the smooth locus of $\mathcal{V}_p$; because of what we have just said we have

$$V(\mathbb{Q}_p) = \mathcal{V}(\mathbb{Z}_p) = \mathcal{U}(\mathbb{Z}_p).$$

Let U be the base change of $\mathcal{U}$ to $\mathrm{Spec}(\mathbb{Q}_p)$. The purity theorem on $\mathcal{U}$ [CTS21, Theorem 3.7.1] gives us the exact sequence

$$\mathrm{Br}(\mathcal{U})[2] \to \mathrm{Br}(U)[2] \xrightarrow{\partial_{D_p}} H^1(k(D_p), \mathbb{Z}/2\mathbb{Z})$$

where D_p is the divisor associated to the special fibre (D_p is the smooth locus of $\mathcal{V}(p)$). We just need to show that $\partial_{D_p}(\mathcal{A}) = 0$. Let ν_p be the valuation corresponding to the prime divisor D_p; then

$$\nu_p\left(\frac{f}{x^3}\right) = 0 \quad \text{and} \quad \nu_p\left(-\frac{z}{x}\right) = 0.$$

Indeed, the point $(1 : 0 : 1 : 0) \in \mathcal{V}(p)(\mathbb{F}_p)$ is smooth, hence it lies in D_p. Moreover

$$\frac{f}{x^3}(1 : 0 : 1 : 0) = 1 \quad \text{and} \quad -\frac{z}{x}(1 : 0 : 1 : 0) = -1.$$

Therefore both $\frac{f}{x^3}$ and $-\frac{z}{x}$ do not vanish on D_p, which implies that $\partial_{D_p}(\mathcal{A}) = 0$. Therefore, $\mathcal{A}$ lies in $\mathrm{Br}(\mathcal{U}) \subseteq \mathrm{Br}(V_p)$ and the evaluation map factors as

$$\begin{array}{ccc} \mathcal{U}(\mathbb{Z}_p) & \xrightarrow{\quad |\mathcal{A}| \quad} & \mathrm{Br}(\mathbb{Q}_p). \\ & \searrow \quad \nearrow & \\ & \mathrm{Br}(\mathbb{Z}_p) & \end{array}$$

Since $\mathrm{Br}(\mathbb{Z}_p)$ is trivial, the evaluation map has to be constant and trivial.

Case $p = \infty$. The evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon V(\mathbb{R}) \to \mathbb{Q}/\mathbb{Z}$$

is constant and equal to 0.

We will show that it is constant on the dense open subset

$$W := \{P \in X(\mathbb{R}) : x(P), z(P), f(P) \neq 0\} \subseteq V(\mathbb{R}).$$

Indeed, from the continuity of the evaluation map it follows that it has to be constant also on the whole of $V(\mathbb{R})$. Let $P = (\alpha : \beta : \gamma : \delta) \in W$, thus $\gamma \neq 0$. First, assume that $-\frac{\gamma}{\alpha} > 0$. Then

$$\mathrm{ev}_{\mathcal{A}}(P) = \left(\frac{f(P)}{x(P)^3}, -\frac{z(P)}{x(P)}\right) = \left(\frac{f(P)}{\alpha^3}, -\frac{\gamma}{\alpha}\right)$$

is trivial in $\mathrm{Br}(\mathbb{R})$. Now, suppose that $-\frac{\gamma}{\alpha} < 0$. Without loss of generality, we can assume that both α and γ are positive. We want to show that in this case $f(P)$ has to be positive:

- if $\delta = 0$, then $P \in V(\mathbb{R})$ implies that $\beta(\alpha^3 + \beta^2\gamma) = 0$. Therefore $\beta = 0$, since $\alpha^3 + \beta^2\gamma \geq \alpha^3 > 0$. Hence, $f(P) = \gamma^3 > 0$;
- if $\delta \neq 0$ then $P \in V(\mathbb{R})$ implies

  $$f(P) = -\frac{\beta}{\delta}(\alpha^3 + \beta^2\gamma).$$

  Hence, since $\alpha^3 + \beta^2\gamma > 0$,

  $$f(P) > 0 \quad \text{if and only if} \quad -\frac{\beta}{\delta} > 0.$$

  Equivalently, β, δ do not have the same sign. Hence, we just need to show that there is no point $P \in W$ with α, γ positive and β, δ with the same sign. First, we observe that β, δ can not be both positive, since otherwise

  $$\alpha^3\beta + \beta^3\gamma + \gamma^3\delta + \delta^3\alpha + \alpha\beta\gamma\delta > 0.$$

  On the other hand β, δ cannot also be both negative. Indeed, we have that $P \in V(\mathbb{R})$ if and only if

  $$\alpha^3(-\beta) + (-\beta)^3\gamma + \gamma^3(-\delta) + (-\delta)^3\alpha = \alpha(-\beta)\gamma(-\delta).$$

  Without loss of generality we may assume that $\alpha \geq \max\{-\beta, \gamma, -\delta\}$; but if $\alpha, -\beta, \gamma, -\delta$ are all positive, then

  $$\alpha^3(-\beta) + (-\beta)^3\gamma + \gamma^3(-\delta) + (-\delta)^3\alpha > \alpha(-\beta)\gamma(-\delta).$$

  Hence, $(\alpha : \beta : \gamma : \delta) \notin V(\mathbb{R})$. □

2.2.1 Remark on the quaternion algebra $\mathcal{A}$

By what we said before Theorem 2.1.6, since 2 is a prime of good reduction and $\mathcal{A}$ is obstructing weak approximation, $\mathcal{A}$ is not an algebraic element. In this section we give a rough idea of the strategy behind the construction of the quaternion algebra $\mathcal{A}$, more details will be provided in Chapter 4.

Let X and $\mathcal{X}$ be the base change of V and $\mathcal{V}$ to $\mathbb{Q}_2$ and $\mathbb{Z}_2$ respectively. Let Y be the special fibre of $\mathcal{X}$,

$$\begin{array}{ccccc} X & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(\mathbb{Q}_2) & \longrightarrow & \operatorname{Spec}(\mathbb{Z}_2) & \longleftarrow & \operatorname{Spec}(\mathbb{F}_2). \end{array} \tag{2.7}$$

Let π be a uniformiser in $\mathbb{Z}_2$. By Corollary 1.3.9(3) we know that there is a non-zero constant $\bar{c}$ in the residue field (that in this case is 1, since the residue field is $\mathbb{F}_2$) such that

$$\operatorname{mult}_{\bar{c}}\left(\operatorname{rsw}_{2,\pi}(\operatorname{Br}(X)[2])\right) \subseteq \mathrm{H}^0(Y, \Omega^2_{Y,\log}) \oplus \mathrm{H}^0(Y, \Omega^1_{Y,\log})$$

Moreover, the special fibre of a K3 surface having good reduction is still a K3 surface, see for example [BN23, Remark 11.5]. Hence, $\mathrm{H}^0(Y, \Omega^2_Y)$ is a one dimensional $\mathbb{F}_2$-vector space. We will prove in Chapter 4, using the results of Section 4.3, that 2 is a prime of good ordinary reduction for X and the only non-trivial two form ω is logarithmic. Finally, in Chapter 4 we will developed the techniques needed to compute the refined Swan conductor on p-torsion elements in the Brauer group and show that $\mathcal{A} \in \operatorname{fil}_2\operatorname{Br}(V)[2]$ and $\operatorname{rsw}_{2,\pi}(\mathcal{A}) = (\omega, 0) \neq (0, 0)$. In particular, this implies that $\mathcal{A} \notin \operatorname{Ev}_{-1}\operatorname{Br}(X)[p]$. See Section 4.4 for the details.

2.3 A family of K3 surfaces with the same property

In this section we will show that the first part of Theorem 2.1.10 can be easily generalised to a family of K3 surfaces that share some properties with our K3 surface V.

Let a, b, c, d, e be odd integers, $\underline{\alpha} = (a, b, c, d, e)$ and $V_{\underline{\alpha}}$ be the K3 surface in $\mathbb{P}^3_{\mathbb{Q}}$ associated to the equation

$$a \cdot x^3 y + b \cdot y^3 z + c \cdot z^3 w + d \cdot w^3 x + e \cdot xyzw = 0. \tag{2.8}$$

Let $\mathcal{V}_{\underline{\alpha}}$ be the projective scheme over $\mathbb{Z}$ defined by the polynomial equation (2.8). Then $\mathcal{V}_{\underline{\alpha}}$ is a $\mathbb{Z}$-model of $V_{\underline{\alpha}}$. Moreover, since a, b, c, d, e are all odd integers, all these varieties have the same reduction, which we will denote by Y, modulo the prime 2. A natural question that arises at this point is if also for all these K3 surfaces there exists an element $\mathcal{A} \in \operatorname{Br}(V_{\underline{\alpha}})[2]$ such that the corresponding evaluation map on $V_{\underline{\alpha}}(\mathbb{Q}_2)$ is non-constant, i.e. that obstruct weak approximation. The following theorem gives a partial answer to the question.

Theorem 2.3.1. *Assume that* $\Delta := abcd \in \mathbb{Q}^{\times 2}$. *Then, the class of the quaternion algebra*

$$\mathcal{A} = \left(d \cdot \frac{c \cdot z^3 + d \cdot w^2 x + e \cdot xyz}{x^3}, -(cd) \cdot \frac{z}{x} \right) \in \operatorname{Br}(\mathbb{Q}(V_{\underline{\alpha}}))$$

defines an element in $\operatorname{Br}(V_{\underline{\alpha}})$. *The evaluation map* $\operatorname{ev}_{\mathcal{A}} \colon V_{\underline{\alpha}}(\mathbb{Q}_2) \to \mathbb{Q}/\mathbb{Z}$ *is non-constant, and therefore gives an obstruction to weak approximation on* V.

Proof. The proof is very similar to the first part of the proof of Theorem 2.1.10. We denote by f the polynomial $c \cdot z^3 + d \cdot w^2 x + e \cdot xyz$. Also in this case, let C_x, C_z, C_f be the closed subsets of $V_{\underline{\alpha}}$ defined by the equations $x = 0, z = 0$ and $f = 0$ respectively. Let U be the open subset of $V_{\underline{\alpha}}$ defined as the complement of $C_x \cup C_z \cup C_f$. Clearly, $\mathcal{A} \in \operatorname{Br}(U)$. Moreover,

- C_x consists of two irreducible components, $D_1 = \{x, z\}$ and $D_2 = \{x, b \cdot y^3 + c \cdot z^2 w\}$.
- C_z consists of two irreducible components, D_1 and $D_3 = \{z, a \cdot x^2 y + d \cdot w^3\}$.
- C_f consists of three irreducible components, D_1, $D_4 = \{y, c \cdot z^3 + d \cdot w^2 x\}$ and $D_5 = \{f, a \cdot x^3 + b \cdot y^2 z, be \cdot y^3 z - ac \cdot x^2 z^2 + bd \cdot y^2 w^2\}$.

In order to show that the quaternion algebra $\mathcal{A}$ lies in the Brauer group of $V_{\underline{\alpha}}$ we will use the exact sequence (4.15) coming from the purity theorem and the explicit description of the residue map given in equation (4.16). We will denote by ν_i the valuation associated to the prime divisor D_i.

1. $\nu_1(f) = \nu_1(x) = \nu_1(z) = 1$, and so $\nu_1\left(\frac{f}{x^3}\right) = -2$ and $\nu_1\left(-\frac{z}{x}\right) = 0$. Hence

$$\begin{aligned} \partial_{D_1}(\mathcal{A}) &= \left[(-1)^0 \left(d \cdot \frac{f}{x^3} \right)^0 \left(-\frac{1}{cd} \cdot \frac{x}{z} \right)^{-2} \right] \\ &= \left[\left(\frac{1}{cd} \cdot \frac{z}{x} \right)^2 \right] = 1 \in \frac{k(D_1)^\times}{k(D_1)^{\times 2}}. \end{aligned}$$

2. $\nu_2(f) = \nu_2(z) = 0$ and $\nu_2(x) = 1$, and so $\nu_2\left(\frac{f}{x^3}\right) = -3$ and $\nu_2\left(-\frac{z}{x}\right) = -1$. Hence

$$\begin{aligned} \partial_{D_2}(\mathcal{A}) &= \left[(-1)^3 \left(d \cdot \frac{f}{x^3} \right)^{-1} \left(-\frac{1}{cd} \cdot \frac{x}{z} \right)^{-3} \right] \\ &= \left[\left(\frac{(cd)^3}{d} \frac{x^3}{c \cdot z^3} \cdot \frac{z^3}{x^3} \right) \right] = 1 \in \frac{k(D_2)^\times}{k(D_2)^{\times 2}} \end{aligned}$$

where the second equality follows from the fact that $f\,|_{D_2} = c \cdot z^3$.

3. $\nu_3(f) = \nu_3(x) = 0$ and $\nu_3(z) = 1$, and so $\nu_3\left(\frac{f}{x^3}\right) = 0$ and $\nu_3\left(-\frac{z}{x}\right) = 1$. Hence
$$\partial_{D_3}(\mathcal{A}) = \left[(-1)^0\left(d\cdot\frac{f}{x^3}\right)^1\left(-\frac{1}{cd}\cdot\frac{x}{z}\right)^0\right] = \left[d\cdot\frac{f}{x^3}\right] = 1 \in \frac{k(D_3)^\times}{k(D_3)^{\times 2}}$$
where the last equality follows from the fact that $f\,|_{D_3} = d\cdot w^2 x$.

4. $\nu_4(f) = 1$ and $\nu_4(x) = \nu_4(z) = 0$, and so $\nu_4\left(\frac{f}{x^3}\right) = 1$ and $\nu_4\left(-\frac{z}{x}\right) = 0$. Hence
$$\partial_{D_4}(\mathcal{A}) = \left[(-1)^0\left(d\cdot\frac{f}{x^3}\right)^0\left(-\frac{1}{cd}\cdot\frac{x}{z}\right)^1\right] = \left[\frac{1}{cd}\cdot\frac{c}{d}\right] = 1 \in \frac{k(D_4)^\times}{k(D_4)^{\times 2}}$$
where the second equality follows from the fact that $-\frac{z}{x} = \frac{d}{c}\left(\frac{w}{z}\right)^2$ on D_4.

5. $\nu_5(f) = 1$ and $\nu_5(x) = \nu_5(z) = 0$, and so $\nu_5\left(\frac{f}{x^3}\right) = 1$ and $\nu_5\left(-\frac{z}{x}\right) = 0$. Hence
$$\partial_{D_5}(\mathcal{A}) = \left[(-1)^0\left(d\cdot\frac{f}{x^3}\right)^0\left(-\frac{1}{cd}\cdot\frac{x}{z}\right)^1\right] = \left[\frac{b}{acd}\right] = 1 \in \frac{k(D_5)^\times}{k(D_5)^{\times 2}}$$
where the last equality follows from the fact that $-\frac{z}{x} = \frac{a}{b}\left(\frac{x}{y}\right)^2$ on D_5 and the assumption that $abcd$ is a square in $\mathbb{Q}$.

The above computations together with the purity theorem show indeed that $\mathcal{A}$ lies in $\mathrm{Br}(V_{\underline{\alpha}})$. Finally, we need to show that the evaluation map on the $\mathbb{Q}_2$-points of $V_{\underline{\alpha}}$ is non-constant. Let
$$P_1 := (1:0:1:0) \in V_{\underline{\alpha}}(\mathbb{Q}).$$
Then, P_1 is such that $\mathrm{ev}_{\mathcal{A}}(P_1) = (dc, -dc)$, which is trivial in $\mathrm{Br}(\mathbb{Q}_2)$. Furthermore, let
$$P_2 := \left(cd : y : 1 : -2\cdot\frac{acde}{2c+cd}\right) \in V_{\underline{\alpha}}(\mathbb{Q}_2)$$
be such that the reduction modulo 8 of y is equal to $2\cdot de$. Then
$$f(P_2) \equiv c + d\cdot 4\cdot(cd) + e\cdot(cd)\cdot 2\cdot ed \equiv 7\cdot c \mod 8$$
and therefore evaluation map at P_2 is
$$\mathrm{ev}_{\mathcal{A}}(P_2) = \left(d\cdot\frac{f(P_2)}{(cd)^3}, -cd\frac{1}{cd}\right) = (g(P_2), -1) \quad \text{with} \quad g(P_2) \equiv 7 \mod 8.$$
Thus, $\mathrm{ev}_{\mathcal{A}}(P_2)$ defines a non-trivial element in $\mathrm{Br}(\mathbb{Q}_2)$. Hence, the element $\mathcal{A} \in \mathrm{Br}(V_{\underline{\alpha}})$ gives an obstruction to weak approximation on $V_{\underline{\alpha}}$. □

A natural question that arises at this point is what happens if $\Delta := abcd$ is not a square in $\mathbb{Q}$. Note that, in this case we can repeat the same computations that we did in the proof of Theorem 2.3.1. That is, for every divisor $D \neq D_5$ we get $\partial_D(\mathcal{A}) = 1$, while for D_5 we have

$$\partial_{D_5}(\mathcal{A}) = [\Delta] \in \frac{k(D_5)^\times}{k(D_5)^{\times 2}}.$$

Hence, in this case, $\mathcal{A}$ defines an element in the Brauer group of the base change of $V_{\underline{\alpha}}$ to $\mathbb{Q}(\sqrt{\Delta})$. With an argument similar to the one sketched in Section 2.2.1, also in this case we expect to be able to find two points P_1, P_2 defined over $\mathbb{Q}_2(\sqrt{\Delta})$ such that $\mathrm{ev}_\mathcal{A}(P_1) \neq \mathrm{ev}_\mathcal{A}(P_2)$.

2.3.1 Final considerations

The results of this chapter arise from the wish to use Theorem 2.1.8 in order to produce example of varieties in which primes of good reduction play a role in the Brauer–Manin obstruction to weak approximation.

In our example,

$$V = \mathrm{Proj}\left(\frac{\mathbb{Q}[x,y,z,w]}{x^3y + y^3z + z^3w + w^3x + xyzw}\right) \subseteq \mathbb{P}^3_\mathbb{Q}$$

is a smooth projective variety defined over the number field $\mathbb{Q}$.

Since V is a K3 surface, the hypotheses of Theorem 2.1.8 are satisfied. In this example, we were able to construct an element $\mathcal{A}$ that satisfies Theorem 2.1.8 which is already defined over the rational numbers and the corresponding evaluation map is non-constant on the 2-adic points. Moreover, $\mathcal{A}$ does not just lie in the 2-primary part of the Brauer group of X, it has order exactly 2.

The next two chapters are devoted to the investigation on whether one can hope to extend this strategy to a more general setting.

CHAPTER 3

The role of primes of good reduction in the Brauer–Manin obstruction

This chapter is based on [Pag23], an article in which we continue to investigate around the following question.

Question 3.0.1. Assume $\mathrm{Pic}(\bar{V})$ to be torsion-free and finitely generated. Which primes can play a role in the Brauer–Manin obstruction to weak approximation on X?

We recall that [BN23, Theorem C] proves that, up to a base change to a finite field extension k'/k, we can always find a prime $\mathfrak{p}'$ of good reduction that plays a role in the Brauer–Manin obstruction to weak approximation on $V_{k'}$.

The main focus of this chapter is on the field extension k'/k appearing in Bright and Newton's result, with a particular emphasis on K3 surfaces. We are going to analyse how the reduction type and the absolute ramification index are involved in the possibility for a prime of good reduction to play a role in the Brauer–Manin obstruction to weak approximation. Let $\mathfrak{p}$ be a prime of good reduction for V, with model $\mathcal{V} \to \mathrm{Spec}(\mathcal{O}_k)$ such that the special fibre $\mathcal{V}(\mathfrak{p})$ has no global 1-form, i.e. $\mathrm{H}^0(\mathcal{V}(\mathfrak{p}), \Omega^1_{\mathcal{V}(\mathfrak{p})}) = 0$. Then, Bright and Newton prove that if the ramification index $e_{\mathfrak{p}}$ is smaller than $(p-1)$, the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V. Hence, already in Bright and Newton paper a link is inferred between the role of a prime $\mathfrak{p}$ of good reduction in the Brauer–Manin obstruction to weak approximation and the ramification index $e_{\mathfrak{p}}$.

In the previous chapter we have shown that there exists a variety V with torsion free and finitely generated geometric Picard group that satisfies the hypothesis of Theorem C of Bright and Newton and for which the field extension k'/k is not needed. In this example V is a K3 surface over the rational numbers and the prime of good ordinary reduction is $\mathfrak{p} = 2$. In particular, we have $e_{\mathfrak{p}} = 1 = 2-1 = p-1$. Hence, one might think that for a prime $\mathfrak{p}$ of good (ordinary) reduction it is enough to satisfy $e_{\mathfrak{p}} \geq (p-1)$, in order to play a role in the Brauer–Manin obstruction to weak approximation. In this chapter we are going to show that this is not the case.

Moreover, inspired by the example of the previous chapter, in Section 3.1.2 we improve the result of Theorem C. In particular, we show that for K3 surfaces it is always possible to find (over a finite field extension k'/k) an element whose order is exactly p and whose evaluation map is non-constant.

On the other hand, in the second part of this chapter we analyse the case of primes of good non-ordinary reduction for K3 surfaces. In particular, we will find also in this case that there is a relation between the possibility for a prime $\mathfrak{p}$ to play a role in the Brauer–Manin obstruction to weak approximation and the ramification index $e_{\mathfrak{p}}$.

3.1 Ordinary good reduction

Let V be a smooth, proper and geometrically integral k-variety and $\mathcal{V}$ an $\mathcal{O}_k$-model of V. In this section we are proving the following theorem.

Theorem 3.1.1. *Let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Assume that the special fibre $\mathcal{V}(\mathfrak{p})$ has no non-trivial global 1-forms, $\mathrm{H}^1(\overline{\mathcal{V}(\mathfrak{p})}, \mathbb{Z}/p\mathbb{Z}) = 0$ and $(p-1) \nmid e_{\mathfrak{p}}$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Let X be the base change to $k_{\mathfrak{p}}$ of V, $\mathcal{X}$ the base change of $\mathcal{V}$ to $\mathcal{O}_{\mathfrak{p}}$ and Y the special fibre $\mathcal{V}(\mathfrak{p})$. As already explained in Section 1.1.3 if we are able to show that for all elements $\mathcal{A} \in \mathrm{Br}(X)$ the evaluation map $\mathrm{ev}_{\mathcal{A}} \colon X(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant, then we get that the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.

We start by giving the definition of ordinary variety.

Definition 3.1.2. *Let Y be a smooth, proper and geometrically integral variety over a perfect field ℓ of positive characteristic. We say that Y is **ordinary** if $\mathrm{H}^i(Y, B^q_Y) = 0$ for every i, q.*

As pointed out in [BK86, Example 7.4] if Y is an abelian variety, then we get back the usual definition of ordinary, namely Y is ordinary if and only if there is an isomorphism between $Y(\bar{\ell})[p]$ and $(\mathbb{Z}/p\mathbb{Z})^{\dim(Y)}$. In general, being ordinary can be defined in many different equivalent ways, see for example [BK86, Proposition 7.3]. In particular, it is related to the slopes of the Newton polygon defined by the action of the Frobenius on the crystalline cohomology groups $\mathrm{H}^q_{\mathrm{cris}}(Y/W)$. Roughly speaking, the role played by the Dieudonné module for abelian varieties is replaced by the F-crystals $\mathrm{H}^q(Y/W)$, with $q \geq 0$. However, since for the aim of this section we can avoid introducing crystalline cohomology and the de Rham-Witt complex we prefer not to add too many technicalities and use the definition of ordinary variety needing to introduce the least number of new concepts. We will see in Section 3.2 how working with the de Rham–Witt complex can be used to get information about the de Rham cohomology of Y and consequently about the Evaluation filtration.

3.1.1 Proof of Theorem 3.1.1

Throughout this section, we will assume that X is such that its special fibre Y is smooth, **ordinary** and such that both $\mathrm{H}^0(Y, \Omega^1_Y)$ and $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z})$ are trivial. In this case, the Cartier operator gives a bijection between the global closed q-forms and the global q-forms on Y. In fact, for every q the short exact sequence $0 \to B^q_Y \to Z^q_Y \xrightarrow{C} \Omega^q_Y \to 0$ induces a long exact sequence in cohomology

$$0 \to \mathrm{H}^0(Y, B^q_Y) \to \mathrm{H}^0(Y, Z^q_Y) \xrightarrow{C} \mathrm{H}^0(Y, \Omega^2_Y) \to \mathrm{H}^1(Y, B^q_Y) \to \dots.$$

The ordinary condition assures the vanishing of $\mathrm{H}^0(Y, B^q_Y)$ and $\mathrm{H}^1(Y, B^q_Y)$ and hence the bijectivity of the Cartier operator $C\colon \mathrm{H}^0(Y, Z^q_Y) \to \mathrm{H}^0(Y, \Omega^q_Y)$.

Lemma 3.1.3. *Let n be an integer such that $0 < n < e'$; then for every $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ we have*

$$\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0$$

Proof. The assumption $\mathrm{H}^0(Y, \Omega^1_Y) = 0$ together with Lemma 1.3.9(1) assure us that if $p \nmid n$, then $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0$. Moreover, as already pointed out in Section 1.2.3.1 if $\mathcal{A} \in \mathrm{Br}(X)$ has order coprime to p, then $\mathcal{A} \in \mathrm{fil}_0\mathrm{Br}(X)$. Therefore we are reduced to the case $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)[p^r]$, with $r \geq 1$ and $p \mid n$. We will prove the lemma by induction on r. If $r = 1$ and $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$, by Corollary 1.3.9(2)

$$(0, 0) = \mathrm{rsw}_{n/p,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha), C(\beta)).$$

By Lemma 1.3.8(2) it follows that $(\alpha, \beta) \in \mathrm{H}^0(Y, B^2_Y) \oplus \mathrm{H}^0(Y, B^1_Y)$, and we are done since by the ordinary assumption there are no non-trivial global exact 1 and 2-forms. Assume the result to be true for $r - 1$ and let $\mathcal{A} \in \mathrm{Br}(X)[p^r]$, then again by Corollary 1.3.9(2) together with the induction hypothesis we have that

$$(0, 0) = \mathrm{rsw}_{n/p,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha), C(\beta))$$

and once again we get that $(\alpha, \beta) \in \mathrm{H}^0(Y, B^2_Y) \oplus \mathrm{H}^0(Y, B^1_Y)$ and therefore both α and β are zero. ☐

Lemma 3.1.4. *Assume that* $(p-1) \nmid e$*; then for every* $n \geq 1$ *and for every* $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ *we have*

$$\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0.$$

Proof. Let $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)[p^r]$. By the previous lemma, we already know that the result is true if $n < e'$ and we know that the result is true if $p \nmid n$, see 1.3.9(1). Again, we work by induction on r. For $r = 1$, we know from Remark 1.2.25(1) that if $n > e'$ then $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0,0)$. Hence, we can assume $n \leq e'$. Since by assumption e' is not an integer $n \leq e'$ is equivalent to $n < e'$, we can therefore conclude by applying the previous lemma. Let $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)[p^r]$ and $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$. By assumption we know that if $n \geq e'$ then $n > e'$. From Corollary 1.3.9(3)

$$\mathrm{rsw}_{n-e}(\mathcal{A}^{\otimes p}) = (\bar{u}\alpha, \bar{u}\beta)$$

with $\bar{u} \in \ell^\times$ and the result follows from the induction hypothesis. □

We get therefore the following theorem.

Theorem 3.1.5. *Assume that* Y *is an ordinary variety with no global* 1*-forms and that* $(p-1) \nmid e$*. Then* $\mathrm{Br}(X) = \mathrm{fil}_0\mathrm{Br}(X)$.

The assumption $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$ implies that $\mathrm{Ev}_{-1}\mathrm{Br}(X)\{p\} = \mathrm{fil}_0\mathrm{Br}(X)\{p\}$, see [BN23, Lemma 11.3]. Hence, combining this result with Theorem 3.1.5 we get the proof of Theorem 3.1.1.

3.1.2 On the existence of a Brauer–Manin obstruction over a field extension

In this section we prove a slightly stronger version of [BN23, Theorem C] for K3 surfaces having good ordinary reduction. We show that it is always possible (after a finite field extension) to find an element of order **exactly** p with non-constant evaluation map.

Theorem 3.1.6. *Let* V *be* K3 *surface over a number field* k *and* $\mathfrak{p}$ *be a prime of good ordinary reduction for* V*. Then there exists a finite field extension* k'/k *and an element* $\mathcal{A} \in \mathrm{Br}(V_{k'})[p]$ *that obstructs weak approximation on* $V_{k'}$.

For the proof we change setting and we work over the p-adic field $L := k_\mathfrak{p}$ and we denote by $X, \mathcal{X}$ the base change of $V, \mathcal{V}$ to $k_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ respectively. Moreover, we denote by Y the base change of $\mathcal{V}$ at $\ell := k(\mathfrak{p})$.

Without loss of generality we can assume that L contains a primitive p-root of unity. We fix, for every q, an isomorphism on $X_{\text{ét}}$ between $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}(q)$. Define

$$\mathcal{M}_1^q := i^*\mathrm{R}^q j_* \left(\mathbb{Z}/p\mathbb{Z}(1)\right)$$

where i and j denote the inclusion of the generic and special fibre (X and Y, respectively) in $\mathcal{X}$, cf. Section 1.3.

Remark 3.1.7. The main reference for this section is [BK86]. In [BK86] Bloch and Kato link a "twisted" version of the sheaves $\mathcal{M}_1^q$ with the sheaf of logarithmic forms on Y. In particular, they work with the sheaves $M_1^q := i^* R^q j_* \big(\mathbb{Z}/p\mathbb{Z}(q)\big)$. Since for every q we fixed an isomorphism between $\mathbb{Z}/p\mathbb{Z}(1)$ and $\mathbb{Z}/p\mathbb{Z}(q)$, the same results apply to the sheaves $\mathcal{M}_1^q$.

The sheaves $\mathcal{M}_1^q$ build the spectral sequence of vanishing cycles [BK86, 0.2]:

$$E_2^{s,t} := \mathrm{H}^s(Y, \mathcal{M}_1^t) \Rightarrow \mathrm{H}^{s+t}\big(X, \mathbb{Z}/p\mathbb{Z}(1)\big)\,.$$

By comparing the spectral sequence of vanishing cycles for X and for K^h (which we recall to be defined as the fraction field of the henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},Y}$) we get the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^2\big(X, \mathbb{Z}/p\mathbb{Z}(1)\big) & \xrightarrow{\quad f \quad} & \mathrm{H}^0(Y, \mathcal{M}_1^2) \\ \downarrow & & \downarrow g \\ \mathrm{Br}(K^h)[p] & \xrightarrow{\ \mathrm{res}\ } & \mathrm{H}^0(K^h, \mathrm{Br}(K^h_{nr})[p]). \end{array}$$

In [BN23, Lemma 3.4] it is proven that the map g is injective. The map f is defined as the composition of the projection $\mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}(1)) \twoheadrightarrow E_\infty^{0,2}$ with inclusion map $E_\infty^{0,2} \hookrightarrow E_2^{0,2}$. Furthermore, the injectivity of g allows us to define a map $f_1\colon \mathrm{Br}(X)[p] \to \mathrm{H}^0(Y, \mathcal{M}_1^2)$ such that the diagram

$$\begin{array}{ccc} \mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}(1)) & \xrightarrow{\ f\ } & \mathrm{H}^0(Y, \mathcal{M}_1^2) \\ \downarrow & \nearrow_{f_1} & \\ \mathrm{Br}(X)[p] & & \end{array}$$

commutes, where the vertical arrow comes from the Kummer exact sequence [CTS21, Section 1.3.4]. In fact, every element in $\mathrm{H}^2\big(X, \mathbb{Z}/p\mathbb{Z}(1)\big)$ coming from an element $\delta \in \mathrm{Pic}(X)$ is sent to 0 by f, since its image in $\mathrm{Br}(K^h)[p]$ comes from an element in $\mathrm{Pic}(K^h)$, which is trivial by Hilbert's theorem 90 [CTS21, Theorem 1.3.2].

By [Kat89, Proposition 6.1] we also know that $\ker(\mathrm{res}) = \mathrm{fil}_0\mathrm{Br}(K^h)[p]$, hence we have the diagram

$$\begin{array}{ccccc} \mathrm{fil}_0\mathrm{Br}(X)[p] & \longrightarrow & \mathrm{Br}(X)[p] & \xrightarrow{\quad f_1 \quad} & \mathrm{H}^0(Y, \mathcal{M}_1^2) \\ \downarrow & & \downarrow & & \downarrow g \\ \mathrm{fil}_0\mathrm{Br}(K^h)[p] & \longrightarrow & \mathrm{Br}(K^h)[p] & \xrightarrow{\ \mathrm{res}\ } & \mathrm{H}^0(K^h, \mathrm{Br}(K^h_{nr})[p]). \end{array} \tag{3.1}$$

Note that we can build a diagram analogous to (3.1) for every finite field extension L'/L.

We are left to prove the existence (over a field extension L' of L) of an element $\mathcal{A}$ in $\mathrm{Br}(X_{L'})[p]$ such that $f_{L'}(\mathcal{A}) \neq 0$. In order to do this, we analyse the spectral sequence of vanishing cycles over an algebraic closure $\bar{L}$ of L. Let Λ be the integral closure of $\mathcal{O}_L$ in $\bar{L}$ and $\bar{\ell}$ the residue field of Λ. Let $\bar{X}$, $\bar{\mathcal{X}}$ and $\bar{Y}$ be the base change of $X, \mathcal{X}$ and Y to $\bar{L}, \Lambda$ and $\bar{\ell}$ respectively. We define $\bar{\mathcal{M}}_1^q := \bar{i}^* \mathrm{R}^q \bar{j}_* \left(\mathbb{Z}/p\mathbb{Z}(1)\right)$, where $\bar{i}$ and $\bar{j}$ denote the inclusion of the generic and special fibre ($\bar{X}$ and $\bar{Y}$, respectively) in $\bar{\mathcal{X}}$. These sheaves build a spectral sequence of vanishing cycles:

$$\bar{E}_2^{s,t} := \mathrm{H}^s(\bar{Y}, \bar{\mathcal{M}}_1^t) \Rightarrow \mathrm{H}^{s+t}\left(\bar{X}, \mathbb{Z}/p\mathbb{Z}(1)\right).$$

We are interested in the map $\bar{f}_1 \colon \mathrm{Br}(\bar{X})[p] \to \bar{E}_\infty^{0,2}$, defined in the same way as the map f_1.

Combining [BK86, Theorem 8.1] with the short exact sequence (8.0.1) and Proposition 7.3 of [BK86] we get that if Y is ordinary, then

$$\mathrm{H}^n(\bar{Y}, \bar{\mathcal{M}}_1^q) \simeq \mathrm{H}^n(\bar{Y}, \Omega^q_{\bar{Y},\log}) \text{ and } \mathrm{H}^n(\bar{Y}, \Omega^q_{\bar{Y},\log}) \otimes_{\mathbb{F}_p} \bar{\ell} \xrightarrow{\sim} \mathrm{H}^n(\bar{Y}, \Omega^q_{\bar{Y}}). \tag{3.2}$$

for every q, n. From equation (3.2) $\bar{E}_2^{2,1} = \mathrm{H}^2(\bar{Y}, \bar{\mathcal{M}}_1^1) \simeq \mathrm{H}^2(\bar{Y}, \Omega^1_{\bar{Y},\log}) = 0$, where the last equality follows from the fact that for K3 surfaces $\mathrm{H}^2(\bar{Y}, \Omega^1_{\bar{Y}}) = 0$. Moreover, since for K3 surfaces $\mathrm{H}^3(\bar{Y}, \Omega^0_{\bar{Y}}) = 0$, we get from equation (3.2) that also $\bar{E}_2^{3,0}$ vanishes. Thus,

$$\bar{E}_\infty^{0,2} = \ker(\bar{E}_2^{0,2} \to \bar{E}_2^{2,1}) = \bar{E}_2^{0,2}$$

and hence (by construction) the map $\bar{f}_1 \colon \mathrm{Br}(\bar{X})[p] \to \mathrm{H}^0(\bar{Y}, \bar{\mathcal{M}}_1^2)$ is surjective. Finally, using again equation (3.2) we get that $\mathrm{H}^0(\bar{Y}, \bar{\mathcal{M}}_1^2)$ is not trivial.

Proof of Theorem 3.1.6. Let V be a K3 surface over a number field k that contains a primitive p-root of unity ζ and let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Let L be the p-adic field $k_\mathfrak{p}$ and X be the base change of V to L. Let $\mathcal{A} \in \mathrm{Br}(\bar{X})[p]$ be such that $\bar{f}_1(\mathcal{A}) \neq 0$, then there is a finite field extension L'/L such that $\mathcal{A}$ is defined over L' (i.e. $\mathcal{A} \in \mathrm{Br}(X_{L'})[p]$) and $f_{1,L'}(\mathcal{A}) \neq 0$. Namely, by what we said above $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X_{L'})[p]$.

Since $\mathrm{Br}(\bar{V})[p] \simeq \mathrm{Br}(V \times_k \overline{k_\mathfrak{p}})[p]$ there exists $\mathcal{A} \in \mathrm{Br}(\bar{V})$ such that $\bar{f}_1(\mathcal{A}) \neq 0$. Hence, if k'/k is a finite field extension such that $\mathcal{A}$ is defined over k', then we know from what we have showed above that there is a prime $\mathfrak{p}'$ above $\mathfrak{p}$ such that $\mathrm{ev}_\mathcal{A}$ is non-constant on $X_{k'}(k'_{\mathfrak{p}'})$. □

3.2 Non-ordinary good reduction

In this section we analyse also what happens, in the case of K3 surfaces, for primes of good non-ordinary reduction.

Theorem 3.2.1. *Let V be a K3 surface and $\mathfrak{p}$ be a prime of good non-ordinary reduction for V with $e_\mathfrak{p} \leq (p-1)$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Like in the previous section, let X be the base change to $k_\mathfrak{p}$ of V, then we denote by $\mathcal{X}$ the base change of $\mathcal{V}$ to $\mathcal{O}_\mathfrak{p}$ and by Y the special fibre $\mathcal{V}(\mathfrak{p})$. As already explained before it is enough to show that for all element $\mathcal{A} \in \mathrm{Br}(X)$ the evaluation map $\mathrm{ev}_\mathcal{A} \colon X(k_\mathfrak{p}) \to \mathbb{Q}/\mathbb{Z}$ is constant.

3.2.1 De Rham–Witt complex and logarithmic forms

The aim of this section is to briefly introduce the de Rham–Witt complex $W\Omega_Y^\bullet$ of a scheme Y over a perfect field ℓ of characteristic p and see how the ordinary condition can be read from it. We will not go into the details of the de Rham–Witt complex nor of crystalline cohomology, we will just use them to study the global sections the sheaf of logarithmic 2-forms on Y.

In [Ill79], Illusie defines the de Rham–Witt complex of Y:

$$W\Omega_Y^\bullet\colon\quad W\Omega_Y^0 = W\mathcal{O}_Y \xrightarrow{d} W\Omega_Y^1 \xrightarrow{d} \dots \xrightarrow{d} W\Omega_Y^q \xrightarrow{d} \dots .$$

To define the de Rham–Witt complex, Illusie begins by building a projective system $\{W_m\Omega_Y^q\}_{m\geq 0}$ equipped with the Frobenius and the Verschiebung maps

$$F\colon W_{m+1}\Omega_Y^q \to W_m\Omega_Y^q \quad \text{and} \quad V\colon W_m\Omega_Y^q \to W_{m+1}\Omega_Y^q$$

for every $m \geq 0$. This projective system has transition maps given by the projection maps $R\colon W_{m+1}\Omega_Y^j \to W_m\Omega_Y^j$. The sheaves $W\Omega_Y^q$ are defined as the inverse limit of the projective system $\{W_m\Omega_Y^q\}_{m\geq 0}$, which is the initial object in a suitable category, see [Ill79, Chapter I]. We will state here some properties of the de Rham–Witt complex, without going into the details.

a. For every non-negative i, j, let $\mathrm{H}^i(Y, W\Omega_Y^j)$ be the corresponding de Rham–Witt cohomology group. The de Rham–Witt cohomology groups are finitely generated W-modules modulo torsion, where $W = W(\ell)$ is the ring of Witt vectors of ℓ.

b. The de Rham–Witt cohomology groups are strictly related to the crystalline cohomology groups of Y (see [BO78] for the definition of the crystalline site and crystalline cohomology). More precisely, there is a spectral sequence, called the the slope spectral sequence:

$$E_1^{i,j} := \mathrm{H}^j(Y, W\Omega_Y^i) \Longrightarrow H^{i+j}_{\mathrm{cris}}(Y/W)$$

and the following degeneracy results [Ill79]:

- the slope spectral sequence always degenerates modulo torsion at E_1;
- the slope spectral sequence degenerates at E_1 if and only if $\mathrm{H}^j(Y, W\Omega_Y^i)$ is a finitely generated W-module for all i, j.

Let σ be the lift of the Frobenius of ℓ to W. The complex $W\Omega_Y^\bullet$ is endowed with a natural σ-semilinear endomorphism $F\colon W\Omega_Y^\bullet \to W\Omega_Y^\bullet$, induced by the Frobenius $F\colon W_{m+1}\Omega_Y^q \to W_m\Omega_Y^q$. Ordinary varieties (see Definition 3.1.2) can also be defined in terms of the action of F on the cohomology groups $\mathrm{H}^j(Y, W\Omega_Y^i)$.

Lemma 3.2.2. *A proper smooth variety Y is ordinary if and only if*

$$F\colon \mathrm{H}^j(Y, W\Omega_Y^i) \to \mathrm{H}^j(Y, W\Omega_Y^i)$$

is bijective for all i, j.

Proof. See [BK86, Section 7]. □

In order to understand how being ordinary is related to logarithmic forms on Y, we first need to introduce the logarithmic Hodge–Witt sheaf.

Definition 3.2.3. *Given two positive integers i, m, the* ***logarithmic Hodge–Witt sheaf*** $W_m\Omega^i_{X,\log}$ *is defined as*

$$W_m\Omega^i_{X,\log} := \mathrm{im}(s\colon (\mathcal{O}_X^\times)^{\otimes i} \to W_m\Omega^i_X),$$

where s is defined by

$$s(x_1 \otimes \cdots \otimes x_i) := d\log \underline{x}_1 \wedge \cdots \wedge d\log \underline{x}_i$$

with $\underline{x} = (x, 0, \ldots, 0) \in W_m\mathcal{O}_X$ the Teichmüller representative of $x \in \mathcal{O}_X$. The system $\{W_m\Omega^i_{Y,\log}\}_{m\geq 0}$ is a sub-system of $\{W_m\Omega^i_Y\}_{m\geq 0}$. Moreover, for every i, q

$$\mathrm{H}^i(Y, W\Omega^q_{Y,\log}) := \varprojlim_m \mathrm{H}^i(Y, W_m\Omega^q_{Y,\log})$$

where the limit is take over the projection maps $R\colon W_{m+1}\Omega^j_{Y,\log} \to W_m\Omega^j_{Y,\log}$.

3.2.2 Mittag-Leffler systems

Assume Y to be smooth and proper over ℓ. A crucial fact that we will use several time in the next section is that for every i, j the inverse systems of abelian groups $\{\mathrm{H}^i(Y, W_m\Omega^j_Y)\}_{m\geq 0}$ and $\{\mathrm{H}^i(Y, W_m\Omega^j_{Y,\log})\}_{m\geq 0}$ both satisfy the Mittag-Leffler (ML) condition [IR83, Corollary 3.5, page 194], [Ill79, proof of Proposition 2.1, page 607] and

$$\mathrm{H}^i(Y, W\Omega^j_Y) = \varprojlim_m \mathrm{H}^i(Y, W_m\Omega^j_Y) \quad \text{[Ill79, Proposition 2.1].}$$

The aim of this section is to briefly recap what Mittag-Leffler systems (ML) are and prove some auxiliary lemmas that we will need afterwards. The main reference for this section is [Sta, 02MY].

Let $\mathcal{C}$ be an abelian category. An inverse system consists of a pair (A_i, φ_{ji}), where for every $i \in \mathbb{N}$, $A_i \in \mathrm{Ob}(\mathcal{C})$ and for each $j > i$ there is a map $\varphi_{ji}\colon A_j \to A_i$ such that $\varphi_{ji} \circ \varphi_{kj} = \varphi_{ki}$ for all $k > j > i$. We will often omit the transition maps from the notation. We will denote by $\varprojlim_i A_i$ the inverse limit of the system (A_i); this limit always exists and can be described as follow

$$\varprojlim_i A_i = \left\{ (a_i) \in \prod_i A_i \;\middle|\; \varphi_{i+1,i}(a_{i+1}) = a_i,\ i = 0, 1, 2, \ldots \right\}.$$

See [Sta, 02MY] for more details.

Definition 3.2.4. *We say that (A_i) satisfies the* ***Mittag-Leffler (ML)*** *condition if for every i there is a $c = c(i) \geq i$ such that*

$$\mathrm{Im}(A_k \xrightarrow{\varphi_{ki}} A_i) = \mathrm{Im}(A_c \xrightarrow{\varphi_{ci}} A_i)$$

for all $k \geq c$.

Lemma 3.2.5. *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups.

(1) *If (B_i) is (ML), then also (C_i) is ML.*

(2) *If (A_i) is (ML), then*

$$0 \to \varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i \to 0$$

is exact.

Proof. This is part (2) and (3) of [Sta, 02N1]. □

Lemma 3.2.6. *Let (A_i, φ_{ji}) and (B_i, ψ_{ji}) be two inverse systems of abelian groups. Assume that there are maps of inverse systems $(\pi_i)\colon (A_i) \to (B_i)$ and $(\lambda_i)\colon (B_i) \to (A_{i-1})$ such that $\varphi_{i(i-1)} = \lambda_i \circ \pi_i$ and $\psi_{i(i-1)} = \pi_{i-1} \circ \lambda_i$. Then both (π_i) and (λ_i) induce an isomorphism between $\varprojlim_i A_i$ and $\varprojlim_i B_i$. Moreover, (A_i) is (ML) if and only if (B_i) is (ML).*

Proof. Given an inverse system (A_i) and a positive integer n, we denote by $(A[n]_i)$ the inverse system given by

$$A[n]_i = A_{i-n} \text{ and } \varphi[n]_{(i+1)i} = \varphi_{(i-n+1)(i-n)}$$

whenever $i > n$ and the trivial group otherwise.

The inverse limits of the systems $A[n]_i$ and $A[m]_i$ are isomorphic. In fact, without loss of generality, we can assume $m > n$. The transition maps $\varphi_{(i-n)(i-m)}$ from $A[n]_i$ to $A_i[m]$ induce a map between inverse systems

$$(\varphi[n,m]_i)\colon (A[n]_i) \to (A[m]_i).$$

This map induces an isomorphism $\varphi[n,m]$ between $\varprojlim_i A[n]_i$ and $\varprojlim_i A[m]_i$ sending an element $(a_{i-n}) \in \varprojlim_i A[n]_i$ to $(\varphi_{i-n,i-m}(a_i)) = (a_{i-m}) \in \varprojlim_i A[m]_i$.

Similarly, for every n, m with $m > n$ we can build maps

$$\psi[n,m]\colon \varprojlim_i B[n]_i \to \varprojlim_i B[m]_i$$
$$\lambda[n]\colon \varprojlim_i B[n]_i \to \varprojlim_i A[n+1]_i$$
$$\pi[n]\colon \varprojlim_i A[n]_i \to \varprojlim B[n]_i$$

that fit in following commutative diagram

$$\begin{array}{ccccc}
\varprojlim_i A[1]_i & \xrightarrow{\varphi[1,2]} & \varprojlim_i A[2]_i & \xrightarrow{\varphi[2,3]} & \varprojlim_i A[3]_i \\
\uparrow{\scriptstyle \lambda[0]} & \searrow^{\pi[1]} & \uparrow{\scriptstyle \lambda[1]} & \searrow^{\pi[2]} & \uparrow{\scriptstyle \lambda[2]} \\
\varprojlim_i B_i & \xrightarrow[\psi[0,1]]{} & \varprojlim_i B[1]_i & \xrightarrow[\psi[1,2]]{} & \varprojlim_i B[2]_i.
\end{array}$$

As the horizontal maps are isomorphism, the maps (π_i) and $(\lambda_{(i+1)i})$ induce the desired isomorphisms.

Assume now that (A_i) is (ML). By definition for every i, there is a $c = c(i)$ such that for all $k \geq c$

$$\mathrm{Im}(A_k \xrightarrow{\varphi_{ki}} A_i) = \mathrm{Im}(A_c \xrightarrow{\varphi_{ci}} A_i).$$

Assume that $k \geq c+1$. Since (B_i) is an inverse system, $\mathrm{Im}(\psi_{ki}) \subseteq \mathrm{Im}(\psi_{(c+1)i})$ always.

At the same time, from the following commutative diagram

$$\begin{array}{ccccc}
A_k & \xrightarrow{\varphi_{kc}} & A_c & \xrightarrow{\varphi_{ci}} & A_i \\
\downarrow{\scriptstyle \pi_k} & & \uparrow{\scriptstyle \lambda_{c+1}} & & \downarrow{\scriptstyle \pi_i} \\
B_k & \xrightarrow[\psi_{k(c+1)}]{} & B_{c+1} & \xrightarrow[\psi_{(c+1)i}]{} & B_i
\end{array}$$

we get that for every $b \in B_{c+1}$

$$\psi_{(c+1)i}(b) = \pi_i \circ \varphi_{ci} \circ \lambda_{c+1}(b).$$

Using that the system (A_i) is (ML) we get that there is an element a in A_k such that $\varphi_{ki}(a) = \varphi_{ci} \circ \lambda_{c+1}(b)$. Putting everything together we get that

$$\psi_{ki}(\pi_k(a)) = \psi_{(c+1)i} \circ \psi_{k(c+1)}(\pi_k(a)) = \pi_i \circ \varphi_{ki}(a) = \pi_i \circ \varphi_{(c)i} \circ \lambda_{c+1}(b) = \psi_{(c+1)i}(b).$$

Hence, for all $k \geq c+1$ every element in the image of $\psi_{(c+1)i}$ lies also in the image of ψ_{ki}, namely (B_i) is (ML). Finally, (B_i) being (ML) implies that (A_i) is (ML) as well just follows from replacing (B_i) with $(B[1]_i)$ and switching the roles of (A_i) and (B_i) in what we just proved. $\square$

Lemma 3.2.7. *Let*

$$0 \to (A_{1,i}) \to (A_{2,i}) \to \dots$$

be a long exact sequence of inverse systems of abelian groups. Assume that for every n, the inverse system $(A_{n,i})$ is (ML). Then we have an induced long exact sequence of abelian groups

$$0 \to \varprojlim_i A_{1,i} \to \varprojlim_i A_{2,i} \to \dots$$

Proof. For every n, we define $B_{n,i} := \mathrm{Im}(A_{n-1,i} \to A_{n,i}) = \mathrm{Ker}(A_{n,i} \to A_{n+1,i})$. For every n we get short exact sequences

$$0 \to (B_{n,i}) \to (A_{n,i}) \to (B_{n+1,i}) \to 0.$$

From Lemma 3.2.5(1) together with the assumption on the $(A_{n,i})$'s we get that, for every n, $(B_{n,i})$ is (ML). Using Lemma 3.2.5(2) we get, for every n, a short exact sequence of abelian groups

$$0 \to \varprojlim_i B_{n,i} \to \varprojlim_i A_{n,i} \to \varprojlim_i B_{n+1,i} \to 0.$$

The result now follows from putting all these short exact sequences together. □

3.2.3 Global logarithmic 2-forms

We are now ready to go back to the inverse systems given by the cohomology groups associated to $\{W_m\Omega^q_Y\}_{m\geq 0}$ and $\{W_m\Omega^q_{Y,\log}\}_{m\geq 0}$

Lemma 3.2.8. *For every positive integer i there is a long exact sequence in cohomology*

$$\begin{array}{ccccc} \mathrm{H}^0(Y, W\Omega^i_{Y,\log}) & \hookrightarrow & \mathrm{H}^0(Y, W\Omega^i_Y) & \xrightarrow{R-F} & \mathrm{H}^0(Y, W\Omega^i_Y) \\ \mathrm{H}^1(Y, W\Omega^i_{Y,\log}) & \longrightarrow & \mathrm{H}^1(Y, W\Omega^i_Y) & \xrightarrow{R-F} & \mathrm{H}^1(Y, W\Omega^i_Y) \\ \mathrm{H}^2(Y, W\Omega^i_{Y,\log}) & \longrightarrow & \dots . & & \end{array}$$

Proof. Following Illusie [Ill79, page 567] we define for every non-negative m

$$\mathrm{Fil}^m W_{m+1}\Omega^i_Y := \ker(R\colon W_{m+1}\Omega^i_Y \to W_m\Omega^i_Y).$$

It is proven in [Ill79, Proposition 3.2, page 568] that $\mathrm{Fil}^m W_{m+1}\Omega^i_Y = V^m\Omega^i_Y + dV^m\Omega^{i-1}_Y$ and hence

$$F(\mathrm{Fil}^m W_{m+1}\Omega^i_Y) = dV^{m-1}\Omega^{i-1}_Y$$

where the last equality follows from the fact that $FV = p$ and $FdV = d$, see [Ill79, Chapter I, page 541].[1] Therefore, the map $F\colon W_{m+1}\Omega^i_Y \to W_m\Omega^i_Y$ induces a map

$$F'\colon W_m\Omega^i_Y \to W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y.$$

In more details: the map R_{m+1} induces an isomorphism between $W_m\Omega^i_Y$ and $W_{m+1}\Omega^i_Y/\ker(R_{m+1})$. We can therefore look at the map induced by F from

[1] In fact, $FV = VF = p$, implies $FV^m = pV^{m-1} = V^{m-1}p$ and $p \cdot \Omega^i_Y = 0$.

$W_{m+1}\Omega^i_Y/\ker(R_{m+1})$ to $W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y$. We denote by π_m the quotient map from $W_m\Omega^i_Y$ to $W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y$ and by $\tilde{\pi}_{m+1}$ the projection from $W_{m+1}\Omega^i_Y$ to $W_{m+1}\Omega^i_Y/\ker(R_{m+1})$. We have the following commutative diagram

$$\begin{array}{ccccc} & & W_{m+1}\Omega^i_Y & & \\ & \swarrow^{R_{m+1}} & \downarrow{\scriptstyle \tilde{\pi}_{m+1}} & \searrow^{F} & \\ W_m\Omega^i_Y & \xrightarrow{\bar{R}^{-1}_{m+1}} & W_{m+1}\Omega^i_Y/\ker(R_{m+1}) & & W_m\Omega^i_Y \\ & \searrow_{F'} & \downarrow{\scriptstyle \bar{F}} & \swarrow_{\pi_m} & \\ & & W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y & & \end{array}$$

Note that, in particular

$$(\pi_m - F') \circ R_{m+1} = \pi_m \circ (R_{m+1} - F). \tag{3.3}$$

By [CTSS83, Lemme 2] for every non-negative integer m there is a short exact sequence

$$0 \to W_m\Omega^i_{Y,\log} \to W_m\Omega^i_Y \xrightarrow{\pi_m - F'} W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y \to 0.$$

These sequences (as m varies) induce long exact sequences of inverse systems of abelian groups

$$\dots \longrightarrow \{\mathrm{H}^j(Y, W_m\Omega^i_Y)\}_{m\geq 0} \xrightarrow{\pi - F'} \{\mathrm{H}^j(Y, W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y)\}_{m\geq 0} \longrightarrow \{\mathrm{H}^{j+1}(Y, W_m\Omega^i_{Y,\log})\}_{m\geq 0} \longrightarrow \dots$$

The projection map $R_m \colon W_m\Omega^i_Y \to W_{m-1}\Omega^i_Y$ factors as

$$W_m\Omega^i_Y \xrightarrow{\pi_m} W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y \xrightarrow{\lambda_m} W_{m-1}\Omega^i_Y,$$

see [Ill79, Proposition 3.2, page 568]. Hence, by Lemma 3.2.6 also the inverse system given by $\{\mathrm{H}^j(Y, W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y)\}_{m\geq 0}$ satisfies the Mittag–Leffler condition and π_m induces an isomorphism π

$$\mathrm{H}^j(Y, W\Omega^i_Y) \simeq \varprojlim_m \mathrm{H}^j(Y, W_m\Omega^i_Y/dV^{m-1}\Omega^{i-1}_Y).$$

Finally, in order to conclude the proof it is enough to apply Lemma 3.2.7 and to notice that equation (3.3) implies that the isomorphism R induced by the transition maps of the system $\{\mathrm{H}^j(Y, W_m\Omega^i_Y)\}_{m\geq 0}$

$$R\colon \varprojlim_m \mathrm{H}^j(Y, W_{m+1}\Omega^i_Y) \xrightarrow{\sim} \varprojlim_m \mathrm{H}^j(Y, W_m\Omega^i_Y)$$

is such that $(\pi - F') \circ R = \pi \circ (R - F)$. $\square$

The following lemma gives a way to relate the $\mathbb{Z}_p$-module $\mathrm{H}^j(Y, W\Omega^i_{Y,\log})$ to the $\mathbb{Z}/p^m\mathbb{Z}$-modules $\mathrm{H}^j(Y, W_m\Omega^i_{Y,\log})$.

Lemma 3.2.9. *For every positive integer n we have a long exact sequence in cohomology*

$$\begin{array}{ccccc} \mathrm{H}^0(Y, W\Omega^i_{Y,\log}) & \xhookrightarrow{p^m} & \mathrm{H}^0(Y, W\Omega^i_{Y,\log}) & \longrightarrow & \mathrm{H}^0(Y, W_m\Omega^i_{Y,\log}) \\ \mathrm{H}^1(Y, W\Omega^i_{Y,\log}) & \xrightarrow{p^m} & \mathrm{H}^1(Y, W\Omega^i_{Y,\log}) & \longrightarrow & \mathrm{H}^1(Y, W_m\Omega^i_{Y,\log}) \\ \mathrm{H}^2(Y, W\Omega^i_{Y,\log}) & \xrightarrow{p^m} & \dots & & \end{array}$$

Proof. In [CTSS83, Lemme 3] the authors show that there is, for every positive m, a short exact sequence of sheaves

$$0 \to W_n\Omega^i_{\log} \xrightarrow{p^m} W_{n+m}\Omega^i_{\log} \to W_m\Omega^i_{\log} \to 0$$

Again, the result follows from taking the long exact sequence of inverse systems of abelian groups in cohomology, then using that the systems involved satisfy the Mittag–Leffler condition and hence applying Lemma 3.2.7. □

Proposition 3.2.10. *Let Y be a smooth, proper variety over a a perfect field ℓ of positive characteristic. Assume that $\mathrm{H}^0(Y, W\Omega^q_Y) = 0$ and $\mathrm{H}^1(Y, W\Omega^q_Y)$ is torsion-free. Then for all $m \geq 1$*

$$\mathrm{H}^0(Y, W_m\Omega^q_{Y,\log}) = 0.$$

Proof. The long exact sequence in cohomology of Lemma 3.2.8 together with the vanishing of $\mathrm{H}^0(Y, W\Omega^q_Y)$ imply that

$$\mathrm{H}^0(Y, W\Omega^q_{Y,\log}) = 0 \quad \text{and} \quad \mathrm{H}^1(Y, W\Omega^q_{Y,\log}) \subseteq \mathrm{H}^1(Y, W\Omega^q_Y).$$

In particular, $\mathrm{H}^1(Y, W\Omega^q_{Y,\log})$ is torsion-free as well. Finally, we can use the long exact sequence of Lemma 3.2.9 to get that

$$\mathrm{H}^0(Y, W_m\Omega^q_{Y,\log}) = \ker(\mathrm{H}^1(Y, W\Omega^q_{Y,\log}) \xrightarrow{p^m} \mathrm{H}^1(Y, W\Omega^q_{Y,\log}))$$

which proves the result. □

The idea is to use the lemma we just proved to show that for non-ordinary K3 surfaces over perfect fields there are no non-trivial global logarithmic 2-forms. We will see at the end of the section how a proof of this fact follows immediately from the following lemma and some results proven by Illusie [Ill79].

Lemma 3.2.11. *Assume that* $\mathrm{H}^0(Y, W\Omega^2_Y)$ *is trivial and* $\mathrm{H}^1(Y, W\Omega^2_Y)$ *is torsion free. Then*

$$\mathrm{H}^0(Y, \Omega^2_{Y,\log}) = 0.$$

If $\mathrm{H}^0(Y, \Omega^2_Y)$ *is a* 1*-dimensional* ℓ*-vector space, then the Cartier operator is zero on the space of global* 2*-forms.*

Proof. The first part is a direct consequence of Proposition 3.2.10 with $q = 2$.

If $\mathrm{H}^0(Y, \Omega^2_Y)$ is a 1-dimensional ℓ-vector space, for every $\omega \in \mathrm{H}^0(Y, \Omega^2_Y)$ there exists $\lambda \in \ell$ such that

$$C(\omega) = \lambda \cdot \omega.$$

Assume that $\lambda \neq 0$, then there exists a finite field extension ℓ'/ℓ that contains an element λ_0 such that $\lambda = \lambda_0^{p-1}$. In particular, we get

$$C(\lambda_0^p \cdot \omega) = \lambda_0 \cdot C(\omega) = (\lambda_0 \lambda) \cdot \omega = \lambda_0^p \cdot \omega.$$

This implies, by definition of logarithmic forms, that $\lambda_0^p \cdot \omega \in \mathrm{H}^0(Y_{\ell'}, \Omega^2_{Y_{\ell'},\log})$, which gives the desired contradiction (in fact the first part of the Lemma applies also to the base change of Y to ℓ'). ☐

We go back to the general setting of Section 1.3 and we work under the extra assumption that the special fibre Y has good non-ordinary reduction and no global 1-forms.

Lemma 3.2.12. *Assume that the special fibre* Y *of the* $\mathcal{O}_L$*-model* $\mathcal{X}$ *is such that there are no non-trivial global logarithmic* 2*-forms and* $\mathrm{H}^0(Y, \Omega^2_Y)$ *is a* 1*-dimensional* ℓ*-vector space. Then, if the absolute ramification index* $e \leq p - 1$*, we have* $\mathrm{Br}(X) = \mathrm{fil}_0\mathrm{Br}(X)$.

Proof. If $e < p-1$ the result is proven by Bright and Newton, [BN23, Lemma 11.2] without any assumption on the global section of the sheaves $\Omega^2_{Y,\log}$ and Ω^2_Y. If $e = p - 1$, let $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ with $\mathrm{rsw}_{n,\pi} = (\alpha, \beta)$. In order to prove the Lemma it is enough to show that $(\alpha, \beta) = (0, 0)$.

- If $n < e'$ then $p \nmid n$ and hence by Corollary 1.3.9(1) and the assumption that $\mathrm{H}^0(Y, \Omega^1_Y) = 0$ we get $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0, 0)$.
- If $n > e'$ then $p \nmid n$ or $p \nmid n - e$ and hence either $\mathrm{rsw}_{n,\pi}(\mathcal{A})$ or $\mathrm{rsw}_{n-e,\pi}(\mathcal{A}^{\otimes p})$ is zero and therefore by Corollary 1.3.9(2) $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0, 0)$.
- If $n = e'$, then by Corollary 1.3.9(3) we have

  $$\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha) - \bar{u}\alpha, C(\beta) - \bar{u}\beta).$$

  However, by assumption on Y there are no non-trivial global 1-forms, hence $\beta = 0$. Moreover, by Lemma 3.2.11 $C(\alpha) = 0$. Hence,

  $$\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = (-\bar{u}\alpha, 0).$$

  Finally, $e' - e < e'$ and hence $\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = 0$, which implies that $\alpha = 0$ and thus also in this case $\mathrm{rsw}_{e',\pi}(\mathcal{A}) = (0, 0)$.

□

In [Ill79, Chapter 7] Illusie shows that if Y is a non-ordinary K3 surface, then Y satisfies the assumptions of Lemma 3.2.11. Moreover, as already mentioned at the end of Section 4.3 from [Ier22, Proposition 2.3] we know that $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X)$ if and only if $\mathrm{ev}_{\mathcal{A}}$ is non-constant on $X(L)$. This shows how the ordinary assumption was needed in the example that we gave in Chapter 2, cf. Theorem 2.1.10.

CHAPTER 4

Computations with the refined Swan conductor

This final chapter is divided in two parts. In the first part we develop some techniques needed to compute the refined Swan conductor of certain elements in the Brauer group. In particular, in Section 4.1, starting from a result from Bright and Newton, we prove a formula that relates the refined Swan conductor with the extension of the base field over which the variety is defined. In Section 4.2 we explain some results of Kato that allows to compute the refined Swan conductor of p-order elements when the base field contains a primitive p-root of unity. In the second part of this final chapter we provide several examples and we use them to show that Theorem 3.1.1 and 3.2.1, are optimal. In particular, we exhibit K3 surfaces V over number fields such that:

(a) V has good ordinary reduction at a prime $\mathfrak{p}$ with ramification index $e_{\mathfrak{p}} = p-1$ and there is an element $\mathcal{A} \in \mathrm{Br}(V)[p]$ whose evaluation map is non-constant on $V(k_{\mathfrak{p}})$;

(b) V has good ordinary reduction at a prime $\mathfrak{p}$ with $e_{\mathfrak{p}} = p-1$ and $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation;

(c) V has good non-ordinary reduction at a prime $\mathfrak{p}$ with $e_{\mathfrak{p}} \geq p$ and there is an element $\mathcal{A} \in \mathrm{Br}(V)[p]$ whose evaluation map is non-constant on $V(k_{\mathfrak{p}})$.

More precisely: from (a) we get that the condition $(p-1) \nmid e_{\mathfrak{p}}$ in Theorem 3.1.1 is necessary; from (b) we get that the inverse of Theorem 3.1.1 does not hold in general; from (c) we get that the bound in Theorem 3.2.1 is optimal.

Finally, as already pointed out in the introduction, we recall that in all the examples of K3 surfaces in which a prime of good reduction plays a role in the Brauer–Manin obstruction of weak approximation, the corresponding element in the Brauer group is of **transcendental** nature, i.e. it does not belong to the algebraic Brauer group, which is defined as the kernel of the natural map from $\mathrm{Br}(V)$ to $\mathrm{Br}(\bar{V})$, where $\bar{V}$ is the base change of V to an algebraic closure of k (cf. Lemma 4.3.4).

Moreover, we prove that if V is a Kummer K3 surface coming from a product of elliptic curves defined over $\mathbb{Q}$ with good ordinary reduction at the prime 2 and full 2-torsion defined over $\mathbb{Q}_2$, then $\mathrm{Br}(V)[2] = \mathrm{Ev}_{-1}\mathrm{Br}(V)[2]$ (cf. Theorem 4.5.6). This theorem proves what was already predict by Ieronymou after some computational evidence, see [Ier23, Remark 2.6].

4.1 Refined Swan conductor and extension of the base field

Let L be a p-adic field with ring of integers $\mathcal{O}_L$, uniformiser π and residue field ℓ. Let X be a proper, smooth and geometrically integral L-variety having a smooth, proper model $\mathcal{X}$ with geometrically integral fibre. We denote by Y its special fibre:

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) & \longleftarrow & \mathrm{Spec}(\ell) \end{array} \tag{4.1}$$

In this section we want to analyse what happens to the refined Swan conductor when we take a field extension L'/L of the base field L. Bright and Newton prove the following result.

Lemma 4.1.1. *Let K'/K be a finite extension of Henselian discrete valuation fields of ramification index e. Let π' be a uniformiser in K', F' be the residue field of K' and define $\bar{a} \in F'$ to be the reduction of $\pi(\pi')^{-e}$. Let $\chi \in \mathrm{fil}_n\mathrm{Br}(K)$, and let*

$$\mathrm{res}\colon \mathrm{Br}(K) \to \mathrm{Br}(K')$$

be the restriction map. Then $\mathrm{res}(\chi) \in \mathrm{fil}_{en}\mathrm{Br}(K')$ and if $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta)$, then

$$\mathrm{rsw}_{en,\pi'}(\mathrm{res}(\chi)) = (\bar{a}^{-n}(\alpha + \beta \wedge d\log(\bar{a}), \bar{a}^{-n}e\beta).$$

Proof. See [BN23, Lemma 2.16]. □

The aim of this section is to use this result to prove the following Lemma.

Lemma 4.1.2 (Base change)**.** *Let L'/L be a finite field extension, with ramification index $e_{L'/L}$. Let π' be an uniformiser in L' and ℓ' its residue field. Let $\mathcal{A} \in \mathrm{Br}(X)$ and let*

$$\mathrm{res}\colon \mathrm{Br}(X) \to \mathrm{Br}(X_{L'})$$

be the restriction map. Then $\mathrm{res}(\mathcal{A}) \in \mathrm{fil}_{e_{L'/L}n}\mathrm{Br}(X_{L'})$ *and if* $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$ *with* $(\alpha, \beta) \in \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y)$*, then*

$$\mathrm{rsw}_{e_{L'/L}n,\pi'}(\mathrm{res}(\mathcal{A})) = (\bar{a}^{-n}\alpha, \bar{a}^{-n}e_{L'/L}\beta) \in \mathrm{H}^0(Y_{\ell'}, \Omega^2_{Y_{\ell'}}) \oplus \mathrm{H}^0(Y_{\ell'}, \Omega^1_{Y_{\ell'}})$$

with $\bar{a} \in \ell'$ *reduction of* $\pi(\pi')^{-e_{L'/L}}$*.*

The refined Swan conductor of an element $\mathcal{A} \in \mathrm{Br}(X)$ is defined through the refined Swan conductor of its image in the discrete henselian valuation field K^h. Namely, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{fil}_n\mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \\ \downarrow & & \downarrow \\ \mathrm{fil}_n\mathrm{Br}(K^h) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \Omega^2_F \oplus \Omega^1_F. \end{array}$$

We recall the construction of K^h: let η be the generic point of $Y \subseteq \mathcal{X}$, then we define R as henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},\eta}$ and K^h as the fraction field of R. The construction of $\mathcal{O}_{\mathcal{X},\eta}$ (and hence of K^h) is local on $\mathcal{X}$. From now on we will therefore assume $\mathcal{X} = \mathrm{Spec}(A)$, with A smooth $\mathcal{O}_L$-algebra, $Y = \mathrm{Spec}(A/\pi A)$; hence $\eta = (\pi) \in \mathrm{Spec}(A)$ and $\mathcal{O}_{\mathcal{X},\eta} = A_{(\pi)}$. We can re-write diagram (4.1) as:

$$\begin{array}{ccccc} A \otimes_{\mathcal{O}_L} L & \longleftarrow & A & \longrightarrow & A/\pi A \\ \uparrow & & \uparrow & & \uparrow \\ L & \longleftarrow & \mathcal{O}_L & \longrightarrow & \ell. \end{array} \tag{4.2}$$

Lemma 4.1.3. *The uniformiser* π *is also a uniformiser for* K^h*. Moreover,* $\mathrm{ord}_{K^h}(p) = \mathrm{ord}_L(p)$*.*

Proof. The uniformiser π is also the generator of the maximal ideal of $A_{(\pi)}$, hence of its henselisation R. The equality $(p) = (\pi)^e$ as ideals on $\mathcal{O}_L$ implies that $p \in (\pi)^e R$, hence $e_1 := \mathrm{ord}_{K^h}(p) \leq e$. The equality between the two orders follows from the fact that for every $m \geq 1$, $(\pi)^m R \cap \mathcal{O}_L = (\pi)^m \mathcal{O}_L$. □

We denote by L' a finite field extension of L, by $\mathcal{O}_{L'}$ its ring of integers with uniformiser π' and residue field ℓ'. Moreover, we denote by X', $\mathcal{X}'$ and Y' the base change of X, $\mathcal{X}$ and Y to $\mathrm{Spec}(L'), \mathrm{Spec}(\mathcal{O}_{L'})$ and $\mathrm{Spec}(\ell')$ respectively. Let $(K')^h$ be the fraction field of R', where R' is the henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X}',Y'}$. In this setting

$$\frac{A}{\pi A} \otimes_\ell \ell' = A \otimes_{\mathcal{O}_L} \ell \otimes_\ell \ell' = A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'} \otimes_{\mathcal{O}_{L'}} \ell' = \frac{A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}}{(1 \otimes \pi')}.$$

Thus, the generic point η' of Y' is the ideal generated by $1 \otimes \pi'$ and $\mathcal{O}_{\mathcal{X}',Y'}$ becomes the ring $(A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'})_{(1\otimes\pi')}$.

Lemma 4.1.4. *The field extension* $(K')^h/K^h$ *is finite with ramification index* $e_{L'/L}$*.*

Proof. We start by noticing that

$$\mathcal{O}_{\mathcal{X}',\eta'} \simeq \mathcal{O}_{\mathcal{X},\eta} \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}.$$

We have that $\mathcal{O}_{\mathcal{X},\eta} \otimes_{\mathcal{O}_L} \mathcal{O}_{L'} = S^{-1}(A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'})$, with $S = (A \setminus (\pi)) \cdot A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$, while $\mathcal{O}_{\mathcal{X}',\eta'} = T^{-1}(A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'})$, with $T = (A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}) \setminus (1 \otimes \pi')$. The isomorphism follows from the equality $(1 \otimes \pi')^{e_{L'/L}} = (1 \otimes \pi)$ together with the fact that $\mathcal{O}_{L'}$ is a free $\mathcal{O}_L$-module with basis $\{1, \pi', \dots, (\pi')^{e_{L'/L}}\}$.

As a second step we show that

$$R' \simeq R \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}.$$

The discrete valuation ring $\mathcal{O}_{L'}$ is a finite $\mathcal{O}_L$-module, hence we get that the natural map

$$R \to R \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$$

is finite; therefore [Sta, 05WS] implies that $R \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$ is henselian and therefore by [Sta, 05WP]

$$R' = R \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}.$$

As a final step, we notice that

$$(K')^h = R'\left[\frac{1}{\pi'}\right] = R'\left[\frac{1}{\pi}\right] = R\left[\frac{1}{\pi}\right] \otimes_L L' = K^h \otimes_L L'.$$

□

Proof of Lemma 4.1.2. It follows immediately from the previous lemma together with Lemma 4.1.1 and the fact that since $\bar{a} \in \ell'$, which is a finite field, $d\log(\bar{a}) = 0$. □

Corollary 4.1.5. *Assume that $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ for some $n \geq 1$ is such that $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$ with $\alpha \neq 0$, then $\mathcal{A} \notin \mathrm{Br}_1(X)$, i.e. $\mathcal{A}$ is a transcendental element in the Brauer group of X.*

Proof. Assume $\mathcal{A}$ to be in $\mathrm{Br}_1(X)$; then by definition of $\mathrm{Br}_1(X)$ there is a finite field extension L'/L such that $\mathrm{res}(\mathcal{A}) = 0$ in $\mathrm{Br}(X_{L'})$, where res is the restriction map from $\mathrm{Br}(X)$ to $\mathrm{Br}(X_{L'})$. Let $e_{L'/L}$ be the ramification index of the extension, π' be a uniformiser of L' and ℓ' its residue field. We know from Lemma 4.1.2 that

$$\mathrm{rsw}_{e(L'/L)n,\pi_{L'}}(\mathrm{res}(\mathcal{A})) = (\bar{a}^{-n} \cdot \alpha, \bar{a}^{-n} e_{L'/L} \cdot \beta)$$

where $\bar{a}^{-n} \in (\ell')^{\times}$. Hence, $\mathrm{rsw}_{e(L'/L)n,\pi_{L'}}(\mathrm{res}(\mathcal{A})) \neq (0,0)$ and therefore $\mathrm{res}(\mathcal{A})$ can not be the trivial element. □

4.2 Computations with the Refined Swan conductor on the p-torsion of the Brauer group

The aim of this section is to collect some results that will allow us to compute later in this chapter the Swan conductor and the refined Swan conductor of elements of order p in the Brauer group of K3 surfaces. In this section we work under the same setting as Section 1.2.3: K is a henselian field of characteristic 0 with ring of integers $\mathcal{O}_K$, uniformiser π and residue field F of positive characteristic p.

We work under the additional assumption that the field K contains a primitive p-root of unity ζ. It follows from the Merkurjev-Suslin Theorem [GS17, Theorem 8.6.5] that in this case $\mathrm{Br}(K)[p]$ is generated by the classes of **cyclic algebras**. In order to do computations with the refined Swan conductor, in this section we will introduce a new filtration on $\mathrm{Br}(K)[p]$, defined by Bloch and Kato in [BK86] and prove that via this filtration it is possible to compute the refined Swan conductor of cyclic algebras in $\mathrm{Br}(K)[p]$.

4.2.1 Cyclic algebras

Let $a, b \in K^\times$, then the K-algebra $(x, y)_p$ defined as

$$(x, y)_p := \langle a, b \mid a^p = x,\, b^p = y,\, ab = \zeta ba \rangle,$$

is a central simple algebra, see [GS17, Section 2.5] for more details. With abuse of notation we will denote by $(x, y)_p$ also the corresponding equivalence class in $\mathrm{Br}(K)[p]$. It is possible to realise $(x, y)_p$ also as the cup product of an element $\chi_x \in \mathrm{H}^1_p(K)$ with $\delta(y)$, where δ is the boundary map $K^\times \to \mathrm{H}^1(K, \mathbb{Z}/p\mathbb{Z}(1))$ coming from the Kummer sequence (a proof can be found in the proof of Proposition 4.7.1 [GS17]).

4.2.2 Another filtration

The map from $\mathbb{Z}/p\mathbb{Z}(1)$ to $\mathbb{Z}/p\mathbb{Z}(2)$ sending 1 to ζ induces an isomorphism

$$\mathrm{Br}(K)[p] \simeq \mathrm{H}^2\left(K, \mathbb{Z}/p\mathbb{Z}(2)\right) =: h^2(K). \tag{4.3}$$

For any two non-zero elements $x, y \in K$ we will denote by $\{x, y\} \in h^2(K)$ the cup product of $\delta(x)$ with $\delta(y)$.

Bloch and Kato [BK86] define a decreasing filtration $\{U^m h^2(K)\}_{m\geq 0}$ on $h^2(K)$ as follows: $U^0 h^2(K) = h^2(K)$ and for $m \geq 1$, $U^m h^2(K)$ is the subgroup of $h^2(K)$ generated by symbols of the form

$$\{1 + \pi^m x, y\}, \text{ with } x \in \mathcal{O}_K \text{ and } y \in K^\times.$$

In this section we are going to prove that for any $0 \leq m \leq e'$ the isomorphism (4.3) induces an isomorphism

$$U^m h^2(K) \simeq \mathrm{fil}_{e'-m}\mathrm{Br}(K)[p].$$

This isomorphism will be crucial in being able to compute the refined Swan conductor. In fact, in [BK86] Bloch and Kato describe the graded pieces of the filtration $\{U^m h^2(K^h)\}_{m\geq 0}$ on $h^2(K)$

$$\mathrm{gr}^m := \frac{U^m h^2(K^h)}{U^{m+1}h^2(K^h)}$$

in terms of differential forms on the residue field F. In [Kat89] Kato strongly relate them to the computation of the refined Swan conductor. We now state the two main results that show how it is possible to calculate the refined Swan conductor of elements of order p in $\mathrm{Br}(K)$.

Proposition 4.2.1. *We have the following description of the graded pieces* gr^m.

(1) $U^m h^2(K^h) = \{0\}$ *for* $m > e'$; $U^{e'}h^2(K)$ *coincides with the image of the injective map*

$$\begin{aligned}\lambda_\pi \colon \mathrm{H}^2_p(F) \oplus \mathrm{H}^1_p(F) &\to \mathrm{gr}^{e'} = h^2(K)\\ \delta_1\left[\bar{x}\cdot d\log\bar{y}\right] &\mapsto \{1+(\zeta-1)^p x, y\}\\ \delta_1\left[\bar{x}\right] &\mapsto \{1+(\zeta-1)^p x, \pi\}\end{aligned}$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K.

(2) Let $0 < m < e'$ *and* $p \nmid m$. *Then we have an isomorphism*

$$\begin{aligned}\rho_m \colon \Omega^1_F &\xrightarrow{\simeq} \mathrm{gr}^m\\ \bar{x}\cdot d\log\bar{y} &\mapsto \{1+\pi^m x, y\}\end{aligned}$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K.

(3) Let $0 < m < e'$ *with* $p \mid m$. *Then we have an isomorphism*

$$\begin{aligned}\rho_m \colon \Omega^1_F/Z^1_F \oplus \Omega^0_F/Z^0_F &\xrightarrow{\simeq} \mathrm{gr}^m\\ (\left[\bar{x}\cdot d\log\bar{y}\right], 0) &\mapsto \{1+\pi^m x, y\}\\ (0, [\bar{x}]) &\mapsto \{1+\pi^m x, \pi\}\end{aligned}$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K.

(4) We have an isomorphism

$$\begin{aligned}\rho_0 \colon \Omega^2_{F,\log} \oplus \Omega^1_{F,\log} &\xrightarrow{\simeq} \mathrm{gr}^0\\ (d\log\bar{y}_1 \wedge d\log\bar{y}_2, 0) &\mapsto \{y_1, y_2\}\\ (0, d\log\bar{y}) &\mapsto \{y, \pi\}\end{aligned}$$

where y, y_1 *and* y_2 *are any lifts of* $\bar{y}, \bar{y}_1$ *and* $\bar{y}_2$ *to* K.

Proof. See [BK86, Section 5]. More precisely, in [BK86, Lemma 5.1] Bloch and Kato prove 4.2.1.(1). However, instead of λ_π (cf. Section 1.2.2.1) they have the morphism $\rho_{e'}$, and they just prove afterwards [BK86, equation (5.15.1)] that $\rho_{e'}$ induces the map λ_π. Finally, in [BK86, Lemma 5.2 and 5.3] the rest of the proposition is proven. □

Note that since by assumption K contains a primitive p-root of unity ζ, the ramification index $e = \mathrm{ord}_K(p)$ is divisible by $(p-1)$ and hence $e' = ep(p-1)^{-1}$ is divisible by p. The key result of this section is the following proposition.

Proposition 4.2.2. *For every $0 \leq m \leq e'$ we have that the isomorphism 4.3 induces an isomorphism*

$$\mathrm{fil}_{e'-m}\mathrm{Br}(K)[p] \simeq U^m h^2(K).$$

Let $\bar{c}$ be the reduction modulo π of $c := \pi^{-e'}(\zeta-1)^p$. For $m < e'$ the compositions $\mathrm{rsw}_{e'-m} \circ \rho_m$ are as follows:

$$\begin{aligned} &\mathrm{rsw}_{e',\pi}(\rho_0(\alpha,\beta)) = (\bar{c}\alpha, \bar{c}\beta) && \\ &\mathrm{rsw}_{e'-m,\pi}(\rho_m(\alpha)) = (\bar{c}d\alpha, (e'-m)\bar{c}\beta), && \text{if } p \nmid m \\ &\mathrm{rsw}_{e'-m,\pi}(\rho_m(\alpha,\beta)) = (\bar{c}d\alpha, \bar{c}d\beta), && \text{if } p \mid m \end{aligned}$$

Warning: The proof of Proposition 4.2.2 is quite technical and will occupy the rest of this section.

4.2.3 Proof of Proposition 4.2.2

We divide the proof into several rather technical lemmas.

Lemma 4.2.3. *Let $a, b \in \mathcal{O}_K$ and n, m be non-negative integers, then the symbol $\{1+\pi^n a, 1+\pi^m b\}$ can be rewritten as*

$$-\left\{1+\pi^{n+m}\frac{ab}{1+\pi^n a}, 1+\pi^m b\right\} - \left\{1+\pi^{n+m}\frac{ab}{1+\pi^n a}, -\pi^n a\right\}.$$

In particular, it lies in $U^{m+n}h^2(K)$.

Proof. This lemma is a reformulation of a special case of [BK86, Lemma 4.1], for which no proof is provided. We have that

$$\begin{aligned} &\{1+\pi^n a, 1+\pi^m b\} + \left\{1+\pi^{n+m}\frac{ab}{1+\pi^n a}, 1+\pi^m b\right\} = \\ &\{1+\pi^n a(1+\pi^m b), 1+\pi^m b\}. \end{aligned}$$

We also have that,

$$\begin{aligned} &\left\{1+\pi^n a(1+\pi^m b), 1+\pi^m b\right\} + \left\{1+\pi^n a(1+\pi^m b), -\pi^n a\right\} = \\ &\left\{1+\pi^n a(1+\pi^m b), -(1+\pi^n b)\pi^n a\right\} = 0 \end{aligned}$$

where the last equality follows from the fact that $\{x, y\} = 0$ if $x + y = 1$. Finally, since $\{1+\pi^n a, -\pi^n a\} = 0$, we have

$$\begin{aligned} &\left\{1+\pi^n a(1+\pi^m b), -\pi^n a\right\} = \left\{1+\pi^n a(1+\pi^m b), -\pi^n a\right\} - \{1+\pi^n a, -\pi^n a\} = \\ &\left\{1+\pi^{n+m}\frac{ab}{1+\pi^n a}, -\pi^n a\right\}. \end{aligned}$$

□

Lemma 4.2.4. *Let $x \in \mathcal{O}_K$, $y \in K^\times$ and n, m be two positive integers; then $\{(y, 1+\pi^m x)_p, 1+\pi^n T\}$ can be written as*

$$-\left(\chi_y \cup \left\{1+\pi^{n+m}\frac{Tx}{1+\pi^n x}, 1+\pi^n T\right\} + \chi_y \cup \left\{1+\pi^{n+m}\frac{Tx}{1+\pi^n x}, -\pi^m x\right\}\right).$$

Proof. As already anticipated at the beginning of this section, we can write $(y, 1+\pi^m x)_p$ as the cup product of $\chi_y \in \mathrm{H}^1_p(K)$ and $\delta(1+\pi^m x)$ with δ boundary map coming from the Kummer sequence, see [GS17, proof of Proposition 4.7.1]. Hence,

$$\{(y, 1+\pi^m x)_p, 1+\pi^n T\} = \chi_y \cup \{1+\pi^m x, 1+\pi^n T\}.$$

The result now follows from Lemma 4.2.3. □

Corollary 4.2.5. *If $m > 0$ and $m + n \geq e'$, we get that*

$$\{(y, 1+\pi^m x)_p, 1+\pi^n T\} = -\chi_y \cup \left\{1+\pi^{n+m}\frac{Tx}{1+\pi^n x}, -\pi^m x\right\}.$$

Proof. If follows from the previous lemma together with the fact that by Lemma 4.2.3

$$\left\{1+\pi^{n+m}\frac{Tx}{1+\pi^n x}, 1+\pi^n T\right\} \in U^{2m+n}h^2(K)$$

and $U^{2m+n}h^2(K) = 0$ from Proposition 4.2.1, since $2m + n > e'$. □

Proposition 4.2.6. *Let $e = ord_K(p)$ and $e' = ep(p-1)^{-1}$. For $0 \leq m \leq e'$ the isomorphism of equation* (4.3) *induces an inclusion*

$$U^m h^2(K) \subseteq \mathrm{fil}_{e'-m}\mathrm{Br}(K)[p].$$

Proof. This is [Kat89, Lemma 4.3(1)], we include a proof of it here. By definition $U^m h^2(K)$ is generated by symbols of the form $\{1+\pi^m x, y\}$, with $x \in \mathcal{O}_K$ and $y \in K^\times$. In order to show that the corresponding element in $\mathrm{Br}(K)[p]$ lies in $\mathrm{fil}_{e'-m}$ we need to check that

$$\{1+\pi^m x, y, 1+\pi^{e'-m+1}T\} = 0.$$

From Corollary 4.2.5 we have that $\{1+\pi^m x, y, 1+\pi^{e'-m+1}T\}$ can be written as

$$\chi_y \cup \left\{1+\pi^{e'+1}\frac{xT}{1+\pi^m x}, -x\pi^m\right\} = \left\{(y, -x\pi^m)_p, 1+\pi^{e'+1}\frac{xT}{1+\pi^m x}\right\}$$

and the latter is zero, since we know that $\mathrm{fil}_{e'}\mathrm{Br}(K)[p] = \mathrm{Br}(K)[p]$, see Section 1.2.3.1. □

We will now show how the inclusion appearing in Proposition 4.2.6 is an equality. We start by recalling the following properties that we proved in Section 1.2.3.1.

Properties 4.2.7. For any $m \geq e'$ we have $\mathrm{fil}_m\mathrm{Br}(K)[p] = \mathrm{Br}(K)[p]$. Moreover,

(1) If $p \nmid m$, then the map

$$\mathrm{fil}_m \mathrm{Br}(K)[p] \xrightarrow{\mathrm{rsw}_{m,\pi}} \Omega^2_F \oplus \Omega^1_F \xrightarrow{\mathrm{pr}_2} \Omega^1_F$$

has also kernel equal to $\mathrm{fil}_{m-1}\mathrm{Br}(K)[p]$.

(2) If $p \mid m$ and $m < e'$, then the map $\mathrm{rsw}_{m,\pi}$ takes values in $B^2_F \oplus B^1_F$.

(3) If $m = e'$, then the map $\mathrm{mult}_{\bar{c}^{-1}} \circ \mathrm{rsw}_{e',\pi}$ takes values in $\Omega^2_{F,\log} \oplus \Omega^1_{F,\log}$, with $\bar{c}$ the reduction modulo π of $\pi^{e'} \cdot (\zeta-1)^{-p}$.

We proceed by induction on $m+1$.

- For $m = 0$ we have by definition $U^0h^2(k) = h^2(K)$, which implies that the inclusion $U^0h^2(K) \subseteq \mathrm{fil}_{e'}\mathrm{Br}(K)[p]$ is indeed an equality.

- For $m = 1$ we have

$$\Omega^2_{F,\log} \oplus \Omega^1_{F,\log} \simeq \mathrm{gr}^0 h^2(K) \subseteq \mathrm{gr}_{e'}\mathrm{Br}(K)[p] \hookrightarrow \Omega^2_{F,\log} \oplus \Omega^1_{F,\log} \tag{4.4}$$

where the first isomorphism is induced by ρ_0, while the inclusion follows from property 4.2.7(3). Moreover, note that given $\alpha = d\log\bar{x} \wedge d\log\bar{y} \in \Omega^2_{F,\log}$, $\rho_0(\alpha, 0) = \{x, y\}$ and

$$\{x, y, 1+\pi^{e'}T\} = \{x, y, 1+(\zeta-1)^p(cT)\} = \lambda_\pi\left(\delta_1\left[\bar{c}T d\log\bar{x} \wedge d\log\bar{y}\right], 0\right).$$

Similarly, if we start with $\beta = d\log\bar{y} \in \Omega^1_{F,\log}$, $\rho_0(0,\beta) = \{y, \pi\}$ and

$$\{y, \pi, 1+\pi^{e'}T\} = \{y, \pi, 1+(\zeta-1)^p(cT)\} = \lambda_\pi\left(0, \delta_1\left[\bar{c}T d\log\bar{y}\right]\right).$$

Hence, $\mathrm{rsw}_{e',\pi}(\rho_0(\alpha,\beta)) = \bar{c}(\alpha,\beta)$ and therefore the chain of maps (4.4) is the identity and we have that the inclusion of $U^1h^2(K)$ in $\mathrm{fil}_{e'-1}\mathrm{Br}(K)[p]$ is in fact an equality.

- Inductive step on $m+1$ when $p \nmid m$ and $m < e'$. In this case we have

$$\Omega^1_F \simeq \mathrm{gr}^m \subseteq \mathrm{gr}_{e'-m} \hookrightarrow \Omega^1_F$$

where the first isomorphism is induced by ρ_m, while the inclusion is the restriction of the refined Swan conductor to Ω^1_F, which is injective because of property 4.2.7(1). In this case, given $\alpha = \bar{x}d\log\bar{y} \in \Omega^1_F$, $\rho_m(\alpha) = \{1 + \pi^m x, y\}$. From corollary 4.2.5

$$\{1+\pi^m x, y, 1+\pi^{e'-m}T\} = -\chi_y \cup \left\{1+\pi^{e'}\frac{xT}{1+\pi^m x}, -x\pi^m\right\}$$

The latter can be rewritten as

$$-\chi_y \cup \left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x\right\} + m\cdot\chi_y \cup \left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, \pi\right\}.$$

which is equal (using isomorphism (4.3)) to

$$\left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x, y\right\} - m\left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, y, \pi\right\}. \tag{4.5}$$

Note that $\frac{x}{1+\pi^m x}$ is a possible lift of $\bar{x} \in F$ to K. Hence, (4.5) is equal to

$$\lambda_\pi\left(\bar{c}\bar{T}\bar{x}d\log\bar{x} \wedge d\log\bar{y}, m\bar{c}\bar{T}\bar{x}d\log\bar{y}\right) = \lambda_\pi\left(\bar{c}\bar{T}d\alpha, m\bar{c}\bar{T}\alpha\right).$$

Thus, in this case the composition

$$\Omega^1_F \simeq \mathrm{gr}^m \subseteq \mathrm{gr}_{e'-m} \hookrightarrow \Omega^1_F$$

sends α to $m\bar{c}\alpha$ and therefore we get again the equality between $U^m h^2(K)$ and $\mathrm{fil}_{e'-m}\mathrm{Br}(K)[p]$. Finally, using that by induction hypothesis we have an isomorphism between $U^m h^2(K)$ and $\mathrm{fil}_{e'-m}\mathrm{Br}(K)[p]$, we get that $U^{m+1}h^2(K)$ is isomorphic to $\mathrm{fil}_{e'-(m+1)}\mathrm{Br}(K)[p]$.

- Inductive step on $m+1$ when $p \mid m$ and $m < e'$. In this case we have

$$\Omega^1_F/Z^1_F \oplus \Omega^0_F/Z^0_F \simeq \mathrm{gr}^{m-1} \hookrightarrow \mathrm{gr}_{e'-m} \hookrightarrow B^2_F \oplus B^1_F$$

where the first isomorphism is induced by ρ_m, while the inclusion comes from the refined Swan conductor property 4.2.7(2). Given $\alpha = [\bar{x}d\log\bar{y}]$ in Ω^1_F/Z^1_F, $\rho_m(\alpha, 0) = \{1+\pi^m x, y\}$ and again using an argument similar to the one used above

$$\begin{aligned}
&\{1+\pi^m x, y, 1+\pi^{e'-m}T\} \\
&= \left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x, y\right\} - m\left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, y, \pi\right\} \\
&= \left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x, y\right\}
\end{aligned}$$

where the last equality follows from the fact that $p \mid m$ and that we are working with groups of order p. Like before, note that $\frac{x}{1+\pi^m x}$ is a possible lift of $\bar{x} \in F$ to K. Hence,

$$\left\{1+(\zeta-1)^p\frac{cxT}{1+\pi^m x}, x, y\right\} = \lambda_\pi\left(\bar{c}\bar{T}d\alpha, 0\right)$$

With very similar computations it is possible to show that starting from $\beta = [\bar{x}] \in \Omega^0_F/Z^0_F$,

$$\{(1+\pi^m x, \pi)_p, 1+\pi^{e'-m}T\} = \lambda_\pi(0, \bar{c}\bar{T}d\beta)$$

Hence, in this case the composition

$$\Omega^1_F/Z^1_F \oplus \Omega^0_F/Z^0_F \simeq \mathrm{gr}^m \hookrightarrow \mathrm{gr}_{e'-m} \xrightarrow{\mathrm{rsw}_{e'-m,\pi}} B^2_F \oplus B^1_F$$

sends (α, β) to $(\bar{c}d\alpha, \bar{c}d\beta)$. Finally, using that by induction hypothesis we have an isomorphism between $U^m h^2(K)$ and $\mathrm{fil}_{e'-m}\mathrm{Br}(K)[p]$, we get an isomorphism between $U^{m+1}h^2(K)$ and $\mathrm{fil}_{e'-(m+1)}\mathrm{Br}(K)[p]$.

4.3 The case of K3 surfaces

In this thesis examples will always be about K3 surfaces with good reduction, cf. Section 2.1.3 for the definition and some properties of K3 surfaces. The special fibre of a K3 surface with good reduction is still a K3 surface, see for example [BN23, Remark 11.5]. We start by stating the following well known result, of which we include the proof as we could not find it in standard literature.

Lemma 4.3.1. *Let p be a prime number and Y a K3 surface over the finite field $\mathbb{F}_{p^n}$ for some non-negative n. Then Y is ordinary if and only if $|Y(\mathbb{F}_{p^n})| \not\equiv 1 \mod p$.*

Proof. The proof is an almost immediate consequence of [BZ09, Section 1]. Let $\bar{Y}$ be the base change of Y to an algebraic closure of $\mathbb{F}_{p^n}$ and l be a prime different from p. The Frobenius endomorphism F of $\bar{Y}$ acts by functoriality on 22-dimensional $\mathbb{Q}_l$-vector space

$$\mathrm{H}^2_{\text{ét}}(\bar{Y}, \mathbb{Q}_l) := \mathrm{H}^2_{\text{ét}}(\bar{Y}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Let λ_i with $i = 1, \dots, 22$ be the corresponding eigenvalues. From the Lefschetz trace formula [Kat81, Section 1] we get

$$|Y(\mathbb{F}_{p^n})| = \sum (-1)^i \mathrm{Tr}(\mathrm{F}, \mathrm{H}^i_{\text{ét}}(\bar{Y}, \mathbb{Q}_l) = 1 + \sum_{i=1}^{22} \lambda_i + p^{2n}. \tag{4.6}$$

The last equality follows from the fact that for K3 surfaces both the first and the third Betti numbers are trivial and $\mathrm{H}^0_{\text{ét}}(\bar{Y}, \mathbb{Q}_l)$ and $\mathrm{H}^4_{\text{ét}}(\bar{Y}, \mathbb{Q}_l)$ are 1-dimensional $\mathbb{Q}_l$-vector spaces with Frobenius eigenvalue equal to 1 and p^{2n} respectively [Del74, Theorem 1.6].

It is proven in [BZ09, Lemma 1.1] that a K3 surface Y is ordinary if and only if $\sum_{i=1}^{22} \lambda_i$ is not divisible by p. It is therefore clear from (4.6) that

$$|Y(\mathbb{F}_{p^n})| \equiv 1 + \sum_{i=1}^{22} \lambda_i \not\equiv 1 \mod p.$$

if and only if Y is ordinary. ☐

If the K3 surface X has good ordinary reduction, then the remark that follows shows that there is a strong link between global logarithmic 2-forms on Y and p-torsion elements on X having non-constant evaluation map.

Remark 4.3.2 (K3 surfaces with good ordinary reduction)**.** Let X be a K3 surface defined over a p-adic field L having absolute ramification index divsible by $p-1$ with good ordinary reduction. Assume that there is an element $\mathcal{A} \in \mathrm{Br}(X)[p]$ that does not belong to $\mathrm{fil}_0\mathrm{Br}(X)$. Then, $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$ for some $n \geq 1$. From Section 1.2.3.1 we can assume $n \leq e' = \frac{ep}{p-1} = p$, since $\mathrm{fil}_{e'}\mathrm{Br}(K)[p] = \mathrm{Br}(K)[p]$. Moreover, for $n < e' = p$ we have from Corollary 1.3.9(1) together with the fact that for K3 surfaces we do not have non-trivial global 1-forms, $\mathrm{fil}_n\mathrm{Br}(X)[p] =$

$\mathrm{fil}_0\mathrm{Br}(X)[p]$. By property 4.2.7(3) we know that there is a constant $\bar{c} \in \ell^\times$ such that

$$\mathrm{mult}_{\bar{c}}\left(\mathrm{fil}_p\mathrm{Br}(X)[p]\right) \subseteq \mathrm{H}^0(Y, \Omega^2_{Y,\log}) \subseteq \mathrm{H}^0(\bar{Y}, \Omega^2_{\bar{Y},\log}).$$

From Proposition 4.2.2 we know that the class of $\mathcal{A}$ in $\frac{\mathrm{fil}_p\mathrm{Br}(K^h)[p]}{\mathrm{fil}_{p-1}\mathrm{Br}(K^h)[p]} \simeq \mathrm{gr}^0$ has to be such that

$$[\mathcal{A}] = \rho_0(\omega, 0)$$

where $\omega \in \Omega^2_{F,\log}$ is the image in Ω^2_F of a non-trivial global logarithmic form, i.e. an element of $\mathrm{H}^0(Y, \Omega^2_{Y,\log})$. Moreover, $\mathrm{H}^0(\bar{Y}, \Omega^2_{\bar{Y},\log})$ is a 1-dimensional $\mathbb{F}_p$-vector space (cf. (3.2)).

If the K3 surface is defined by an homogeneous polynomial of degree 4, the following lemma gives us a way to write down explicitly a generator for the ℓ-vector space of global 2-forms.

Lemma 4.3.3. *Let ℓ be a field and $f(x_0, x_1, x_2, x_3) \in \ell[x_0, x_1, x_2, x_3]$ be a homogeneous polynomial of degree 4. Assume that the corresponding projective variety Y is smooth. Then Y is a K3 surface and the 1-dimensional ℓ-vector space of global 2-forms is generated by the 2-form*

$$\omega = \frac{d\left(\frac{x_1}{x_0}\right) \wedge d\left(\frac{x_2}{x_0}\right)}{\frac{1}{x_0^3} \cdot \frac{\partial f}{\partial x_3}}.$$

Proof. The first part follows from [Huy16, Example 1.3(i)]. For every permutation $\{p, q, i, j\}$ of $\{0, 1, 2, 3\}$ we define $W_{p,q} \subseteq Y$ as the open subset of Y where $x_p \cdot \frac{\partial f}{\partial x_q}$ does not vanish. We define

$$\omega_{p,q} := (-1)^{p+q+1} \cdot \frac{d\left(\frac{x_i}{x_p}\right) \wedge d\left(\frac{x_j}{x_p}\right)}{\frac{1}{x_p^3} \cdot \frac{\partial f}{\partial x_q}} \in \mathrm{H}^0(W_{p,q}, \Omega^2_Y).$$

Since Y is smooth, the open sets $\{W_{p,q}\}$ cover it. We are left to show that for every $(p, q) \neq (p', q')$, $\omega_{p,q} = \omega_{p',q'}$ on $W_{p,q} \cap W_{p',q'}$.

It is enough to show the equality in the following two cases (the complete proof follows from the symmetry among the variables):

- $(p, q) = (0, 1)$ and $(p', q') = (0, 2)$.

  Let $s_1 := x_1(x_0)^{-1}$, $s_2 := x_2(x_0)^{-1}$, $s_3 := x_3(x_0)^{-1}$ and $f_0 := f(1, s_1, s_2, s_3)$. We can then rewrite $\omega_{0,1}$ and $\omega_{0,2}$ as

  $$\omega_{0,1} = \frac{ds_2 \wedge ds_3}{\partial f_0/\partial s_1} \quad \text{and} \quad \omega_{0,2} = \frac{ds_1 \wedge ds_3}{\partial f_0/\partial s_2}.$$

  From the equation $f_0 = 0$ we get

  $$\frac{\partial f_0}{\partial s_1} ds_1 + \frac{\partial f_0}{\partial s_2} ds_2 + \frac{\partial f_0}{\partial s_3} ds_3 = 0.$$

In particular,

$$0 = \left(\frac{\partial f_0}{\partial s_1} ds_1 + \frac{\partial f_0}{\partial s_2} ds_2 + \frac{\partial f_0}{\partial s_3} ds_3\right) \wedge ds_3 = \frac{\partial f_0}{\partial s_1}(ds_1 \wedge ds_3) + \frac{\partial f_0}{\partial s_2}(ds_2 \wedge ds_3).$$

therefore, on $W_{0,1} \cap W_{0,2}$

$$\omega_{0,1} = \frac{ds_2 \wedge ds_3}{\partial f_0/\partial s_1} = (-1) \cdot \frac{ds_1 \wedge ds_3}{\partial f_0/\partial s_2} = \omega_{0,2}.$$

- $(p, q) = (0, 2)$ and $(p', q') = (1, 2)$.

  Let $t_1 := x_0(x_1)^{-1}$, $t_2 := x_2(x_1)^{-1}$, $t_3 := x_3(x_1)^{-1}$ and $f_1 := f(t_1, 1, t_2, t_3)$. On $W_{0,2} \cap W_{1,2}$ we have

$$s_1^{-1} = t_1, \quad s_2 \cdot s_1^{-1} = t_2 \quad s_3 \cdot s_1^{-1} = t_3 \quad s_1^4 \cdot f_1 = f_0.$$

  In particular,

$$\omega_{1,2} = \frac{dt_1 \wedge dt_3}{\partial f_1/\partial t_2} = \frac{d(s_1^{-1}) \wedge d(s_3 \cdot s_1^{-1})}{\partial s_1^{-4} f_0/\partial(s_2 \cdot s_1^{-1})} = (-1) \cdot \frac{s_1^{-3} ds_1 \wedge ds_3}{s_1^{-3} \partial f_0/\partial s_2} = \omega_{0,2}$$

  where the second last equality comes from the equality

$$\frac{\partial s_1^{-4} f_0}{\partial(s_2 s_1^{-1})} = s_1^{-3} \frac{\partial f_0}{\partial s_2}.$$

□

We point out that in [Ier23, Proposition 2.3] it is proven that for K3 surface an element $\mathcal{A}$ lies in $\mathrm{fil}_0\mathrm{Br}(X)$ if and only if $\mathrm{ev}_{\mathcal{A}} : X(L) \to \mathrm{Br}(L)$ is constant. It is already known from [BN23, Lemma 11.3] that, since for K3 surface $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$, then $\mathrm{fil}_0\mathrm{Br}(X) = \mathrm{Ev}_0\mathrm{Br}(X) = \mathrm{Ev}_{-1}\mathrm{Br}(X)$. Hence, from the result proven by Ieronymou we know that in order to detect whether $\mathcal{A}$ belongs to $\mathrm{fil}_0\mathrm{Br}(X)$ it is enough to look at the corresponding evaluation map on the L-points, $X(L)$.

Lemma 4.3.4. *Let X be a K3 surface and $\mathcal{A} \in \mathrm{Br}(X)$ be such that $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X)$. Then $\mathcal{A} \notin \mathrm{Br}_1(X)$.*

Proof. As we just pointed out, if $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X)$, then the evaluation map attached to $\mathcal{A}$ in non-constant on $X(L)$. The result now follows from the fact that in [CTS13, Proposition 2.3] Colliot-Thélène and Skorobogatov prove that for every element in the algebraic Brauer group the associated evaluation map at a prime with good reduction has to be constant. □

4.4 Example of Chapter 2 revisited

We start with recalling the following example, which is the central result of Chapter 2.

Example 4.4.1 ([Pag22]). Let $V \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the projective K3 surface defined by the equation

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \tag{4.7}$$

Then V has good ordinary reduction at 2 and the class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right) \in \operatorname{Br} \mathbb{Q}(V)$$

defines an element in $\operatorname{Br}(V)$. The evaluation map $\operatorname{ev}_{\mathcal{A}} : V(\mathbb{Q}_2) \to \operatorname{Br}(\mathbb{Q}_2)$ is non-constant, and therefore gives an obstruction to weak approximation on X.

In this case, $\mathrm{H}^0(Y, \Omega^2_Y)$ is a one dimensional $\mathbb{F}_2$ vector space, let ω be the only non-trivial element, then $C(\omega) = 0$ or $C(\omega) = \omega$. However, Y being ordinary implies that $\mathrm{H}^0(Y, B^2_Y) = 0$, hence by Lemma 1.3.8 and Corollary 1.2.4 $C(\omega) = \omega$ and $\mathrm{H}^0(Y, \Omega^2_{Y,\log})$ is a 1-dimensional $\mathbb{F}_2$-vector space. From Lemma 4.3.3 we get that the non-zero global logarithmic 2-form ω can be written (locally) as:

$$\omega = \frac{d\left(\frac{z^3+w^2x+xyz}{x^3}\right)}{\left(\frac{z^3+w^2x+xyz}{x^3}\right)} \wedge \frac{d\left(\frac{z}{x}\right)}{\left(\frac{z}{x}\right)}.$$

If we denote by f and g the functions $\frac{z^3+w^2x+xyz}{x^3}$ and $\frac{z}{x}$ seen as element in the function field F of Y, then we see that the two functions appearing in the definition of $\mathcal{A}$ are lifts to characteristic 0 of f and g, and hence from Proposition 4.2.1

$$\rho_0(\omega, 0) = \left[\left\{ \frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right\} \right] \in \operatorname{gr}^0.$$

Using Proposition 4.2.2,

$$\operatorname{rsw}_{2,\pi}(\mathcal{A}) = (\omega, 0) \neq (0, 0)$$

and $\mathcal{A} \notin \operatorname{fil}_1 \operatorname{Br}(X)[2] \supseteq \operatorname{Ev}_{-1}\operatorname{Br}(X)[2]$.

Note that, by Remark 4.3.2 we already know that since the K3 surface has good ordinary reduction at 2, the only way for the prime 2 to play a role in the Brauer–Manin obstruction to weak approximation via an 2-torsion element $\mathcal{A} \in \operatorname{Br}(X)[2]$ is if $\mathcal{A}$ comes from a logarithmic 2-form through ρ_0.

4.5 Kummer K3 surfaces over 2-adic fields

In this section we are going to treat Kummer K3 surfaces. As already mentioned in Section 2.1.3 those surfaces arise as the resolution of singularities of the quotient of an abelian surface by its involution map. The details about the construction for fields of characteristic different from 2 can be found in [B0̆1, Section 10.5].

Let L be a 2-adic field. The recent papers [LS23], [Mat23] allow us to know whether the Kummer K3 surface attached to an abelian variety A/L with good reduction is still a K3 surface with good reduction (this was already known for K3 surfaces over p-adic fields with $p \neq 2$). In [SZ12] Skorobogatov and Zarhin link the transcendental part of the Brauer group of a Kummer K3 surface to the one of the corresponding abelian variety (cf. Section 4.5.1). All these results open up the possibility of building examples of K3 surfaces with good reduction at the prime 2 and for which we are able to study the Brauer group. In this section, we are going to show that for every pair of elliptic curves E_1, E_2 over $\mathbb{Q}$ with good ordinary reduction at $p = 2$ and full 2-torsion defined over $\mathbb{Q}_2$, the 2-torsion elements in the Brauer group of the corresponding Kummer K3 surface X do not play a role in the Brauer–Manin obstruction to weak approximation. In particular, this shows that the field extension in Theorem 3.1.6 is needed. We will then use these computations to exhibit an example of a K3 surface over $\mathbb{Q}_2$ with good ordinary reduction and such that $\mathrm{Br}(X) = \mathrm{Ev}_{-1}\mathrm{Br}(X)$, showing that the inverse of Theorem 3.1.1 does not hold in general.

4.5.1 Kummer K3 surfaces and their Brauer group: generalities

Let A be an abelian surface over a field k of characteristic different from 2 and $V = \mathrm{Kum}(A)$ the corresponding Kummer surface, Skorobogatov and Zarhin [SZ12] prove that there is a well-defined map

$$\pi^* \colon \mathrm{Br}(V) \to \mathrm{Br}(A)$$

that induces an injection of $\mathrm{Br}(V)/\mathrm{Br}_1(V)$ into $\mathrm{Br}(A)/\mathrm{Br}_1(A)$. They also prove that this injection is an isomorphism on the p-torsion for all odd primes, see [SZ12, Theorem 2.4]. We say that an element $\mathcal{A} \in \mathrm{Br}(A)$ **descends** to $\mathrm{Br}(V)$ if there exists $\mathcal{C} \in \mathrm{Br}(V)$ such that $\pi^*(\mathcal{C}) = \mathcal{A}$.

Lemma 4.5.1. *Let $V = \mathrm{Kum}(A)$, $\mathcal{C} \in \mathrm{Br}(V)$ and $\mathcal{B} := \pi^*(\mathcal{C}) \in \mathrm{Br}(A)$. Let $\mathfrak{p}$ be a prime in $\mathcal{O}_k$; if the image of $\mathcal{C}$ in $\mathrm{Br}(V_\mathfrak{p})$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(V_\mathfrak{p})$ then the image of $\mathcal{B}$ in $\mathrm{Br}(A_\mathfrak{p})$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(A_\mathfrak{p})$.*

Proof. The result follows from the fact that any finite field extension $M/k_\mathfrak{p}$ and $P \in A(M)$ we have

$$\mathrm{ev}_{\mathcal{B}}(P) = \mathrm{ev}_{\pi^*(\mathcal{C})}(P) = \mathrm{ev}_{\mathcal{C}}(\pi(P)).$$

□

Moreover, Skorobogatov and Zarhin [SZ12] show that given two elliptic curves E_1 and E_2 with Weierstrass equations

$$E_1 : v_1^2 = u_1 \cdot (u_1 - \gamma_{1,1}) \cdot (u_1 - \gamma_{1,2}), \quad E_2 : v_2^2 = u_2 \cdot (u_2 - \gamma_{2,1}) \cdot (u_2 - \gamma_{2,2})$$

the quotient $\mathrm{Br}(E_1 \times E_2)[2]/\mathrm{Br}_1(E_1 \times E_2)[2]$ is generated by the classes of the four Azumaya algebras

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = ((u_1 - \epsilon_1)(u_1 - \gamma_{1,2}), (u_2 - \epsilon_2)(u_2 - \gamma_{2,2})) \text{ with } \epsilon_i \in \{0, \gamma_{i,1}\}.$$

Finally, if M is the matrix

$$M = \begin{pmatrix} 1 & \gamma_{1,1} \cdot \gamma_{1,2} & \gamma_{2,1} \cdot \gamma_{2,2} & -\gamma_{1,1} \cdot \gamma_{2,1} \\ \gamma_{1,1} \cdot \gamma_{1,2} & 1 & \gamma_{1,1} \cdot \gamma_{2,1} & \gamma_{2,1} \cdot (\gamma_{2,1} - \gamma_{2,2}) \\ \gamma_{2,1} \cdot \gamma_{2,2} & \gamma_{1,1} \cdot \gamma_{2,1} & 1 & \gamma_{1,1} \cdot (\gamma_{1,1} - \gamma_{1,2}) \\ -\gamma_{1,1} \cdot \gamma_{2,1} & \gamma_{2,1} \cdot (\gamma_{2,1} - \gamma_{2,2}) & \gamma_{1,1} \cdot (\gamma_{1,1} - \gamma_{1,2}) & 1 \end{pmatrix}$$

then by [SZ12, Lemma 3.6]:

1. $\mathcal{A}_{\gamma_{1,1},\gamma_{2,1}}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the first row of M are all squares;
2. $\mathcal{A}_{\gamma_{1,1},0}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the second row of M are all squares;
3. $\mathcal{A}_{0,\gamma_{2,1}}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the third row of M are all squares;
4. $\mathcal{A}_{0,0}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the last row of M are all squares.

4.5.2 Product of elliptic curves with good reduction at 2 and full 2-torsion

In order to use the results summarised in the previous section we need to analyse what the 2-torsion points of an elliptic curve with good ordinary reduction at 2 look like. Let $E/\mathbb{Q}$ be the elliptic curve defined by the minimal Weierstrass equation

$$y^2 + xy + \delta y = x^3 + ax^2 + bx + c \tag{4.8}$$

with $\delta \in \{0, 1\}$ and $a, b, c \in \mathbb{Z}$ such that E has good reduction at 2. Assume furthermore that the 2-torsion of E is defined over $\mathbb{Q}_2$, i.e. $E(\mathbb{Q}_2)[2] = E(\bar{\mathbb{Q}}_2)[2]$. Let $\alpha_i, \beta_i \in \mathbb{Q}_2$ be such that $E(\mathbb{Q}_2)[2] = \{\mathcal{O}, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\}$, with $\mathcal{O}$ the point at infinity of E.

Lemma 4.5.2. *Assume that $\beta_1, \beta_2, \beta_3$ are ordered as*

$$\mathrm{ord}_2(\beta_1) \leq \mathrm{ord}_2(\beta_2) \leq \mathrm{ord}_2(\beta_3).$$

Then $\mathrm{ord}_2(\alpha_1) = -2$ *and* $\alpha_2, \alpha_3 \in \mathbb{Z}_2$.

Proof. The 2-torsion points on E can be computed through the 2-division polynomial of E, which is $\psi_2(x, y) = 2y + x + \delta$. In particular, $\alpha_i = -2\beta_i - \delta$ with β_i solution of

$$\Phi(y) := y^2 + (-2y - \delta)y + \delta y - ((-2y - \delta)^3 + a(-2y - \delta)^2 + b(-2y - \delta) + c).$$

The polynomial $\Phi(y)$ can be rewritten as

$$\Phi(y) = 8y^3 - (1 - 12\delta + 4a)y^2 - (-6\delta^2 + 4a\delta - 2b)y + \delta^3 - a\delta^2 + b\delta - c. \quad (4.9)$$

Looking at the coefficients of $\Phi(y)$ we get that

$$\begin{cases} \mathrm{ord}_2(\beta_1 + \beta_2 + \beta_3) = \mathrm{ord}_2(1 - 12\delta + 4a) - \mathrm{ord}_2(8) \\ \mathrm{ord}_2(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3) = \mathrm{ord}_2(-6\delta^2 + 4a\delta - 2b) - \mathrm{ord}_2(8) \\ \mathrm{ord}_2(\beta_1\beta_2\beta_3) = \mathrm{ord}_2(\delta^3 - a\delta^2 + b\delta - c) - \mathrm{ord}_2(8). \end{cases}$$

From the first equation, we get $\mathrm{ord}_2(\beta_1) \le -3$ that combined with [Sil86, Theorem VIII.7.1] tells us that $\mathrm{ord}_2(\beta_1) = -3$. From the third equation, we get $\mathrm{ord}_2(\beta_2) + \mathrm{ord}_2(\beta_3) \ge 0$. Hence, if $\mathrm{ord}_2(\beta_2) = \mathrm{ord}_2(\beta_3)$ then β_2 and β_3 have both non-negative 2-adic valuation; otherwise, if $\mathrm{ord}_2(\beta_2) < \mathrm{ord}_2(\beta_3)$ then, from the second equation, we get $\mathrm{ord}_2(\beta_2) \ge 1$ which implies again that both β_2 and β_3 have non-negative 2-adic valuation. The result now follows from the fact that $\alpha_i = -2\beta_i - \delta$, with $\delta \in \{0, 1\}$. □

Lemma 4.5.3. *The change of variables given by*

$$\begin{cases} u = 4x - 4\alpha_1 \\ v = 4(2y + x + \delta) \end{cases} \quad (4.10)$$

induces an isomorphism between E and the elliptic curve given by the equation

$$v^2 = u(u - \gamma_1)(u - \gamma_2) \quad (4.11)$$

where $\gamma_1 = 4 \cdot (\alpha_2 - \alpha_1)$ and $\gamma_2 = 4 \cdot (\alpha_3 - \alpha_1)$.

Proof. The change of variables

$$\begin{cases} u_1 = 4x \\ v_1 = 4(2y + x + \delta) \end{cases}$$

sends the elliptic curve given by the equation

$$v_1^2 = u_1^3 + (4a + 1)u_1^2 + (16b + 8\delta)u_1 + 16c + 16\delta^2 \quad (4.12)$$

to the elliptic curve given by equation (4.8). Moreover, the 2-division polynomial of E is given by $2y + x + \delta$. Hence the non-trivial 2-torsion points on the elliptic curve given by equation (4.12) are sent to non-trivial 2-torsion points on E. It is therefore enough to consider the extra translation $u = u_1 - 4\alpha_1$ and $v = v_1$ to get the desired equation. □

Let E_1 and E_2 be two elliptic curves with equations of the form (4.8). We denote by $(\delta_i, a_i, b_i, c_i)$ the parameters that determine the equation attached to E_i, by $(\alpha_{i,j}, \beta_{i,j})$, $j \in \{1, 2, 3\}$ the non-trivial 2-torsion points of E_i and by A the abelian surface given by the product of E_1 with E_2. We denote by $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2] \rangle$ the subgroup of $\mathrm{Br}(A)[2]$ generated by $\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$ and $\mathrm{Br}_1(A)[2]$, where $\mathrm{Br}_1(A)[2]$ is the algebraic Brauer group of A.

Lemma 4.5.4. *Assume that* ϵ_1 *and* ϵ_2 *are as in Section 4.5.1; then the class of the quaternion algebra* $\mathcal{A}_{\epsilon_1,\epsilon_2}$ *lies in* $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2]\rangle$ *if and only if at least one among* ϵ_1 *and* ϵ_2 *is different from* 0.

Proof. We fix $\pi = 2$ as a uniformiser and $\xi = -1$ as a primitive 2-root of unity. We start by assuming that at least one among ϵ_1 and ϵ_2 is different from 0. By the symmetry of the statement, we can assume without loss of generality that $\epsilon_1 \neq 0$. Then

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = ((u_1 - \gamma_{1,1}) \cdot (u_1 - \gamma_{1,2})\, , \, (u_2 - \epsilon_2) \cdot (u_2 - \gamma_{2,2})) = (u_1, f_{\epsilon_2}(u_2))$$

where $f_{\epsilon_2}(u_2) = (u_2 - \epsilon_2) \cdot (u_2 - \gamma_{2,2})$. The quaternion algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ corresponds via the change of variables of Lemma 4.5.3 to

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = (4 \cdot (x_1 - \alpha_{1,1})\, , \, f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})) = (x_1 - \alpha_{1,1}\, , \, f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})).$$

We define

$$g_{\epsilon_2}(x_2) := \begin{cases} (x_2 - \alpha_{2,2}) \cdot (x_2 - \alpha_{2,3}) \text{ if } \epsilon_2 = \gamma_{2,1}; \\ (x_2 - \alpha_{2,1}) \cdot (x_2 - \alpha_{2,3}) \text{ if } \epsilon_2 = 0. \end{cases}$$

Then, $16 \cdot g_{\epsilon_2}(x_2) = f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})$. Thus we can rewrite $\mathcal{A}_{\epsilon_1,\epsilon_2}$ as

$$(-\alpha_{1,1}\, , \, g_{\epsilon_2}(x_2)) \otimes (1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, , \, g_{\epsilon_2}(x_2)).$$

Since $(-\alpha_{1,1}\, , \, g_{\epsilon_2}(x_2))$ lies in $\mathrm{Br}_1(A)[2]$, we are left to show that the class of the quaternion algebra $(1 + (-\alpha_{1,1})^{-1} \cdot x_1\, , \, g_{\epsilon_2}(x_2))$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$. By Lemma 4.5.2 we know that $\mathrm{ord}_2(\alpha_{1,1}^{-1}) = 2$ and therefore by Proposition 4.2.2

$$(1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, , \, g_{\epsilon_2}(x_2)) \in \mathrm{fil}_0\mathrm{Br}(A)[2].$$

By [BN23, Theorem C] in order to establish whether $(1 + (-\alpha_{1,1})^{-1} \cdot x_1\, , \, g_{\epsilon_2}(x_2))$ belongs to $\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$ we need to compute $\partial(1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, , \, g_{\epsilon_2}(x_2))$. We have that $g_{\epsilon_2}(x_2) \not\equiv 0 \mod 2$ and from Proposition 4.2.1.(1) together with Proposition 4.2.2 we get

$$\lambda_\pi \left(\bar{x}_1 \cdot d\log(\bar{g}_{\epsilon_2}(\bar{x}_2)), 0\right) = \left(1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, , \, g_{\epsilon_2}(x_2)\right)$$

since $1 + (-\alpha_{1,1}^{-1}) \cdot x_1 = 1 + 4 \cdot (s^{-1} \cdot x_1)$ with $s = -4 \cdot \alpha_{1,1} \in \mathbb{Z}_2^\times$ and hence $s^{-1} \cdot x_1$ is a lift to characteristic 0 of $\bar{x}_1$. Therefore, by definition of the residue map ∂, we get

$$\partial((1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, , \, g_{\epsilon_2}(x_2))) = 0$$

which by Theorem 1.3.1 implies that

$$(1 + (-\alpha_{1,1}^{-1}) \cdot x_1\, , \, g_{\epsilon_2}(x_2)) \in \mathrm{Ev}_{-2}\mathrm{Br}(A)[2] \subseteq \mathrm{Ev}_{-1}\mathrm{Br}(A)[2].$$

In order to end the proof we are left to show that $\mathcal{A}_{0,0} \notin \langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2]\rangle$. The change of variables of Lemma 4.5.3 sends the class of the quaternion algebra

$$\mathcal{A}_{0,0} = (u_1 \cdot (u_1 - \gamma_{1,2})\, , \, u_2 \cdot (u_2 - \gamma_{2,2})) = (u_1 - \gamma_{1,1}\, , \, u_2 - \gamma_{2,1})$$

to the class of the quaternion algebra

$$(4\cdot(x_1-\alpha_{1,2})\,,\,4\cdot(x_2-\alpha_{2,2}))=(x_1+2\beta_{1,2}+\delta\,,\,x_2+2\beta_{2,2}+\delta).$$

From Proposition 4.2.1(d) the latter is such that

$$\rho_0\left(\frac{d(\bar{x}_1+\delta)}{\bar{x}_1+\delta}\wedge\frac{d(\bar{x}_2+\delta)}{\bar{x}_2+\delta}\right)=\Big[\big\{x_1+2\beta_{2,1}+\delta,x_2+2\beta_{2,2}+\delta\big\}\Big]\in\mathrm{gr}^0.$$

In fact, $x_1+2\beta_{2,1}+\delta, x_2+2\beta_{2,2}+\delta$ and $x_2+2\beta_{2,2}+\delta$ are lifts to characteristic 0 of $\bar{x}_1+\delta$ and $\bar{x}_2+\delta$ respectively. Note that, $\frac{d(\bar{x}_1+\delta)}{\bar{x}_1+\delta}\wedge\frac{d(\bar{x}_2+\delta)}{\bar{x}_2+\delta}$ comes from a global 2-form on the special fibre Y of A and hence it is non-zero in its function field. Finally, using Proposition 4.2.2 we get that

$$\mathrm{rsw}_{2,\pi}((x_1+2\beta_{2,1}+\delta,x_2+2\beta_{2,2}+\delta))=\left(\frac{d(\bar{x}_1+\delta)}{\bar{x}_1+\delta}\wedge\frac{d(\bar{x}_2+\delta)}{\bar{x}_2+\delta},0\right)\neq(0,0)$$

and hence $(x_1+2\beta_{1,2}+\delta,x_2+2\beta_{2,2}+\delta)\notin\mathrm{fil}_1\mathrm{Br}(A)[2]\supseteq\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$. Moreover, as a consequence of Corollary 4.1.5 we get that $\mathcal{A}_{0,0}\notin\langle\mathrm{Ev}_{-1}\mathrm{Br}(A)[2],\mathrm{Br}_1(A)[2]\rangle$. In fact, otherwise, there would be an element $\mathcal{A}_1\in\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$ such that $\mathcal{A}_{0,0}\otimes\mathcal{A}_1\in\mathrm{Br}_1(A)[2]$, but $\mathcal{A}_{0,0}\otimes\mathcal{A}_1$ has the same refined Swan conductor as $\mathcal{A}_{0,0}$. □

Remark 4.5.5. We will later use a slightly stronger statement of the theorem above. Let $L/\mathbb{Q}_2$ be any field extension and res the natural map from $\mathrm{Br}(A)[2]$ to $\mathrm{Br}(A_L)[2]$. Then, $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2})\in\langle\mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2],\mathrm{Br}_1(A_L)[2]\rangle$ if and only if at least one among ϵ_1 and ϵ_2 is different from 0. We clearly have that $\mathcal{A}_{\epsilon_1,\epsilon_2}$ in $\langle\mathrm{Ev}_{-1}\mathrm{Br}(A)[2],\mathrm{Br}_1(A)[2]\rangle$ implies $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2})$ in $\langle\mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2],\mathrm{Br}_1(A_L)[2]\rangle$. Moreover, we have proven that the first component of $\mathrm{rsw}_{2,\pi}(\mathcal{A}_{0,0})$ is different from 0, and hence using Lemma 4.1.2 we get that $\mathrm{rsw}_{e_{L'/L}2,\pi}(\mathrm{res}(\mathcal{A}_{0,0}))\neq(0,0)$ and therefore in particular $\mathrm{res}(\mathcal{A}_{0,0})\notin\langle\mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2],\mathrm{Br}_1(A_L)[2]\rangle$

4.5.3 No Brauer–Manin obstruction from 2-torsion elements in Kum(A)

In this section, we show how, from the results of the previous section, we can deduce information on the 2-torsion elements in the Brauer group of the corresponding Kummer surface $V=\mathrm{Kum}(A_{\mathbb{Q}})$. We denote by A and X the base change of the abelian surface $A_{\mathbb{Q}}$ and the corresponding Kummer surface X to $\mathbb{Q}_2$. By Section 4.5.1 we know that $\mathcal{A}_{0,0}$ descends to X if and only if

$$\big[-\gamma_{1,1}\cdot\gamma_{2,1},\gamma_{2,1}(\gamma_{2,1}-\gamma_{2,2}),\gamma_{1,1}(\gamma_{1,1}-\gamma_{1,2}),1\big]\in(\mathbb{Q}_2^{\times 2})^4.$$

By construction, $\gamma_{1,1}=4\cdot(\alpha_{1,2}-\alpha_{1,1})=8\beta_{1,1}-8\beta_{1,2}$ and therefore

$$\gamma_{1,1}\equiv 8\beta_{1,1}\equiv 1-12\delta_1+4a_1\equiv 1-4(3\delta_1-a_1)\mod 8.$$

In fact, from Lemma 4.5.2 and more precisely from equation (4.9) we know that $8\beta_{1,1}+8\beta_{1,2}+8\beta_{1,3}=1-12\delta_1+4a_1$ and both $\mathrm{ord}_2(\beta_{1,2})$ and $\mathrm{ord}_2(\beta_{1,3})$ are non-negative. Similarly,

$$\gamma_{2,1}\equiv 8\beta_{2,1}\equiv 1-12\delta_2+4a_2\equiv 1-4(3\delta_2-a_2)\mod 8.$$

In particular, both $\gamma_{1,1}$ and $\gamma_{2,1}$ are either 1 or 5 modulo 8; hence $-\gamma_{1,1}\cdot\gamma_{2,1}$ is either -1 or 3 and therefore it is never a square. Summing up: we have shown $\mathcal{A}_{0,0}$ never descends to $\mathrm{Br}(X)$. We are ready to prove the main theorem of this section.

Theorem 4.5.6. *Let $X = \mathrm{Kum}(A)$, where $A = E_1 \times E_2$ is as in Section 4.5.1; then* $\mathrm{Br}(X)[2] = \mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$.

Proof. We recall that if $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)[2]$, we denote by $\mathcal{C}_{\epsilon_1,\epsilon_2}$ the corresponding element in $\mathrm{Br}(X)[2]$, i.e. $\mathcal{C}_{\epsilon_1,\epsilon_2}$ is such that $\pi^*(\mathcal{C}_{\epsilon_1,\epsilon_2}) = \mathcal{A}_{\epsilon_1,\epsilon_2}$. We need to prove that if $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)$, then $\mathcal{C}_{\epsilon_1,\epsilon_2}$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(X)$. Since we have already shown at the beginning of this section that $\mathcal{A}_{0,0}$ never descends to $\mathrm{Br}(X)$ we are left to show it for $(\epsilon_1,\epsilon_2) \neq (0,0)$.

Let $L/\mathbb{Q}_2$ be such that all elements appearing in the matrix M of Section 4.5.1 are squares, i.e. the injective map

$$\pi^*\colon \frac{\mathrm{Br}(X)[2]}{\mathrm{Br}_1(X)[2]} \hookrightarrow \frac{\mathrm{Br}(A)[2]}{\mathrm{Br}_1(A)[2]}$$

is an isomorphism.

With abuse of notation, we denote by res both

$$\mathrm{res}\colon \mathrm{Br}(A) \to \mathrm{Br}(A_L) \quad \text{and} \quad \mathrm{res}\colon \mathrm{Br}(X) \to \mathrm{Br}(X_L).$$

We denote by $(\mathcal{C}_{\epsilon_1,\epsilon_2})_L$ the pre-image of $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2}) \in \mathrm{Br}(A_L)[2]$.

From [LS23, Theorem 2] we know that the reduction of X is an ordinary K3 surface. Let e be the ramification index of $L/\mathbb{Q}_2$ and π_L the an uniformiser of $\mathcal{O}_L$; then we have $\mathrm{fil}_n\mathrm{Br}(X_L)[p] = \mathrm{fil}_0\mathrm{Br}(X_L)[p] = \mathrm{Ev}_{-1}\mathrm{Br}(X_L)[p]$ if $n < e' := 2e$ and $\mathrm{fil}_{e'}\mathrm{Br}(X)[p] = \mathrm{Br}(X_L)[p]$, see Remark 4.3.2. Hence, using Corollary 1.3.9(3) we have an injection

$$\mathrm{mult}_{\overline{c}} \circ \mathrm{rsw}_{e',\pi_L} : \frac{\mathrm{Br}(X_L)[2]}{\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]} \hookrightarrow \mathrm{H}^0(Y_\ell, \Omega^2_{Y_\ell,\log})$$

where ℓ is the residue field of L.

Since X (and hence all its base change) has good ordinary reduction, we know that $\mathrm{H}^0(\bar{Y}, \Omega^2_{\bar{Y},\log}) \otimes_{\mathbb{F}_2} \bar{\ell}$ is a one dimensional $\bar{\ell}$-vector space (cf. equation (3.2)) and hence $\mathrm{Br}(X_L)[2]/\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]$ is a vector space of dimension at most 1 over $\mathbb{F}_2$.

From Lemma 4.5.4 we know that $\mathcal{A}_{0,0} \notin \mathrm{fil}_1\mathrm{Br}(A)[2]$. Applying Lemma 4.1.2 we get that $\mathrm{res}(\mathcal{A}_{0,0}) \notin \mathrm{fil}_e\mathrm{Br}(A_L)[2]$ and by Lemma 4.5.1 $(\mathcal{C}_{0,0})_L \notin \mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]$. Therefore

$$\langle[(\mathcal{C}_{0,0})_L]\rangle = \frac{\mathrm{Br}(X_L)[2]}{\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]}.$$

Assume that there exists $(\epsilon_1,\epsilon_2) \neq (0,0)$ such that $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)$ and the corresponding element $\mathcal{C}_{\epsilon_1,\epsilon_2}$ does not lie in $\mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$. By Lemma 4.1.2 $\mathrm{res}(\mathcal{C}_{\epsilon_1,\epsilon_2})$ does not lie in $\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]$ and therefore, since the quotient $\mathrm{Br}(X_L)[2]/\mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2]$ is a 1-dimensional $\mathbb{F}_2$-vector space, we have that also the product $\mathrm{res}(\mathcal{C}_{\epsilon_1,\epsilon_2}) \otimes (\mathcal{C}_{0,0})_L$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$. This implies that also the corresponding element in $\mathrm{Br}(A_L)[2]$, $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2} \otimes \mathcal{A}_{0,0})$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(A_L)$. However, since by Remark 4.5.5 $\mathrm{res}(A_{\epsilon_1,\epsilon_2})$ lies in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(X_L)[2], \mathrm{Br}_1(X_L)[2]\rangle$, we get that also $\mathrm{res}(\mathcal{A}_{0,0})$ has to lie in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2], \mathrm{Br}_1(A_L)[2]\rangle$ which gives us the desired contradiction. $\square$

Finally, we give an example of a K3 surface over $\mathbb{Q}$ with good ordinary reduction at 2 and such that $\mathrm{Br}(X) = \mathrm{Ev}_{-1}\mathrm{Br}(X)$. The existence of such an example shows that the converse of Theorem 3.1.1 is not true, i.e. it is not enough to have that $p-1 \mid e$ in order to find an element in $\mathrm{Br}(X)$ that does not lie in $\mathrm{Ev}_{-1}\mathrm{Br}(X)$.

Example 4.5.7. Let $A = E \times E$, where E is the elliptic curve given by the minimal Weierstrass equation

$$y^2 + xy + y = x^3 - 7 \cdot x + 5.$$

Then, with the same notation as in the previous sections $\beta_1 = -11/8$, $\beta_2 = -1$ and $\beta_3 = 1$. Hence

$$\alpha_1 = 7/4, \quad \alpha_2 = 1, \quad \alpha_3 = -3 \quad \text{and} \quad \gamma_1 = -3, \quad \gamma_2 = -21.$$

The matrix M is of the form

$$\begin{pmatrix} 1 & 3\cdot 21 & 3\cdot 21 & -9 \\ 3\cdot 21 & 1 & 9 & -3\cdot 18 \\ 3\cdot 21 & 9 & 1 & -3\cdot 18 \\ -9 & -3\cdot 18 & -3\cdot 18 & 1 \end{pmatrix}.$$

In particular, all the rows of M have at least one term which does not lie in $\mathbb{Q}_2^{\times 2}$. Moreover, using [SZ12, Proposition 3.7] we can compute the dimension as an $\mathbb{F}_2$-vector space of the quotient of $\mathrm{Br}(X)[2]$ by $\mathrm{Br}(\mathbb{Q}_2)[2]$ and in this case particular example:

$$\dim_{\mathbb{F}_2}\left(\frac{\mathrm{Br}(X)[2]}{\mathrm{Br}(\mathbb{Q}_2)[2]}\right) = 0.$$

We want to show that $\mathrm{Br}(X)\{2\} = \mathrm{Br}(\mathbb{Q}_2)\{2\}$. We work by induction on n; let $\mathcal{A}$ be in $\mathrm{Br}(X)[2^n]$, then

$$\mathcal{A}^{\otimes 2^{n-1}} \in \mathrm{Br}(X)[2] = \mathrm{Br}(\mathbb{Q}_2)[2].$$

In particular, given $P \in X(\mathbb{Q}_2)$, we have that

$$(\mathcal{A} \otimes \mathrm{ev}_{\mathcal{A}}(P))^{\otimes 2^{n-1}} = \mathcal{A}^{\otimes 2^{n-1}} \otimes \mathrm{ev}_{\mathcal{A}^{\otimes 2^{n-1}}}(P) = 0$$

hence $\mathcal{A} \otimes \mathrm{ev}_{\mathcal{A}}(P) \in \mathrm{Br}(X)[2^{n-1}]$, and by induction hypothesis that $\mathcal{A} \in \mathrm{Br}(\mathbb{Q}_2)[2^n]$.

4.5.4 Examples of Brauer–Manin obstruction

We continue this section by giving new examples of primes of good reduction that plays a role in the Brauer–Manin obstruction to weak approximation in the case $p = 3$ and $p = 5$.

Example 4.5.8. Let $L = \mathbb{Q}_3(\zeta)$ with ζ primitive 3-root of unity. Let π be a uniformiser of $\mathcal{O}_L$; then $e = e(L/\mathbb{Q}_3) = 2$ and the residue field ℓ is equal to $\mathbb{F}_3$.

We define X to be the Kummer K3 surface over L attached to the abelian surface $A = E \times E$, with E the elliptic curve over L defined by the Weierstrass equation

$$y^2 = x^3 + 4 \cdot x^2 + 3 \cdot x + 1.$$

The elliptic curve E (and hence A and X) has good ordinary reduction at the prime $\mathfrak{p} = (\pi)$. We will denote by $\{x, y, z\}$ and $\{u, v, w\}$ the variables corresponding to the embedding of respectively the first and the second copy of E in $\mathbb{P}^2_L$. We define the cyclic algebra

$$\mathcal{A} := \left(\frac{v-u}{w}, \frac{y-x}{z}\right)_\zeta \in \mathrm{Br}(L(A))[3].$$

Claim 1: $\mathcal{A}$ belongs to $\mathrm{Br}(A)[3]$.

Proof. First of all, notice that from $y^2z = x^3 + 4x^2z + 3xz^2 + z^3$ we get

$$z(y-x)(y+x) = (x+z)^3 \quad \text{and} \quad z(y^2 - 4x^2 - 3xz - z^2) = x^3.$$

Then:

- if $z = 0$, then $x = 0$ and therefore $y^2 - 4x^2 - 3xz - z^2 \neq 0$ and $x \neq y$ and from the equation above $\mathcal{A}$ is equivalent to

$$\left(\frac{v-u}{(v^2 - 4u^2 - 3uw - w^2)^{-1}}, \frac{y-x}{(y^2 - 4x^2 - 3xz - z^2)^{-1}}\right)_\zeta;$$

- if $x = y$, then $x \neq -y$ and $z \neq 0$ (since $z = 0$ implies $x \neq y$) and from the equation above $\mathcal{A}$ is equivalent to

$$\left(\frac{w}{v+u}, \frac{z}{x+y}\right)_\zeta.$$

Thus, we see that along all the divisors over which $\mathcal{A}$ is not well defined we are able to find an equivalent Azumaya algebra which on those divisors is well defined. Hence, $\mathcal{A}$ defines an element in $\mathrm{Br}(A)[3]$. □

Claim 2: $\mathcal{A}$ and does not lie in $\mathrm{Ev}_{-1}\mathrm{Br}(A)[3]$.

Proof. The regular global 1-form on the reduction of E modulo $\mathfrak{p}$ is given by the (local) formula

$$\frac{dx}{2y} = -\frac{1}{2} \cdot \frac{dx \cdot (x-y)}{y(y-x)} = -\frac{1}{2} \cdot \frac{dx \cdot \frac{x}{y} - dx}{y-x} = \frac{d(y-x)}{y-x}$$

where the last equality follows from the fact that on the special fibre $\frac{dx}{y} = \frac{dy}{x}$ and since we are in characteristic 3, $\frac{1}{2} = -1$. Hence, if we denote by Y the reduction modulo $\mathfrak{p}$ of A, we have that the global 2-form on Y is given by

$$\omega = \frac{d\left(\frac{v-u}{w}\right)}{\left(\frac{v-u}{w}\right)} \wedge \frac{d\left(\frac{y-x}{z}\right)}{\left(\frac{y-x}{z}\right)}.$$

Finally, $\rho_0(\omega, 0) = \left[\left\{\frac{v-u}{w}, \frac{y-x}{z}\right\}\right]$ and hence again by Proposition 4.2.2 we get that

$$\mathrm{rsw}_{3,\pi}(\mathcal{A}) \neq (\bar{c}^{-1} \cdot \omega, 0)$$

and therefore $\mathcal{A} \notin \mathrm{fil}_2\mathrm{Br}(A)[3] \supseteq \mathrm{Ev}_{-1}\mathrm{Br}(A)[3]$. □

Claim 3: The cyclic algebra $\mathcal{A}$ in $\mathrm{Br}(A)[3]$ is not algebraic, i.e. $\mathcal{A} \notin \mathrm{Br}_1(A)[3]$.

Proof. If follows directly from Corollary 4.1.5. □

Finally, from [SZ12, Theorem 2.4] the map

$$\pi^* \colon \frac{\mathrm{Br}(X)[3]}{\mathrm{Br}_1(X)[3]} \hookrightarrow \frac{\mathrm{Br}(A)[3]}{\mathrm{Br}_1(A)[3]}$$

is an isomorphism. Let $\mathcal{B} \in \mathrm{Br}(X)[3]$ be such that $\pi^*(\mathcal{B}) = \mathcal{A} \in \mathrm{Br}(A)[3]$. Then, from Lemma 4.5.1 we get that $\mathcal{B} \notin \mathrm{Ev}_{-1}\mathrm{Br}(X)[3]$, namely (since for K3 surfaces $\mathrm{Ev}_0\mathrm{Br}(X) = \mathrm{Ev}_{-1}\mathrm{Br}(X)$, see Section 4.3) the corresponding evaluation map on $X(L)$ is non-constant.

Before proceeding with the next example we need a lemma that shows how the evaluation map behaves under base change without the assumption of good reduction for X.

Lemma 4.5.9. *Let X be a variety over a p-adic field L, not necessarily having good reduction, and let $\mathcal{A} \in \mathrm{Br}(X)\{p\}$ be such that $\mathrm{ev}_{\mathcal{A}} \colon X(L) \to \mathrm{Br}(L)$ is non-constant. Then for every field extension L'/L with degree co-prime to p we have that $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(X_{L'})$ has also a non-constant evaluation map $\mathrm{ev}_{\mathrm{res}(\mathcal{A})} \colon X_{L'}(L') \to \mathrm{Br}(L')$.*

Proof. Let $P, Q \in X(L)$ be such that

$$\mathrm{ev}_{\mathcal{A}}(P) \neq \mathrm{ev}_{\mathcal{A}}(Q).$$

Denote by P' and Q' the base change of P and Q to L', i.e. we have the following commutative diagrams

$$\begin{array}{ccc} X_{L'} & \xrightarrow{\quad\psi\quad} & X_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(L') & \xrightarrow{\varphi} & \mathrm{Spec}(L) \end{array} \qquad \begin{array}{ccc} \mathrm{Spec}(L') & \xrightarrow{P'} & X_{L'} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \mathrm{Spec}(L) & \xrightarrow{P} & X \end{array} \qquad \begin{array}{ccc} \mathrm{Spec}(L') & \xrightarrow{Q'} & X_{L'} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \mathrm{Spec}(L) & \xrightarrow{Q} & X. \end{array}$$

Then

$$\mathrm{ev}_{\mathrm{res}(\mathcal{A})}(P') = \mathrm{Br}(P')(\mathrm{Br}(\psi_L)(\mathcal{A}_0)) = \mathrm{Br}(\varphi_L)\mathrm{Br}(P)(\mathcal{A}) = \mathrm{Br}(\varphi_L)(\mathrm{ev}_{\mathcal{A}}(P)).$$

Finally since L'/L has degree co-prime to p the map $\mathrm{Br}(\varphi)\colon \mathrm{Br}(L) \to \mathrm{Br}(L')$ is injective on elements of p-order; hence

$$\mathrm{ev}_{\mathrm{res}(A)}(P') \neq \mathrm{ev}_{\mathrm{res}(A)}(Q').$$

□

Example 4.5.10. Let X be the diagonal quartic surface over $\mathbb{Q}_5$ defined by the equation:

$$5x^4 - 4y^4 = z^4 + w^4.$$

Skorobogatov and Ieronymou prove [IS15, Theorem 1.1], [IS15, Proposition 5.12] that there exists an element $\mathcal{A} \in \mathrm{Br}(X)[5]$ with surjective evaluation map. Let $L = \mathbb{Q}_5(\sqrt[4]{5})$, $e(L/\mathbb{Q}_5) = 4$ and $\alpha \in L$ be such that $\alpha^4 = 5$. Then the change of variables:

$$(x, y, z, w) \mapsto \left(\frac{x_1}{\alpha}, y_1, z_1, w_1\right)$$

sends X_L to the diagonal quartic $\tilde{X}/L$ given by the equation

$$x_1^4 - 4y_1^4 = z_1^4 + w_1^4.$$

The surface $\tilde{X}$ has good ordinary reduction over L. Finally, by Lemma 4.5.9 we know that $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(X_{L'}) = \mathrm{Br}(\tilde{X})$ has non-constant evaluation map.

Note that, at this point we have not write the algebra $\mathcal{A}$ as a cyclic algebra, and hence we are not able to show explicitly the link with the global logarithmic 2-forms on the special fibre and hence compute the refined Swan conductor of $\mathcal{A}$, even if from Remark 4.3.2 we know that $\mathcal{A} \in \mathrm{fil}_p\mathrm{Br}(X)[5]$.

4.6 Family of examples

We end this thesis by giving an example of a family of K3 surfaces.

Let $\alpha \in \bar{\mathbb{Q}}$ be such that $\alpha^2 \in \mathbb{Z}$ and let V_α be the K3 surface over $k := \mathbb{Q}(\alpha)$ defined by the equation

$$x^3y + y^3z + z^3w - w^4 + \alpha^2 \cdot xyzw - 2 \cdot \alpha^{-1} \cdot xzw^2 = 0. \tag{4.13}$$

Lemma 4.6.1. *The class of the quaternion algebra*

$$\mathcal{A} := \left(\frac{z^2 + \alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right) \in \mathrm{Br}(k(V_\alpha))$$

lies in $\mathrm{Br}(V_\alpha)[2]$.

Proof. Let $f := z^2 + \alpha^2 xy$ and C_x,C_z,C_f be the closed subsets of V_α defined by the equations $x = 0$, $z = 0$ and $f = 0$ respectively. The quaternion algebra $\mathcal{A}$ defines an element in $\mathrm{Br}(U)$, where $U := V_\alpha \setminus (C_x \cup C_z \cup C_f)$. The purity theorem for the Brauer group [CTS21, Theorem 3.7.2], assures us of the existence of the exact sequence

$$0 \to \mathrm{Br}(V_\alpha)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus\partial_D} \bigoplus_D \mathrm{H}^1(k(D), \mathbb{Z}/2) \tag{4.14}$$

where D ranges over the irreducible divisors of V_α with support in $X \setminus U$ and $k(D)$ denotes the residue field at the generic point of D.

In order to use the exact sequence (4.14) we need to understand what the prime divisors of V_α with support in $V_\alpha \setminus U = C_x \cup C_z \cup C_f$ look like. Using MAGMA [BCP97] it is possible to check the following:

- C_x has one irreducible component D_1 defined by the equations $\{x = 0, y^3z + z^3w + w^4 = 0\}$;
- C_z has one irreducible component D_2, defined by the equations $\{z = 0, x^3y - w^4 = 0\}$;
- C_f has one irreducible component D_3, defined by the equations $\{\alpha^2xy + z^2 = 0, x^3z^2 + y^2z^3 + 2\alpha x^2zw^2 + \alpha^2xw^4 = 0, \alpha^2y^3z - x^2z^2 - 2\alpha xzw^2 - \alpha^2w^4 = 0\}$.

Therefore, we can rewrite (4.14) in the following way:

$$0 \to \mathrm{Br}(V_\alpha)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus\partial_{D_i}} \bigoplus_{i=1}^{3} H^1(k(D_i), \mathbb{Z}/2). \tag{4.15}$$

Moreover, we have an explicit description of the residue map on quaternion algebras: for an element $(a, b) \in \mathrm{Br}(U)[2]$ we have

$$\partial_{D_i}(a, b) = \left[(-1)^{\nu_i(a)\nu_i(b)} \frac{a^{\nu_i(b)}}{b^{\nu_i(a)}}\right] \in \frac{k(D_i)^\times}{k(D_i)^{\times 2}} \simeq H^1(k(D_i), \mathbb{Z}/2) \tag{4.16}$$

where ν_i is the valuation associated to the prime divisor D_i. This follows from the definition of the tame symbols in Milnor K-theory together with the compatibility of the residue map with the tame symbols given by the Galois symbols (see [GS17], Proposition 7.5.1).

We can proceed with the computation of the residue maps ∂_{D_i} for $i = 1, \ldots, 3$:

1. $\nu_1(x) = 1$ and $\nu_1(f) = \nu_1(z) = 0$. Hence,

$$\partial_{D_1}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[\left(\frac{f}{z^2}\right)^{-1}\right] = 1 \in \frac{k(D_1)^\times}{k(D_1)^{\times 2}}$$

where the last equality follows from the fact that $x = 0$ on D_1, thus $f\mid_{D_1} = z^2$.

2. $\nu_2(z) = 1$ and $\nu_2(f) = \nu_2(x) = 0$. Hence,

$$\partial_{D_2}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[\left(\frac{f}{z^2}\right)\left(-\frac{z}{x}\right)^2\right] = \left[\frac{f}{x^2}\right] = 1 \in \frac{k(D_2)^\times}{k(D_2)^{\times 2}}$$

where the last equality follows from the fact that $z = 0$ and $x^3y = w^4$ on D_2, thus $f\mid_{D_2} = \alpha^2 xy$ and $\frac{\alpha^2 y}{x} = \alpha^2\left(\frac{w}{x}\right)^4 = \left(\alpha\frac{w^2}{x^2}\right)^2$.

3. $\nu_3(f) = 1$ and $\nu_3(x) = \nu_3(z) = 0$. Hence,

$$\partial_{D_3}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[-\frac{z}{x}\right] = \left[\left(\frac{\alpha^3 x}{z^3}\left(w^2 + \frac{xz}{\alpha}\right)\right)^2\right] = 1 \in \frac{k(D_3)^\times}{k(D_3)^{\times 2}}$$

where the last equality follows from the following equalities on D_3:

- $y^3z = w^4 + \frac{2}{\alpha}xzw^2 + \frac{x^2z^2}{\alpha^2} = \left(w^2 + \frac{xz}{\alpha}\right)^2$;
- $xy = -\frac{1}{\alpha^2}z^2$ implies that $y^3z = (xy)^3\frac{z}{x}\frac{1}{x^2} = -\frac{z}{x}\left(\frac{z^2}{\alpha^2}\right)^3\frac{1}{x^2} = \frac{-z}{x}\left(\frac{z^3}{\alpha^3 x}\right)^2$

Therefore, $\partial_{D_i}(\mathcal{A}) = 0$ for all $i \in \{1,2,3\}$, hence $\mathcal{A} \in \mathrm{Br}(V_\alpha)$. □

Let $\mathfrak{p}$ be a prime above 2 and $\mathcal{O}_\mathfrak{p}$ be the valuation ring of $k_\mathfrak{p}$. We have that $2\cdot\alpha^{-1} \in \mathcal{O}_\mathfrak{p}$ if and only if $\alpha^2 \not\equiv 0 \mod 8$; we can define $\mathcal{X}_\alpha$ to be the $\mathcal{O}_\mathfrak{p}$-scheme defined by equation (4.13). If $\alpha^2 \not\equiv 0 \mod 8$, then $\mathcal{X}_\alpha$ is smooth and hence V_α has good reduction at $\mathfrak{p}$, we denote by X_α the base change of V_α to $k_\mathfrak{p}$.

Theorem 4.6.2. *Assume that $\alpha^2 \not\equiv 0 \mod 8$. Then, $\mathcal{X}_\alpha$ has good ordinary reduction if and only if $\alpha^2 \equiv 1 \mod 2$. The evaluation map attached to $\mathcal{A}$*

$$\mathrm{ev}_\mathcal{A}\colon X_\alpha(k_\mathfrak{p}) \to \mathbb{Q}/\mathbb{Z}$$

is non-constant if and only if

$$\alpha^2 \not\equiv 0 \mod 4.$$

Proof. Recall that we know that for K3 surfaces the evaluation map attached to $\mathcal{A}$ is non-constant if and only if $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X_\alpha)$ (cf.Section 4.3).

- If $\alpha^2 \equiv 1 \mod 2$, then the special fibre Y_α is defined by the equation
$$x^3y + y^3z + z^3w + w^4 + xyzw.$$
From Lemma 4.3.1 we get that Y_α is an ordinary K3 surface. From Lemma 4.3.3 we know that a generator (as $k(\mathfrak{p})$-vector space) of $\mathrm{H}^0(Y_\alpha, \Omega^2_{Y_\alpha})$ is given by the global 2-form ω that can be written (locally) as
$$\frac{d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)}{\frac{z^3+xyz}{x^3}} = \frac{x^2}{z^2+xy} d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)\frac{x}{z} =$$
$$\left(\frac{x^2}{z^2+xy}\right) \cdot d\left(\frac{z^2+xy}{x^2}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right)$$
where the last equality follows from $d\left(\frac{z^2+xy}{x^2}\right) \wedge d\left(\frac{z}{x}\right) = d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)$. Hence, we can write ω as $\frac{df}{f} \wedge \frac{dg}{g}$, with $f = \frac{z^2+xy}{x^2}$, $g = \frac{z}{x}$. Finally, we see that, by Proposition 4.2.1
$$[\mathcal{A}] = \left[\left\{\frac{z^2+\alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right\}\right] = \left[\left\{\frac{z^2+\alpha^2 \cdot xy}{x^2}, -\frac{z}{x}\right\}\right] = \rho_0(\omega, 0)$$
since $\frac{z^2+\alpha^2 \cdot xy}{x^2}$ and $-\frac{z}{x}$ are lifts to characteristic 0 of f and g, respectively. Hence, using Proposition 4.2.2 we get that $\mathrm{rsw}_{e',\pi}(\mathcal{A}) \neq (0,0)$ and $\mathcal{A} \notin \mathrm{fil}_{e'-1}\mathrm{Br}(X_\alpha) \supseteq \mathrm{fil}_0\mathrm{Br}(X_\alpha)$.

- If $\alpha^2 \equiv 2 \mod 4$, then the special fibre Y_α is defined by the equation
$$x^3y + y^3z + z^3w + w^4$$
From Lemma 4.3.1 we get that Y_α is a non-ordinary K3 surface over $k(\mathfrak{p})$. From Lemma 4.3.3 we know that $\mathrm{H}^0(Y_\alpha, \Omega^2_{Y_\alpha})$ is generated (as a $k(\mathfrak{p})$-vector space) by the 2-form ω that can be written (locally) as
$$\frac{d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)}{\frac{z^3}{x^3}} = \left(\frac{x^2}{z^2}\right) \cdot d\left(\frac{y}{x}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right) = d\left(\frac{xy}{z^2}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right)$$
where the last equality follow from the fact that, since we are working over a field of characteristic 2, $\left(\frac{x}{z}\right)^2 d\left(\frac{y}{x}\right) = d\left(\frac{xy}{z^2}\right)$. Hence, in this case we can write ω as $d\left(f \cdot \frac{dg}{g}\right)$, with $f = \frac{xy}{z^2}$, $g = \frac{z}{x}$. Since $\alpha^2 \equiv 2 \mod 4$, the prime ideal (2) is ramified in the field extension $\mathbb{Q}_2(\alpha)/\mathbb{Q}_2$ and $\pi = \alpha$ is a uniformiser. From Proposition 4.2.1 we get that
$$\left[\left\{\frac{z^2+\alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right\}\right] = \left[\left\{1+\alpha^2\frac{xy}{z^2}, -\frac{z}{x}\right\}\right] = \rho_2\left(f \cdot \frac{dg}{g}\right)$$
since $\frac{xy}{z^2}$ and $-\frac{z}{x}$ are two lifts to characteristic 0 of f and g, respectively. Hence, by Proposition 4.2.2 $\mathrm{rsw}_{2,\pi}(\mathcal{A}) \neq (0,0)$ and hence $\mathcal{A} \notin \mathrm{fil}_1\mathrm{Br}(X_\alpha)$ and thus $\mathcal{A} \notin \mathrm{fil}_0\mathrm{Br}(X_\alpha) \subseteq \mathrm{fil}_1\mathrm{Br}(X_\alpha)$.

- If $\alpha^2 \equiv 0 \mod 4$, then the special fibre Y_α is defined by the equation

$$x^3y + y^3z + z^3w + w^4 + xzw^2.$$

Again, from Lemma 4.3.1 we get that Y_α is a non-ordinary K3 surface over $k(\mathfrak{p})$. If $\mathbb{Q}_2(\alpha)/\mathbb{Q}_2$ is unramified then we are done by Theorem 3.2.1. If the field extension is ramified and π is a uniformiser, then since $\alpha^2 \equiv 0 \mod 4$ and $\alpha^2 \not\equiv 0 \mod 8$, we have that $\alpha^2 = \pi^4\beta$ with $\beta \in \mathcal{O}_\mathfrak{p}^\times$. Hence, if we look at the corresponding element in $k^2(K^h)$ via the isomorphism of equation (4.3), we have that

$$\mathcal{A} \mapsto \left\{ \frac{z^2 + \alpha^2 \cdot xy}{z^2}, -\frac{z}{x} \right\} = \left\{ 1 + \pi^4 \cdot \frac{\beta \cdot xy}{z^2}, -\frac{z}{x} \right\} \in U^4 k^2(K^h).$$

Hence, using again Proposition 4.2.2, $\mathcal{A} \in \mathrm{fil}_0\mathrm{Br}(X_\alpha)[2]$.

□

Remark 4.6.3. The case $\alpha^2 \equiv 2 \mod 4$ proves that the bound appearing in Theorem 3.2.1 is optimal. In fact, we are able to find examples of K3 surface V over a quadratic field extensions of $\mathbb{Q}$ such that there is a prime above 2 whose ramification index is $e(\mathfrak{p}/2) = 2$ and that plays a role in the Brauer–Manin obstruction to weak approximation.

Finally, note that when $\alpha^2 \equiv 0 \mod 4$ and $e(\mathfrak{p}/2) = 1$, we know from Theorem 3.2.1 that there is an equality $\mathrm{Ev}_{-1}\mathrm{Br}(X_\alpha) = \mathrm{Br}(X_\alpha)$. However, if $e(\mathfrak{p}/2) = 2$, we just showed that the element $\mathcal{A}$ of the previous theorem lies in $\mathrm{Ev}_{-1}\mathrm{Br}(X_\alpha)[2]$ and not that $\mathrm{Br}(X_\alpha) = \mathrm{Ev}_{-1}\mathrm{Br}(X_\alpha)$.

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Summary

This thesis is on the study of rational points on varieties. This kind of problem stems from the desire to be able to describe the rational solutions of a polynomial, i.e. given a polynomial $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, what can we say about

$$\mathrm{Z}(f)(\mathbb{Q}) := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n \mid f(\alpha_1, \ldots, \alpha_n) = 0\}?$$

A first way to study this set is by looking at the polynomial equation over the real numbers. In fact, if there is no $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $f(\alpha_1, \ldots, \alpha_n) = 0$ then $\mathrm{Z}(f)(\mathbb{Q})$ is also the empty set. The real numbers are a complete field, which makes the study of the zeros of functions defined over it much more accessible. However, $\mathbb{R}$ is just one of the possible completions of $\mathbb{Q}$, the one with respect to the euclidean metric. The p-adic metrics give for every prime p a complete field $\mathbb{Q}_p$ that contains $\mathbb{Q}$. Putting all this together gives a natural inclusion

$$\mathrm{Z}(f)(\mathbb{Q}) \hookrightarrow \prod_{p \text{ prime}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R}).$$

Moreover, the sets $\mathrm{Z}(f)(\mathbb{Q}_p)$ and $\mathrm{Z}(f)(\mathbb{R})$ are subsets of $\mathbb{Q}_p^n$ and $\mathbb{R}^n$, hence they inherit a topology coming from the topology on $\mathbb{Q}_p$ and $\mathbb{R}$ respectively. I am particularly interested in understanding the closure of the set $\mathrm{Z}(f)(\mathbb{Q})$ inside the product $\prod_{p \text{ prime}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R})$.

In the language of algebraic geometry, the zero set of the polynomial f defines a variety X over the rational numbers. The biggest advantage of looking at the image of $X(\mathbb{Q})$ inside $\prod_{p \text{ prime}} X(\mathbb{Q}_p) \times X(\mathbb{R})$ is that the latter is more accessible. In fact, both over $\mathbb{R}$ and over $\mathbb{Q}_p$, we have Newton's methods that allows to build solutions to polynomial equations. Moreover, if $X(\mathbb{Q})$ is dense in $X(\mathbb{Q}_p)$ for a given prime p, then combining this with Hensel's Lemma we get that we can lift solutions modulo p to solutions over the rational numbers. For example in the case of $\mathrm{Z}(f)$, with $f \in \mathbb{Z}[x_1, \ldots, x_n]$ a solution modulo p is an n-uple $(\beta_1, \ldots, \beta_n) \in (\mathbb{Z}/p\mathbb{Z})^n$ such that

$$f(\beta_1, \ldots, \beta_n) = 0 \mod p.$$

Unfortunately, very often it is too optimistic to hope for the density of $X(\mathbb{Q})$ in $\prod_{p \text{ prime}} X(\mathbb{Q}_p) \times X(\mathbb{R})$.

In this thesis we work with smooth and proper varieties over number fields. In this case, instead of the euclidean metric we have the archimedean places, while instead of p-adic metric we have the non-archimedean places associated to prime ideals in the ring of integers of k. We denote by Ω_k the set of places of k and for every place $\nu \in \Omega_k$, we denote by k_ν the completion of k at ν.

In 1970 Manin introduced the use of the Brauer group of a variety X, to build a subset $X(k_\Omega)^{\mathrm{Br}}$ of

$$\prod_{\nu \in \Omega_k} X(k_\nu) =: X(k_\Omega)$$

such that

$$X(k) \subseteq \overline{X(k)} \subseteq X(k_\Omega)^{\mathrm{Br}} \subseteq X(k_\Omega).$$

The key point is that this set gives a better approximation of $X(k)$ inside the product $X(k_\Omega)$ and even if its construction is more involved then the on of $X(k_\Omega)$, it is still more accessible then the set of rational points $X(k)$. Assume that the Brauer–Manin set is non-empty and that a prime $\mathfrak{p}$ is such that

$$X(k_\Omega)^{\mathrm{Br}} = X(k_\mathfrak{p}) \times Z$$

with $Z \subseteq \prod_{\nu \neq \mathfrak{p}} X(k_\nu)$. Then we say that the prime ideal $\mathfrak{p}$ **does not play a role** in the Brauer–Manin obstruction.

My research is inspired by the following question:

Question. *Assume* $\mathrm{Pic}(\bar{X})$ *to be torsion-free and finitely generated. Which primes can play a role in the Brauer–Manin obstruction to weak approximation on* X*?*

This question is inspired by a question originally asked by Swinnerton–Dyer; he asked whether under the assumption of question above the only places that can play a role in the Brauer–Manin obstruction to weak approximation are the one of bad reduction for the variety. Roughly speaking, if X is defined over the rational numbers, a prime p is of good reduction for X if we can define X by polynomial with integer coefficients whose reduction modulo p define a smooth variety over the finite field $\mathbb{F}_p$. The wish is to be able to identify all the primes that play a role in the Brauer–Manin obstruction to weak approximation in order to describe the Brauer–Manin set.

The second chapter of this thesis is devoted to exhibiting the first example of a K3 surface[1] defined over $\mathbb{Q}$ for which a prime of good reduction plays a role in the Brauer–Manin obstruction to weak approximation.

Theorem. *Let* $X \subseteq \mathbb{P}^3_{\mathbb{Q}}$ *be the projective* K3 *surface defined by the equation*

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0.$$

[1]K3 surfaces are varieties that satisfy the assumption of the question.

Then the prime 2 *is of good reduction for* X *and plays a role in the Brauer–Manin obstruction to weak approximation.*

At this point some natural questions arise:

1. In the theorem above the prime 2 is of good ordinary reduction. Is the ordinary condition needed?

2. What happens over number fields?

Answering these two questions is the aim of Chapter 3 and 4. In particular, in Chapter 3 I proved the following results.

Theorem. *Let* X *be a* $K3$ *surface and* $\mathfrak{p}$ *be a prime of good non-ordinary reduction for* X *with* $e_{\mathfrak{p}} \leq (p-1)$. *Then the prime* $\mathfrak{p}$ *does not play a role in the Brauer–Manin obstruction to weak approximation on* X.

In Chapter 4, I show that the bound $e_{\mathfrak{p}} \leq (p-1)$ appearing in the theorem above is optimal.

Theorem. *Let* $\mathfrak{p}$ *be a prime of good ordinary reduction for* X *of residue characteristic* p. *Assume that the special fibre* Y *has no non-trivial global* 1*-forms,* $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$ *and* $(p-1) \nmid e_{\mathfrak{p}}$. *Then the prime* $\mathfrak{p}$ *does not play a role in the Brauer–Manin obstruction to weak approximation on* X.

Moreover, in Chapter 4 I also show that the condition $(p-1) \nmid e_{\mathfrak{p}}$ in the last theorem is sufficient but not necessary.

All these results go in the direction of having a better understanding of the Brauer–Manin set and hence of the set of rational points on varieties. In my results there is an emphasis on K3 surfaces, which are one of the first kind of varieties (in terms of complexity of the geometry) for which very little is known about the arithmetic (i.e. the set of rational points).

Samenvatting

Dit proefschrift gaat over de studie van rationale punten op variëteiten. Dit soort problemen ontstaat uit de behoefte om de rationale oplossingen van een polynoom te kunnen beschrijven: gegeven een polynoom $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, wat kunnen we zeggen over

$$\mathrm{Z}(f)(\mathbb{Q}) := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n \mid f(\alpha_1, \ldots, \alpha_n) = 0\}?$$

Een manier om deze verzameling te bestuderen is door te kijken naar de polynoomvergelijking over de reële getallen. In feite, als er geen $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ is zodat $f(\alpha_1, \ldots, \alpha_n) = 0$, dan is $\mathrm{Z}(f)(\mathbb{Q})$ ook de lege verzameling. De reële getallen vormen een compleet lichaam, wat de studie van de nulpunten van functies die erover gedefinieerd zijn veel toegankelijker maakt. Echter, $\mathbb{R}$ is slechts een van de mogelijke complete lichamen van $\mathbb{Q}$, namelijk die met betrekking tot de euclidische metriek. De p-adische metrieken geven voor elke priemgetal p een compleet lichaam, dat $\mathbb{Q}$ bevat. Als we dit allemaal samenvoegen, krijgen we een natuurlijke inclusie

$$\mathrm{Z}(f)(\mathbb{Q}) \hookrightarrow \prod_{p \text{ priem}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R}).$$

Bovendien zijn de verzamelingen $\mathrm{Z}(f)(\mathbb{Q}_p)$ en $\mathrm{Z}(f)(\mathbb{R})$ deelverzamelingen van respectievelijk $\mathbb{Q}_p^n$ en $\mathbb{R}^n$, dus erven ze een topologie die voortkomt uit de topologie op $\mathbb{Q}_p$ en $\mathbb{R}$ respectievelijk. Ik ben met name geïnteresseerd in het begrijpen van de afsluiting van de verzameling $\mathrm{Z}(f)(\mathbb{Q})$ binnen het product $\prod_{p \text{ priem}} \mathrm{Z}(f)(\mathbb{Q}_p) \times \mathrm{Z}(f)(\mathbb{R})$.

In de taal van de algebraïsche meetkunde, definieert de nulpuntsverzameling van het polynoom f een variëteit X over de rationale getallen. Het grootste voordeel van het kijken naar het beeld van $X(\mathbb{Q})$ binnen $\prod_{p \text{ priem}} X(\mathbb{Q}_p) \times X(\mathbb{R})$ is dat de tweede gemakkelijker te begrijpen is. In feite hebben we zowel over $\mathbb{R}$ als over $\mathbb{Q}_p$ Newton-methoden die het mogelijk maken oplossingen voor polynoomvergelijkingen te construeren. Bovendien, als $X(Q)$ dicht is in $X(Q_p)$ voor een gegeven priemgetal p, dan krijgen we in combinatie met het lemma van Hensel dat we

de oplossing modulo p kunnen liften naar oplossingen over de rationale getallen. Bijvoorbeeld in het geval van $Z(f)$, met $f \in \mathbb{Z}[x_1, \ldots, x_n]$, is een oplossing modulo p een n-tupel $(\beta_1, \ldots, \beta_n) \in (\mathbb{Z}/p\mathbb{Z})^n$ zodat

$$f(\beta_1, \ldots, \beta_n) = 0 \mod p.$$

Helaas is het vaak te optimistisch om te hopen op de dichtheid van $X(\mathbb{Q})$ in $\prod_{p \text{ priem}} X(\mathbb{Q}_p) \times X(\mathbb{R})$.

In dit proefschrift werken we met gladde en propere variëteiten over getallenlichamen. In dit geval hebben we in plaats van de euclidische metriek de archimedische plaatsen, en in plaats van de p-adische metriek hebben we de niet-archimedische plaatsen die geassocieerd zijn met priemidealen in de ring van gehele getallen van k. We duiden met Ω_k de verzameling plaatsen van k aan en voor elke plaats $\nu \in \Omega_k$ duiden we met k_ν de completering van k bij ν aan.

In 1970 introduceerde Manin het gebruik van de Brauer groep van een variëteit X, om een deelverzameling $X(k_\Omega)^{\mathrm{Br}}$ te construeren van

$$\prod_{\nu \in \Omega_k} X(k_\nu) =: X(k_\Omega)$$

zodanig dat

$$X(k) \subseteq \overline{X(k)} \subseteq X(k_\Omega)^{\mathrm{Br}} \subseteq X(k_\Omega).$$

Het sleutelpunt is dat de Brauer-Manin verzameling een betere approximatie geeft van $X(k)$ binnen het product $X(k_\Omega)$ en zelfs als de constructie ervan ingewikkelder is dan die van $X(k_\Omega)$, het toegankelijker is dan de verzameling rationale punten $X(k)$. Stel dat de Brauer-Manin-verzameling niet leeg is. Als $\mathfrak{p}$ een priemgetal is zodanig dat

$$X(k_\Omega)^{\mathrm{Br}} = X(k_\mathfrak{p}) \times \prod_{\nu \neq \mathfrak{p}} Z$$

met $Z \subseteq \prod_{\nu \neq \mathfrak{p}} X(k_\nu)$. Dan zeggen we dat het priemideaal $\mathfrak{p}$ **geen rol speelt** in de Brauer–Manin-obstructie.

Mijn onderzoek is geïnspireerd door de volgende vraag.

Vraag. *Stel* $\mathrm{Pic}(\bar{X})$ *torsievrij en eindig voortgebracht. Welke priemgetallen kunnen een rol spelen in de Brauer–Manin-obstructie voor zwakke approximatie op* X*?*

Deze vraag is geïnspireerd door een vraag die oorspronkelijk gesteld werd door Swinnerton–Dyer; hij vroeg zich af of onder de aanname van de vraag de enige plaatsen die een rol kunnen spelen in de Brauer–Manin obstructie voor zwakke approximatie degene zijn met slechte reductie voor de variëteit. In grote lijnen, als X gedefinieerd is over de rationale getallen, is een priemgetal p van goede reductie voor X als we X kunnen definiëren door polynomen met gehele coëfficiënten waarvan de reductie modulo p een gladde variëteit definieert over het eindige lichaam $\mathbb{F}_p$. De wens is om alle priemgetallen te kunnen identificeren die een rol spelen in de Brauer–Manin-obstructie voor zwakke approximatie om zo de Brauer–Manin-verzameling te kunnen beschrijven.

Het tweede hoofdstuk van dit proefschrift is gewijd aan het presenteren van het eerste voorbeeld van een K3-oppervlak[2] gedefinieerd over $\mathbb{Q}$ waarin een priemgetal van goede reductie een rol speelt in de Brauer–Manin-obstructie voor zwakke approximatie.

Stelling. *Laat $X \subseteq \mathbb{P}^3_{\mathbb{Q}}$ het projectieve K3-oppervlak zijn gedefinieerd door de vergelijking*

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0.$$

Dan is het priemgetal 2 van goede reductie voor X en speelt het een rol in de Brauer–Manin-obstructie voor zwakke approximatie.

Op dit punt komen enkele natuurlijke vragen naar boven:

1. In het bovenstaande stelling is het priemgetal 2 van goede gewone reductie. Is de gewone voorwaarde nodig?

2. Wat gebeurt er over getallenlichamen?

Het beantwoorden van deze twee vragen is het doel van Hoofdstuk 3 en 4. In het bijzonder bewijs ik in Hoofdstuk 3 de volgende resultaten.

Stelling. *Laat X een K3-oppervlak zijn en $\mathfrak{p}$ een priemgetal van goede niet-gewone reductie voor X met $e_{\mathfrak{p}} \leq (p-1)$. Dan speelt het priemgetal $\mathfrak{p}$ geen rol in de Brauer–Manin-obstructie voor zwakke approximatie op X.*

In Hoofdstuk 4 laat ik zien dat de grens $e_{\mathfrak{p}} \leq (p-1)$ zoals vermeld in de bovenstaande stelling optimaal is.

Stelling. *Laat $\mathfrak{p}$ een priemgetal van goede gewone reductie zijn voor X van restklassekarakteristiek p. Stel dat de speciale vezel Y geen niet-triviale globale 1-vormen heeft, $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$ en $(p-1) \nmid e_{\mathfrak{p}}$. Dan speelt het priemgetal $\mathfrak{p}$ geen rol in de Brauer–Manin-obstructie voor zwakke approximatie op X.*

In hoofdstuk 4 laat ik ook zien dat de voorwaarde $(p-1) \nmid e_{\mathfrak{p}}$ in de laatste stelling voldoende maar niet noodzakelijk is.

Al deze resultaten gaan in de richting van een beter begrip van de Brauer–Manin-verzameling en dus van de verzameling rationale punten op variëteiten. In mijn resultaten ligt de nadruk op K3-oppervlakken, die een van de eerste soorten variëteiten zijn (in termen van complexiteit van de meetkunde) waarover heel weinig bekend is over de aritmetiek (d.w.z. de verzameling rationale punten).

[2] K3-oppervlakken zijn variëteiten die voldoen aan de aanname in de vraag.

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The tiger on the cover is from a shot by Martin Bright in Kabini, India. The cover was realised by the illustrator Nicole Vascotto. Many thanks to both of them.

Curriculum Vitae

Margherita Pagano was born in Rome, Italy in 1997. She got her high school diploma from the European School of Varese.

In 2015 she enrolled at the University of Milano Bicocca, where she obtained her bachelor degree *with honour* with a thesis on the complexification of compact Lie groups under the supervision of Prof. Roberto Paoletti.

She then joined the Algant Master Program, spending one year at the University of Milan and one year at Leiden University. She wrote her thesis under the supervision of Dr. Martin Bright, entitled *Strong approximation on some punctured affine cones*. She was awarded her Masters diploma *cum laude* by the University of Milano and *summa cum laude* by Leiden University.

She continued her studies by starting a PhD in 2020 at Leiden University under the supervision of Dr. Martin Bright.

After her PhD she will start a post-doc at the Imperial College of London.