Motivic invariants of character stacks

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Chapter 7

Motivic stability

Let Γ_n be a sequence of finitely generated groups, and let G be an algebraic group over a field k. One can wonder whether the invariants of the corresponding sequence of character stacks $\mathfrak{X}_G(\Gamma_n)$ are related. We will mainly focus on the sequence $\Gamma_n = \mathbb{Z}^n$ of free abelian groups and the sequence $\Gamma_n = F_n$ of free groups, for which the character stacks parametrize (commuting) tuples of elements in G up to conjugation.

Geometric invariants of these and related spaces have been studied extensively [Bai07, AC07, PS13, FL14]. For X_n the sequence of G-representation varieties or G-character varieties of $\mathbb{Z}^n$, the homology groups $H_k(X_n)$ were computed in [RS19], and their mixed Hodge structures in [FS21]. A pattern emerged: fixing n and varying G through sequences G_r of classical groups (such as GL_r or U_r), the homology groups $H_k(X_n)$ remain constant for sufficiently large r, that is, they *stabilize*. This pattern was proved in [RS21], as well as for fixed G and increasing n, and for many related sequences X_n. Moreover, taking into account the action of the symmetric group S_n on $\mathbb{Z}^n$ by permutation, inducing an action of S_n on X_n and in turn on $H_k(X_n)$, they showed the homology groups stabilize as S_n-representations. This type of stability, called *representation stability*, was formulated in [CF13]: a sequence V_n of S_n-representations is representation stable, roughly speaking, if the multiplicities of the irreducible representations V_λ, corresponding to the partitions λ of n, stabilize. Partitions for n and $n+1$ are related by increasing the first number.

In this chapter, we combine the notion of representation stability with that of *motivic stability*. Completing the Grothendieck ring of varieties, one can study the convergence of a sequence of virtual classes. Such convergence was studied in [VW15] for sequences of symmetric powers $\operatorname{Sym}^n X$ (as an algebraic analogue of the Dold–Thom theorem) and sequences of configuration spaces $\operatorname{Conf}^n X$.

Using the theory of Section 3.6, we will generalize the notion of motivic stability, and introduce the concept of *motivic representation stability.* As an application, we will show that the sequence of GL_r-character stacks $\mathfrak{X}_{\mathrm{GL}_r}(\mathbb{Z}^n)$, with the action of S_n, is motivically representation stable.

7.1 Motivic stability

Motivic stability is a property of a sequence of varieties, which amounts to the convergence (in some sense) of their virtual classes in the topological ring $\widehat{\mathcal{M}_{\mathbb{L}}}$, which is the completion of the localization $\mathrm{K}_0(\mathbf{Var}_k)[\mathbb{L}^{-1}]$ of the Grothendieck ring of varieties. This topological ring was originally constructed by Kontsevich in the context of motivic integration [Kon95]. For more information on this object, we refer to [Bou11, Loo02, VW15].

For our applications, we adapt the standard definitions to the equivariant setting. Throughout, fix an algebraic group G over k, and denote by $\mathbb{L}$ the class $[\mathbb{A}^1_k] \in \mathrm{K}_0(\mathbf{Var}^G_k)$ of $\mathbb{A}^1_k$ on which G acts trivially.

Definition 7.1.1. Write $\mathcal{M}^G_{\mathbb{L}}$ for the localization $\mathrm{K}_0(\mathbf{Var}^G_k)[\mathbb{L}^{-1}]$. Consider the increasing filtration on $\mathcal{M}^G_{\mathbb{L}}$,

$$0 \subseteq \cdots \subseteq F_n\mathcal{M}^G_{\mathbb{L}} \subseteq F_{n+1}\mathcal{M}^G_{\mathbb{L}} \subseteq \cdots \subseteq \mathcal{M}^G_{\mathbb{L}},$$

where $F_n\mathcal{M}^G_{\mathbb{L}}$ is the subgroup of $\mathcal{M}^G_{\mathbb{L}}$ generated by all elements of the form $[X]/\mathbb{L}^m$ with $\dim X - m \le n$. Note that $\bigcup_{n\in\mathbb{Z}} F_n\widehat{\mathcal{M}^G_{\mathbb{L}}} = \mathcal{M}^G_{\mathbb{L}}$. The completion with respect to this filtration is denoted

$$\widehat{\mathcal{M}^G_{\mathbb{L}}} = \varprojlim_n \mathcal{M}^G_{\mathbb{L}}/F_n\mathcal{M}^G_{\mathbb{L}}.$$

An element $x \in \widehat{\mathcal{M}^G_{\mathbb{L}}}$ can be represented as a tuple $(x_n) \in \prod_{n\in\mathbb{Z}} \mathcal{M}^G_{\mathbb{L}}/F_n\mathcal{M}^G_{\mathbb{L}}$ such that $x_n \equiv x_m \mod F_n\mathcal{M}^G_{\mathbb{L}}$ for all $m \le n$.

The completion $\widehat{\mathcal{M}^G_{\mathbb{L}}}$ inherits, a priori, only the group structure from $\mathcal{M}^G_{\mathbb{L}}$. Multiplication is defined as follows. Let $x = (x_n)$ and $y = (y_n)$ be elements of $\widehat{\mathcal{M}^G_{\mathbb{L}}}$. Note that there exists a sufficiently large N such that $x_n = y_n = 0$ for all $n \ge N$. Now define xy by $(xy)_n = x'_{n-N}y'_{n-N} \mod F_n\mathcal{M}^G_{\mathbb{L}}$, where $x'_{n-N}, y'_{n-N} \in \mathcal{M}^G_{\mathbb{L}}$ are lifts x_{n-N} and y_{n-N}, respectively. This is independent of the choice of lift since, for any other lift x''_{n-N}, we have $x''_{n-N}y'_{n-N} - x'_{n-N}y'_{n-N} = (x''_{n-N} - x'_{n-N})y'_{n-N} \in F_{n-N}\mathcal{M}^G_{\mathbb{L}} \cdot F_N\mathcal{M}^G_{\mathbb{L}} \subseteq F_n\mathcal{M}^G_{\mathbb{L}}$. Similarly, it is independent of the choice of lift y'_{n-M}. This gives $\widehat{\mathcal{M}^G_{\mathbb{L}}}$ a ring structure.

Definition 7.1.2. Let X be a G-variety over k. For any $n \ge 0$, the *n-th G-symmetric power* of X, denoted $\mathrm{Sym}^n_G X$, is the G-variety given by the ordinary symmetric power $\operatorname{Sym}^n X = X^n /\!\!/ S_n$ with the action of G induced by the diagonal action on X^n.

Definition 7.1.3. Let X be a G-variety over k. The symmetric powers $\operatorname{Sym}^n_G X$ of X are called *motivically stable* if the limit

$$\lim_{n\to\infty} \frac{[\operatorname{Sym}^n_G X]}{\mathbb{L}^{n \dim X}}$$

exists in $\widehat{\mathcal{M}^G_{\mathbb{L}}}$. More generally, a sequence X_n of G-varieties over k is *motivically stable* if the limit

$$\lim_{n\to\infty} \frac{[X_n]}{\mathbb{L}^{\dim X_n}}$$

exists in $\widehat{\mathcal{M}^G_{\mathbb{L}}}$.

Example 7.1.4. When $G = 1$ is the trivial group, we simply write $\widehat{\mathcal{M}_{\mathbb{L}}}$ instead of $\widehat{\mathcal{M}^G_{\mathbb{L}}}$. In this case, the following sequences are motivically stable.

- From Example 3.3.5, we see that the sequence $X_n = \mathrm{GL}_n$ is motivically stable, with limit $\lim_{n\to\infty}[\mathrm{GL}_n]/\mathbb{L}^{n^2} = \prod_{i\geq 1}(1 - \mathbb{L}^{-i})$.
- Similarly, $X_n = \mathrm{SL}_n$ is motivically stable with limit $\lim_{n\to\infty}[\mathrm{SL}_n]/\mathbb{L}^{n^2-1} = \prod_{i\geq 2}(1 - \mathbb{L}^{-i})$. Since $[\mathrm{PGL}_n] = [\mathrm{SL}_n]$, the sequence $X_n = \mathrm{PGL}_n$ is also motivically stable, with the same limit.
- It is still an open conjecture [VW15, Conjecture 1.25] whether the symmetric powers of all geometrically irreducible varieties are motivically stable. However, some evidence has been presented against it [Lit14].

Example 7.1.5. Let us give some intuition for what motivic stabilization implies about the cohomology of X_n. Suppose X_n is a sequence of varieties over $k = \mathbb{C}$. Note that the E-polynomial descends to a continuous morphism

$$e\colon \widehat{\mathcal{M}^G_{\mathbb{L}}} \to \mathbb{Z}[u, v][\![(uv)^{-1}]\!]$$

where the target is equipped with the $(uv)^{-1}$-adic topology. Since

$$e([X_n]/\mathbb{L}^{\dim X_n}) = \sum_{k,p,q\in\mathbb{Z}} (-1)^k h_c^{k;p,q}(X_n)\, u^{p-\dim X_n} v^{q-\dim X_n},$$

it follows, if the sequence X_n motivically stabilizes, that, for all p and q, the numbers $h_c^{k;\dim X_n - p, \dim X_n - q}(X_n)$ are eventually constant as $n \to \infty$. If the X_n are smooth projective, then evaluating in $u = v = t$, it also follows that the dimensions $\dim_{\mathbb{C}} H_c^{\dim X_n - k}(X_n; \mathbb{C})$ are eventually constant as $n \to \infty$, as well as the dimensions $\dim_{\mathbb{C}} H_k(X_n; \mathbb{C})$ by Poincaré duality.

In the context of motivic stability, an important source of sequences of varieties are the symmetric powers of a variety X. In order to keep track of the virtual classes of these symmetric powers, we collect them as the coefficients of a power series, as first done by [Kap00].

Definition 7.1.6. Let X be a G-variety over k. The *motivic zeta function* of X is

$$Z_G(X,t) = \sum_{n\geq 0} [\operatorname{Sym}^n_G X]\, t^n \in 1 + t\cdot \mathrm{K}_0(\mathbf{Var}^G_k)[\![t]\!].$$

Lemma 7.1.7. *Let X be a G-variety over k, and $Y \subseteq X$ a G-invariant closed subvariety with open complement U. Then $Z_G(X,t) = Z_G(Y,t)\, Z_G(U,t)$, and hence $Z_G(-,t)$ descends to a group morphism*

$$Z_G(-,t)\colon \mathrm{K}_0(\mathbf{Var}^G_k) \to 1 + t\cdot \mathrm{K}_0(\mathbf{Var}^G_k)[\![t]\!]$$

with the multiplicative group structure on the right. In particular, Sym^n_G *descends to a map*

$$\operatorname{Sym}^n_G\colon \mathrm{K}_0(\mathbf{Var}^G_k) \to \mathrm{K}_0(\mathbf{Var}^G_k).$$

Proof. From

$$\begin{aligned}[\operatorname{Sym}^n_G X] = [X^n /\!\!/ S_n] &= \sum_{i+j=n} [(S_n\cdot (Y^i \times U^j)) /\!\!/ S_n] \\ &= \sum_{i+j=n} [(Y^i /\!\!/ S_i)\times (U^j /\!\!/ S_j)] = \sum_{i+j=n} [\operatorname{Sym}^i_G Y][\operatorname{Sym}^j_G U]\end{aligned}$$

follows that

$$Z_G(X,t) = \sum_{\substack{n\geq 0\\ i+j=n}} [\operatorname{Sym}^i_G Y][\operatorname{Sym}^j_G U]\, t^n = Z_G(Y,t)\, Z_G(U,t). \qquad \square$$

The following lemma is a variation of [Göt01, Lemma 4.4], adapted to the equivariant setting.

Proposition 7.1.8. *Let G be a finite group, and let X be a G-variety over k. For any $r \geq 0$, we have*

$$Z_G(\mathbb{L}^r[X], t) = Z_G([X], \mathbb{L}^r t).$$

Proof. It suffices to treat the case $r = 1$. Denote by $\pi\colon \operatorname{Sym}^n(X\times \mathbb{A}^1_k) \to \operatorname{Sym}^n X$ the obvious projection. Note that $\operatorname{Sym}^n X$ is naturally stratified by locally closed subvarieties $(\operatorname{Sym}^n X)_\lambda$ according to the partitions λ of n. For every such partition λ, we consider the cartesian diagram

$$\begin{array}{ccc} X_*^{\ell(\lambda)} \times \prod_{i=1}^n (\mathbb{A}_k^i /\!\!/ S_i)^{a_i(\lambda)} & \longrightarrow & \pi^{-1}\left((\operatorname{Sym}^n X)_\lambda\right) \\ \downarrow & & \downarrow \pi_\lambda \\ X_*^{\ell(\lambda)} & \longrightarrow & (\operatorname{Sym}^n X)_\lambda \end{array}$$

where $a_i(\lambda)$ denotes the number of times i appears in λ, and $X_*^{\ell(\lambda)}$ the space of $\ell(\lambda) = \sum_i a_i(\lambda)$ distinct ordered points of X. Since $\prod_{i=1}^n (\mathbb{A}_k^i /\!\!/ S_i)^{a_i(\lambda)} \cong \mathbb{A}_k^n$, the diagram defines an étale trivialization of π_λ. The transition functions are given by the action of the group $S_{a_1(\lambda)} \times \cdots \times S_{a_n(\lambda)}$, which acts linearly. Hence, π_λ is a vector bundle which is étale-locally trivial, so by Hilbert's Theorem 90 [Ser58, Theorem 2] also Zariski-locally trivial. However, note that a stratification of $(\operatorname{Sym}^n X)_\lambda$ trivializing π_λ need not necessarily be G-invariant. Nevertheless, using that G is finite, any such stratification can be intersected with all of its translations by $g \in G$, in order to obtain a G-invariant stratification. Hence, we conclude that $[\operatorname{Sym}_G^n(X \times \mathbb{A}_k^1)] = \mathbb{L}^n[\operatorname{Sym}_G^n X]$. □

From the Chevalley–Shephard–Todd theorem [Che55], it is easy to see that $\operatorname{Sym}^n \mathbb{A}_k^r$ is not isomorphic to $\mathbb{A}_k^{nr}$ for $n, r > 1$. Nevertheless, the above proposition yields the following corollary.

Corollary 7.1.9. *For any $n, r \geq 0$, we have $\operatorname{Sym}^n \mathbb{L}^r = \mathbb{L}^{nr}$. In particular, $Z_G(\mathbb{L}^r, t) = 1/(1 - \mathbb{L}^r t)$.* □

Lemma 7.1.10. *Let X be a d-dimensional G-variety over k, and suppose that the symmetric powers $\operatorname{Sym}_G^n X$ are motivically stable. Then*

$$\lim_{n \to \infty} \frac{[\operatorname{Sym}_G^n X]}{\mathbb{L}^{nd}} = \Big[(1 - t) Z_G(X, t/\mathbb{L}^d)\Big]_{t=1}.$$

Proof. As

$$\Big[(1 - t) Z_G(X, t/\mathbb{L}^d)\Big]_{t=1} = \left[1 + \sum_{n \geq 1} \left(\frac{[\operatorname{Sym}_G^n X]}{\mathbb{L}^{nd}} - \frac{[\operatorname{Sym}_G^{n-1} X]}{\mathbb{L}^{(n-1)d}} \right) t^n \right]_{t=1}$$

evaluates to a telescoping series, it is equal to $\lim_{n \to \infty} [\operatorname{Sym}_G^n X]/\mathbb{L}^{nd}$. □

Example 7.1.11. Let X be a variety over k such that $[X] \in \mathrm{K}_0(\mathbf{Var}_k)$ is a polynomial in $\mathbb{L}$. Then the sequence of symmetric powers $X_n = \operatorname{Sym}^n X$ is motivically stable if and only if $[X]$ is monic in $\mathbb{L}$. Namely, writing $[X] = \sum_{i=0}^d a_i \mathbb{L}^i$

with $a_d \neq 0$, it follows from Lemma 7.1.7 and Corollary 7.1.9 that

$$Z_G([X],t) = \prod_{i=0}^{d} \left(\frac{1}{1-\mathbb{L}^i t} \right)^{a_i}.$$

Hence, $[\operatorname{Sym}^n X]/\mathbb{L}^{nd}$ is the n-th coefficient of

$$Z_G(X,t/\mathbb{L}^d) = \prod_{i=0}^{d} \left(\frac{1}{1-\mathbb{L}^{i-d} t} \right)^{a_i}.$$

Therefore, for $a_d = 1$, we find that

$$\lim_{n\to\infty} \frac{[\operatorname{Sym}^n X]}{\mathbb{L}^{nd}} = \Big[(1-t) Z_G(X, t/\mathbb{L}^d)\Big]_{t=1} = \prod_{i=0}^{d-1} \left(\frac{1}{1-\mathbb{L}^{i-d}} \right)^{a_i},$$

and for $a_d > 1$, the limit is easily seen to not exist.

Proposition 7.1.12 ([VW15, Proposition 4.2])**.** *Let X be a G-variety over k, and $Y \subseteq X$ a G-invariant closed subvariety of dimension $\dim Y < \dim X$, with open complement $U = X \backslash Y$. Then the symmetric powers $\operatorname{Sym}^n_G X$ are motivically stable if and only if the symmetric powers $\operatorname{Sym}^n_G U$ are motivically stable, and in this case*

$$\lim_{n\to\infty} \frac{[\operatorname{Sym}^n_G X]}{\mathbb{L}^{n \dim X}} = Z_G(Y, \mathbb{L}^{-\dim X}) \lim_{n\to\infty} \frac{[\operatorname{Sym}^n_G U]}{\mathbb{L}^{n \dim X}}.$$

Proof. Let us prove the result modulo $F_{-m}\mathcal{M}^G_{\mathbb{L}}$ for all $m \geq 0$, by induction on m. The case $m = 0$ is trivial as $[\operatorname{Sym}^n_G X]/\mathbb{L}^{n \dim X} \equiv 0 \bmod F_0 \mathcal{M}^G_{\mathbb{L}}$, and similarly for U. For $m > 0$ we find, as in Lemma 7.1.7, that, for all $n \geq 1$,

$$\frac{[\operatorname{Sym}^n_G X]}{\mathbb{L}^{n \dim X}} \equiv \sum_{i=0}^{m-1} \frac{[\operatorname{Sym}^{n-i}_G U]}{\mathbb{L}^{(n-i)\dim X}} \frac{[\operatorname{Sym}^i_G Y]}{\mathbb{L}^{i \dim X}} \bmod F_{-m}\mathcal{M}^G_{\mathbb{L}} \qquad (*)$$

since $[\operatorname{Sym}^{n-i}_G U][\operatorname{Sym}^i_G Y]/\mathbb{L}^{n \dim X} \in F_{-m}\mathcal{M}^G_{\mathbb{L}}$ for $i \geq m$ as $\dim Y < \dim X$. Now, if the symmetric powers of U stabilize modulo $F_{-m}\mathcal{M}^G_{\mathbb{L}}$, say to $\ell = \lim_{n\to\infty} [\operatorname{Sym}^n_G U]/\mathbb{L}^{\dim X}$, then the right-hand side of equation $(*)$ stabilizes modulo $F_{-m}\mathcal{M}^G_{\mathbb{L}}$ to $\ell\, Z_G(Y, \mathbb{L}^{-\dim X})$. Conversely, if the symmetric powers of X stabilize modulo $F_{-m}\mathcal{M}^G_{\mathbb{L}}$, then the symmetric powers of U stabilize modulo $F_{-m+1}\mathcal{M}^G_{\mathbb{L}}$ (by the induction hypothesis), so every term on the right-hand side of $(*)$ with $i \geq 1$ stabilizes modulo $F_{-m}\mathcal{M}^G_{\mathbb{L}}$. But then also the term with $i = 0$ must stabilize, which shows that the symmetric powers of U stabilize modulo $F_{-m}\mathcal{M}^G_{\mathbb{L}}$. □

Remark 7.1.13. Suppose G is the trivial group, and write $Z(-,t)$ for $Z_G(-,t)$. The definition of $Z(-,t)$ can be extended to the Grothendieck ring of stacks

$\mathrm{K}_0(\mathbf{Stck}_k)$. Since $\mathrm{K}_0(\mathbf{Stck}_k) \cong \mathrm{K}_0(\mathbf{Var}_k)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1}]$ by Theorem 3.5.7, it suffices to recursively define $Z(x/\mathbb{L}, t)$ and $Z(x/(\mathbb{L}^n - 1), t)$ in terms of $Z(x, t)$ for all elements $x \in \mathrm{K}_0(\mathbf{Stck}_k)$, using that $Z(x, t)$ is determined for $x \in \mathrm{K}_0(\mathbf{Var}_k)$. This is done as follows.

$$Z(x/\mathbb{L}, t) = Z(x, \mathbb{L}^{-1}t)$$
$$Z(x/(\mathbb{L}^n - 1), t) = \prod_{i \geq 0} Z(x, \mathbb{L}^{in}t)^{-1}$$

Note that this gives a well-defined map

$$Z(-, t)\colon \mathrm{K}_0(\mathbf{Stck}_k) \to 1 + t \cdot \mathrm{K}_0(\mathbf{Stck}_k)[\![t]\!]$$

since

$$Z(x\mathbb{L}/\mathbb{L}, t) = Z(x, t) \quad \text{and}$$
$$Z(x(\mathbb{L}^n - 1)/(\mathbb{L}^n - 1)) = \prod_{k \geq 0} Z(x(\mathbb{L}^n - 1), \mathbb{L}^{kn}t) = \prod_{k \geq 0} \frac{Z(x, \mathbb{L}^{kn}t)}{Z(x, \mathbb{L}^{(k+1)n}t)} = Z(x, t)$$

which is easily seen to still be group morphism. In particular, looking at the n-th coefficient of $Z(-, t)$, we find that Sym^n descends to a map

$$\mathrm{Sym}^n\colon \mathrm{K}_0(\mathbf{Stck}_k) \to \mathrm{K}_0(\mathbf{Stck}_k).$$

The definition of symmetric powers does not naturally extend from varieties to stacks. However, as shown in [Eke09b], the class $\mathrm{Sym}^n[\mathfrak{X}]$ coincides with the virtual class of the stacky symmetric power $[\mathfrak{X}^n/S_n]$ for objects $\mathfrak{X}$ of $\mathbf{Stck}_k$ when $\mathrm{char}(k) = 0$ and $\mathrm{char}(k) > n$.

7.2 Equivariant stability

In this section we will show various stability results, for non-trivial algebraic groups G. Let us start by considering one of the simplest actions.

Proposition 7.2.1. *Let $G = \mathbb{G}_m$ act on $\mathbb{A}^1_k$ via $\alpha \cdot x = \alpha x$. Then*

$$\lim_{n \to \infty} \frac{\mathrm{Sym}^n_G[\mathbb{A}^1_k]}{\mathbb{L}^n} = \frac{[\mathbb{G}_m]}{\mathbb{L} - 1}$$

in $\widehat{\mathcal{M}^G_{\mathbb{L}}}$, where on the right $\mathbb{G}_m$ acts transitively on itself.

Proof. Write $X = [\mathbb{A}^1_k]$ for the described action of $\mathbb{G}_m$ on $\mathbb{A}^1_k$. Since $\mathrm{Sym}^n \mathbb{A}^1_k \cong \mathbb{A}^n_k$ has basis of coordinates given by the elementary symmetric polynomials, we have

$$\mathrm{Sym}^n_G X = \prod_{i=1}^n [\mathbb{A}^1_k] = \prod_{i=1}^n (1 + Y_i),$$

where, for any $i \geq 1$, we denote $Y_i = [\mathbb{G}_m]$ for the action $\alpha \cdot x = \alpha^i x$. Note that $Y_i Y_j = (\mathbb{L} - 1) Y_{\gcd(i,j)}$ for any $i, j \geq 1$. Indeed, there exist $a, b \in \mathbb{Z}$ such that $ai + bj = d \coloneqq \gcd(i, j)$, so the equality follows from the isomorphism

$$\begin{aligned} \mathbb{G}_m \times \mathbb{G}_m &\cong \mathbb{G}_m \times \mathbb{G}_m \\ (x, y) &\mapsto (x^a y^b, x^{j/\gcd(i,j)} y^{-i/d}) \\ (z^{i/d} w^b, z^{j/d} w^{-a}) &\leftarrow\!\shortmid (z, w), \end{aligned}$$

where $\alpha \cdot (x, y, z, w) = (\alpha^i x, \alpha^j y, \alpha^d z, w)$. Now, it follows that

$$\operatorname{Sym}^n_G X = 1 + \sum_{i \geq 1} a_{n,i} Y_i \quad \text{with} \quad a_{n,i} = \sum_S (\mathbb{L} - 1)^{|S|-1},$$

where the latter sum runs over all non-empty subsets $S \subseteq \{1, 2, \ldots, n\}$ such that $\gcd(S) = i$. Now, for any $i \geq 2$, we see that any S appearing in this sum must have $|S| \leq n/i$, so that $\deg_{\mathbb{L}}(a_{n,i}) \leq n/i - 1$. In particular,

$$\lim_{n \to \infty} \frac{a_{n,i}}{\mathbb{L}^n} = 0$$

for $i \geq 2$. Furthermore, from the equality $1 + \sum_{i=1}^n a_{n,i}(\mathbb{L} - 1) = \mathbb{L}^n$ follows that

$$\lim_{n \to \infty} \frac{a_{n,1}}{\mathbb{L}^n} = \lim_{n \to \infty} \frac{1}{\mathbb{L}^n} \left(\frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \sum_{i=2}^n a_{n,i} \right) = \frac{1}{\mathbb{L} - 1},$$

and therefore

$$\lim_{n \to \infty} \frac{\operatorname{Sym}^n_G X}{\mathbb{L}^n} = \frac{1}{\mathbb{L} - 1} Y_1.$$ □

Corollary 7.2.2. *The action of $G = \mathbb{G}_m$ on $\mathbb{A}^1_k$ given by $\alpha \cdot x = \alpha x$ extends to $\mathbb{P}^1_k$ and restricts to $\mathbb{G}_m$. The symmetric powers of $\mathbb{P}^1_k$ and $\mathbb{G}_m$ are motivically stable with limits*

$$\begin{aligned} \lim_{n \to \infty} \frac{\operatorname{Sym}^n_G [\mathbb{P}^1]}{\mathbb{L}^n} &= \frac{\mathbb{L}}{(\mathbb{L} - 1)^2} [\mathbb{G}_m] \\ \lim_{n \to \infty} \frac{\operatorname{Sym}^n_G [\mathbb{G}_m]}{\mathbb{L}^n} &= \frac{1}{\mathbb{L}} [\mathbb{G}_m]. \end{aligned}$$

Proof. This follows from Proposition 7.2.1 together with Proposition 7.1.12 and the fact that $Z_G(1, t) = 1/(1 - t)$. □

Next, we will generalize this result to the groups $G = \mathrm{GL}_r$ acting on affine space. In doing so, the following definition will be useful.

Definition 7.2.3. Let X_n be a sequence of G-varieties over k. A family of G-invariant subvarieties $Y_n \subseteq X_n$ is *negligible* if $\lim_{n \to \infty} \dim X_n - \dim Y_n = \infty$. In particular, X_n is motivically stable with limit $\ell = \lim_{n \to \infty} [X_n] / \mathbb{L}^{\dim X_n}$ if and only if $Z_n = X_n \setminus Y_n$ is motivically stable with the same limit.

Proposition 7.2.4. *Let $G = \mathrm{GL}_r$ act naturally on $\mathbb{A}^r_k$ for some $r \geq 1$. Then*

$$\lim_{n\to\infty} \frac{\mathrm{Sym}^n_G[\mathbb{A}^r_k]}{\mathbb{L}^{nr}} = \frac{[\mathrm{GL}_r]}{\prod_{i=1}^r(\mathbb{L}^r - \mathbb{L}^{i-1})} = \prod_{i=1}^r \frac{[\mathbb{A}^r_k] - \mathbb{L}^{i-1}}{\mathbb{L}^r - \mathbb{L}^{i-1}}$$

with GL_r *acting transitively on itself.*

Proof. Let $X_n \subseteq \mathrm{Sym}^n_G \mathbb{A}^r_k$ be the strata where GL_r acts freely, that is, the strata of points whose stabilizer is trivial. Then $X_n \to X_n /\!\!/ \mathrm{GL}_r$ is a GL_r-torsor, so $[X_n] = [X_n /\!\!/ \mathrm{GL}_r][\mathrm{GL}_r]$ since GL_r is a special group. In particular, if we show that the complement $Y_n = (\mathrm{Sym}^n_G \mathbb{A}^r_k) \setminus X_n$ of points with non-trivial stabilizer is negligible, then the result follows as

$$\lim_{n\to\infty} \frac{\mathrm{Sym}^n_G[\mathbb{A}^r_k]}{\mathbb{L}^{nr}} = \lim_{n\to\infty} \frac{[X_n]}{\mathbb{L}^{nr}} = \lim_{n\to\infty} \frac{[X_n /\!\!/ \mathrm{GL}_r]}{\mathbb{L}^{nr}}[\mathrm{GL}_r]$$

where

$$\lim_{n\to\infty} \frac{[X_n /\!\!/ \mathrm{GL}_r]}{\mathbb{L}^{nr}} = \lim_{n\to\infty} \frac{[X_n]}{\mathbb{L}^{nr}}[\mathrm{GL}_r]^{-1} = [\mathrm{GL}_r]^{-1} = \frac{1}{\prod_{i=1}^r(\mathbb{L}^r - \mathbb{L}^{i-1})}.$$

To show that Y_n is negligible, suppose $(x_1, \ldots, x_n) \in (\mathbb{A}^r_k)^n$ is a point which (in passing to the quotient by S_n) is stabilized by some non-trivial $A \in \mathrm{GL}_r$. Then there is a permutation $\sigma \in S_n$ such that $Ax_i = x_{\sigma(i)}$ for all $i = 1, \ldots, n$. Hence, there is a surjection

$$\bigsqcup_{\sigma\in S_n} Z_\sigma \to Y_n$$

with $Z_\sigma = \left\{(A, x_1, \ldots, x_n) \in (\mathrm{GL}_r \setminus \{1\}) \times (\mathbb{A}^r_k)^n \mid Ax_i = x_{\sigma(i)}\right\}$. We claim that $\dim Z_\sigma \leq \dim \mathrm{GL}_r + nr - n$ for all $\sigma \in S_n$, from which it follows that $\dim Y_n \leq \dim \mathrm{GL}_r + nr - n$, which in turn implies Y_n is negligible. To prove this claim, fix some $\sigma \in S_n$ and write $\sigma = \tau_1\tau_2\ldots\tau_s$ in canonical cycle notation (in particular, we do not omit 1-cycles). Then for every cycle $\tau = (i_1\ i_2\ \ldots\ i_m)$, let

$$Z_\tau = \left\{(A, x_{i_1}, \ldots, x_{i_m}) \mid Ax_{i_j} = x_{\tau(i_j)}\right\}.$$

If τ is a 1-cycle, then $\dim Z_\tau \leq \dim \mathrm{GL}_r + r - 1$ since A is non-trivial. If τ is an $(m \geq 2)$-cycle, then $\dim Z_\tau \leq \dim \mathrm{GL}_r + r$. Simple combinatorics now yields

$$\dim Z_\sigma = \dim(Z_{\tau_1} \times_{\mathrm{GL}_r\setminus\{1\}} \cdots \times_{\mathrm{GL}_r\setminus\{1\}} Z_{\tau_s}) \leq \dim \mathrm{GL}_r + nr - n. \qquad \square$$

Remark 7.2.5. Note that Proposition 7.2.1 is a special case of this proposition, but with an alternative proof.

Finally, we want to extend this result to any linear algebraic group G acting linearly on affine space. In order to relate $\widehat{\mathcal{M}^G_{\mathbb{L}}}$ for various G, consider the following lemma.

Lemma 7.2.6. *Let G be an algebraic group over k with subgroup $H \subseteq G$. The morphisms Res_H^G and Ind_H^G of Definition 3.6.5 extend to continuous morphisms*

$$\operatorname{Res}_H^G\colon \widehat{\mathcal{M}_{\mathbb{L}}^G} \to \widehat{\mathcal{M}_{\mathbb{L}}^H} \quad \text{and} \quad \operatorname{Ind}_H^G\colon \widehat{\mathcal{M}_{\mathbb{L}}^H} \to \widehat{\mathcal{M}_{\mathbb{L}}^G}.$$

In fact, Res_H^G is defined for any morphism $H \to G$ of algebraic groups over k.

Proof. Since $\operatorname{Res}_H^G(F_m \mathcal{M}_{\mathbb{L}}^G) \subseteq F_m \mathcal{M}_{\mathbb{L}}^H$ and $\operatorname{Ind}_H^G(F_m \mathcal{M}_{\mathbb{L}}^H) \subseteq F_{m'} \mathcal{M}_{\mathbb{L}}^G$, with $m' = m + \dim G - \dim H$, both Ind_H^G and Res_H^G extend to the completions. □

Corollary 7.2.7. *Let G be an algebraic group over k acting on $\mathbb{A}_k^r$ via some morphism $\rho\colon G \to \mathrm{GL}_r$ of algebraic groups. Then*

$$\lim_{n\to\infty} \frac{\operatorname{Sym}_G^n[\mathbb{A}_k^r]}{\mathbb{L}^{nr}} = \frac{[\mathrm{GL}_r]}{\prod_{i=1}^r (\mathbb{L}^r - \mathbb{L}^{i-1})}$$

where G acts on GL_r by multiplication via ρ.

Proof. Use Proposition 7.2.4 and that $\operatorname{Res}_G^{\mathrm{GL}_r} \circ \operatorname{Sym}_{\mathrm{GL}_r}^n = \operatorname{Sym}_G^n \circ \operatorname{Res}_G^{\mathrm{GL}_r}$. □

7.3 Motivic representation stability

In the context of motivic stability, it is typical to consider a sequence of symmetric powers $\operatorname{Sym}^n X = X^n /\!\!/ S_n$ of a variety X over k. However, one can more generally consider the whole X^n together with the action of S_n by permutation. One can then attempt to study the stability of the S_n-virtual class of X^n, as in Definition 3.6.12.

However, two problems arise. First of all, the group S_n depends on n, so to talk about stability, we must identify the irreducible representations of S_n for varying n. Recall that the irreducible representations of S_n are parametrized by the partitions of n [FH91]. Write V_λ for the irreducible representation of S_n corresponding to a partition λ of n. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ and any integer $n \geq |\lambda| + \lambda_1$, denote by $\lambda[n]$ the partition of n given by

$$\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \ldots).$$

Then, we think of the family $V_{\lambda[n]}$ of irreducible representations of S_n as corresponding to each other.

The second problem is that the S_n-virtual class depends on the choice of a set $\mathcal{H}$ of subgroups of S_n. One could, as in Example 3.6.15, take set of Young subgroups

$$\mathcal{H} = \{S_{\lambda_1} \times \cdots \times S_{\lambda_k} \mid \lambda \text{ is a partition of } n\}. \tag{7.1}$$

This idea will give rise to Definition 7.3.4. However, to get rid of the choice, we will first consider invariants in $\mathrm{K}_0(\mathcal{A})$ instead of $\mathrm{K}_0(\mathbf{Var}_k)$ for some suitable category $\mathcal{A}$ and functor $\mathcal{X}\colon \mathbf{Var}_k \to \mathcal{A}$. We will assume the following:

- $\mathcal{A}$ is a K-linear idempotent complete tensor triangulated category, with K a field of characteristic zero.
- The functor $\mathcal{X}$ induces a ring morphism $\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathcal{A})$. For any element $x \in \mathrm{K}_0(\mathbf{Var}_k)$, we will denote its image in $\mathrm{K}_0(\mathcal{A})$ also by x.
- For any finite group G and G-variety X over k, the coefficient of $[\mathcal{X}(X)]^G \in \mathrm{K}_0(\mathcal{A}) \otimes R_K(G)$ corresponding to the trivial representation equals $[\mathcal{X}(X /\!/ G)]$.

Inspired by [CF13, Definition 2.3], we introduce the following definition.

Definition 7.3.1. Let X_n be a sequence of varieties over k with an action of S_n. The sequence is *$\mathcal{A}$-representation stable* if, writing

$$[X_n]^{S_n} = \sum_{\lambda[n]} [X_n]_{\lambda[n]} \otimes [V_{\lambda[n]}] \in \mathrm{K}_0(\mathcal{A}) \otimes R_{\mathbb{Q}}(S_n),$$

the coefficients $[X_n]_{\lambda[n]} / \mathbb{L}^{\dim X_n}$ are eventually independent of n.

One way to compute the coefficients $[X_n]_\mu$, for partitions μ of n, is to look at the virtual classes of the quotients $X_n /\!/ S_\lambda$ with $S_\lambda \in \mathcal{H}$. This way, one inevitably encounters the Kostka numbers $K_{\mu\lambda}$. We will need the following lemma.

Lemma 7.3.2. *Let λ and μ be partitions. The Kostka number $K_{\mu[n]\lambda[n]}$ is independent of n for $n \geq |\lambda| + \mu_1$.*

Proof. Recall that $K_{\mu\lambda}$ is equal to the number of ways to fill the Young diagram of μ with λ_1 1's, λ_2 2's, etc., such that the resulting tableau is non-decreasing along rows and strictly increasing along columns [FH91]. Denote by $A_{\mu\lambda}$ the set of such tableaux. In particular, $K_{\mu\lambda} = |A_{\mu\lambda}|$.

For $|\mu| > |\lambda|$, we have $\mu[n] < \lambda[n]$, and hence $K_{\mu[n]\lambda[n]} = 0$. Now suppose $|\mu| \leq |\lambda|$. Considering $A_{\mu[n]\lambda[n]}$, note that all $(n - |\lambda|)$ 1's must be placed on the first row of the Young diagram of $\mu[n]$. Therefore, any Young tableau in $A_{\mu[n]\lambda[n]}$ is completely determined by the second through last rows and the last $|\lambda| - |\mu|$ entries of the first row. Note that, for $n \geq |\lambda| + \mu_1$, these last $|\lambda| - |\mu|$ entries do not put any restrictions on the entries of the second through last rows. Hence, we obtain a bijection between $A_{\mu[n]\lambda[n]}$ and $A_{\mu[n'],\lambda[n']}$ for all $n, n' \geq |\lambda| + \mu_1$, which shows that $K_{\mu[n]\lambda[n]} = K_{\mu[n']\lambda[n']}$. ☐

Proposition 7.3.3. *Suppose the sequences $[X_n /\!/ S_{\lambda[n]}] / \mathbb{L}^{\dim X_n} \in \mathrm{K}_0(\mathcal{A})$ stabilize for all partitions λ. Then, the sequence X_n is $\mathcal{A}$-representation stable.*

Proof. Write $[X_n]^{S_n} = \sum_{\lambda[n]} [X_n]_{\lambda[n]} \otimes [V_{\lambda[n]}]$. For any λ, we have, similar to Example 3.6.15,

$$\begin{aligned} [X_n /\!\!/ S_{\lambda[n]}] &= \left\langle T_{S_{\lambda[n]}}, \operatorname{Res}^{S_n}_{S_{\lambda[n]}} [X_n]^{S_n} \right\rangle \\ &= \left\langle \operatorname{Ind}^{S_n}_{S_{\lambda[n]}} T_{S_{\lambda[n]}}, [X_n]^{S_n} \right\rangle \\ &= \sum_{\mu[n] \geq \lambda[n]} K_{\mu[n]\lambda[n]} [X_n]^{\mu[n]}. \end{aligned}$$

Note that there are, independent of n, only finitely many partitions μ such that $\mu[n] \geq \lambda[n]$: those μ with $|\mu| < |\lambda|$, and those with $|\mu| = |\lambda|$ and $\mu > \lambda$.

By Lemma 7.3.2, the numbers $K_{\mu[n]\lambda[n]}$ are, for sufficiently large n, independent of n. Hence, $[X_n]_{\lambda[n]}$ can be expressed as a linear combination of $[X /\!\!/ S_{\mu[n]}]$ with $\mu[n] \geq \lambda[n]$, where the coefficients do not change for sufficiently large $n \geq 2|\lambda| \geq |\lambda| + \mu_1$. □

This motivates the following definition. Also agrees with stabilization of G-virtual class with $\mathcal{H}$ given by (7.1).

Definition 7.3.4. Let X_n be a sequence of varieties over k with an action of S_n. The sequence is said to be *motivically representation stable* if the sequences $[X_n /\!\!/ S_{\lambda[n]}]$ are motivically stable for all partitions λ. In particular, this implies X_n is $\mathcal{A}$-representation stable for all $\mathcal{A}$ and $\mathcal{X}\colon \mathbf{Var}_k \to \mathcal{A}$ as above. Also, in particular, the sequence $[X_n /\!\!/ S_n]$ is motivically stable.

More generally, a sequence X_n of $(G \times S_n)$-varieties over k is *motivically representation stable* if the sequences $[X_n /\!\!/ S_{\lambda[n]}]$ are motivically stable, as sequences of G-varieties, for all partitions λ.

Example 7.3.5. Let X be a variety over k whose sequence of symmetric powers $\operatorname{Sym}^n X$ is motivically stable, and let $X_n = X^n$ with S_n acting by permutation. Then, for any partition λ, the sequence

$$[X_n /\!\!/ S_{\lambda[n]}] = \operatorname{Sym}^{n-|\lambda|} X \times \prod_{i \geq 1} \operatorname{Sym}^{\lambda_i} X$$

is motivically stable. In particular, X_n is motivically representation stable.

7.4 $\mathbf{GL}_r$-character stacks

The goal of this section is to show the sequences of character stacks

$$\mathfrak{X}_n = \mathfrak{X}_G(\Gamma_n) = [R_G(\Gamma_n)/G]$$

of the free groups $\Gamma_n = F_n$ and the free abelian groups $\Gamma_n = \mathbb{Z}^n$ are motivic representation stable for the general linear groups $G = \mathrm{GL}_r$ of any rank $r \geq 0$ over a field k, where the action of S_n is induced from the action of S_n on Γ_n by permutation. However, since the notion of motivic representation stability is only defined for (G-)varieties, we will instead prove that the sequences of representation varieties $X_n = R_G(\Gamma_n)$ are motivically representation stable as sequences of G-varieties. Indeed, note that the action of G by conjugation and the action of S_n by permutation commute.

The case of $\Gamma_n = F_n$ turns out to be a quick consequence of the theory developed in the previous sections.

Theorem 7.4.1. *For every $r \geq 0$, the sequence of GL_r-representation varieties*

$$X_n = R_{\mathrm{GL}_r}(F_n)$$

with the action of GL_r by conjugation, and the action of S_n by permutation, is motivically representation stable.

Proof. For any $n \geq 1$, write $X_n = R_{\mathrm{GL}_r}(F_n) = (\mathrm{GL}_r)^n$. Given any partition λ, we find

$$X_n /\!\!/ S_{\lambda[n]} = \mathrm{Sym}^{n-|\lambda|}_{\mathrm{GL}_r} \mathrm{GL}_r \times \prod_{i \geq 1} \mathrm{Sym}^{\lambda_i}_{\mathrm{GL}_r} \mathrm{GL}_r,$$

where GL_r acts on itself by conjugation. Viewing GL_r as a dense open subset of $\mathbb{A}^{r^2}_k$, the action of GL_r on itself is linear, and hence the sequence $X_n /\!\!/ S_{\lambda[n]}$ is motivically stable by Corollary 7.2.7 and Proposition 7.1.12. □

For the remainder of this section, we will focus on the case $\Gamma_n = \mathbb{Z}^n$, and assume that k is algebraically closed.

Theorem 7.4.2. *For every $r \geq 0$, the sequence of GL_r-representation varieties*

$$X_n = R_{\mathrm{GL}_r}(\mathbb{Z}^n)$$

with the action of GL_r by conjugation, and the action of S_n by permutation, is motivically representation stable.

Notation-wise, we will use the following presentation of X_n, as the closed subvariety of $(\mathrm{GL}_r)^n$ given by commuting tuples of elements $A_i \in \mathrm{GL}_r$.

$$X_n = \left\{ (A_1, \ldots, A_n) \in (\mathrm{GL}_r)^n \;\middle|\; \text{all } A_i \text{ commute} \right\}$$

Interestingly, it turns out the cases $r \leq 3$ should be treated differently from the general case $r > 3$. We will first treat the cases $r = 2, 3$.

Proposition 7.4.3. *The* GL_2*-representation varieties* $X_n = R_{\mathrm{GL}_2}(\mathbb{Z}^n)$ *are motivically representation stable.*

Proof. Consider the possible Jordan normal forms of an element $A \in \mathrm{GL}_2$.

$$\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$$

$$\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right) \qquad \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$$

In particular, note that a matrix of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, with $\lambda \neq \mu$, only commutes with diagonal matrices, and that a matrix of the form $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ only commutes with matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$. Therefore, X_n can be stratified by the subvarieties

$$\begin{aligned} Y_n &= \{A \in X_n \mid \text{all } A_i \text{ are scalar}\}, \\ J_n &= \left\{A \in X_n \mid \text{some } A_i \text{ is conjugate to } \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)\right\} \\ \text{and} \quad M_n &= \left\{A \in X_n \mid \text{some } A_i \text{ is conjugate to } \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)\right\}. \end{aligned}$$

Simultaneously conjugating the A_i into normal form, we can express J_n as

$$J_n = \mathrm{Ind}_H^{\mathrm{GL}_2}(J^n \setminus Y_n)$$

where $J = \{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \mid x \neq 0\}$ and $H = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, c \neq 0\}$ the stabilizer of J. Similarly, we have

$$M_n = \mathrm{Ind}_K^{\mathrm{GL}_2}(M^n \setminus Y_n)$$

where $M = \left\{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) \mid x, y \neq 0\right\}$ and $K = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right) \mid a, b \neq 0\}$ the stabilizer of M. Clearly $\dim Y_n = n$ while $\dim J_n, \dim M_n \geq 2n$, implying $Y_n \subseteq X_n$ is negligible. Hence, it suffices to show that the sequences J'_n and M'_n, given by

$$J'_n = \mathrm{Ind}_H^{\mathrm{GL}_2}(J^n) \qquad \text{and} \qquad M'_n = \mathrm{Ind}_K^{\mathrm{GL}_2}(M^n)$$

are motivically representation stable.

First we consider J'_n. Note that, for any partition λ, the actions of $S_{\lambda[n]}$ and GL_2 on J^n commute, so that

$$J'_n /\!\!/ S_{\lambda[n]} = \mathrm{Ind}_H^{\mathrm{GL}_2}\Big(\mathrm{Sym}_H^{n-|\lambda|} J \times \prod_{i \geq 1} \mathrm{Sym}_H^{\lambda_i} J\Big).$$

Note that the action of H on J is linear, viewing J as an open dense subvariety of $\mathbb{A}^2_k$. Hence, the sequence $J'_n /\!\!/ S_{\lambda[n]}$ is motivically stable as a result of Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6.

The argument regarding M_n is analogous: for any partition λ, we have

$$M'_n /\!\!/ S_{\lambda[n]} = \mathrm{Ind}_K^{\mathrm{GL}_2}\Big(\mathrm{Sym}_K^{n-|\lambda|} M \times \prod_{i \geq 1} \mathrm{Sym}_K^{\lambda_i} M\Big).$$

Again, the action of K on $M \subseteq \mathbb{A}_k^2$ is linear, so the sequence $M'_n /\!/ S_{\lambda[n]}$ is also motivically stable. $\square$

Proposition 7.4.4. *The* GL_3*-representation varieties* $X_n = R_{\mathrm{GL}_3}(\mathbb{Z}^n)$ *are motivically stable.*

Proof. The proof is very similar to that of Proposition 7.4.3. Consider the possible Jordan normal forms of an element $A \in \mathrm{GL}_3$.

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \rho \end{pmatrix}$$

Having analyzed which matrices commute with each Jordan type, we stratify X_n by the subvarieties

$$\begin{aligned} Y_n^0 &= \left\{ A \in X_n \mid \text{ all } A_i \text{ are conjugate to } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \right\}, \\ Y_n^1 &= \left\{ A \in X_n \mid \text{ some } A_i \text{ is conjugate to } \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \right\}, \\ Y_n^2 &= \left\{ A \in X_n \mid \text{ some } A_i \text{ is conjugate to } \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \right\}, \\ Y_n^3 &= \left\{ A \in X_n \mid \text{ some } A_i \text{ is conjugate to } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \rho \end{pmatrix} \right\}. \end{aligned}$$

Note that the sequences Y_n^1, Y_n^2 and Y_n^3 do not intersect since matrices that have different Jordan type in the bottom row never commute. As in the proof of Proposition 7.4.3, to show motivic representation stability of the strata Y_n^i, it suffices to show motivic representation stability of the sequences

$$Y'^i_n = \mathrm{Ind}_{H_i}^{\mathrm{GL}_3}(J_i^n) \qquad \text{for } i = 1, 2, 3,$$

where

$$\begin{aligned} J_1 &= \left\{ \begin{pmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \mid x \neq 0 \right\} & H_1 &= \left\{ \begin{pmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1/a \end{pmatrix} \mid a \neq 0 \right\} \\ J_2 &= \left\{ \begin{pmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, z \neq 0 \right\} & H_2 &= \left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} \mid a, c, d \neq 0 \right\} \\ J_3 &= \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \neq 0 \right\} & H_3 &= \mathbb{G}_m^3 \rtimes S_3. \end{aligned}$$

More precisely, $H_3 \subseteq \mathrm{GL}_r$ is the subgroup generated by the diagonal matrices and the permutation matrices.

Now, for any partition λ, we find

$$Y'^{i}_{n} /\!\!/ S_{\lambda[n]} = \mathrm{Ind}_{H_i}^{\mathrm{GL}_3} \Big(\mathrm{Sym}_{H_i}^{n-|\lambda|} J_i \times \prod_{j \geq 1} \mathrm{Sym}_{H_i}^{\lambda_j} J_i \Big).$$

For all i, the group H_i acts linearly on J_i, a dense open of $\mathbb{A}_k^3$, so it follows from Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6 that the limits $\lim_{n\to\infty} Y_n^i/\mathbb{L}^{3n}$ exist. Knowing that $\dim X_n \geq 3n$, we see that $Y_n^0 \subseteq X_n$ is negligible, and the result follows. □

Looking at the proofs of Proposition 7.4.3 and Proposition 7.4.4, it might be tempting to think that in the general case the non-negligible strata are those containing matrices with maximal Jordan type. However, this turns out to be the case only for $r \leq 3$.

For the general case we use a result initially proved by Schur [Sch05], and later reproved by Jacobson [Jac44], about the maximum number of linearly independent commuting matrices. This leads to the idea of stratifying the representation varieties $R_{\mathrm{GL}_r}(\mathbb{Z}^n)$ by the dimension of the linear subspace inside $\mathrm{Mat}_{r\times r}$ spanned by the matrices A_i.

Proof of Theorem 7.4.2. The case $r = 0$ is obvious, and the case $r = 1$ follows from motivic representation stability of $\mathbb{G}_m^n$, see Example 7.1.11 and Example 7.3.5. The cases $r = 2$ and $r = 3$ were treated in Proposition 7.4.3 and Proposition 7.4.4, so we can assume $r > 3$.

As usual, write $X_n = R_{\mathrm{GL}_r}(\mathbb{Z}^n)$ for all $n \geq 1$. For any point $A \in X_n$ corresponding to a tuple $(A_1, \ldots, A_n)$ of commuting elements in GL_r, define

$$d_A = \dim_k \langle A_1, \ldots, A_n \rangle$$

to be the dimension of the linear subspace of $\mathrm{Mat}_{r\times r}(k)$ spanned by the A_i. By [Jac44, Theorem 1], we have $d_A \leq m$ with

$$m = \begin{cases} r^2/4 + 1 & \text{if } r \text{ is even,} \\ (r^2-1)/4 + 1 & \text{if } r \text{ is odd.} \end{cases}$$

Note that d_A is invariant under the actions of S_n and GL_r, so X_n can be stratified equivariantly by

$$X_{n,d} = \{A \in X_n \mid d_A = d\} \qquad \text{for } 1 \leq d \leq m.$$

Now, we will show that $X_{n,d} \subseteq X_n$ is negligible for $d < m$, so that we solely need to focus on $X_{n,m}$. Note that the dimension of X_n is at least nm, as it contains the family of commuting matrices given by

$$A_1 = \left(\begin{array}{c|c} \lambda_1 I & M_1 \\ \hline 0 & \lambda_1 I \end{array}\right), \dots, A_n = \left(\begin{array}{c|c} \lambda_n I & M_n \\ \hline 0 & \lambda_n I \end{array}\right) \tag{$*$}$$

with $\lambda_i \neq 0$, $M_i \in \mathrm{Mat}_{\frac{r}{2} \times \frac{r}{2}}$ if r is even, and $M_i \in \mathrm{Mat}_{\frac{r+1}{2} \times \frac{r-1}{2}}$ if r is odd. To see why the strata $X_{n,d}$ with $d < m$ are negligible, observe that $X_{n,d}$ can be covered by a dense open of $X_d \times (\mathbb{A}^d_k)^n$, that is, there is a surjective morphism from a dense open $Y_{n,d} \subseteq X_d \times (\mathbb{A}^d_k)^n$ given by

$$Y_{n,d} \to X_{n,d}, \quad \left((A_i)_{i=1}^d, (\alpha_{ij})_{i,j=1}^{n,d}\right) \mapsto \left(\sum_{j=1}^d \alpha_{ij} A_j\right)_{i=1}^n.$$

In particular, $\dim X_{n,d} \leq \dim Y_{n,d} \leq r^2 d + nd$, and hence $\lim_{n\to\infty} \dim X_n - \dim X_{n,d} = \infty$ for $d < m$, so it follows that $X_{n,d} \subseteq X_n$ is negligible.

By [Jac44, Theorem 3], every $A \in X_{n,m}$ can be conjugated to a tuple of the form $(*)$. Hence, to show motivic representation stability of $X_{n,m}$ it suffices to show motivic representation stability of

$$X'_{n,m} = \mathrm{Ind}_H^{\mathrm{GL}_r}\left(J^n\right) \quad \text{with} \quad J = \left\{ \left(\begin{array}{c|c} \lambda I & M \\ \hline 0 & \lambda I \end{array}\right) \;\middle|\; \begin{array}{c} \lambda \neq 0 \text{ and} \\ M \in \mathrm{Mat}_{\lceil \frac{r}{2} \rceil \times \lfloor \frac{r}{2} \rfloor} \end{array} \right\},$$

where the stabilizer

$$H = \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \right\} \subseteq \mathrm{GL}_r$$

acts trivially on λ, and acts on M via the linear action

$$\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \cdot M = AMC^{-1}.$$

Now, from Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6, it follows that $\lim_{n\to\infty} [X_{n,m}]/\mathbb{L}^{\dim X_{n,m}}$ exists. Moreover, as all $X_{n,d}$ with $d < m$ are negligible, this limit is equal to $\lim_{n\to\infty} [X_n]/\mathbb{L}^{\dim X_n}$. □