Motivic invariants of character stacks

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Summary

This thesis studies the geometry of representation varieties and character stacks. These are spaces that parametrize the representations of a finitely generated group Γ into an algebraic group G. More precisely, the representation variety parametrizes all such representations, whereas the character stack parametrizes them up to isomorphism. Usually, the finitely generated group Γ is the fundamental group of a compact manifold M, in which case the representation variety and character stack equivalently parametrize G-local systems on M. This thesis contains a number of methods to study these spaces through their invariants. Besides providing theoretical descriptions, another aim of this thesis is to explicitly compute these invariants in specific cases. Motivated by these applications, we develop a number of new computational tools.

In Chapter 1, we review the necessary background on groupoids and algebraic stacks, focusing in particular on quotient stacks and stabilizers. We use this theory in Chapter 2, where we give precise definitions of representation varieties and character stacks. Furthermore, we show that these spaces admit a number of functorial properties that are crucial for the later parts of the thesis.

In Chapter 3, we study *motivic invariants*, which are invariants χ of varieties that are additive and multiplicative in the sense that $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ and $\chi(X \times Y) = \chi(X)\,\chi(Y)$ for all varieties X and Y and closed subvarieties $Z \subseteq X$. We discuss various motivic invariants and their properties, with a special focus on the universal motivic invariant, called the *virtual class*, which takes values in the Grothendieck ring of varieties. This Grothendieck ring has a natural generalization to algebraic stacks, allowing us to talk about the virtual class, and other motivic invariants, of character stacks. Furthermore, we develop tools for computing motivic invariants, such as an algorithm to compute virtual classes of certain varieties, and we study how motivic invariants behave with respect to finite group actions.

In Chapter 4, we describe two known methods for computing motivic invariants of representation varieties and character stacks. We show how both the *arithmetic*

method, which studies the character stacks of compact orientable surfaces through counting points over finite fields, and the *geometric method*, which studies the same character stacks using clever stratifications, can be expressed in terms of *Topological Quantum Field Theories (TQFTs)*. Originating from physics, TQFTs are monoidal functors from the category of bordisms to the category of modules over a fixed commutative ring. The TQFTs associated to both methods can be expressed as the composite of a field theory and a quantization functor. Comparing the field theories and quantization functors of both methods, we show that the TQFTs of both methods can be related through natural transformations.

In Chapter 5, we apply the theory of Chapter 4 to explicitly compute the virtual classes of the SL_2-character stacks of orientable and non-orientable surfaces. This results in many intricate computations. Even though similar computations already exist that compute the E-polynomial (an invariant coarser than the virtual class, reflecting the mixed Hodge structure) of these character stacks, adapting these computations to the Grothendieck ring of varieties introduces many subtle problems which we deal with.

In Chapter 6, we focus on the groups G of $n \times n$ upper triangular matrices and unipotent upper triangular matrices. By means of computer-assisted calculations, we compute the virtual classes of the G-character stacks of orientable surfaces for $n \leq 5$ through the geometric method, and their E-polynomials for $n \leq 10$ through the arithmetic method. This task, which is already difficult for small n, was made possible by introducing the notion of algebraic representatives, and using the theory of special algebraic groups. Comparing the arithmetic and geometric method, we show how the geometric method can be simplified significantly using the results from the arithmetic method, that is, using the representation theory of the groups of upper triangular matrices over finite fields.

Finally, in Chapter 7, we turn our attention to the representation varieties and character stacks of the free groups F_n and free abelian groups $\mathbb{Z}^n$. These spaces parametrize tuples (resp. commuting tuples) of elements of G. It is known that the homology of these spaces, and many variations thereof, stabilize as n tends to infinity, in a well-defined sense known as *representation stability*. Inspired by this notion, we define an analogous notion of *motivic representation stability* for stability in the Grothendieck ring of varieties. As an application, we show that the character stacks of F_n and $\mathbb{Z}^n$ stabilize in this sense for the linear groups $G = \mathrm{GL}_r$.