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Motivic invariants of character stacks

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Chapter 7

Motivic stability

Let Γ_n be a sequence of finitely generated groups, and let G be an algebraic group over a field k . One can wonder whether the invariants of the corresponding sequence of character stacks $\mathfrak{X}_G(\Gamma_n)$ are related. We will mainly focus on the sequence $\Gamma_n = \mathbb{Z}^n$ of free abelian groups and the sequence $\Gamma_n = F_n$ of free groups, for which the character stacks parametrize (commuting) tuples of elements in G up to conjugation.

Geometric invariants of these and related spaces have been studied extensively [Bai07, AC07, PS13, FL14]. For X_n the sequence of G -representation varieties or G -character varieties of \mathbb{Z}^n , the homology groups $H_k(X_n)$ were computed in [RS19], and their mixed Hodge structures in [FS21]. A pattern emerged: fixing n and varying G through sequences G_r of classical groups (such as GL_r or U_r), the homology groups $H_k(X_n)$ remain constant for sufficiently large r , that is, they *stabilize*. This pattern was proved in [RS21], as well as for fixed G and increasing n , and for many related sequences X_n . Moreover, taking into account the action of the symmetric group S_n on \mathbb{Z}^n by permutation, inducing an action of S_n on X_n and in turn on $H_k(X_n)$, they showed the homology groups stabilize as S_n -representations. This type of stability, called *representation stability*, was formulated in [CF13]: a sequence V_n of S_n -representations is representation stable, roughly speaking, if the multiplicities of the irreducible representations V_λ , corresponding to the partitions λ of n , stabilize. Partitions for n and $n + 1$ are related by increasing the first number.

In this chapter, we combine the notion of representation stability with that of *motivic stability*. Completing the Grothendieck ring of varieties, one can study the convergence of a sequence of virtual classes. Such convergence was studied in [VW15] for sequences of symmetric powers $\mathrm{Sym}^n X$ (as an algebraic analogue of the Dold–Thom theorem) and sequences of configuration spaces $\mathrm{Conf}^n X$.

Using the theory of Section 3.6, we will generalize the notion of motivic stability, and introduce the concept of *motivic representation stability*. As an application, we will show that the sequence of GL_r -character stacks $\mathfrak{X}_{\mathrm{GL}_r}(\mathbb{Z}^n)$, with the action of S_n , is motivically representation stable.

7.1 Motivic stability

Motivic stability is a property of a sequence of varieties, which amounts to the convergence (in some sense) of their virtual classes in the topological ring $\widehat{\mathcal{M}}_{\mathbb{L}}$, which is the completion of the localization $\mathrm{K}_0(\mathbf{Var}_k)[\mathbb{L}^{-1}]$ of the Grothendieck ring of varieties. This topological ring was originally constructed by Kontsevich in the context of motivic integration [Kon95]. For more information on this object, we refer to [Bou11, Loo02, VW15].

For our applications, we adapt the standard definitions to the equivariant setting. Throughout, fix an algebraic group G over k , and denote by \mathbb{L} the class $[\mathbb{A}_k^1] \in \mathrm{K}_0(\mathbf{Var}_k^G)$ of \mathbb{A}_k^1 on which G acts trivially.

Definition 7.1.1. Write $\mathcal{M}_{\mathbb{L}}^G$ for the localization $\mathrm{K}_0(\mathbf{Var}_k^G)[\mathbb{L}^{-1}]$. Consider the increasing filtration on $\mathcal{M}_{\mathbb{L}}^G$,

$$0 \subseteq \cdots \subseteq F_n \mathcal{M}_{\mathbb{L}}^G \subseteq F_{n+1} \mathcal{M}_{\mathbb{L}}^G \subseteq \cdots \subseteq \mathcal{M}_{\mathbb{L}}^G,$$

where $F_n \mathcal{M}_{\mathbb{L}}^G$ is the subgroup of $\mathcal{M}_{\mathbb{L}}^G$ generated by all elements of the form $[X]/\mathbb{L}^m$ with $\dim X - m \leq n$. Note that $\bigcup_{n \in \mathbb{Z}} F_n \widehat{\mathcal{M}}_{\mathbb{L}}^G = \widehat{\mathcal{M}}_{\mathbb{L}}^G$. The completion with respect to this filtration is denoted

$$\widehat{\mathcal{M}}_{\mathbb{L}}^G = \varprojlim_n \mathcal{M}_{\mathbb{L}}^G / F_n \mathcal{M}_{\mathbb{L}}^G.$$

An element $x \in \widehat{\mathcal{M}}_{\mathbb{L}}^G$ can be represented as a tuple $(x_n) \in \prod_{n \in \mathbb{Z}} \mathcal{M}_{\mathbb{L}}^G / F_n \mathcal{M}_{\mathbb{L}}^G$ such that $x_n \equiv x_m \pmod{F_n \mathcal{M}_{\mathbb{L}}^G}$ for all $m \leq n$.

The completion $\widehat{\mathcal{M}}_{\mathbb{L}}^G$ inherits, a priori, only the group structure from $\mathcal{M}_{\mathbb{L}}^G$. Multiplication is defined as follows. Let $x = (x_n)$ and $y = (y_n)$ be elements of $\widehat{\mathcal{M}}_{\mathbb{L}}^G$. Note that there exists a sufficiently large N such that $x_n = y_n = 0$ for all $n \geq N$. Now define xy by $(xy)_n = x'_{n-N} y'_{n-N} \pmod{F_n \mathcal{M}_{\mathbb{L}}^G}$, where $x'_{n-N}, y'_{n-N} \in \mathcal{M}_{\mathbb{L}}^G$ are lifts x_{n-N} and y_{n-N} , respectively. This is independent of the choice of lift since, for any other lift x''_{n-N} , we have $x''_{n-N} y'_{n-N} - x'_{n-N} y'_{n-N} = (x''_{n-N} - x'_{n-N}) y'_{n-N} \in F_{n-N} \mathcal{M}_{\mathbb{L}}^G \cdot F_N \mathcal{M}_{\mathbb{L}}^G \subseteq F_n \mathcal{M}_{\mathbb{L}}^G$. Similarly, it is independent of the choice of lift y'_{n-N} . This gives $\widehat{\mathcal{M}}_{\mathbb{L}}^G$ a ring structure.

Definition 7.1.2. Let X be a G -variety over k . For any $n \geq 0$, the n -th G -symmetric power of X , denoted $\mathrm{Sym}_G^n X$, is the G -variety given by the ordinary

symmetric power $\text{Sym}^n X = X^n // S_n$ with the action of G induced by the diagonal action on X^n .

Definition 7.1.3. Let X be a G -variety over k . The symmetric powers $\text{Sym}_G^n X$ of X are called *motivically stable* if the limit

$$\lim_{n \rightarrow \infty} \frac{[\text{Sym}_G^n X]}{\mathbb{L}^{n \dim X}}$$

exists in $\widehat{\mathcal{M}}_{\mathbb{L}}^G$. More generally, a sequence X_n of G -varieties over k is *motivically stable* if the limit

$$\lim_{n \rightarrow \infty} \frac{[X_n]}{\mathbb{L}^{\dim X_n}}$$

exists in $\widehat{\mathcal{M}}_{\mathbb{L}}^G$.

Example 7.1.4. When $G = 1$ is the trivial group, we simply write $\widehat{\mathcal{M}}_{\mathbb{L}}$ instead of $\widehat{\mathcal{M}}_{\mathbb{L}}^G$. In this case, the following sequences are motivically stable.

- From Example 3.3.5, we see that the sequence $X_n = \text{GL}_n$ is motivically stable, with limit $\lim_{n \rightarrow \infty} [\text{GL}_n] / \mathbb{L}^{n^2} = \prod_{i \geq 1} (1 - \mathbb{L}^{-i})$.
- Similarly, $X_n = \text{SL}_n$ is motivically stable with limit $\lim_{n \rightarrow \infty} [\text{SL}_n] / \mathbb{L}^{n^2-1} = \prod_{i \geq 2} (1 - \mathbb{L}^{-i})$. Since $[\text{PGL}_n] = [\text{SL}_n]$, the sequence $X_n = \text{PGL}_n$ is also motivically stable, with the same limit.
- It is still an open conjecture [VW15, Conjecture 1.25] whether the symmetric powers of all geometrically irreducible varieties are motivically stable. However, some evidence has been presented against it [Lit14].

Example 7.1.5. Let us give some intuition for what motivic stabilization implies about the cohomology of X_n . Suppose X_n is a sequence of varieties over $k = \mathbb{C}$. Note that the E -polynomial descends to a continuous morphism

$$e: \widehat{\mathcal{M}}_{\mathbb{L}}^G \rightarrow \mathbb{Z}[u, v][[(uv)^{-1}]]$$

where the target is equipped with the $(uv)^{-1}$ -adic topology. Since

$$e([X_n] / \mathbb{L}^{\dim X_n}) = \sum_{k,p,q \in \mathbb{Z}} (-1)^k h_c^{k;p,q}(X_n) u^{p-\dim X_n} v^{q-\dim X_n},$$

it follows, if the sequence X_n motivically stabilizes, that, for all p and q , the numbers $h_c^{k;\dim X_n-p, \dim X_n-q}(X_n)$ are eventually constant as $n \rightarrow \infty$. If the X_n are smooth projective, then evaluating in $u = v = t$, it also follows that the dimensions $\dim_{\mathbb{C}} H_c^{\dim X_n-k}(X_n; \mathbb{C})$ are eventually constant as $n \rightarrow \infty$, as well as the dimensions $\dim_{\mathbb{C}} H_k(X_n; \mathbb{C})$ by Poincaré duality.

In the context of motivic stability, an important source of sequences of varieties are the symmetric powers of a variety X . In order to keep track of the virtual classes of these symmetric powers, we collect them as the coefficients of a power series, as first done by [Kap00].

Definition 7.1.6. Let X be a G -variety over k . The *motivic zeta function* of X is

$$Z_G(X, t) = \sum_{n \geq 0} [\mathrm{Sym}_G^n X] t^n \in 1 + t \cdot \mathrm{K}_0(\mathbf{Var}_k^G)[[t]].$$

Lemma 7.1.7. Let X be a G -variety over k , and $Y \subseteq X$ a G -invariant closed subvariety with open complement U . Then $Z_G(X, t) = Z_G(Y, t) Z_G(U, t)$, and hence $Z_G(-, t)$ descends to a group morphism

$$Z_G(-, t): \mathrm{K}_0(\mathbf{Var}_k^G) \rightarrow 1 + t \cdot \mathrm{K}_0(\mathbf{Var}_k^G)[[t]]$$

with the multiplicative group structure on the right. In particular, Sym_G^n descends to a map

$$\mathrm{Sym}_G^n: \mathrm{K}_0(\mathbf{Var}_k^G) \rightarrow \mathrm{K}_0(\mathbf{Var}_k^G).$$

Proof. From

$$\begin{aligned} [\mathrm{Sym}_G^n X] &= [X^n // S_n] = \sum_{i+j=n} [(S_n \cdot (Y^i \times U^j)) // S_n] \\ &= \sum_{i+j=n} [(Y^i // S_i) \times (U^j // S_j)] = \sum_{i+j=n} [\mathrm{Sym}_G^i Y][\mathrm{Sym}_G^j U] \end{aligned}$$

follows that

$$Z_G(X, t) = \sum_{\substack{n \geq 0 \\ i+j=n}} [\mathrm{Sym}_G^i Y][\mathrm{Sym}_G^j U] t^n = Z_G(Y, t) Z_G(U, t). \quad \square$$

The following lemma is a variation of [Göt01, Lemma 4.4], adapted to the equivariant setting.

Proposition 7.1.8. Let G be a finite group, and let X be a G -variety over k . For any $r \geq 0$, we have

$$Z_G(\mathbb{L}^r[X], t) = Z_G([X], \mathbb{L}^r t).$$

Proof. It suffices to treat the case $r = 1$. Denote by $\pi: \mathrm{Sym}^n(X \times \mathbb{A}_k^1) \rightarrow \mathrm{Sym}^n X$ the obvious projection. Note that $\mathrm{Sym}^n X$ is naturally stratified by locally closed

subvarieties $(\mathrm{Sym}^n X)_\lambda$ according to the partitions λ of n . For every such partition λ , we consider the cartesian diagram

$$\begin{array}{ccc} X_*^{\ell(\lambda)} \times \prod_{i=1}^n (\mathbb{A}_k^i // S_i)^{a_i(\lambda)} & \longrightarrow & \pi^{-1}((\mathrm{Sym}^n X)_\lambda) \\ \downarrow & & \downarrow \pi_\lambda \\ X_*^{\ell(\lambda)} & \longrightarrow & (\mathrm{Sym}^n X)_\lambda \end{array}$$

where $a_i(\lambda)$ denotes the number of times i appears in λ , and $X_*^{\ell(\lambda)}$ the space of $\ell(\lambda) = \sum_i a_i(\lambda)$ distinct ordered points of X . Since $\prod_{i=1}^n (\mathbb{A}_k^i // S_i)^{a_i(\lambda)} \cong \mathbb{A}_k^n$, the diagram defines an étale trivialization of π_λ . The transition functions are given by the action of the group $S_{a_1(\lambda)} \times \cdots \times S_{a_n(\lambda)}$, which acts linearly. Hence, π_λ is a vector bundle which is étale-locally trivial, so by Hilbert’s Theorem 90 [Ser58, Theorem 2] also Zariski-locally trivial. However, note that a stratification of $(\mathrm{Sym}^n X)_\lambda$ trivializing π_λ need not necessarily be G -invariant. Nevertheless, using that G is finite, any such stratification can be intersected with all of its translations by $g \in G$, in order to obtain a G -invariant stratification. Hence, we conclude that $[\mathrm{Sym}_G^n(X \times \mathbb{A}_k^1)] = \mathbb{L}^n[\mathrm{Sym}_G^n X]$. \square

From the Chevalley–Shephard–Todd theorem [Che55], it is easy to see that $\mathrm{Sym}^n \mathbb{A}_k^r$ is not isomorphic to \mathbb{A}_k^{nr} for $n, r > 1$. Nevertheless, the above proposition yields the following corollary.

Corollary 7.1.9. *For any $n, r \geq 0$, we have $\mathrm{Sym}^n \mathbb{L}^r = \mathbb{L}^{nr}$. In particular, $Z_G(\mathbb{L}^r, t) = 1/(1 - \mathbb{L}^r t)$.* \square

Lemma 7.1.10. *Let X be a d -dimensional G -variety over k , and suppose that the symmetric powers $\mathrm{Sym}_G^n X$ are motivically stable. Then*

$$\lim_{n \rightarrow \infty} \frac{[\mathrm{Sym}_G^n X]}{\mathbb{L}^{nd}} = \left[(1-t)Z_G(X, t/\mathbb{L}^d) \right]_{t=1}.$$

Proof. As

$$\left[(1-t)Z_G(X, t/\mathbb{L}^d) \right]_{t=1} = \left[1 + \sum_{n \geq 1} \left(\frac{[\mathrm{Sym}_G^n X]}{\mathbb{L}^{nd}} - \frac{[\mathrm{Sym}_G^{n-1} X]}{\mathbb{L}^{(n-1)d}} \right) t^n \right]_{t=1}$$

evaluates to a telescoping series, it is equal to $\lim_{n \rightarrow \infty} [\mathrm{Sym}_G^n X]/\mathbb{L}^{nd}$. \square

Example 7.1.11. Let X be a variety over k such that $[X] \in K_0(\mathbf{Var}_k)$ is a polynomial in \mathbb{L} . Then the sequence of symmetric powers $X_n = \mathrm{Sym}^n X$ is motivically stable if and only if $[X]$ is monic in \mathbb{L} . Namely, writing $[X] = \sum_{i=0}^d a_i \mathbb{L}^i$

with $a_d \neq 0$, it follows from Lemma 7.1.7 and Corollary 7.1.9 that

$$Z_G([X], t) = \prod_{i=0}^d \left(\frac{1}{1 - \mathbb{L}^i t} \right)^{a_i}.$$

Hence, $[\mathrm{Sym}^n X]/\mathbb{L}^{nd}$ is the n -th coefficient of

$$Z_G(X, t/\mathbb{L}^d) = \prod_{i=0}^d \left(\frac{1}{1 - \mathbb{L}^{i-d} t} \right)^{a_i}.$$

Therefore, for $a_d = 1$, we find that

$$\lim_{n \rightarrow \infty} \frac{[\mathrm{Sym}^n X]}{\mathbb{L}^{nd}} = \left[(1-t) Z_G(X, t/\mathbb{L}^d) \right]_{t=1} = \prod_{i=0}^{d-1} \left(\frac{1}{1 - \mathbb{L}^{i-d}} \right)^{a_i},$$

and for $a_d > 1$, the limit is easily seen to not exist.

Proposition 7.1.12 ([VW15, Proposition 4.2]). *Let X be a G -variety over k , and $Y \subseteq X$ a G -invariant closed subvariety of dimension $\dim Y < \dim X$, with open complement $U = X \setminus Y$. Then the symmetric powers $\mathrm{Sym}_G^n X$ are motivically stable if and only if the symmetric powers $\mathrm{Sym}_G^n U$ are motivically stable, and in this case*

$$\lim_{n \rightarrow \infty} \frac{[\mathrm{Sym}_G^n X]}{\mathbb{L}^{n \dim X}} = Z_G(Y, \mathbb{L}^{-\dim X}) \lim_{n \rightarrow \infty} \frac{[\mathrm{Sym}_G^n U]}{\mathbb{L}^{n \dim X}}.$$

Proof. Let us prove the result modulo $F_{-m} \mathcal{M}_{\mathbb{L}}^G$ for all $m \geq 0$, by induction on m . The case $m = 0$ is trivial as $[\mathrm{Sym}_G^n X]/\mathbb{L}^{n \dim X} \equiv 0 \pmod{F_0 \mathcal{M}_{\mathbb{L}}^G}$, and similarly for U . For $m > 0$ we find, as in Lemma 7.1.7, that, for all $n \geq 1$,

$$\frac{[\mathrm{Sym}_G^n X]}{\mathbb{L}^{n \dim X}} \equiv \sum_{i=0}^{m-1} \frac{[\mathrm{Sym}_G^{n-i} U]}{\mathbb{L}^{(n-i) \dim X}} \frac{[\mathrm{Sym}_G^i Y]}{\mathbb{L}^{i \dim X}} \pmod{F_{-m} \mathcal{M}_{\mathbb{L}}^G} \quad (*)$$

since $[\mathrm{Sym}_G^{n-i} U][\mathrm{Sym}_G^i Y]/\mathbb{L}^{n \dim X} \in F_{-m} \mathcal{M}_{\mathbb{L}}^G$ for $i \geq m$ as $\dim Y < \dim X$. Now, if the symmetric powers of U stabilize modulo $F_{-m} \mathcal{M}_{\mathbb{L}}^G$, say to $\ell = \lim_{n \rightarrow \infty} [\mathrm{Sym}_G^n U]/\mathbb{L}^{n \dim X}$, then the right-hand side of equation (*) stabilizes modulo $F_{-m} \mathcal{M}_{\mathbb{L}}^G$ to $\ell Z_G(Y, \mathbb{L}^{-\dim X})$. Conversely, if the symmetric powers of X stabilize modulo $F_{-m} \mathcal{M}_{\mathbb{L}}^G$, then the symmetric powers of U stabilize modulo $F_{-m+1} \mathcal{M}_{\mathbb{L}}^G$ (by the induction hypothesis), so every term on the right-hand side of (*) with $i \geq 1$ stabilizes modulo $F_{-m} \mathcal{M}_{\mathbb{L}}^G$. But then also the term with $i = 0$ must stabilize, which shows that the symmetric powers of U stabilize modulo $F_{-m} \mathcal{M}_{\mathbb{L}}^G$. \square

Remark 7.1.13. Suppose G is the trivial group, and write $Z(-, t)$ for $Z_G(-, t)$. The definition of $Z(-, t)$ can be extended to the Grothendieck ring of stacks

$K_0(\mathbf{Stck}_k)$. Since $K_0(\mathbf{Stck}_k) \cong K_0(\mathbf{Var}_k)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1}]$ by Theorem 3.5.7, it suffices to recursively define $Z(x/\mathbb{L}, t)$ and $Z(x/(\mathbb{L}^n - 1), t)$ in terms of $Z(x, t)$ for all elements $x \in K_0(\mathbf{Stck}_k)$, using that $Z(x, t)$ is determined for $x \in K_0(\mathbf{Var}_k)$. This is done as follows.

$$\begin{aligned} Z(x/\mathbb{L}, t) &= Z(x, \mathbb{L}^{-1}t) \\ Z(x/(\mathbb{L}^n - 1), t) &= \prod_{i \geq 0} Z(x, \mathbb{L}^{in}t)^{-1} \end{aligned}$$

Note that this gives a well-defined map

$$Z(-, t): K_0(\mathbf{Stck}_k) \rightarrow 1 + t \cdot K_0(\mathbf{Stck}_k)[[t]]$$

since

$$\begin{aligned} Z(x\mathbb{L}/\mathbb{L}, t) &= Z(x, t) \quad \text{and} \\ Z(x(\mathbb{L}^n - 1)/(\mathbb{L}^n - 1), t) &= \prod_{k \geq 0} Z(x(\mathbb{L}^n - 1), \mathbb{L}^{kn}t) = \prod_{k \geq 0} \frac{Z(x, \mathbb{L}^{kn}t)}{Z(x, \mathbb{L}^{(k+1)n}t)} = Z(x, t) \end{aligned}$$

which is easily seen to still be group morphism. In particular, looking at the n -th coefficient of $Z(-, t)$, we find that Sym^n descends to a map

$$\mathrm{Sym}^n: K_0(\mathbf{Stck}_k) \rightarrow K_0(\mathbf{Stck}_k).$$

The definition of symmetric powers does not naturally extend from varieties to stacks. However, as shown in [Eke09b], the class $\mathrm{Sym}^n[\mathfrak{X}]$ coincides with the virtual class of the stacky symmetric power $[\mathfrak{X}^n/S_n]$ for objects \mathfrak{X} of \mathbf{Stck}_k when $\mathrm{char}(k) = 0$ and $\mathrm{char}(k) > n$.

7.2 Equivariant stability

In this section we will show various stability results, for non-trivial algebraic groups G . Let us start by considering one of the simplest actions.

Proposition 7.2.1. *Let $G = \mathbb{G}_m$ act on \mathbb{A}_k^1 via $\alpha \cdot x = \alpha x$. Then*

$$\lim_{n \rightarrow \infty} \frac{\mathrm{Sym}_G^n[\mathbb{A}_k^1]}{\mathbb{L}^n} = \frac{[\mathbb{G}_m]}{\mathbb{L} - 1}$$

in $\widehat{\mathcal{M}}_{\mathbb{L}}^G$, where on the right \mathbb{G}_m acts transitively on itself.

Proof. Write $X = [\mathbb{A}_k^1]$ for the described action of \mathbb{G}_m on \mathbb{A}_k^1 . Since $\mathrm{Sym}^n \mathbb{A}_k^1 \cong \mathbb{A}_k^n$ has basis of coordinates given by the elementary symmetric polynomials, we have

$$\mathrm{Sym}_G^n X = \prod_{i=1}^n [\mathbb{A}_k^1] = \prod_{i=1}^n (1 + Y_i),$$

where, for any $i \geq 1$, we denote $Y_i = [\mathbb{G}_m]$ for the action $\alpha \cdot x = \alpha^i x$. Note that $Y_i Y_j = (\mathbb{L} - 1) Y_{\gcd(i,j)}$ for any $i, j \geq 1$. Indeed, there exist $a, b \in \mathbb{Z}$ such that $ai + bj = d := \gcd(i, j)$, so the equality follows from the isomorphism

$$\begin{aligned} \mathbb{G}_m \times \mathbb{G}_m &\cong \mathbb{G}_m \times \mathbb{G}_m \\ (x, y) &\mapsto (x^a y^b, x^{j/\gcd(i,j)} y^{-i/d}) \\ (z^{i/d} w^b, z^{j/d} w^{-a}) &\leftarrow (z, w), \end{aligned}$$

where $\alpha \cdot (x, y, z, w) = (\alpha^i x, \alpha^j y, \alpha^d z, w)$. Now, it follows that

$$\mathrm{Sym}_G^n X = 1 + \sum_{i \geq 1} a_{n,i} Y_i \quad \text{with} \quad a_{n,i} = \sum_S (\mathbb{L} - 1)^{|S|-1},$$

where the latter sum runs over all non-empty subsets $S \subseteq \{1, 2, \dots, n\}$ such that $\gcd(S) = i$. Now, for any $i \geq 2$, we see that any S appearing in this sum must have $|S| \leq n/i$, so that $\deg_{\mathbb{L}}(a_{n,i}) \leq n/i - 1$. In particular,

$$\lim_{n \rightarrow \infty} \frac{a_{n,i}}{\mathbb{L}^n} = 0$$

for $i \geq 2$. Furthermore, from the equality $1 + \sum_{i=1}^n a_{n,i} (\mathbb{L} - 1) = \mathbb{L}^n$ follows that

$$\lim_{n \rightarrow \infty} \frac{a_{n,1}}{\mathbb{L}^n} = \lim_{n \rightarrow \infty} \frac{1}{\mathbb{L}^n} \left(\frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \sum_{i=2}^n a_{n,i} \right) = \frac{1}{\mathbb{L} - 1},$$

and therefore

$$\lim_{n \rightarrow \infty} \frac{\mathrm{Sym}_G^n X}{\mathbb{L}^n} = \frac{1}{\mathbb{L} - 1} Y_1. \quad \square$$

Corollary 7.2.2. *The action of $G = \mathbb{G}_m$ on \mathbb{A}_k^1 given by $\alpha \cdot x = \alpha x$ extends to \mathbb{P}_k^1 and restricts to \mathbb{G}_m . The symmetric powers of \mathbb{P}_k^1 and \mathbb{G}_m are motivically stable with limits*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathrm{Sym}_G^n[\mathbb{P}^1]}{\mathbb{L}^n} &= \frac{\mathbb{L}}{(\mathbb{L} - 1)^2} [\mathbb{G}_m] \\ \lim_{n \rightarrow \infty} \frac{\mathrm{Sym}_G^n[\mathbb{G}_m]}{\mathbb{L}^n} &= \frac{1}{\mathbb{L}} [\mathbb{G}_m]. \end{aligned}$$

Proof. This follows from Proposition 7.2.1 together with Proposition 7.1.12 and the fact that $Z_G(1, t) = 1/(1 - t)$. \square

Next, we will generalize this result to the groups $G = \mathrm{GL}_r$ acting on affine space. In doing so, the following definition will be useful.

Definition 7.2.3. Let X_n be a sequence of G -varieties over k . A family of G -invariant subvarieties $Y_n \subseteq X_n$ is *negligible* if $\lim_{n \rightarrow \infty} \dim X_n - \dim Y_n = \infty$. In particular, X_n is motivically stable with limit $\ell = \lim_{n \rightarrow \infty} [X_n]/\mathbb{L}^{\dim X_n}$ if and only if $Z_n = X_n \setminus Y_n$ is motivically stable with the same limit.

Proposition 7.2.4. *Let $G = \mathrm{GL}_r$ act naturally on \mathbb{A}_k^r for some $r \geq 1$. Then*

$$\lim_{n \rightarrow \infty} \frac{\mathrm{Sym}_G^n[\mathbb{A}_k^r]}{\mathbb{L}^{nr}} = \frac{[\mathrm{GL}_r]}{\prod_{i=1}^r (\mathbb{L}^r - \mathbb{L}^{i-1})} = \prod_{i=1}^r \frac{[\mathbb{A}_k^r] - \mathbb{L}^{i-1}}{\mathbb{L}^r - \mathbb{L}^{i-1}}$$

with GL_r acting transitively on itself.

Proof. Let $X_n \subseteq \mathrm{Sym}_G^n \mathbb{A}_k^r$ be the strata where GL_r acts freely, that is, the strata of points whose stabilizer is trivial. Then $X_n \rightarrow X_n // \mathrm{GL}_r$ is a GL_r -torsor, so $[X_n] = [X_n // \mathrm{GL}_r][\mathrm{GL}_r]$ since GL_r is a special group. In particular, if we show that the complement $Y_n = (\mathrm{Sym}_G^n \mathbb{A}_k^r) \setminus X_n$ of points with non-trivial stabilizer is negligible, then the result follows as

$$\lim_{n \rightarrow \infty} \frac{\mathrm{Sym}_G^n[\mathbb{A}_k^r]}{\mathbb{L}^{nr}} = \lim_{n \rightarrow \infty} \frac{[X_n]}{\mathbb{L}^{nr}} = \lim_{n \rightarrow \infty} \frac{[X_n // \mathrm{GL}_r]}{\mathbb{L}^{nr}} [\mathrm{GL}_r]$$

where

$$\lim_{n \rightarrow \infty} \frac{[X_n // \mathrm{GL}_r]}{\mathbb{L}^{nr}} = \lim_{n \rightarrow \infty} \frac{[X_n]}{\mathbb{L}^{nr}} [\mathrm{GL}_r]^{-1} = [\mathrm{GL}_r]^{-1} = \frac{1}{\prod_{i=1}^r (\mathbb{L}^r - \mathbb{L}^{i-1})}.$$

To show that Y_n is negligible, suppose $(x_1, \dots, x_n) \in (\mathbb{A}_k^r)^n$ is a point which (in passing to the quotient by S_n) is stabilized by some non-trivial $A \in \mathrm{GL}_r$. Then there is a permutation $\sigma \in S_n$ such that $Ax_i = x_{\sigma(i)}$ for all $i = 1, \dots, n$. Hence, there is a surjection

$$\bigsqcup_{\sigma \in S_n} Z_\sigma \rightarrow Y_n$$

with $Z_\sigma = \{(A, x_1, \dots, x_n) \in (\mathrm{GL}_r \setminus \{1\}) \times (\mathbb{A}_k^r)^n \mid Ax_i = x_{\sigma(i)}\}$. We claim that $\dim Z_\sigma \leq \dim \mathrm{GL}_r + nr - n$ for all $\sigma \in S_n$, from which it follows that $\dim Y_n \leq \dim \mathrm{GL}_r + nr - n$, which in turn implies Y_n is negligible. To prove this claim, fix some $\sigma \in S_n$ and write $\sigma = \tau_1 \tau_2 \dots \tau_s$ in canonical cycle notation (in particular, we do not omit 1-cycles). Then for every cycle $\tau = (i_1 i_2 \dots i_m)$, let

$$Z_\tau = \{(A, x_{i_1}, \dots, x_{i_m}) \mid Ax_{i_j} = x_{\tau(i_j)}\}.$$

If τ is a 1-cycle, then $\dim Z_\tau \leq \dim \mathrm{GL}_r + r - 1$ since A is non-trivial. If τ is an $(m \geq 2)$ -cycle, then $\dim Z_\tau \leq \dim \mathrm{GL}_r + r$. Simple combinatorics now yields

$$\dim Z_\sigma = \dim(Z_{\tau_1} \times_{\mathrm{GL}_r \setminus \{1\}} \dots \times_{\mathrm{GL}_r \setminus \{1\}} Z_{\tau_s}) \leq \dim \mathrm{GL}_r + nr - n. \quad \square$$

Remark 7.2.5. Note that Proposition 7.2.1 is a special case of this proposition, but with an alternative proof.

Finally, we want to extend this result to any linear algebraic group G acting linearly on affine space. In order to relate $\widehat{\mathcal{M}}_{\mathbb{L}}^G$ for various G , consider the following lemma.

Lemma 7.2.6. *Let G be an algebraic group over k with subgroup $H \subseteq G$. The morphisms Res_H^G and Ind_H^G of Definition 3.6.5 extend to continuous morphisms*

$$\text{Res}_H^G: \widehat{\mathcal{M}}_{\mathbb{L}}^G \rightarrow \widehat{\mathcal{M}}_{\mathbb{L}}^H \quad \text{and} \quad \text{Ind}_H^G: \widehat{\mathcal{M}}_{\mathbb{L}}^H \rightarrow \widehat{\mathcal{M}}_{\mathbb{L}}^G.$$

In fact, Res_H^G is defined for any morphism $H \rightarrow G$ of algebraic groups over k .

Proof. Since $\text{Res}_H^G(F_m \mathcal{M}_{\mathbb{L}}^G) \subseteq F_m \mathcal{M}_{\mathbb{L}}^H$ and $\text{Ind}_H^G(F_m \mathcal{M}_{\mathbb{L}}^H) \subseteq F_{m'} \mathcal{M}_{\mathbb{L}}^G$, with $m' = m + \dim G - \dim H$, both Ind_H^G and Res_H^G extend to the completions. \square

Corollary 7.2.7. *Let G be an algebraic group over k acting on \mathbb{A}_k^r via some morphism $\rho: G \rightarrow \text{GL}_r$ of algebraic groups. Then*

$$\lim_{n \rightarrow \infty} \frac{\text{Sym}_G^n[\mathbb{A}_k^r]}{\mathbb{L}^{nr}} = \frac{[\text{GL}_r]}{\prod_{i=1}^r (\mathbb{L}^r - \mathbb{L}^{i-1})}$$

where G acts on GL_r by multiplication via ρ .

Proof. Use Proposition 7.2.4 and that $\text{Res}_G^{\text{GL}_r} \circ \text{Sym}_{\text{GL}_r}^n = \text{Sym}_G^n \circ \text{Res}_G^{\text{GL}_r}$. \square

7.3 Motivic representation stability

In the context of motivic stability, it is typical to consider a sequence of symmetric powers $\text{Sym}^n X = X^n // S_n$ of a variety X over k . However, one can more generally consider the whole X^n together with the action of S_n by permutation. One can then attempt to study the stability of the S_n -virtual class of X^n , as in Definition 3.6.12.

However, two problems arise. First of all, the group S_n depends on n , so to talk about stability, we must identify the irreducible representations of S_n for varying n . Recall that the irreducible representations of S_n are parametrized by the partitions of n [FH91]. Write V_λ for the irreducible representation of S_n corresponding to a partition λ of n . For any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and any integer $n \geq |\lambda| + \lambda_1$, denote by $\lambda[n]$ the partition of n given by

$$\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \dots).$$

Then, we think of the family $V_{\lambda[n]}$ of irreducible representations of S_n as corresponding to each other.

The second problem is that the S_n -virtual class depends on the choice of a set \mathcal{H} of subgroups of S_n . One could, as in Example 3.6.15, take set of Young subgroups

$$\mathcal{H} = \{S_{\lambda_1} \times \dots \times S_{\lambda_k} \mid \lambda \text{ is a partition of } n\}. \quad (7.1)$$

This idea will give rise to Definition 7.3.4. However, to get rid of the choice, we will first consider invariants in $K_0(\mathcal{A})$ instead of $K_0(\mathbf{Var}_k)$ for some suitable category \mathcal{A} and functor $\mathcal{X}: \mathbf{Var}_k \rightarrow \mathcal{A}$. We will assume the following:

- \mathcal{A} is a K -linear idempotent complete tensor triangulated category, with K a field of characteristic zero.
- The functor \mathcal{X} induces a ring morphism $K_0(\mathbf{Var}_k) \rightarrow K_0(\mathcal{A})$. For any element $x \in K_0(\mathbf{Var}_k)$, we will denote its image in $K_0(\mathcal{A})$ also by x .
- For any finite group G and G -variety X over k , the coefficient of $[\mathcal{X}(X)]^G \in K_0(\mathcal{A}) \otimes R_K(G)$ corresponding to the trivial representation equals $[\mathcal{X}(X // G)]$.

Inspired by [CF13, Definition 2.3], we introduce the following definition.

Definition 7.3.1. Let X_n be a sequence of varieties over k with an action of S_n . The sequence is *\mathcal{A} -representation stable* if, writing

$$[X_n]^{S_n} = \sum_{\lambda[n]} [X_n]_{\lambda[n]} \otimes [V_{\lambda[n]}] \in K_0(\mathcal{A}) \otimes R_{\mathbb{Q}}(S_n),$$

the coefficients $[X_n]_{\lambda[n]} / \mathbb{L}^{\dim X_n}$ are eventually independent of n .

One way to compute the coefficients $[X_n]_{\mu}$, for partitions μ of n , is to look at the virtual classes of the quotients $X_n // S_{\lambda}$ with $S_{\lambda} \in \mathcal{H}$. This way, one inevitably encounters the Kostka numbers $K_{\mu\lambda}$. We will need the following lemma.

Lemma 7.3.2. *Let λ and μ be partitions. The Kostka number $K_{\mu[n]\lambda[n]}$ is independent of n for $n \geq |\lambda| + \mu_1$.*

Proof. Recall that $K_{\mu\lambda}$ is equal to the number of ways to fill the Young diagram of μ with λ_1 1's, λ_2 2's, etc., such that the resulting tableau is non-decreasing along rows and strictly increasing along columns [FH91]. Denote by $A_{\mu\lambda}$ the set of such tableaux. In particular, $K_{\mu\lambda} = |A_{\mu\lambda}|$.

For $|\mu| > |\lambda|$, we have $\mu[n] < \lambda[n]$, and hence $K_{\mu[n]\lambda[n]} = 0$. Now suppose $|\mu| \leq |\lambda|$. Considering $A_{\mu[n]\lambda[n]}$, note that all $(n - |\lambda|)$ 1's must be placed on the first row of the Young diagram of $\mu[n]$. Therefore, any Young tableau in $A_{\mu[n]\lambda[n]}$ is completely determined by the second through last rows and the last $|\lambda| - |\mu|$ entries of the first row. Note that, for $n \geq |\lambda| + \mu_1$, these last $|\lambda| - |\mu|$ entries do not put any restrictions on the entries of the second through last rows. Hence, we obtain a bijection between $A_{\mu[n]\lambda[n]}$ and $A_{\mu[n']\lambda[n']}$ for all $n, n' \geq |\lambda| + \mu_1$, which shows that $K_{\mu[n]\lambda[n]} = K_{\mu[n']\lambda[n']}$. \square

Proposition 7.3.3. *Suppose the sequences $[X_n // S_{\lambda[n]}] / \mathbb{L}^{\dim X_n} \in K_0(\mathcal{A})$ stabilize for all partitions λ . Then, the sequence X_n is \mathcal{A} -representation stable.*

Proof. Write $[X_n]^{S_n} = \sum_{\lambda[n]} [X_n]_{\lambda[n]} \otimes [V_{\lambda[n]}]$. For any λ , we have, similar to Example 3.6.15,

$$\begin{aligned} [X_n // S_{\lambda[n]}] &= \left\langle T_{S_{\lambda[n]}}, \text{Res}_{S_{\lambda[n]}}^{S_n} [X_n]^{S_n} \right\rangle \\ &= \left\langle \text{Ind}_{S_{\lambda[n]}}^{S_n} T_{S_{\lambda[n]}}, [X_n]^{S_n} \right\rangle \\ &= \sum_{\mu[n] \geq \lambda[n]} K_{\mu[n]\lambda[n]} [X_n]^{\mu[n]}. \end{aligned}$$

Note that there are, independent of n , only finitely many partitions μ such that $\mu[n] \geq \lambda[n]$: those μ with $|\mu| < |\lambda|$, and those with $|\mu| = |\lambda|$ and $\mu > \lambda$.

By Lemma 7.3.2, the numbers $K_{\mu[n]\lambda[n]}$ are, for sufficiently large n , independent of n . Hence, $[X_n]_{\lambda[n]}$ can be expressed as a linear combination of $[X // S_{\mu[n]}]$ with $\mu[n] \geq \lambda[n]$, where the coefficients do not change for sufficiently large $n \geq 2|\lambda| \geq |\lambda| + \mu_1$. \square

This motivates the following definition. Also agrees with stabilization of G -virtual class with \mathcal{H} given by (7.1).

Definition 7.3.4. Let X_n be a sequence of varieties over k with an action of S_n . The sequence is said to be *motivically representation stable* if the sequences $[X_n // S_{\lambda[n]}]$ are motivically stable for all partitions λ . In particular, this implies X_n is \mathcal{A} -representation stable for all \mathcal{A} and $\mathcal{X}: \mathbf{Var}_k \rightarrow \mathcal{A}$ as above. Also, in particular, the sequence $[X_n // S_n]$ is motivically stable.

More generally, a sequence X_n of $(G \times S_n)$ -varieties over k is *motivically representation stable* if the sequences $[X_n // S_{\lambda[n]}]$ are motivically stable, as sequences of G -varieties, for all partitions λ .

Example 7.3.5. Let X be a variety over k whose sequence of symmetric powers $\text{Sym}^n X$ is motivically stable, and let $X_n = X^n$ with S_n acting by permutation. Then, for any partition λ , the sequence

$$[X_n // S_{\lambda[n]}] = \text{Sym}^{n-|\lambda|} X \times \prod_{i \geq 1} \text{Sym}^{\lambda_i} X$$

is motivically stable. In particular, X_n is motivically representation stable.

7.4 GL_r -character stacks

The goal of this section is to show the sequences of character stacks

$$\mathfrak{X}_n = \mathfrak{X}_G(\Gamma_n) = [R_G(\Gamma_n)/G]$$

of the free groups $\Gamma_n = F_n$ and the free abelian groups $\Gamma_n = \mathbb{Z}^n$ are motivic representation stable for the general linear groups $G = \mathrm{GL}_r$ of any rank $r \geq 0$ over a field k , where the action of S_n is induced from the action of S_n on Γ_n by permutation. However, since the notion of motivic representation stability is only defined for (G -)varieties, we will instead prove that the sequences of representation varieties $X_n = R_G(\Gamma_n)$ are motivically representation stable as sequences of G -varieties. Indeed, note that the action of G by conjugation and the action of S_n by permutation commute.

The case of $\Gamma_n = F_n$ turns out to be a quick consequence of the theory developed in the previous sections.

Theorem 7.4.1. *For every $r \geq 0$, the sequence of GL_r -representation varieties*

$$X_n = R_{\mathrm{GL}_r}(F_n)$$

with the action of GL_r by conjugation, and the action of S_n by permutation, is motivically representation stable.

Proof. For any $n \geq 1$, write $X_n = R_{\mathrm{GL}_r}(F_n) = (\mathrm{GL}_r)^n$. Given any partition λ , we find

$$X_n // S_{\lambda[n]} = \mathrm{Sym}_{\mathrm{GL}_r}^{n-|\lambda|} \mathrm{GL}_r \times \prod_{i \geq 1} \mathrm{Sym}_{\mathrm{GL}_r}^{\lambda_i} \mathrm{GL}_r,$$

where GL_r acts on itself by conjugation. Viewing GL_r as a dense open subset of $\mathbb{A}_k^{r^2}$, the action of GL_r on itself is linear, and hence the sequence $X_n // S_{\lambda[n]}$ is motivically stable by Corollary 7.2.7 and Proposition 7.1.12. \square

For the remainder of this section, we will focus on the case $\Gamma_n = \mathbb{Z}^n$, and assume that k is algebraically closed.

Theorem 7.4.2. *For every $r \geq 0$, the sequence of GL_r -representation varieties*

$$X_n = R_{\mathrm{GL}_r}(\mathbb{Z}^n)$$

with the action of GL_r by conjugation, and the action of S_n by permutation, is motivically representation stable.

Notation-wise, we will use the following presentation of X_n , as the closed subvariety of $(\mathrm{GL}_r)^n$ given by commuting tuples of elements $A_i \in \mathrm{GL}_r$.

$$X_n = \{(A_1, \dots, A_n) \in (\mathrm{GL}_r)^n \mid \text{all } A_i \text{ commute}\}$$

Interestingly, it turns out the cases $r \leq 3$ should be treated differently from the general case $r > 3$. We will first treat the cases $r = 2, 3$.

Proposition 7.4.3. *The GL_2 -representation varieties $X_n = R_{\mathrm{GL}_2}(\mathbb{Z}^n)$ are motivically representation stable.*

Proof. Consider the possible Jordan normal forms of an element $A \in \mathrm{GL}_2$.

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

In particular, note that a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, with $\lambda \neq \mu$, only commutes with diagonal matrices, and that a matrix of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ only commutes with matrices of the form $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$. Therefore, X_n can be stratified by the subvarieties

$$\begin{aligned} Y_n &= \{A \in X_n \mid \text{all } A_i \text{ are scalar}\}, \\ J_n &= \{A \in X_n \mid \text{some } A_i \text{ is conjugate to } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\} \\ \text{and } M_n &= \{A \in X_n \mid \text{some } A_i \text{ is conjugate to } \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\}. \end{aligned}$$

Simultaneously conjugating the A_i into normal form, we can express J_n as

$$J_n = \mathrm{Ind}_H^{\mathrm{GL}_2} (J^n \setminus Y_n)$$

where $J = \{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x \neq 0\}$ and $H = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \neq 0\}$ the stabilizer of J . Similarly, we have

$$M_n = \mathrm{Ind}_K^{\mathrm{GL}_2} (M^n \setminus Y_n)$$

where $M = \{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \neq 0\}$ and $K = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a, b \neq 0\}$ the stabilizer of M . Clearly $\dim Y_n = n$ while $\dim J_n, \dim M_n \geq 2n$, implying $Y_n \subseteq X_n$ is negligible. Hence, it suffices to show that the sequences J'_n and M'_n , given by

$$J'_n = \mathrm{Ind}_H^{\mathrm{GL}_2} (J^n) \quad \text{and} \quad M'_n = \mathrm{Ind}_K^{\mathrm{GL}_2} (M^n)$$

are motivically representation stable.

First we consider J'_n . Note that, for any partition λ , the actions of $S_{\lambda[n]}$ and GL_2 on J^n commute, so that

$$J'_n // S_{\lambda[n]} = \mathrm{Ind}_H^{\mathrm{GL}_2} \left(\mathrm{Sym}_H^{n-|\lambda|} J \times \prod_{i \geq 1} \mathrm{Sym}_H^{\lambda_i} J \right).$$

Note that the action of H on J is linear, viewing J as an open dense subvariety of \mathbb{A}_k^2 . Hence, the sequence $J'_n // S_{\lambda[n]}$ is motivically stable as a result of Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6.

The argument regarding M_n is analogous: for any partition λ , we have

$$M'_n // S_{\lambda[n]} = \mathrm{Ind}_K^{\mathrm{GL}_2} \left(\mathrm{Sym}_K^{n-|\lambda|} M \times \prod_{i \geq 1} \mathrm{Sym}_K^{\lambda_i} M \right).$$

Again, the action of K on $M \subseteq \mathbb{A}_k^2$ is linear, so the sequence $M'_n // S_{\lambda[n]}$ is also motivically stable. \square

Proposition 7.4.4. *The GL_3 -representation varieties $X_n = R_{GL_3}(\mathbb{Z}^n)$ are motivically stable.*

Proof. The proof is very similar to that of Proposition 7.4.3. Consider the possible Jordan normal forms of an element $A \in GL_3$.

$$\begin{aligned} & \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \\ & \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \\ & \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix} \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \rho \end{pmatrix} \end{aligned}$$

Having analyzed which matrices commute with each Jordan type, we stratify X_n by the subvarieties

$$\begin{aligned} Y_n^0 &= \left\{ A \in X_n \mid \text{all } A_i \text{ are conjugate to } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \right\}, \\ Y_n^1 &= \left\{ A \in X_n \mid \text{some } A_i \text{ is conjugate to } \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \right\}, \\ Y_n^2 &= \left\{ A \in X_n \mid \text{some } A_i \text{ is conjugate to } \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \right\}, \\ Y_n^3 &= \left\{ A \in X_n \mid \text{some } A_i \text{ is conjugate to } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \rho \end{pmatrix} \right\}. \end{aligned}$$

Note that the sequences Y_n^1, Y_n^2 and Y_n^3 do not intersect since matrices that have different Jordan type in the bottom row never commute. As in the proof of Proposition 7.4.3, to show motivic representation stability of the strata Y_n^i , it suffices to show motivic representation stability of the sequences

$$Y_n^i = \text{Ind}_{H_i}^{GL_3} (J_i^n) \quad \text{for } i = 1, 2, 3,$$

where

$$\begin{aligned} J_1 &= \left\{ \begin{pmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \mid x \neq 0 \right\} & H_1 &= \left\{ \begin{pmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1/a \end{pmatrix} \mid a \neq 0 \right\} \\ J_2 &= \left\{ \begin{pmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, z \neq 0 \right\} & H_2 &= \left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} \mid a, c, d \neq 0 \right\} \\ J_3 &= \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \neq 0 \right\} & H_3 &= \mathbb{G}_m^3 \rtimes S_3. \end{aligned}$$

More precisely, $H_3 \subseteq GL_r$ is the subgroup generated by the diagonal matrices and the permutation matrices.

Now, for any partition λ , we find

$$Y_n^i // S_{\lambda[n]} = \text{Ind}_{H_i}^{\text{GL}_3} \left(\text{Sym}_{H_i}^{n-|\lambda|} J_i \times \prod_{j \geq 1} \text{Sym}_{H_i}^{\lambda_j} J_i \right).$$

For all i , the group H_i acts linearly on J_i , a dense open of \mathbb{A}_k^3 , so it follows from Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6 that the limits $\lim_{n \rightarrow \infty} Y_n^i / \mathbb{L}^{3n}$ exist. Knowing that $\dim X_n \geq 3n$, we see that $Y_n^0 \subseteq X_n$ is negligible, and the result follows. \square

Looking at the proofs of Proposition 7.4.3 and Proposition 7.4.4, it might be tempting to think that in the general case the non-negligible strata are those containing matrices with maximal Jordan type. However, this turns out to be the case only for $r \leq 3$.

For the general case we use a result initially proved by Schur [Sch05], and later reproved by Jacobson [Jac44], about the maximum number of linearly independent commuting matrices. This leads to the idea of stratifying the representation varieties $R_{\text{GL}_r}(\mathbb{Z}^n)$ by the dimension of the linear subspace inside $\text{Mat}_{r \times r}$ spanned by the matrices A_i .

Proof of Theorem 7.4.2. The case $r = 0$ is obvious, and the case $r = 1$ follows from motivic representation stability of \mathbb{G}_m^n , see Example 7.1.11 and Example 7.3.5. The cases $r = 2$ and $r = 3$ were treated in Proposition 7.4.3 and Proposition 7.4.4, so we can assume $r > 3$.

As usual, write $X_n = R_{\text{GL}_r}(\mathbb{Z}^n)$ for all $n \geq 1$. For any point $A \in X_n$ corresponding to a tuple (A_1, \dots, A_n) of commuting elements in GL_r , define

$$d_A = \dim_k \langle A_1, \dots, A_n \rangle$$

to be the dimension of the linear subspace of $\text{Mat}_{r \times r}(k)$ spanned by the A_i . By [Jac44, Theorem 1], we have $d_A \leq m$ with

$$m = \begin{cases} r^2/4 + 1 & \text{if } r \text{ is even,} \\ (r^2 - 1)/4 + 1 & \text{if } r \text{ is odd.} \end{cases}$$

Note that d_A is invariant under the actions of S_n and GL_r , so X_n can be stratified equivariantly by

$$X_{n,d} = \{A \in X_n \mid d_A = d\} \quad \text{for } 1 \leq d \leq m.$$

Now, we will show that $X_{n,d} \subseteq X_n$ is negligible for $d < m$, so that we solely need to focus on $X_{n,m}$. Note that the dimension of X_n is at least nm , as it contains

the family of commuting matrices given by

$$A_1 = \left(\begin{array}{c|c} \lambda_1 I & M_1 \\ \hline 0 & \lambda_1 I \end{array} \right), \dots, A_n = \left(\begin{array}{c|c} \lambda_n I & M_n \\ \hline 0 & \lambda_n I \end{array} \right) \quad (*)$$

with $\lambda_i \neq 0$, $M_i \in \mathrm{Mat}_{\frac{r}{2} \times \frac{r}{2}}$ if r is even, and $M_i \in \mathrm{Mat}_{\frac{r+1}{2} \times \frac{r-1}{2}}$ if r is odd. To see why the strata $X_{n,d}$ with $d < m$ are negligible, observe that $X_{n,d}$ can be covered by a dense open of $X_d \times (\mathbb{A}_k^d)^n$, that is, there is a surjective morphism from a dense open $Y_{n,d} \subseteq X_d \times (\mathbb{A}_k^d)^n$ given by

$$Y_{n,d} \rightarrow X_{n,d}, \quad \left((A_i)_{i=1}^d, (\alpha_{ij})_{i,j=1}^{n,d} \right) \mapsto \left(\sum_{j=1}^d \alpha_{ij} A_j \right)_{i=1}^n.$$

In particular, $\dim X_{n,d} \leq \dim Y_{n,d} \leq r^2 d + nd$, and hence $\lim_{n \rightarrow \infty} \dim X_n - \dim X_{n,d} = \infty$ for $d < m$, so it follows that $X_{n,d} \subseteq X_n$ is negligible.

By [Jac44, Theorem 3], every $A \in X_{n,m}$ can be conjugated to a tuple of the form (*). Hence, to show motivic representation stability of $X_{n,m}$ it suffices to show motivic representation stability of

$$X'_{n,m} = \mathrm{Ind}_H^{\mathrm{GL}_r} (J^n) \quad \text{with} \quad J = \left\{ \left(\begin{array}{c|c} \lambda I & M \\ \hline 0 & \lambda I \end{array} \right) \mid \begin{array}{l} \lambda \neq 0 \text{ and} \\ M \in \mathrm{Mat}_{\lceil \frac{r}{2} \rceil \times \lfloor \frac{r}{2} \rfloor} \end{array} \right\},$$

where the stabilizer

$$H = \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) \right\} \subseteq \mathrm{GL}_r$$

acts trivially on λ , and acts on M via the linear action

$$\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) \cdot M = AMC^{-1}.$$

Now, from Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6, it follows that $\lim_{n \rightarrow \infty} [X_{n,m}] / \mathbb{L}^{\dim X_{n,m}}$ exists. Moreover, as all $X_{n,d}$ with $d < m$ are negligible, this limit is equal to $\lim_{n \rightarrow \infty} [X_n] / \mathbb{L}^{\dim X_n}$. \square