Motivic invariants of character stacks

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Chapter 6

Upper triangular matrices

In this chapter we apply the theory of the Chapter 4 in order to study the G-character stacks of the closed orientable surfaces Σ_g, for G equal to one of the following algebraic groups, over any field k:

- the group $\mathbb{T}_n = \{A \in \mathrm{GL}_n \mid A_{ij} = 0 \text{ for } 1 \leq j < i \leq n\} \subseteq \mathrm{GL}_n$ of $n \times n$ upper triangular matrices, and
- its subgroup $\mathbb{U}_n = \{A \in \mathbb{T}_n \mid A_{ii} = 1 \text{ for } 1 \leq i \leq n\}$ of unipotent matrices.

These groups can be realized as semidirect products of copies of $\mathbb{G}_a$ and $\mathbb{G}_m$ and are therefore all special, see Proposition 3.3.16 and Example 3.3.17. In particular, the virtual class of the G-character stack of Σ_g in the Grothendieck ring of stacks is simply given by the quotient

$$[\mathfrak{X}_G(\Sigma_g)] = [R_G(\Sigma_g)]/[G] \tag{6.1}$$

as in Proposition 3.5.5. Hence, it suffices to apply the theory of Section 4.12, and work on the level of the G-representation variety.

Furthermore, these groups G are all connected. Therefore, the geometric TQFT and the arithmetic TQFT can be compared, as there is a natural transformation between them, see Corollary 4.10.5. We will consider both the geometric and the arithmetic method, and compare the results.

Note that the algebraic groups $\mathbb{T}_n$, for all $n \geq 1$, decompose as a product

$$\mathbb{T}_n = \mathbb{G}_m \times \tilde{\mathbb{T}}_n \quad \text{where} \quad \tilde{\mathbb{T}}_n = \{A \in \mathbb{T}_n \mid A_{nn} = 1\}.$$

In turn, this induces a decomposition of representation varieties

$$R_{\mathbb{T}_n}(M) \cong R_{\tilde{\mathbb{T}}_n}(M) \times R_{\mathbb{G}_m}(M). \tag{6.2}$$

As $\mathbb{G}_m$ is abelian, we have $R_{\mathbb{G}_m}(\Sigma_g) \cong \mathbb{G}_m^{2g}$, and we can focus on $\tilde{\mathbb{T}}_n$ instead. This slightly simplifies the computations as dimension is lower.

6.1 Algebraic representatives

Before doing computations, we first introduce the notion of *algebraic representatives*, which are crucial to doing computations in the later sections.

Definition 6.1.1. Let G be an algebraic group over k, and let X be a variety over k with a transitive G-action. A point $\xi \in X(k)$ is an *algebraic representative* for X if the $\operatorname{Stab}(\xi)$-torsor

$$G \to X, \quad g \mapsto g \cdot \xi$$

is Zariski-locally trivial. Equivalently, ξ is an algebraic representative for X if every point of X has an open neighborhood U and a morphism $\gamma \colon U \to G$ such that $x = \gamma(x) \cdot \xi$ for all $x \in U$.

Remark 6.1.2.
- If there exists an algebraic representative ξ for X, then every $\xi' \in X(k)$ is an algebraic representative for X. Namely, if $\xi = g \cdot \xi'$, then one takes $\gamma'(x) = \gamma(x)g$.
- Algebraic representatives need not always exist. For example, consider the group $G = \mathbb{G}_m$ acting on $X = \mathbb{A}^1_k \setminus \{0\}$ via $t \cdot x = t^2 x$. Then X does not have an algebraic representative as $\mathbb{G}_m \to \mathbb{A}^1_k \setminus \{0\}$ given by $t \mapsto t^2$ is not Zariski-locally trivial.
- The proof of Corollary 3.3.15 shows that $\xi \in X(k)$ is an algebraic representative for X if $\operatorname{Stab}(\xi)$ is special. However, it is possible that ξ is an algebraic representative even when $\operatorname{Stab}(\xi)$ is not special. For example, consider any non-special group G acting trivially on a point.

For us, the main example of algebraic representatives are for any of the groups $\mathbb{U}_n, \mathbb{T}_n$ or $\tilde{\mathbb{T}}_n$ acting by conjugation on a conjugacy class.

Proposition 6.1.3. *Let G be $\mathbb{U}_n, \mathbb{T}_n$ or $\tilde{\mathbb{T}}_n$, for some $n \geq 1$, acting on itself by conjugation. Then the stabilizer* $\operatorname{Stab}(A)$ *of any point $A \in G(k)$ is special. In particular, A is an algebraic representative for its conjugacy class.*

Proof. If $G = \mathbb{T}_n$, the stabilizer $\operatorname{Stab}(A) \subseteq \mathbb{T}_n$ is triangularizable, and can be written as an extension

$$1 \to U \to \operatorname{Stab}(A) \to D \to 1$$

of $D = \{B \in \operatorname{Stab}(A) \mid B \text{ is diagonal}\}$ by the maximal normal unipotent subgroup $U = \operatorname{Stab}(A) \cap \mathbb{U}_n$. We will show that both U and D are special, so that the result follows from Proposition 3.3.16 *(i)*. If $G = \mathbb{U}_n$, we have $\operatorname{Stab}(A) = U$, so the result follows from the same proof. If $G = \tilde{\mathbb{T}}_n$, then the action of $\tilde{\mathbb{T}}_n$ on itself by conjugation can be extended to an action of $\mathbb{T}_n$, and the corresponding stabilizers are related by $\mathrm{Stab}_{\mathbb{T}_n}(A) \cong \mathbb{G}_m \times \mathrm{Stab}_{\tilde{\mathbb{T}}_n}(A)$. The result then follows from Proposition 3.3.16 *(ii)* or *(iii)* and the fact that $\mathrm{Stab}_{\mathbb{T}_n}(A)$ is special.

Note that $U = \{B \in \mathbb{U}_n \mid AB - BA = 0\}$ is a subgroup of $\mathbb{U}_n$ given by a linear subspace, identifying $\mathbb{U}_n \cong \mathbb{A}_k^{n(n-1)/2}$ in the usual way. From [Mil15, Example 8.46] we know that $\mathbb{U}_n$ admits a normal series

$$\mathbb{U}_n = U_n^{(0)} \supseteq \cdots \supseteq U_n^{(r)} \supseteq U_n^{(r+1)} \supseteq \cdots \supseteq U_n^{(n(n-1)/2)} = \{1\},$$

where each $U_n^{(r)} \subseteq \mathbb{U}_n$ is a normal subgroup given by a linear subspace of $\mathbb{U}_n$, whose quotients $U_n^{(r)}/U_n^{(r+1)}$ are canonically isomorphic to $\mathbb{G}_a$. Therefore, intersecting this normal series with U yields a normal series of U where each quotient is either $\mathbb{G}_a$ or 0. Hence, U is an extension of copies of $\mathbb{G}_a$, which is special by Proposition 3.3.16 *(i)*.

Furthermore, D can be identified with

$$\begin{aligned} D &= \{B \in G \mid B \text{ is diagonal and } AB - BA = 0\} \\ &= \{(B_{11}, \ldots, B_{nn}) \in \mathbb{G}_m^n \mid A_{ij}(B_{ii} - B_{jj}) = 0 \text{ for all } 1 \leq i \leq j \leq n\} \\ &= \{(B_{11}, \ldots, B_{nn}) \in \mathbb{G}_m^n \mid B_{ii} = B_{jj} \text{ whenever } A_{ij} \neq 0\}, \end{aligned}$$

which, being a product of copies of $\mathbb{G}_m$, is also special. □

The notion of algebraic representatives can be generalized to a relative setting as follows. This generalization is useful when the G-action is not transitive, and we will use it in the later sections.

Definition 6.1.4. Let G be an algebraic group over k, acting on a variety X over k, and let $\pi\colon X \to T$ be a G-invariant morphism. A *family of algebraic representatives* for X over T is a morphism $\xi\colon T \to X$ over T (that is, $\pi \circ \xi = \mathrm{id}_T$) such that the $\mathrm{Stab}(\xi)$-torsor

$$G \times T \to X, \quad (g, t) \mapsto g \cdot \xi(t)$$

of varieties over T is Zariski-locally trivial. Note that by $\mathrm{Stab}(\xi)$ we understand $(G \times T) \times_X T$ as a group over T. Equivalently, ξ is a family of algebraic representatives for X over T if every point of X has an open neighborhood U and a morphism $\gamma\colon U \to G$ such that $x = \gamma(x) \cdot \xi(\pi(x))$ for all $x \in U$.

Example 6.1.5. Consider the group $G = \mathbb{T}_2 = \left\{ \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \;\middle|\; x, z \neq 0 \right\}$ of 2×2 upper triangular matrices acting on $X = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; a \neq 0, 1 \right\}$ by conjugation. Then X has a family of algebraic representatives over $T = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; a \neq 0, 1 \right\}$, with π and ξ the projection and inclusion, respectively, as one can take

$$\gamma \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{1-a} \\ 0 & 1 \end{pmatrix}.$$

The following lemma shows why it is useful to have algebraic representatives in the context of computing virtual classes.

Proposition 6.1.6. *Let G be an algebraic group over k, acting on a variety S over k. Let $\pi\colon S \to T$ be a G-invariant morphism, and let $\xi\colon T \to S$ be a family of algebraic representatives for S over T. Then for any morphism $f\colon Y \to S$ and G-equivariant morphism $g\colon X \to S$, we have*

$$[X \times_S Y]_S = [(X \times_S T) \times_T Y]_S \in \mathrm{K}_0(\mathbf{Var}_S),$$

where $(X \times_S T) \times_T Y$ is seen as a variety over S via the composite $f \circ \pi_Y$.

Proof. Locally on X, there is a commutative diagram

$$\begin{array}{ccc} X \times_S Y & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & (X \times_S T) \times_T Y \\ & \searrow \quad S \quad \swarrow_{f \circ \pi_Y} & \end{array}$$

where $\varphi(x, y) = (\gamma(f(y)) \cdot x, \pi(f(y)), y)$ and $\psi((x, t), y) = (\gamma(f(y)) \cdot x, y)$. One easily sees that φ and ψ are well-defined over S and inverse to each other. □

In the case of algebraic representatives, that is, when T is a point, we obtain the following corollaries.

Corollary 6.1.7. *Let G be an algebraic group over k, acting on a variety S over k with algebraic representative $\xi \in S(k)$. Then for any morphism $f\colon Y \to S$ and G-equivariant morphism $g\colon X \to S$, we have*

$$[X \times_S Y]_S = [X \times_S \{\xi\}] \cdot [Y]_S \in \mathrm{K}_0(\mathbf{Var}_S). \qquad \square$$

Corollary 6.1.8. *Let G be an algebraic group over k, acting on a variety S over k with algebraic representative $\xi \in S(k)$. Then for any G-equivariant morphism $g\colon X \to S$, we have*

$$[S] \cdot [X]_S = [X] \cdot [S]_S \in \mathrm{K}_0(\mathbf{Var}_S).$$

Proof. Apply Corollary 6.1.7 with $f = \mathrm{id}_S$ to find that

$$[X]_S = [X \times_S \{\xi\}] \cdot [S]_S.$$

Applying $c_!$ to both sides of this equation, for $c\colon S \to \operatorname{Spec} k$ the final morphism, yields $[S][X \times_S \{\xi\}] = [X]$ in $\mathrm{K}_0(\mathbf{Var}_k)$. □

6.2 Geometric method

The virtual classes of the G-representation varieties $R_G(\Sigma_g)$ in the Grothendieck ring of varieties can be computed, as was shown in Section 4.12, using the morphisms

Z_G^{rep}  $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_k)$, $\quad Z_G^{\mathrm{rep}}$  $: \mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{Var}_G)$

and $\quad Z_G^{\mathrm{rep}}$  $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_G)$

of $\mathrm{K}_0(\mathbf{Var}_k)$-modules. Note that all varieties over G that we consider are naturally equipped with a G-action such that the morphism to G is G-equivariant, even though the ring $\mathrm{K}_0(\mathbf{Var}_G)$ does not remember this information. For this reason, it turns out that $\mathrm{K}_0(\mathbf{Var}_G)$ is best understood via a decomposition

$$\mathrm{K}_0(\mathbf{Var}_G) \cong \bigoplus_{i=1}^{N} \mathrm{K}_0(\mathbf{Var}_{\mathcal{C}_i}),$$

as in Proposition 3.3.6, where the $\mathcal{C}_i$ are locally closed subvarieties of G given by families of conjugacy classes. We will show that each $\mathcal{C}_i$ has a family of algebraic representatives. As a result, the submodule of $\mathrm{K}_0(\mathbf{Var}_G)$ generated by the units $\mathbf{1}_{\mathcal{C}_i} \in \mathrm{K}_0(\mathbf{Var}_{\mathcal{C}_i})$ will be invariant under Z_G^{rep} .

Conjugacy classes of $\tilde{\mathbb{T}}_n$

Let us start by describing the conjugacy classes of $\tilde{\mathbb{T}}_n$, with a focus on the unipotent conjugacy classes. We will give algebraic representatives for the unipotent conjugacy classes, and families of algebraic representatives for (families of) non-unipotent conjugacy classes. Furthermore, we will determine equations describing the unipotent conjugacy classes, and finally, compute the virtual classes of the unipotent conjugacy classes as well as those of the stabilizers of their representatives. All of this data will be used to compute Z_G^{rep}  as a matrix with respect to the generators given by the unipotent classes.

Unipotent conjugacy classes. To find the number of unipotent conjugacy classes of $\tilde{\mathbb{T}}_n$, and a representative for each one, one can use Belitskii's algorithm as described in [Kob05]. Given a unipotent matrix $A \in \tilde{\mathbb{T}}_n$ over any field k, Belitskii's algorithm outputs a canonical representative of the conjugacy class of A. It achieves this by repeatedly conjugating A by certain elementary matrices in order to make as many entries of A as possible equal to 0 or 1, see [Kob05] for details. For $n = 1, \ldots, 5$, it turns out there are only finitely many unipotent conjugacy classes $\mathcal{U}_1, \ldots, \mathcal{U}_M$, and the canonical representatives $\xi_1, \ldots, \xi_M$ only have entries with 0's and 1's [Kob05]. The number M of unipotent conjugacy classes is given by the following table.

n	1	2	3	4	5
M	1	2	5	16	61

We will use the convention that $\mathcal{U}_1$ is the conjugacy class of the identity. Note that the representatives will be automatically algebraic by Proposition 6.1.3.

Remark 6.2.1. The qualitative result of Belitskii's algorithm, that for $n = 1, \ldots, 5$ every unipotent matrix in $\tilde{\mathbb{T}}_n$ can be conjugated to a matrix containing only 0's and 1's, is enough to find representatives. Only finitely many such matrices exist ($2^{n(n-1)/2}$) and they are easily partitioned by whether they are conjugate. Then, one simply chooses one representative in each conjugacy class.

Non-unipotent conjugacy classes. Next, we describe the non-unipotent conjugacy classes of $\tilde{\mathbb{T}}_n$ in terms of families depending on their diagonal. Define a *diagonal pattern* to be a partition of the set $\{1, 2, \ldots, n\}$. Then, for any matrix $A \in \tilde{\mathbb{T}}_n$, the diagonal pattern δ_A of A is the partition such that i and j are equivalent if $A_{ii} = A_{jj}$. Note that two matrices A and B in $\tilde{\mathbb{T}}_n$ are conjugate only if their diagonals coincide, but not necessarily if. Now, we look at the following families of conjugacy classes:

$$\mathcal{C}_{\delta,i} = \{A \in \tilde{\mathbb{T}}_n \mid \delta_A = \delta \text{ and } A \sim \operatorname{diag}(A) + \xi_i - 1\},$$

for any diagonal pattern δ and $i = 1, \ldots, M$, where $\operatorname{diag}(A)$ denotes the diagonal part of A. We claim that any such $\mathcal{C}_{\delta,i}$ has a family of representatives over

$$C_{\delta,i} = \{A \in \tilde{\mathbb{T}}_n \mid \delta_A = \delta \text{ and } A = \operatorname{diag}(A) + \xi_i - 1\}$$

where $\pi_{\delta,i} \colon \mathcal{C}_{\delta,i} \to C_{\delta,i}$ and $\xi_{\delta,i} \colon C_{\delta,i} \to \mathcal{C}_{\delta,i}$ are given by $\pi_{\delta,i}(A) = \operatorname{diag}(A) + \xi_i - 1$ and $\xi_{\delta,i}(A) = A$. This is proved in Lemma 6.2.2 below. Of course, some $\mathcal{C}_{\delta,i}$ may be equal to $\mathcal{C}_{\delta,j}$ while $i \neq j$, but one can explicitly check whether any of the representatives are conjugate in order to remove any such duplicates. In particular, one can ensure $A_{ij} = 0$ for all $A \in C_{\delta,k}$ whenever $A_{ii} \neq A_{jj}$, after appropriate conjugation. In the end, we obtain families of conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_N$ with families of algebraic representatives over $C_1, \ldots, C_N$, where the number N is given by the following table.

n	1	2	3	4	5
N	2	3	12	61	372

We will choose our indices in such a way that the ξ_i coincide with the unipotent representatives for $i = 1, \dots, M$.

Lemma 6.2.2. *For every $i = 1, \dots, N$, the following statements hold:*

(i) the stabilizer $H_i \coloneqq \mathrm{Stab}(\xi_i(t))$ is independent of $t \in C_i$,

(ii) ξ_i is a family of representatives of $\mathcal{C}_i$ over C_i,

(iii) the map $G/H_i \times C_i \to \mathcal{C}_i$ given by $(g, t) \mapsto g\xi_i(t)g^{-1}$ is an isomorphism.

Proof. *(i)* The statement can easily be verified by a computer, as there are only a finite number of cases to consider. Alternatively, write $A = \xi_i(t)$ and note that $B \in \mathrm{Stab}(A)$ if and only if for all $1 \leq i \leq j \leq n$,

$$B_{ij}(A_{ii} - A_{jj}) + \sum_{k=i+1}^{j} B_{kj}A_{ik} - \sum_{k=i}^{j-1} B_{ik}A_{kj} = 0. \qquad (*)$$

We claim that $B_{ij} = 0$ for all $i \leq j$ such that $A_{ii} \neq A_{jj}$. The result follows from this claim, because A_{ij} is independent of t for $i \neq j$ (by definition of C_i), so the solutions to $(*)$ will be independent of t. We proof the claim by induction on $j - i$, the case $j - i = 0$ being trivial. For the general case, take $i \leq j$ such that $A_{ii} \neq A_{jj}$. Now, for every $k \in \{i+1, \dots, j\}$ such that $A_{ik} \neq 0$, we have $A_{kk} = A_{ii} \neq A_{jj}$, so $B_{kj} = 0$ by the induction hypothesis. Similarly, for every $k \in \{i, \dots, j-1\}$ such that $A_{kj} \neq 0$, we have $A_{kk} = A_{jj} \neq A_{ii}$ so $B_{ik} = 0$ by the induction hypothesis. Therefore, $(*)$ reduces to $B_{ij} = 0$.

(ii) From *(i)* follows that the map

$$G \times C_i \to \mathcal{C}_i, \quad (g, t) \mapsto g\xi_i(t)g^{-1}$$

is an H_i-torsor, which is Zariski-locally trivial because H_i is special by Proposition 6.1.3. Hence, it follows that ξ_i is a family of algebraic representatives for $\mathcal{C}_i$ over C_i. This also proves *(iii)*. □

Equations. Next, we want to find equations describing the unipotent conjugacy classes $\mathcal{U}_i$ for $i = 1, \dots, M$. For simplicity, we will compute equations for the closures $\overline{\mathcal{U}}_i$ rather than $\mathcal{U}_i$. This is sufficient using the inclusion-exclusion matrix of Section 3.3. The closure $\overline{\mathcal{U}}_i$ is the closure of the image of the morphism

$$f_i \colon G \to G, \quad g \mapsto g\xi_i g^{-1}.$$

Since $G = \tilde{\mathbb{T}}_n$ is affine, f_i can equivalently be described by the corresponding morphism on the coordinate ring of G,

$$f_i^{\#} \colon \mathcal{O}_G(G) \to \mathcal{O}_G(G).$$

In particular, the closure $\overline{\mathcal{U}}_i$ corresponds to the ideal $I_i \subseteq \mathcal{O}_G(G)$ which is the kernel of $f_i^{\#}$. Generators for these ideals can be computed using Gröbner basis [AL94], and this gives us the desired equations. In particular, we use [AL94, Theorem 2.4.2] in order to compute the kernel of $f_i^{\#}$.

Example 6.2.3. Consider the unipotent conjugacy class $\mathcal{U}$ of $\xi = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ in $G = \tilde{\mathbb{T}}_3$. The morphism $f \colon G \to G$, $g \mapsto g\xi g^{-1}$ is given by

$$f \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

that is, $f^{\#}(a) = f^{\#}(d) = 1$, $f^{\#}(b) = f^{\#}(e) = 0$ and $f^{\#}(c) = a$. Indeed, we find that the ideal $\ker f^{\#} = (a-1, b, d-1, e)$ describes the closure of the conjugacy class $\mathcal{U} = \left\{ \left(\begin{smallmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \;\middle|\; c \neq 0 \right\}$.

Orbits and stabilizers. For any $A \in \tilde{\mathbb{T}}_n$, in order to compute the virtual class of the conjugacy class of A, we can use Corollary 3.3.15. Indeed, the stabilizer $\operatorname{Stab}(A)$ of any $A \in \tilde{\mathbb{T}}_n$ is special by Proposition 6.1.3. To compute the virtual class of the stabilizer of A, we apply Algorithm 3.4.3, using the explicit description

$$\operatorname{Stab}(A) = \{B \in \tilde{\mathbb{T}}_n \mid AB - BA = 0\}.$$

Computing the TQFT

Let us return to the problem of computing the matrix associated to Z () with respect to the generators $\mathbf{1}_{\mathcal{U}_i} = [\mathcal{U}_i]_G \in \mathrm{K}_0(\mathbf{Var}_G)$. We start by computing the first column of this matrix. Recall that by convention $\mathcal{U}_1 = \{1\}$ is the conjugacy class of the identity, and that c denotes the final morphism to $\operatorname{Spec} k$. To compute the first column of this matrix, we write

$c_!\big(Z_G^{\mathrm{rep}}$ ()$(\mathbf{1}_{\mathcal{U}_1})|_{\mathcal{U}_i}\big)$

$$\begin{aligned} &= [\{(A,B) \in G^2 \mid [A,B] \in \mathcal{U}_i\}] \\ &= \sum_{j=1}^{N} [\{(A,B) \in G \times \mathcal{C}_j \mid [A,B] \in \mathcal{U}_i\}] \\ &= \sum_{j=1}^{N} [\{(A,t) \in G \times C_j \mid [A,\xi_j(t)] \in \mathcal{U}_i\}] \times [\tilde{\mathbb{T}}_n / \operatorname{Stab}(\xi_j(t_0))] \\ &= \sum_{j=1}^{N} E_{ij}\, [\operatorname{Orbit}(\xi_j(t_0))], \end{aligned}$$

where $E_{ij} = [\{(A,t) \in G \times C_j \mid [A, \xi_j(t)] \in \mathcal{U}_i\}]$ and $t_0 \in C_j$ is any closed point. For the third equality, we used Proposition 6.1.6 with $Y = S = \mathcal{C}_j$ and $X = \{(A,B) \in G \times \mathcal{C}_j \mid [A,B] \in \mathcal{U}_i\}$, in combination with Lemma 6.2.2 *(iii)*.

Since we have computed equations describing the closures $\overline{\mathcal{U}}_i$, it is in fact easier to compute the classes $\overline{E}_{ij} = [\{(A,t) \in G \times C_j \mid [A, \xi_j(t)] \in \overline{\mathcal{U}}_i\}]$ rather than the E_{ij}. By Corollary 3.3.11, they are related through the inclusion-exclusion matrix C of the stratification by

$$E_{ij} = \sum_{k=1}^{M} C_{ik} \overline{E}_{kj}.$$

The coefficients $\overline{E}_{ij}$ can be computed using Algorithm 3.4.3. Then, using Corollary 6.1.8, we obtain

$$Z_G^{\text{rep}}\left(\bigcirc\!\!\!\!\bigcirc\right)(\mathbf{1}_{\mathcal{U}_1}) = \sum_{i,k=1}^{M} \sum_{j=1}^{N} C_{ik} \overline{E}_{kj} [\text{Orbit}(\xi_j(t_0))]/[\mathcal{U}_i] \cdot \mathbf{1}_{\mathcal{U}_i}.$$

Next, to compute the other columns of the matrix associated to $Z_G^{\text{rep}}\left(\bigcirc\!\!\!\!\bigcirc\right)$, we will make use of the already computed first column. In particular, we have

$$\begin{aligned}
&c_!\left(Z_G^{\text{rep}}\left(\bigcirc\!\!\!\!\bigcirc\right)(\mathbf{1}_{\mathcal{U}_j})|_{\mathcal{U}_i}\right) \\
&\qquad = [\{(g,A,B) \in \mathcal{U}_j \times G^2 \mid g[A,B] \in \mathcal{U}_i\}] \\
&\qquad = \sum_{k=1}^{M} [\{(g,A,B) \in \mathcal{U}_j \times G^2 \mid g[A,B] \in \mathcal{U}_i, [A,B] \in \mathcal{U}_k\}] \\
&\qquad = \sum_{k=1}^{M} [\{g \in \mathcal{U}_j \mid g\xi_k \in \mathcal{U}_i\}][\{(A,B) \in G^2 \mid [A,B] \in \mathcal{U}_k\}] \\
&\qquad = \sum_{k=1}^{M} F_{ijk}\, c_!\left(Z_G^{\text{rep}}\left(\bigcirc\!\!\!\!\bigcirc\right)(\mathbf{1}_{\mathcal{U}_1})|_{\mathcal{U}_k}\right),
\end{aligned}$$

where $F_{ijk} = [\{g \in \mathcal{U}_j \mid g\xi_k \in \mathcal{U}_i\}]$. Note that the third equality follows from Corollary 6.1.7 applied to $S = \mathcal{U}_k$ and $X = \{(g,h) \in \mathcal{U}_j \times \mathcal{U}_k \mid gh \in \mathcal{U}_i\}$ and $Y = \{(A,B) \in G^2 \mid [A,B] \in \mathcal{U}_k\}$.

As for the coefficients E_{ij}, it is easier to compute $\overline{F}_{ijk} = [\{g \in \overline{\mathcal{U}}_j \mid g\xi_k \in \overline{\mathcal{U}}_i\}]$ rather than F_{ijk}, and they are related through the inclusion-exclusion matrix of the stratification by

$$F_{ijk} = \sum_{m,\ell=1}^{M} C_{im} C_{j\ell} \overline{F}_{m\ell k}.$$

The coefficients $\overline{F}_{ijk}$ can be computed using Algorithm 3.4.3. Finally, using Corollary 6.1.8 we obtain

$$Z_G^{\mathrm{rep}}\left(\bigcirc\right)(\mathbf{1}_{\mathcal{U}_j}) = \sum_{i,k,\ell,m=1}^{M} C_{im} C_{j\ell} \overline{F}_{m\ell k}\, c_!\left(Z_G^{\mathrm{rep}}\left(\bigcirc\right)(\mathbf{1}_{\mathcal{U}_1})|_{\mathcal{U}_k}\right)/[\mathcal{U}_i] \cdot \mathbf{1}_{\mathcal{U}_i}.$$

Remark 6.2.4. Naively computing the coefficients of the matrix representing $Z_G^{\mathrm{rep}}\left(\bigcirc\right)$ would require computing the virtual class of M^2 varieties, each of which being a subvariety of G^3, with equations being mostly quadratic due to the commutator $[A, B]$. With this new setup, one needs to compute the virtual class of $MN + M^3$ varieties to obtain the coefficients $\overline{E}_{ij}$ and $\overline{F}_{ijk}$. However, the advantage of this approach is that these varieties will now be subvarieties of $G \times C_j$ and $\mathcal{U}_j$, respectively, with equations being mostly linear. In practice, the simplification of these systems of equations far outweighs the number of such systems. It is due to the (families of) algebraic representatives of the conjugacy classes $\mathcal{C}_i$ that these simplifications can be made.

Remark 6.2.5. Let us make a few computational remarks. First, to speed up the computation of the coefficients $\overline{E}_{ij}$ and $\overline{F}_{ijk}$, which are done by Algorithm 3.4.3, note that these can be performed in parallel as they are independent. Second, there are some checks one can perform to detect obvious errors. In particular, one can assert that the following equalities hold:

- $\sum_{i=1}^{M} c_!\left(Z_G^{\mathrm{rep}}\left(\bigcirc\right)(\mathbf{1}_{\mathcal{U}_j})|_{\mathcal{U}_i}\right) = [G]^2\,[\mathcal{U}_j]$ for all j,
- $\sum_{i=1}^{M} F_{ijk} = [\mathcal{U}_j]$ for all j, k,
- $\sum_{j=1}^{M} F_{ijk} = [\mathcal{U}_i]$ for all i, k,
- $\sum_{i=1}^{M} E_{ij} = [G]\,[C_j]$ for all j.

Results

The code to perform the computations as described in this section can be found in [Vog22]. For every $n = 1, \ldots, 5$, the resulting matrix associated to $Z_G^{\mathrm{rep}}\left(\bigcirc\right)$, with respect to the generators $\mathbf{1}_{\mathcal{U}_i}$, is a matrix whose coefficients are polynomials in the Lefschetz class $\mathbb{L}$. These matrices can be diagonalized over the field $\mathbb{Q}(\mathbb{L})$ of rational functions in $\mathbb{L}$, and the resulting eigenvalues and eigenvectors are recorded in Appendix A.

Applying equations (4.10), (6.2) and (6.1), we obtain the following theorem.

Theorem 6.2.6. *The virtual classes of the $\mathbb{T}_n$-character stacks of Σ_g in the Grothendieck ring of stacks for $n = 2, 3, 4, 5$ are given by* *(i)* $[\mathfrak{X}_{\mathbb{T}_2}(\Sigma_g)] = \mathbb{L}^{2g-2}(\mathbb{L}-1)^{2g-1} + \mathbb{L}^{2g-2}(\mathbb{L}-1)^{4g-2}$

(ii) $[\mathfrak{X}_{\mathbb{T}_3}(\Sigma_g)] = \mathbb{L}^{4g-4}(\mathbb{L}-1)^{4g-2} + \mathbb{L}^{6g-6}(\mathbb{L}-1)^{2g-1} + 2\mathbb{L}^{6g-6}(\mathbb{L}-1)^{4g-2} + \mathbb{L}^{6g-6}(\mathbb{L}-1)^{6g-3}$

(iii) $[\mathfrak{X}_{\mathbb{T}_4}(\Sigma_g)] = \mathbb{L}^{8g-8}(\mathbb{L}-1)^{4g-2} + \mathbb{L}^{8g-8}(\mathbb{L}-1)^{6g-3} + \mathbb{L}^{10g-10}(\mathbb{L}-1)^{2g-1} + 3\mathbb{L}^{10g-10}(\mathbb{L}-1)^{4g-2} + 2\mathbb{L}^{10g-10}(\mathbb{L}-1)^{6g-3} + \mathbb{L}^{12g-12}(\mathbb{L}-1)^{2g-1} + 3\mathbb{L}^{12g-12}(\mathbb{L}-1)^{4g-2} + 3\mathbb{L}^{12g-12}(\mathbb{L}-1)^{6g-3} + \mathbb{L}^{12g-12}(\mathbb{L}-1)^{8g-4}$

(iv) $[\mathfrak{X}_{\mathbb{T}_5}(\Sigma_g)] = \mathbb{L}^{12g-12}(\mathbb{L}-1)^{6g-3} + 2\mathbb{L}^{14g-14}(\mathbb{L}-1)^{4g-2} + 3\mathbb{L}^{14g-14}(\mathbb{L}-1)^{6g-3} + \mathbb{L}^{14g-14}(\mathbb{L}-1)^{8g-4} + 2\mathbb{L}^{16g-16}(\mathbb{L}-1)^{2g-1} + 7\mathbb{L}^{16g-16}(\mathbb{L}-1)^{4g-2} + 7\mathbb{L}^{16g-16}(\mathbb{L}-1)^{6g-3} + 2\mathbb{L}^{16g-16}(\mathbb{L}-1)^{8g-4} + 2\mathbb{L}^{18g-18}(\mathbb{L}-1)^{2g-1} + 7\mathbb{L}^{18g-18}(\mathbb{L}-1)^{4g-2} + 8\mathbb{L}^{18g-18}(\mathbb{L}-1)^{6g-3} + 3\mathbb{L}^{18g-18}(\mathbb{L}-1)^{8g-4} + \mathbb{L}^{20g-20}(\mathbb{L}-1)^{2g-1} + 4\mathbb{L}^{20g-20}(\mathbb{L}-1)^{4g-2} + 6\mathbb{L}^{20g-20}(\mathbb{L}-1)^{6g-3} + 4\mathbb{L}^{20g-20}(\mathbb{L}-1)^{8g-4} + \mathbb{L}^{20g-20}(\mathbb{L}-1)^{10g-5}$. □

The computation times[1] for $Z_G^{\mathrm{rep}}(\ominus)$ for the groups $G = \tilde{\mathbb{T}}_n$ are listed in the table below. These times do not include the diagonalization of $Z_G^{\mathrm{rep}}(\ominus)$, as this was done by hand.

n	2	3	4	5
world time	1.92s	5.17s	1m19s	1h38m
CPU time	2.11s	31.50s	28m12s	50h9m

Finally, we note that precisely the same method can be applied to the groups $G = \mathbb{U}_n$ for $n = 1, \ldots, 5$. In fact, the coefficients F_{ijk} can be reused. For these groups, the map $Z_G^{\mathrm{rep}}(\ominus)$ is given by

$$Z_G^{\mathrm{rep}}(\ominus)(\mathbf{1}_{\mathcal{U}_1}) = \sum_{i,j,k=1}^{M} C_{ik}\overline{E}_{kj}[\mathrm{Orbit}(\xi_j)]/[\mathcal{U}_i] \cdot \mathbf{1}_{\mathcal{U}_i}$$

$$Z_G^{\mathrm{rep}}(\ominus)(\mathbf{1}_{\mathcal{U}_j}) = \sum_{i,k,\ell,m=1}^{M} C_{im}C_{j\ell}\overline{F}_{m\ell k}\, c_!\left(Z_G^{\mathrm{rep}}(\ominus)(\mathbf{1}_{\mathcal{U}_1})|_{\mathcal{U}_k}\right)/[\mathcal{U}_i] \cdot \mathbf{1}_{\mathcal{U}_i}$$

where now $\overline{E}_{ij} = [\{A \in \mathbb{U}_n \mid [A, \xi_j] \in \overline{\mathcal{U}}_i\}]$, and $\overline{F}_{ijk}$ are the same as for $G = \tilde{\mathbb{T}}_n$. Importantly, we still consider the action of $\tilde{\mathbb{T}}_n$ on $\mathbb{U}_n$ by conjugation so that the orbits and stabilizers, such as $\mathrm{Orbit}(\xi_j)$, remain unchanged. This yields the following theorem.

Theorem 6.2.7. *The virtual classes of the $\mathbb{U}_n$-character stacks of Σ_g in the Grothendieck ring of stacks for $n = 2, 3, 4, 5$ are given by*

[1] As performed on an Intel®Xeon®CPU E5-4640 0 @ 2.40GHz. Since the computations were performed in parallel (64 cores), both the world time and the CPU time were recorded.

(i) $[\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)] = \mathbb{L}^{2g-1}$

(ii) $[\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)] = \mathbb{L}^{4g-4}(\mathbb{L}-1) + \mathbb{L}^{6g-4}$

(iii) $[\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)] = \mathbb{L}^{8g-7}(\mathbb{L}-1) + \mathbb{L}^{10g-9}(\mathbb{L}-1)(\mathbb{L}+1) + \mathbb{L}^{12g-9}$

(iv) $[\mathfrak{X}_{\mathbb{U}_5}(\Sigma_g)] = \mathbb{L}^{12g-12}(\mathbb{L}-1)^2 + \mathbb{L}^{14g-13}(\mathbb{L}-1)(2\mathbb{L}-1) +$
$\mathbb{L}^{16g-15}(\mathbb{L}-1)(\mathbb{L}+1)(2\mathbb{L}-1) + \mathbb{L}^{18g-16}(\mathbb{L}-1)(2\mathbb{L}+1) + \mathbb{L}^{20g-16}$. □

6.3 Arithmetic method

Let us now consider the arithmetic side of the same story, applying the theory of Section 4.5. That is, we will study the representation theory of the groups $\mathbb{T}_n$ and $\mathbb{U}_n$ over finite fields $\mathbb{F}_q$, and, in particular, we want to determine the dimensions of the irreducible representations of these finite groups G. We will encode these values in the *representation zeta function*

$$\zeta_G(s) = \sum_{\chi \in \hat{G}} \chi(1)^{-s}$$

where $\hat{G}$ denotes the set of irreducible complex characters of G. Theorem 4.5.3 shows that $\zeta_G(s)$ contains precisely enough information about the point count of the G-character groupoid of Σ_g, since the given equation can be rewritten to

$$|\mathfrak{X}_G(\Sigma_g)| = |G|^{-\chi(\Sigma_g)} \zeta_G(-\chi(\Sigma_g)), \tag{6.3}$$

where $\chi(\Sigma_g) = 2 - 2g$ denotes the Euler characteristic of Σ_g. Finally, these point counts will turn out to be polynomial in q, so that by Katz' theorem 4.6.1, these polynomials determine the E-polynomials of the character stacks.

The representation zeta functions of these finite groups will be computed algorithmically. Roughly speaking, the algorithm, which we describe below, computes representation zeta functions recursively by decomposing subgroups of $\mathbb{T}_n$ as semidirect subgroups $N \rtimes H$ with $H \subseteq \mathbb{T}_{n-1}$ and $N \subseteq \mathbb{G}_a^{n-1}$. Hence, let us recall how the representation theory of semidirect products is related to that of its factors.

Semidirect products

Consider a finite group $G = N \rtimes H$, with $N \subseteq G$ an abelian normal subgroup. Following [Ser77, Section 8.2], we describe how the representation theory of G is related to that of N and H. As N is abelian, its irreducible representations are one-dimensional and given by $X = \mathrm{Hom}(N, \mathbb{C}^\times)$. The group H acts on X via

$$(h \cdot \chi)(n) = \chi(h^{-1} n h) \quad \text{for all } \chi \in X,\ h \in H \text{ and } n \in N.$$

Let $(\chi_i)_{i\in X/H}$ be a collection of representatives for the orbits in X under H. For each $i \in X/H$, let $H_i = \{h \in H \mid h \cdot \chi_i = \chi_i\}$ denote the stabilizer of χ_i, and let $G_i = N \rtimes H_i \subseteq G$ be the corresponding subgroup of G. We can extend χ_i to G_i by setting

$$\chi_i((n,h)) = \chi_i(n) \quad \text{for all } n \in N \text{ and } h \in H_i.$$

Indeed, this defines a (1-dimensional) character of G_i as

$$\begin{aligned}\chi_i((n_1,h_1)(n_2,h_2)) &= \chi_i((n_1(h_1n_2h_1^{-1}),h_1h_2)) \\ &= \chi_i(n_1h_1n_2h_1^{-1}) = \chi_i(n_1)\chi_i(n_2) = \chi_i((n_1,h_1))\chi_i((n_2,h_2))\end{aligned}$$

for all $n_1, n_2 \in N$ and $h_1, h_2 \in H_i$. Now, any irreducible representation ρ of H_i induces a representation $\tilde{\rho}$ of G_i by composing with the projection $G_i \to G_i/N = H_i$, and we define

$$\theta_{i,\rho} = \operatorname{Ind}_{G_i}^{G}\left(\chi_i \otimes \tilde{\rho}\right).$$

It turns out that these are precisely all the irreducible representations of G.

Proposition 6.3.1 ([Ser77, Proposition 25])**.** *(i) $\theta_{i,\rho}$ is irreducible.*

(ii) If $\theta_{i,\rho}$ is isomorphic to $\theta_{i',\rho'}$, then $i = i'$ and ρ is isomorphic to ρ'.

(iii) Every irreducible representation of G is isomorphic to some $\theta_{i,\rho}$.

In terms of representation zeta functions, this proposition translates to the following corollary, using the fact that $\dim\left(\operatorname{Ind}_H^G(\rho)\right) = \dim(\rho)[G:H]$.

Corollary 6.3.2. *The representation zeta function of G is given by*

$$\zeta_G(s) = \sum_{i\in X/H} \zeta_{H_i}(s)[G:G_i]^{-s} = \sum_{i\in X/H} \zeta_{H_i}(s)[H:H_i]^{-s}. \qquad \square$$

Decomposing triangles

Consider the group $\mathbb{U}_n(\mathbb{F}_q) = \{A \in \mathrm{GL}_n(\mathbb{F}_q) \mid A_{ii} = 1 \text{ and } A_{ij} = 0 \text{ for } i > j\}$ of $n \times n$ unipotent upper triangular matrices over a finite field $\mathbb{F}_q$. Let N be the kernel of

$$\mathbb{U}_n(\mathbb{F}_q) \to \mathbb{U}_{n-1}(\mathbb{F}_q), \quad A \mapsto (A_{ij})_{i,j=1}^{n-1},$$

so that the quotient $\mathbb{U}_n(\mathbb{F}_q)/N$ is isomorphic to $\mathbb{U}_{n-1}(\mathbb{F}_q)$. Now we have a split exact sequence

$$1 \longrightarrow N \longrightarrow \mathbb{U}_n(\mathbb{F}_q) \longrightarrow \mathbb{U}_{n-1}(\mathbb{F}_q) \longrightarrow 1,$$

which yields a semidirect decomposition $\mathbb{U}_n(\mathbb{F}_q) = N \rtimes \mathbb{U}_{n-1}(\mathbb{F}_q)$, where N is abelian. Moreover, for any unipotent subgroup $U \subseteq \mathbb{U}_n(\mathbb{F}_q)$, the above exact sequence can be intersected with U to obtain

$$1 \longrightarrow U \cap N \longrightarrow U \xrightarrow{\curvearrowleft} U \cap \mathbb{U}_{n-1}(\mathbb{F}_q) \longrightarrow 1,$$

yielding a semidirect decomposition $U = (U \cap N) \rtimes (U \cap \mathbb{U}_{n-1}(\mathbb{F}_q))$.

We identify $N \cong \mathbb{G}_a^{n-1}(\mathbb{F}_q) = \mathbb{F}_q^{n-1}$, where $\mathbb{F}_q$ as additive group is equal to $(\mathbb{Z}/p\mathbb{Z})^m$ for $q = p^m$. The irreducible characters $\chi_\alpha \in X = \mathrm{Hom}(N, \mathbb{C}^\times)$ of N can now also be identified with vectors $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{F}_q^{n-1}$, via

$$\chi_\alpha(x) = \zeta_p^{\langle \alpha, x\rangle} \qquad \text{for all } x \in N,$$

where ζ_p is a primitive p^{th} root of unity, and $\langle -, - \rangle$ denotes the trace form given by $\langle \alpha, x \rangle = \sum_{i=1}^{n-1} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha_i x_i) \in \mathbb{F}_p$, which is a non-degenerate bilinear form. Since $\mathbb{U}_{n-1}(\mathbb{F}_q)$ acts on $N \cong \mathbb{F}_q^{n-1}$ by left multiplication, it acts on $X \cong \mathbb{F}_q^{n-1}$ by right multiplication, because $\langle \alpha A, x \rangle = \langle \alpha, Ax \rangle$ for all $\alpha, x \in \mathbb{F}_q^{n-1}$ and $A \in \mathrm{GL}_{n-1}(\mathbb{F}_q)$.

From now on, to unclutter the notation, we will omit the field $\mathbb{F}_q$ from the group, simply writing G instead of $G(\mathbb{F}_q)$, and $\mathbb{U}_n$ instead of $\mathbb{U}_n(\mathbb{F}_q)$, etc.

Example 6.3.3. Consider the group $\mathbb{U}_3 \cong \mathbb{G}_a^2 \rtimes \mathbb{U}_2$. As discussed above, $H = \mathbb{U}_2$ acts on $X = \mathrm{Hom}(\mathbb{G}_a^2, \mathbb{C}^\times) \cong \mathbb{G}_a^2$ by right-multiplication, that is,

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta + a\alpha \end{pmatrix} \qquad \text{for all } \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in H \text{ and } \begin{pmatrix} \alpha & \beta \end{pmatrix} \in X.$$

Hence, the orbits in X under H are given by $\left\{ \begin{pmatrix} \alpha & \beta \end{pmatrix} : \beta \in \mathbb{F}_q \right\}$ for all $\alpha \in \mathbb{F}_q^\times$ and $\left\{ \begin{pmatrix} 0 & \beta \end{pmatrix} \right\}$ for all $\beta \in \mathbb{F}_q$. We choose the following representatives:

- $\chi_\alpha = \begin{pmatrix} \alpha & 0 \end{pmatrix}$, for which $H_\alpha = \{1\}$, so the contribution to the zeta function is

$$(q-1)\, \zeta_{\{1\}}(s)\, [H : H_\alpha]^{-s} = (q-1)\, q^{-s}.$$

- $\chi_\beta = \begin{pmatrix} 0 & \beta \end{pmatrix}$, for which $H_\beta = \mathbb{U}_2$, so the contribution to the zeta function is

$$q\, \zeta_{\mathbb{U}_2}(s)\, [H : H_\beta]^{-s} = q^2.$$

Adding up the contributions, it follows from Corollary 6.3.2 that $\zeta_{\mathbb{U}_3}(s) = q^2 + (q-1)\, q^{-s}$.

Example 6.3.4. Consider $\mathbb{U}_4 \cong \mathbb{G}_a^3 \rtimes \mathbb{U}_3$, for which $X = \mathrm{Hom}(\mathbb{G}_a^3, \mathbb{C}^\times) \cong \mathbb{G}_a^3$, and $H = \mathbb{U}_3$ acts on $\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix} \in X$ by right-multiplication, that is,

$$\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & \beta + a\alpha & \gamma + b\alpha + c\beta\end{pmatrix}.$$

Hence, the orbits in X under H are given by $\left\{\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix} : \beta, \gamma \in \mathbb{F}_q\right\}$ for all $\alpha \in \mathbb{F}_q^\times$, $\left\{\begin{pmatrix}0 & \beta & \gamma\end{pmatrix} : \gamma \in \mathbb{F}_q\right\}$ for all $\beta \in \mathbb{F}_q^\times$, and $\left\{\begin{pmatrix}0 & 0 & \gamma\end{pmatrix}\right\}$ for all $\gamma \in \mathbb{F}_q$. We choose the following representatives:

- $\chi_\alpha = \begin{pmatrix}\alpha & 0 & 0\end{pmatrix}$ with $H_\alpha \cong \mathbb{G}_a$, contributing
$$(q-1)\,\zeta_{\mathbb{G}_a}(s)\,[H : H_\alpha]^{-s} = q^{1-2s}(q-1).$$
- $\chi_\beta = \begin{pmatrix}0 & \beta & 0\end{pmatrix}$ with $H_\beta \cong \mathbb{G}_a^2$, contributing
$$(q-1)\,\zeta_{\mathbb{G}_a^2}(s)\,[H : H_\beta]^{-s} = q^{2-s}(q-1).$$
- $\chi_\gamma = \begin{pmatrix}0 & 0 & \gamma\end{pmatrix}$ with $H_\gamma = \mathbb{U}_3$, contributing
$$q\,\zeta_{\mathbb{U}_3}(s)\,[H : H_\gamma]^{-s} = q^3 + (q-1)\,q^{1-s}.$$

In total, $\zeta_{\mathbb{U}_4}(s) = q^3 + q^{1-s}(q-1)(q+1) + q^{1-2s}(q-1)$.

The construction as described above can be applied more generally to any connected algebraic subgroup $G \subseteq \mathbb{T}_n$ as follows. Let G' be the image of the map $G \to \tilde{\mathbb{T}}_n$ given by $A \mapsto A/A_{nn}$. Then either $G \cong G'$ or $G \cong \mathbb{G}_m \times G'$, because the only connected subgroups of $\mathbb{G}_m$ are $\{1\}$ and $\mathbb{G}_m$ itself. Since $\zeta_{\mathbb{G}_m}(s) = q - 1$ is known, we may assume $G \subseteq \tilde{\mathbb{T}}_n$. The group $\tilde{\mathbb{T}}_n$ can be decomposed, similar to $\mathbb{U}_n$, as

$$1 \longrightarrow \mathbb{G}_a^{n-1} \longrightarrow \tilde{\mathbb{T}}_n \overset{\curvearrowleft}{\longrightarrow} \mathbb{T}_{n-1} \longrightarrow 1,$$

where the map $\tilde{\mathbb{T}}_n \to \mathbb{T}_{n-1}$ is given by $A \mapsto (A_{ij})_{i,j=1}^{n-1}$. Intersecting with G, we obtain $G = N \rtimes H$ with $N = G \cap \mathbb{G}_a^{n-1}$ abelian and $H = G \cap \mathbb{T}_{n-1}$, to which we can apply Corollary 6.3.2.

Example 6.3.5. Consider $G = \mathbb{T}_2 \cong \mathbb{G}_m \times \tilde{\mathbb{T}}_2$ with $\tilde{\mathbb{T}}_2 \cong \mathbb{G}_a \rtimes \mathbb{G}_m$, for which $X = \mathrm{Hom}(\mathbb{G}_a, \mathbb{C}^\times) \cong \mathbb{G}_a$ and $H = \mathbb{G}_m$ acts on $\alpha \in X$ by multiplication. Hence, the orbits in X under H are given by $\{0\}$ and $\{\alpha : \alpha \in \mathbb{F}_q^\times\}$. We choose the following representatives:

- $\chi_0 = 0$ yields $H_0 = \mathbb{G}_m$, contributing $\zeta_{\mathbb{G}_m}(s) = q - 1$,

- $\chi_1 = 1$ yields $H_1 = \{1\}$, contributing $(q-1)^{-s}$.

In total,

$$\zeta_{\mathbb{T}_2}(s) = \zeta_{\mathbb{G}_m}(s)\,\zeta_{\tilde{\mathbb{T}}_2}(s) = (q-1)((q-1)+(q-1)^{-s}) = (q-1)^2 + (q-1)^{1-s}.$$

These examples illustrate how one computes the representation zeta function in a recursive manner using Proposition 6.3.1. Note that in all examples, the stabilizers H_α of χ_α are independent of α (and similarly for H_β, H_γ, ...). However, the following example shows that this need not always be the case: we obtain a family of stabilizers $H_{\alpha,\beta}$ which depend explicitly on the parameters α and β.

Example 6.3.6. Consider $G = \mathbb{G}_a^3 \rtimes H$ with $H = \left\{\left(\begin{smallmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)\right\}$ acting naturally on $\mathbb{G}_a^3$. Then H acts on $X = \operatorname{Hom}(\mathbb{G}_a^3, \mathbb{C}^\times) \cong \mathbb{G}_a^3$ by

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma + a\alpha + b\beta \end{pmatrix}.$$

Hence, the orbits in X under H are given by $\left\{\begin{pmatrix} 0 & 0 & \gamma \end{pmatrix}\right\}$ for all $\gamma \in \mathbb{F}_q$ and $\left\{\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} : \gamma \in \mathbb{F}_q\right\}$ for all $\alpha, \beta \in \mathbb{F}_q$ with $(\alpha, \beta) \neq (0, 0)$. We choose the following representatives:

- $\chi_\gamma = \begin{pmatrix} 0 & 0 & \gamma \end{pmatrix}$ with $H_\gamma = H$, contributing $q\,\zeta_H(s) = q^3$,
- $\chi_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta & 0 \end{pmatrix}$ with $H_{\alpha,\beta} = \left\{\left(\begin{smallmatrix} 1 & 0 & x\beta \\ 0 & 1 & -x\alpha \\ 0 & 0 & 1 \end{smallmatrix}\right) : x \in \mathbb{F}_q\right\}$. Note that $H_{\alpha,\beta}$ depends explicitly on α and β, even though $H_{\alpha,\beta} \cong \mathbb{G}_a$ for all α and β. These representatives contribute

$$(q^2-1)\,\zeta_{\mathbb{G}_a}(s)\,[H : H_{\alpha,\beta}]^{-s} = q^{1-s}(q-1)(q+1).$$

In total, $\zeta_G(s) = q^3 + q^{1-s}(q-1)(q+1)$.

Algorithmically computing $\zeta_G(s)$

Now we will describe an algorithm to compute $\zeta_G(s)$ for connected algebraic groups $G \subseteq \mathbb{T}_n$, in the style of examples 6.3.3, 6.3.4 and 6.3.6. An implementation of this algorithm can be found at [Vog22], together with the code for computing $\zeta_{\mathbb{U}_n}(s)$ and $\zeta_{\mathbb{T}_n}(s)$ for $n = 1, \ldots, 10$. The resulting zeta functions are given in Theorem 6.3.11 and Theorem 6.3.10.

Before discussing the algorithm, let us give some remarks.

The algorithm is divided into two parts. The main part, Algorithm 6.3.7, finds a semidirect decomposition $G \cong N \rtimes H$ and applies Corollary 6.3.2 in order to compute $\zeta_G(s)$. Finding representatives for the orbits in X under H is a more intricate step, and is described separately in Algorithm 6.3.8.

As highlighted in Example 6.3.6, it is possible for the stabilizers H_α to depend explicitly on the parameter α. Therefore, in order for the algorithm to work recursively, we allow the input of the algorithm to be a family G of algebraic groups $G_t \subseteq \mathbb{T}_n$ parametrized by a variety T over $\mathbb{F}_q$, that is, a subgroup $G \subseteq \mathbb{T}_n \times T$ over T. We then understand the representation zeta function of G to be

$$\zeta_G(s) = \sum_{t \in T(\mathbb{F}_q)} \zeta_{G_t}(s).$$

As we want the computations to hold over general a ground field $\mathbb{F}_q$, we will in practice work over $\mathbb{Z}$. Then $|T(\mathbb{F}_q)|$ can be computed as a polynomial in q whenever $[T] \in \mathrm{K}_0(\mathbf{Var}_\mathbb{Z})$ can be computed as a polynomial in $q = [\mathbb{A}^1_\mathbb{Z}]$ using Algorithm 3.4.3.

There are steps in the algorithm containing conditions that depend on the value of $t \in T$. At such steps, we stratify T into the stratum where the condition holds and the stratum where it does not hold, and continue the algorithm on both strata separately.

Algorithm 6.3.7. Input: A family of connected algebraic groups $G \subseteq \mathbb{T}_n \times T$ over a variety T.

Output: The representation zeta function $\zeta_G(s)$ as a polynomial in q, q^{-s} and $(q-1)^{-s}$.

1. If $n = 0$, then G is trivial, so that $\zeta_G(s) = |T(\mathbb{F}_q)|$. Hence, we can assume $n \geq 1$.

2. Since G is connected, the image of the map $G \to \mathbb{G}_m^n \times T$ given by $(A, t) \mapsto ((A_{ii})_{i=1}^n, t)$ is isomorphic to $\mathbb{G}_m^d \times T$ for some $0 \leq d \leq n$, at least locally on T, so after stratifying T we can assume this to be the case. If $d = n$, then there is an isomorphism $G \cong \mathbb{G}_m \times G'$ with $G' \subseteq \tilde{\mathbb{T}}_n \times T$ given by $(A, t) \mapsto (A_{nn}, (A/A_{nn}, t))$, so that $\zeta_G(s) = (q-1)\,\zeta_{G'}(s)$. If $d < n$, then $G \cong G' \subseteq \tilde{\mathbb{T}}_n$ via the map $A \mapsto A/A_{nn}$. Either way, we can assume $G \subseteq \tilde{\mathbb{T}}_n \times T$.

3. Write $G = N \rtimes H$ as discussed above. The group $H \subseteq \mathbb{T}_{n-1} \times T$ can be obtained as the group of minors $H = \big\{((A_{ij})_{i,j=1}^{n-1}, t) : (A, t) \in G\big\}$, and N can be obtained as the closed subgroup of G given by $A_{ij} = 0$ for $1 \leq i < j \leq n-1$ and $A_{ii} = 1$ for $1 \leq i \leq n-1$.

4. Identify $N \cong \mathbb{G}_a^r \times T$ for some $0 \leq r \leq n-1$, possibly after stratifying T. Consider induced action of H on the space of characters $X = \mathrm{Hom}_T(N, \mathbb{G}_m \times T) \cong \mathbb{G}_a^r \times T$.

5. Use Algorithm 6.3.8 to find families of representatives $\chi_i \colon T_i \to X$, parametrized by varieties T_i over T, for the orbits in X under H, together with their stabilizer $H_i \subseteq H \times_T T_i$ over T_i and index $[H \times_T T_i : H_i]$.

6. Repeat the algorithm to compute $\zeta_{H_i}(s)$ for all i, from which $\zeta_G(s)$ can be computed using Corollary 6.3.2.

Algorithm 6.3.8. Input: A family of connected algebraic groups $H \subseteq \mathbb{T}_n \times T$ over a variety T, acting linearly on a subvariety $X \subseteq \mathbb{G}_a^r \times T$ over T. Write $\alpha_1, \ldots, \alpha_r$ for the coordinates on $\mathbb{G}_a^r$.

Output: A stratification of X by H-invariant locally closed subvarieties X_i; families of representatives $\chi_i \colon T_i \to X_i$ with T_i varieties over T; the stabilizers $H_i \subseteq H \times_T T_i$ of the χ_i; such that the index $[H \times_T T_i : H_i]$ is polynomial in q.

1. Repeat steps 2 and 3 until H acts trivially on X. Then $\chi \coloneqq \mathrm{id}_X \colon X \to X$ is a family of representatives, with stabilizer $H \times_T X$ and index $[H \times_T X : H \times_T X] = 1$. If both step 2 and 3 do not apply, fail.

2. If $\alpha_i \overset{H}{\mapsto} a\alpha_i$ for some coordinate a on H, then a must be a diagonal entry of H. Stratify X based on α_i:

   (i) Case $\alpha_i = 0$. Continue with the action of H restricted to the closed subvariety $X' = X \cap \{\alpha_i = 0\}$.

   (ii) Case $\alpha_i \neq 0$. By choosing representatives with $\alpha_i = 1$, we can replace X by $X' = X \cap \{\alpha_i = 1\}$ and H by $H' = H \cap \{a = 1\}$. Continue with the action of H' on X', and keep track of the index $[H : H'] = q - 1$. In the end, compose the families of representatives $\chi_i' \colon T_i \to X_i'$ with the inclusion $X_i' \to X_i = H \cdot X_i'$.

3. Write $\alpha_i \overset{H}{\mapsto} \sum_j a_j f_j$, where a_j are coordinates on H and f_j are functions on X which are not identically zero. If some f_ℓ is invariant under the action of H, then stratify X based on f_ℓ:

   (i) Case $f_\ell = 0$. Continue with the action of H restricted to the closed subvariety $X' = X \cap \{f_\ell = 0\}$.

   (ii) Case $f_\ell \neq 0$. By choosing representatives with $\alpha_i = 0$, we can replace X by $X' = X \cap \{\alpha_i = 0, f_\ell \neq 0\}$ and H by $H' = H \cap \{a_\ell = -f_\ell^{-1} \sum_{j \neq \ell} a_j f_j\}$. Continue with the action of H' on X', and keep track of the index $[H : H'] = q - 1$. In the end, compose the families of representatives $\chi_i' \colon T_i \to X_i'$ with the inclusion $X_i' \to X_i = H \cdot X_i'$.

Remark 6.3.9. Unfortunately, this algorithm might possibly fail. In fact, if this algorithm were to never fail, then the representation zeta function $\zeta_G(s)$ is always a polynomial in q, q^{-s} and $(q-1)^{-s}$. Then, evaluating at $s = 0$, this would imply that the number of conjugacy classes of G is a polynomial in q. In particular, this would imply Higman's Conjecture [PS15, Conjecture 1.1]. For us, the algorithm does not fail when applied to $G = \mathbb{U}_n$ or $G = \mathbb{T}_n$ for $n = 1, \ldots, 10$.

Results

The representation zeta functions of $\mathbb{U}_n$ and $\mathbb{T}_n$ were computed using Algorithm 6.3.7, and are presented in Theorem 6.3.10 and Theorem 6.3.11 below. One can evaluate these zeta functions at $s = 0$ in order to obtain the number of conjugacy classes of the groups over finite fields $\mathbb{F}_q$. For $G = \mathbb{U}_n$, the resulting polynomials in q can be seen to agree with [PS15, Appendix A], where $t = q - 1$. In this sense, these zeta functions are a generalization of the polynomials $k(\mathbb{U}_n(\mathbb{F}_q))$ as in [PS15]. Furthermore, the E-polynomials of $R_{\mathbb{U}_n}(\Sigma_g)$ and $R_{\mathbb{T}_n}(\Sigma_g)$ over $k = \mathbb{C}$ can be obtained through Theorem 4.6.1 and (6.3). Indeed, one can verify that for $1 \le n \le 5$ these E-polynomials agree with the virtual classes as given by Theorem 6.2.6 and Theorem 6.2.7, via the map (3.6).

Theorem 6.3.10. *The representation zeta functions $\zeta_{\mathbb{U}_n}(s)$ for $n = 1, \ldots, 10$ are given by*

(i) $\zeta_{\mathbb{U}_1}(s) = 1$

(ii) $\zeta_{\mathbb{U}_2}(s) = q$

(iii) $\zeta_{\mathbb{U}_3}(s) = q^{-s}(q-1) + q^2$

(iv) $\zeta_{\mathbb{U}_4}(s) = q^{1-s}(q-1)(q+1) + q^{1-2s}(q-1) + q^3$

(v) $\zeta_{\mathbb{U}_5}(s) = q^{1-2s}(q-1)(q+1)(2q-1) + q^{2-s}(q-1)(2q+1) + q^{1-3s}(q-1)(2q-1) + q^{-4s}(q-1)^2 + q^4$

(vi) $\zeta_{\mathbb{U}_6}(s) = q^{2-2s}(q-1)(q+2)(q^2+q-1) + q^{2-3s}(q-1)(q+1)(4q-3) + q^{-4s}(q-1)(2q^2-1)(q^2+q-1) + q^{3-s}(q-1)(3q+1) + q^{1-5s}(q-1)^2(2q+1) + q^{1-6s}(q-1)^2 + q^5$

(vii) $\zeta_{\mathbb{U}_7}(s) = q^{3-2s}(q-1)(q+1)(2q^2+3q-3) + q^{1-4s}(q-1)(2q-1)(q^4+5q^3-3q-1) + q^{4-s}(q-1)(4q+1) + q^{2-3s}(q-1)(3q^4+6q^3-2q^2-5q+1) + q^{1-5s}(q-1)(q^5+7q^4-2q^3-9q^2+3q+1) + q^{1-6s}(q-1)^2(4q^3+7q^2-3q-1) + q^{1-8s}(q-1)^2(3q-2) + q^{-7s}(q-1)^2(5q^3-3q+1) + q^{-9s}(q-1)^3 + q^6$

(viii) $\zeta_{\mathbb{U}_8}(s) = q^{4-2s}(q-1)(3q+2)(q^2+2q-2) + q^{5-s}(q-1)(5q+1) + q^{3-3s}(q-1)(q^5+5q^4+10q^3-7q^2-8q+3) + q^{3-6s}(q-1)(q^5+7q^4+16q^3-24q^2-14q+15) + q^{2-4s}(q-1)(12q^5+9q^4-16q^3-9q^2+6q+1) + q^{1-5s}(q-1)(2q^7+8q^6+13q^5-23q^4-9q^3+12q^2-1) + q^{1-7s}(q-1)^2(6q^5+18q^4+4q^3-19q^2+q+3) + q^{1-8s}(q-1)^2(q^5+13q^4+8q^3-14q^2-4q+3) + q^{1-11s}(q-1)^3(3q+1) + q^{-9s}(q-1)^2(4q^5+10q^4-7q^3-8q^2+3q+1) + q^{-10s}(q-1)^2(5q^4+q^3-6q^2+1) + q^{1-12s}(q-1)^3 + q^7$

(ix) $\zeta_{\mathbb{U}_9}(s) = q^{5-2s}(q-1)(2q+1)(2q^2+5q-5) + q^{6-s}(q-1)(6q+1) + q^{4-3s}(q-1)(2q^5+9q^4+14q^3-15q^2-11q+6) + q^{4-4s}(q-1)(4q^5+19q^4+11q^3-34q^2-10q+14) + q^{2-5s}(q-1)(q^8+5q^7+29q^6+q^5-53q^4-2q^3+27q^2-3q-2) + q^{2-6s}(q-1)(10q^7+33q^6-9q^5-68q^4+10q^3+38q^2-11q-1) + q^{1-7s}(q-1)(2q^9+8q^8+27q^7+2q^6-87q^5+20q^4+46q^3-15q^2-3q+1) + q^{1-8s}(q-1)^2(9q^7+33q^6+40q^5-45q^4-40q^3+21q^2+5q-1) + q^{1-9s}(q-1)^2(2q^7+30q^6+42q^5-44q^4-48q^3+25q^2+7q-1) + q^{1-11s}(q-1)^2(4q^6+25q^5+5q^4-48q^3+7q^2+9q+1) + q^{1-12s}(q-1)^2(10q^5+18q^4-32q^3-10q^2+18q-3) + q^{1-15s}(q-1)^3(4q-3) + q^{-10s}(q-1)^2(2q^8+13q^7+38q^6-24q^5-49q^4+20q^3+11q^2-3q-1) + q^{-13s}(q-1)^3(12q^4+10q^3-13q^2+q+1) + q^{-14s}(q-1)^3(9q^3-2q^2-5q+2) + q^{-16s}(q-1)^4 + q^8$

(x) $\zeta_{\mathbb{U}_{10}}(s) = q^{6-2s}(q-1)(5q+2)(q^2+3q-3) + q^{7-s}(q-1)(7q+1) + q^{5-3s}(q-1)(3q^5+15q^4+19q^3-28q^2-13q+10) + q^{4-4s}(q-1)(q^7+7q^6+32q^5+12q^4-65q^3-6q^2+27q-3) + q^{3-5s}(q-1)(2q^8+21q^7+42q^6-16q^5-103q^4+24q^3+50q^2-13q-3) + q^{2-6s}(q-1)(6q^9+27q^8+64q^7-73q^6-118q^5+64q^4+70q^3-39q^2+q+1) + q^{2-7s}(q-1)(2q^{10}+5q^9+39q^8+74q^7-130q^6-133q^5+128q^4+74q^3-60q^2+2q+1) + q^{2-8s}(q-1)(q^{10}+12q^9+39q^8+67q^7-137q^6-172q^5+200q^4+63q^3-80q^2+2q+6) + q^{2-9s}(q-1)^2(10q^8+65q^7+117q^6-36q^5-221q^4+18q^3+98q^2-11q-6) + q^{1-11s}(q-1)^2(6q^9+31q^8+109q^7+8q^6-240q^5-10q^4+135q^3-17q^2-8q-1) + q^{1-12s}(q-1)^2(2q^9+22q^8+77q^7+46q^6-217q^5-48q^4+156q^3-12q^2-20q+1) + q^{1-13s}(q-1)^2(10q^8+50q^7+60q^6-138q^5-110q^4+146q^3+8q^2-25q+2) + q^{1-15s}(q-1)^3(4q^6+42q^5+46q^4-51q^3-44q^2+23q+5) + q^{1-19s}(q-1)^4(4q+1) + q^{-10s}(q-1)^2(2q^{11}+8q^{10}+50q^9+112q^8-29q^7-227q^6+17q^5+123q^4-24q^3-12q^2+q+1) + q^{-14s}(q-1)^2(2q^9+24q^8+53q^7-52q^6-127q^5+84q^4+49q^3-32q^2-3q+3) + q^{-16s}(q-1)^3(10q^6+37q^5-9q^4-42q^3+6q^2+10q-1) + q^{-17s}(q-1)^3(12q^5+14q^4-21q^3-8q^2+6q+1) + q^{-18s}(q-1)^3(9q^4-q^3-9q^2+q+1) + q^{1-20s}(q-1)^4 + q^9.$ □

Theorem 6.3.11. *The representation zeta functions $\zeta_{\mathbb{T}_n}(s)$ for $n = 1, \ldots, 10$ are given by*

(i) $\zeta_{\mathbb{T}_1}(s) = q-1$

(ii) $\zeta_{\mathbb{T}_2}(s) = (q-1)^{1-s} + (q-1)^2$

(iii) $\zeta_{\mathbb{T}_3}(s) = q^{-s}(q-1)^{2-s} + 2(q-1)^{2-s} + (q-1)^{1-2s} + (q-1)^3$

(iv) $\zeta_{\mathbb{T}_4}(s) = 3q^{-s}(q-1)^{2-2s} + 2q^{-s}(q-1)^{3-s} + q^{-2s}(q-1)^{3-s} + q^{-2s}(q-1)^{2-2s} + q^{-s}(q-1)^{1-3s} + 3(q-1)^{3-s} + 3(q-1)^{2-2s} + (q-1)^{1-3s} + (q-1)^4$

(v) $\zeta_{\mathbb{T}_5}(s) = 8q^{-s}(q-1)^{3-2s} + 7q^{-2s}(q-1)^{3-2s} + 7q^{-2s}(q-1)^{2-3s} + 7q^{-s}(q-1)^{2-3s} + 3q^{-3s}(q-1)^{3-2s} + 3q^{-s}(q-1)^{4-s} + 2q^{-2s}(q-1)^{4-s} + 2q^{-2s}(q-1)^{1-4s} + 2q^{-3s}(q-1)^{2-3s} + 2q^{-s}(q-1)^{1-4s} + q^{-3s}(q-1)^{4-s} + q^{-4s}(q-1)^{3-2s} + 6(q-1)^{3-2s} + 4(q-1)^{4-s} + 4(q-1)^{2-3s} + (q-1)^{1-4s} + (q-1)^5$

(vi) $\zeta_{\mathbb{T}_6}(s) = q^{-2s}(q-1)^{1-5s}(q+7) + 29q^{-2s}(q-1)^{3-3s} + 24q^{-3s}(q-1)^{3-3s} + 23q^{-2s}(q-1)^{2-4s} + 21q^{-s}(q-1)^{3-3s} + 17q^{-3s}(q-1)^{2-4s} + 16q^{-2s}(q-1)^{4-2s} + 15q^{-4s}(q-1)^{3-3s} + 15q^{-s}(q-1)^{4-2s} + 13q^{-3s}(q-1)^{4-2s} + 13q^{-s}(q-1)^{2-4s} + 10q^{-4s}(q-1)^{2-4s} + 7q^{-4s}(q-1)^{4-2s} + 5q^{-5s}(q-1)^{3-3s} + 4q^{-3s}(q-1)^{1-5s} + 4q^{-s}(q-1)^{5-s} + 3q^{-2s}(q-1)^{5-s} + 3q^{-5s}(q-1)^{4-2s} + 3q^{-s}(q-1)^{1-5s} + 2q^{-3s}(q-1)^{5-s} + 2q^{-4s}(q-1)^{1-5s} + 2q^{-5s}(q-1)^{2-4s} + q^{-4s}(q-1)^{5-s} + q^{-6s}(q-1)^{4-2s} + q^{-6s}(q-1)^{3-3s} + 10(q-1)^{4-2s} + 10(q-1)^{3-3s} + 5(q-1)^{5-s} + 5(q-1)^{2-4s} + (q-1)^{1-5s} + (q-1)^6$

(vii) $\zeta_{\mathbb{T}_7}(s) = 2q^{-4s}(q-1)^{1-6s}(q+8) + q^{-2s}(q-1)^{2-5s}(2q+53) + q^{-2s}(q-1)^{1-6s}(2q+13) + q^{-3s}(q-1)^{2-5s}(2q+71) + q^{-3s}(q-1)^{1-6s}(3q+19) + q^{-4s}(q-1)^{2-5s}(3q+67) + q^{-5s}(q-1)^{1-6s}(q+12) + 107q^{-3s}(q-1)^{3-4s} + 104q^{-4s}(q-1)^{3-4s} + 87q^{-2s}(q-1)^{3-4s} + 79q^{-3s}(q-1)^{4-3s} + 73q^{-4s}(q-1)^{4-3s} + 73q^{-5s}(q-1)^{3-4s} + 71q^{-2s}(q-1)^{4-3s} + 49q^{-5s}(q-1)^{4-3s} + 48q^{-5s}(q-1)^{2-5s} + 46q^{-s}(q-1)^{4-3s} + 44q^{-s}(q-1)^{3-4s} + 42q^{-6s}(q-1)^{3-4s} + 30q^{-6s}(q-1)^{4-3s} + 28q^{-2s}(q-1)^{5-2s} + 27q^{-3s}(q-1)^{5-2s} + 24q^{-s}(q-1)^{5-2s} + 23q^{-6s}(q-1)^{2-5s} + 22q^{-4s}(q-1)^{5-2s} + 21q^{-s}(q-1)^{2-5s} + 15q^{-7s}(q-1)^{3-4s} + 13q^{-5s}(q-1)^{5-2s} + 12q^{-7s}(q-1)^{4-3s} + 7q^{-6s}(q-1)^{5-2s} + 5q^{-7s}(q-1)^{2-5s} + 5q^{-s}(q-1)^{6-s} + 4q^{-2s}(q-1)^{6-s} + 4q^{-6s}(q-1)^{1-6s} + 4q^{-8s}(q-1)^{4-3s} + 4q^{-s}(q-1)^{1-6s} + 3q^{-3s}(q-1)^{6-s} + 3q^{-7s}(q-1)^{5-2s} + 3q^{-8s}(q-1)^{3-4s} + 2q^{-4s}(q-1)^{6-s} + q^{-5s}(q-1)^{6-s} + q^{-8s}(q-1)^{5-2s} + q^{-9s}(q-1)^{4-3s} + 20(q-1)^{4-3s} + 15(q-1)^{5-2s} + 15(q-1)^{3-4s} + 6(q-1)^{6-s} + 6(q-1)^{2-5s} + (q-1)^{1-6s} + (q-1)^7$

(viii) $\zeta_{\mathbb{T}_8}(s) = 14q^{-3s}(q-1)^{2-6s}(q+14) + 6q^{-7s}(q-1)^{1-7s}(q+7) + 4q^{-3s}(q-1)^{3-5s}(q+87) + 2q^{-2s}(q-1)^{2-6s}(3q+50) + 2q^{-7s}(q-1)^{2-6s}(3q+89) + q^{-2s}(q-1)^{3-5s}(3q+208) + q^{-2s}(q-1)^{1-7s}(3q+20) + q^{-3s}(q-1)^{1-7s}(q^2+11q+47) + q^{-4s}(q-1)^{3-5s}(9q+457) + q^{-4s}(q-1)^{2-6s}(20q+261) + q^{-4s}(q-1)^{1-7s}(12q+61) + q^{-5s}(q-1)^{3-5s}(6q+485) + q^{-5s}(q-1)^{2-6s}(24q+305) + q^{-5s}(q-1)^{1-7s}(2q^2+20q+79) + q^{-6s}(q-1)^{3-5s}(6q+415) + q^{-6s}(q-1)^{2-6s}(19q+250) + q^{-6s}(q-1)^{1-7s}(q^2+13q+60) + q^{-8s}(q-1)^{2-6s}(3q+85) + q^{-8s}(q-1)^{1-7s}(q+15) + 410q^{-4s}(q-1)^{4-4s} + 398q^{-5s}(q-1)^{4-4s} + 340q^{-6s}(q-1)^{4-4s} + 332q^{-3s}(q-1)^{4-4s} + 297q^{-7s}(q-1)^{3-5s} + 238q^{-7s}(q-1)^{4-4s} + 229q^{-2s}(q-1)^{4-4s} + 192q^{-4s}(q-1)^{5-3s} + 174q^{-3s}(q-1)^{5-3s} + 171q^{-5s}(q-1)^{5-3s} + 168q^{-8s}(q-1)^{3-5s} + 147q^{-8s}(q-1)^{4-4s} + 139q^{-2s}(q-1)^{5-3s} + 136q^{-6s}(q-1)^{5-3s} + 110q^{-s}(q-1)^{4-4s} + 90q^{-7s}(q-1)^{5-3s} + 85q^{-s}(q-1)^{5-3s} + 80q^{-s}(q-1)^{3-5s} + 73q^{-9s}(q-1)^{3-5s} + 71q^{-9s}(q-1)^{4-4s} + 56q^{-8s}(q-1)^{5-3s} + 45q^{-3s}(q-1)^{6-2s} + 43q^{-2s}(q-1)^{6-2s} + 42q^{-4s}(q-1)^{6-2s} + 35q^{-s}(q-1)^{6-2s} + 34q^{-5s}(q-1)^{6-2s} + 31q^{-s}(q-1)^{2-6s} + 30q^{-9s}(q-1)^{2-6s} + 27q^{-10s}(q-1)^{4-4s} + 26q^{-9s}(q-1)^{5-3s} + 22q^{-6s}(q-1)^{6-2s} + 21q^{-10s}(q-1)^{3-5s} + 13q^{-7s}(q-1)^{6-2s} + 11q^{-10s}(q-1)^{5-3s} + 7q^{-8s}(q-1)^{6-2s} + 7q^{-11s}(q-1)^{4-4s} + 6q^{-s}(q-1)^{7-s} + 5q^{-2s}(q-1)^{7-s} + 5q^{-10s}(q-1)^{2-6s} + 5q^{-s}(q-1)^{1-7s} + 4q^{-3s}(q-1)^{7-s} + 4q^{-9s}(q-1)^{1-7s} + 4q^{-11s}(q-1)^{5-3s} + 3q^{-4s}(q-1)^{7-s} + 3q^{-9s}(q-1)^{6-2s} + 3q^{-11s}(q-1)^{3-5s} + 2q^{-5s}(q-1)^{7-s} + q^{-6s}(q-$

$1)^{7-s}+q^{-10s}(q-1)^{6-2s}+q^{-12s}(q-1)^{5-3s}+q^{-12s}(q-1)^{4-4s}+35(q-1)^{5-3s}+35(q-1)^{4-4s}+21(q-1)^{6-2s}+21(q-1)^{3-5s}+7(q-1)^{7-s}+7(q-1)^{2-6s}+(q-1)^{1-7s}+(q-1)^8$

(ix) $\zeta_{\mathbb{T}_9}(s) = 10q^{-8s}(q-1)^{4-5s}(q+186)+10q^{-8s}(q-1)^{3-6s}(7q+204)+6q^{-3s}(q-1)^{4-5s}(q+182)+6q^{-11s}(q-1)^{3-6s}(q+81)+4q^{-2s}(q-1)^{4-5s}(q+149)+4q^{-2s}(q-1)^{1-8s}(q+7)+4q^{-6s}(q-1)^{3-6s}(27q+598)+4q^{-9s}(q-1)^{3-6s}(11q+377)+3q^{-7s}(q-1)^{4-5s}(4q+739)+2q^{-3s}(q-1)^{2-7s}(q^2+23q+212)+2q^{-4s}(q-1)^{3-6s}(31q+754)+2q^{-7s}(q-1)^{2-7s}(4q^2+103q+730)+2q^{-8s}(q-1)^{2-7s}(3q^2+72q+586)+2q^{-10s}(q-1)^{3-6s}(7q+491)+2q^{-11s}(q-1)^{1-8s}(2q+19)+2q^{-12s}(q-1)^{2-7s}(q+38)+q^{-2s}(q-1)^{3-6s}(12q+425)+q^{-2s}(q-1)^{2-7s}(12q+167)+q^{-3s}(q-1)^{3-6s}(31q+912)+q^{-3s}(q-1)^{1-8s}(2q^2+21q+85)+q^{-4s}(q-1)^{4-5s}(15q+1658)+q^{-4s}(q-1)^{2-7s}(2q^2+89q+758)+q^{-4s}(q-1)^{1-8s}(4q^2+45q+165)+q^{-5s}(q-1)^{4-5s}(16q+2111)+q^{-5s}(q-1)^{3-6s}(92q+2099)+q^{-5s}(q-1)^{2-7s}(9q^2+157q+1138)+q^{-5s}(q-1)^{1-8s}(q^3+12q^2+83q+262)+q^{-6s}(q-1)^{4-5s}(21q+2302)+q^{-6s}(q-1)^{2-7s}(6q^2+180q+1339)+q^{-6s}(q-1)^{1-8s}(10q^2+97q+316)+q^{-7s}(q-1)^{3-6s}(101q+2451)+q^{-7s}(q-1)^{1-8s}(2q^3+22q^2+131q+369)+q^{-8s}(q-1)^{1-8s}(9q^2+81q+277)+q^{-9s}(q-1)^{2-7s}(3q^2+80q+823)+q^{-9s}(q-1)^{1-8s}(2q^2+39q+181)+q^{-10s}(q-1)^{2-7s}(39q+536)+q^{-10s}(q-1)^{1-8s}(2q^2+25q+119)+q^{-11s}(q-1)^{2-7s}(11q+222)+1395q^{-9s}(q-1)^{4-5s}+1254q^{-6s}(q-1)^{5-4s}+1224q^{-5s}(q-1)^{5-4s}+1123q^{-7s}(q-1)^{5-4s}+1061q^{-4s}(q-1)^{5-4s}+923q^{-8s}(q-1)^{5-4s}+911q^{-10s}(q-1)^{4-5s}+776q^{-3s}(q-1)^{5-4s}+670q^{-9s}(q-1)^{5-4s}+505q^{-11s}(q-1)^{4-5s}+494q^{-2s}(q-1)^{5-4s}+436q^{-10s}(q-1)^{5-4s}+394q^{-5s}(q-1)^{6-3s}+381q^{-4s}(q-1)^{6-3s}+369q^{-6s}(q-1)^{6-3s}+319q^{-3s}(q-1)^{6-3s}+299q^{-7s}(q-1)^{6-3s}+251q^{-11s}(q-1)^{5-4s}+242q^{-12s}(q-1)^{4-5s}+239q^{-2s}(q-1)^{6-3s}+230q^{-8s}(q-1)^{6-3s}+230q^{-s}(q-1)^{5-4s}+225q^{-s}(q-1)^{4-5s}+208q^{-12s}(q-1)^{3-6s}+154q^{-9s}(q-1)^{6-3s}+141q^{-s}(q-1)^{6-3s}+132q^{-s}(q-1)^{3-6s}+126q^{-12s}(q-1)^{5-4s}+98q^{-10s}(q-1)^{6-3s}+89q^{-13s}(q-1)^{4-5s}+67q^{-3s}(q-1)^{7-2s}+67q^{-4s}(q-1)^{7-2s}+61q^{-2s}(q-1)^{7-2s}+61q^{-5s}(q-1)^{7-2s}+58q^{-13s}(q-1)^{3-6s}+53q^{-13s}(q-1)^{5-4s}+51q^{-11s}(q-1)^{6-3s}+50q^{-6s}(q-1)^{7-2s}+48q^{-s}(q-1)^{7-2s}+43q^{-s}(q-1)^{2-7s}+34q^{-7s}(q-1)^{7-2s}+25q^{-12s}(q-1)^{6-3s}+25q^{-14s}(q-1)^{4-5s}+22q^{-8s}(q-1)^{7-2s}+18q^{-14s}(q-1)^{5-4s}+13q^{-9s}(q-1)^{7-2s}+12q^{-13s}(q-1)^{2-7s}+11q^{-13s}(q-1)^{6-3s}+9q^{-14s}(q-1)^{3-6s}+8q^{-12s}(q-1)^{1-8s}+7q^{-10s}(q-1)^{7-2s}+7q^{-s}(q-1)^{8-s}+6q^{-2s}(q-1)^{8-s}+6q^{-s}(q-1)^{1-8s}+5q^{-3s}(q-1)^{8-s}+5q^{-15s}(q-1)^{5-4s}+4q^{-4s}(q-1)^{8-s}+4q^{-14s}(q-1)^{6-3s}+4q^{-15s}(q-1)^{4-5s}+3q^{-5s}(q-1)^{8-s}+3q^{-11s}(q-1)^{7-2s}+2q^{-6s}(q-1)^{8-s}+q^{-7s}(q-1)^{8-s}+q^{-12s}(q-1)^{7-2s}+q^{-15s}(q-1)^{6-3s}+q^{-16s}(q-1)^{5-4s}+70(q-1)^{5-4s}+56(q-1)^{6-3s}+56(q-1)^{4-5s}+28(q-1)^{7-2s}+28(q-1)^{3-6s}+8(q-1)^{8-s}+8(q-1)^{2-7s}+(q-1)^{1-8s}+(q-1)^9$

(x) $\zeta_{\mathbb{T}_{10}}(s) = q^{-6s}(q-1)^{1-9s}(3q+17)(2q^2+13q+63)+12q^{-13s}(q-1)^{3-7s}(14q+349)+10q^{-14s}(q-1)^{3-7s}(7q+219)+7q^{-5s}(q-1)^{4-6s}(30q+1201)+7q^{-12s}(q-1)^{4-6s}(12q+967)+5q^{-2s}(q-1)^{5-5s}(q+282)+5q^{-10s}(q-1)^{5-5s}(3q+1366)+4q^{-2s}(q-1)^{4-6s}(5q+333)+4q^{-15s}(q-1)^{1-9s}(q+11)+3q^{-2s}(q-1)^{3-7s}(10q+259)+3q^{-7s}(q-1)^{4-6s}(118q+4475)+3q^{-8s}(q-1)^{3-7s}(9q^2+385q+4636)+3q^{-11s}(q-1)^{3-7s}(4q^2+205q+3159)+2q^{-3s}(q-1)^{4-6s}(27q+1496)+2q^{-4s}(q-1)^{2-8s}(8q^2+139q+889)+2q^{-5s}(q-1)^{5-5s}(13q+$

$3104) + 2q^{-5s}(q-1)^{2-8s}(q^3+25q^2+293q+1620) + 2q^{-8s}(q-1)^{5-5s}(19q+4337) +$
$2q^{-12s}(q-1)^{3-7s}(3q^2+176q+3381) + 2q^{-13s}(q-1)^{4-6s}(13q+2180) + 2q^{-14s}(q-$
$1)^{4-6s}(5q+1223)+2q^{-15s}(q-1)^{2-8s}(11q+170)+2q^{-16s}(q-1)^{3-7s}(2q+161)+q^{-2s}(q-$
$1)^{2-8s}(20q+257)+q^{-2s}(q-1)^{1-9s}(5q+37)+q^{-3s}(q-1)^{5-5s}(8q+2717)+q^{-3s}(q-$
$1)^{3-7s}(3q^2+117q+2039)+q^{-3s}(q-1)^{2-8s}(6q^2+104q+791)+q^{-3s}(q-1)^{1-9s}(3q^2+33q+$
$134)+q^{-4s}(q-1)^{5-5s}(21q+4418)+q^{-4s}(q-1)^{4-6s}(122q+5413)+q^{-4s}(q-1)^{3-7s}(4q^2+$
$273q+4096)+q^{-4s}(q-1)^{1-9s}(q^3+15q^2+108q+342)+q^{-5s}(q-1)^{3-7s}(19q^2+545q+$
$6943)+q^{-5s}(q-1)^{1-9s}(2q^3+35q^2+230q+659)+q^{-6s}(q-1)^{5-5s}(41q+7704)+q^{-6s}(q-$
$1)^{4-6s}(299q+11188)+q^{-6s}(q-1)^{3-7s}(21q^2+809q+9883)+q^{-6s}(q-1)^{2-8s}(2q^3+$
$84q^2+953q+4932)+q^{-7s}(q-1)^{5-5s}(36q+8593)+q^{-7s}(q-1)^{3-7s}(37q^2+1103q+$
$12627)+q^{-7s}(q-1)^{2-8s}(8q^3+144q^2+1413q+6663)+q^{-7s}(q-1)^{1-9s}(2q^4+21q^3+$
$143q^2+654q+1524)+q^{-8s}(q-1)^{4-6s}(356q+14191)+q^{-8s}(q-1)^{2-8s}(6q^3+135q^2+$
$1587q+7639)+q^{-8s}(q-1)^{1-9s}(q^4+20q^3+165q^2+790q+1818)+q^{-9s}(q-1)^{5-5s}(20q+$
$8073)+q^{-9s}(q-1)^{4-6s}(322q+13751)+q^{-9s}(q-1)^{3-7s}(30q^2+1125q+13721)+q^{-9s}(q-$
$1)^{2-8s}(4q^3+125q^2+1513q+7536)+q^{-9s}(q-1)^{1-9s}(10q^3+131q^2+723q+1771)+q^{-10s}(q-$
$1)^{4-6s}(227q+12026)+q^{-10s}(q-1)^{3-7s}(21q^2+934q+12376)+q^{-10s}(q-1)^{2-8s}(8q^3+$
$142q^2+1407q+6977)+q^{-10s}(q-1)^{1-9s}(2q^4+22q^3+154q^2+704q+1669)+q^{-11s}(q-$
$1)^{4-6s}(136q+9425)+q^{-11s}(q-1)^{2-8s}(61q^2+889q+5143)+q^{-11s}(q-1)^{1-9s}(6q^3+73q^2+$
$433q+1181)+q^{-12s}(q-1)^{2-8s}(32q^2+526q+3607)+q^{-12s}(q-1)^{1-9s}(2q^3+36q^2+255q+$
$805)+q^{-13s}(q-1)^{2-8s}(10q^2+249q+2095)+q^{-13s}(q-1)^{1-9s}(10q^2+110q+431)+q^{-14s}(q-$
$1)^{2-8s}(5q^2+101q+983)+q^{-14s}(q-1)^{1-9s}(2q^2+33q+171)+q^{-15s}(q-1)^{3-7s}(20q+$
$929)+q^{-16s}(q-1)^{2-8s}(2q+91)+5331q^{-11s}(q-1)^{5-5s}+3802q^{-12s}(q-1)^{5-5s}+$
$3288q^{-7s}(q-1)^{6-4s}+3213q^{-6s}(q-1)^{6-4s}+3128q^{-8s}(q-1)^{6-4s}+2802q^{-5s}(q-1)^{6-4s}+$
$2724q^{-9s}(q-1)^{6-4s}+2472q^{-13s}(q-1)^{5-5s}+2228q^{-4s}(q-1)^{6-4s}+2216q^{-10s}(q-1)^{6-4s}+$
$1657q^{-11s}(q-1)^{6-4s}+1544q^{-3s}(q-1)^{6-4s}+1442q^{-14s}(q-1)^{5-5s}+1192q^{-15s}(q-$
$1)^{4-6s}+1149q^{-12s}(q-1)^{6-4s}+937q^{-2s}(q-1)^{6-4s}+759q^{-15s}(q-1)^{5-5s}+757q^{-6s}(q-$
$1)^{7-3s}+729q^{-5s}(q-1)^{7-3s}+725q^{-13s}(q-1)^{6-4s}+700q^{-7s}(q-1)^{7-3s}+655q^{-4s}(q-$
$1)^{7-3s}+607q^{-8s}(q-1)^{7-3s}+525q^{-s}(q-1)^{5-5s}+524q^{-3s}(q-1)^{7-3s}+492q^{-16s}(q-$
$1)^{4-6s}+480q^{-9s}(q-1)^{7-3s}+427q^{-s}(q-1)^{6-4s}+426q^{-14s}(q-1)^{6-4s}+413q^{-s}(q-$
$1)^{4-6s}+377q^{-2s}(q-1)^{7-3s}+365q^{-10s}(q-1)^{7-3s}+346q^{-16s}(q-1)^{5-5s}+249q^{-11s}(q-$
$1)^{7-3s}+225q^{-15s}(q-1)^{6-4s}+217q^{-s}(q-1)^{7-3s}+203q^{-s}(q-1)^{3-7s}+163q^{-12s}(q-$
$1)^{7-3s}+155q^{-17s}(q-1)^{4-6s}+133q^{-17s}(q-1)^{5-5s}+105q^{-16s}(q-1)^{6-4s}+97q^{-4s}(q-$
$1)^{8-2s}+94q^{-5s}(q-1)^{8-2s}+93q^{-3s}(q-1)^{8-2s}+92q^{-13s}(q-1)^{7-3s}+85q^{-6s}(q-1)^{8-2s}+$
$82q^{-2s}(q-1)^{8-2s}+74q^{-17s}(q-1)^{3-7s}+70q^{-7s}(q-1)^{8-2s}+63q^{-s}(q-1)^{8-2s}+57q^{-s}(q-$
$1)^{2-8s}+50q^{-8s}(q-1)^{8-2s}+50q^{-14s}(q-1)^{7-3s}+43q^{-17s}(q-1)^{6-4s}+42q^{-18s}(q-$
$1)^{5-5s}+35q^{-18s}(q-1)^{4-6s}+34q^{-9s}(q-1)^{8-2s}+25q^{-15s}(q-1)^{7-3s}+22q^{-10s}(q-1)^{8-2s}+$
$16q^{-18s}(q-1)^{6-4s}+13q^{-11s}(q-1)^{8-2s}+12q^{-17s}(q-1)^{2-8s}+11q^{-16s}(q-1)^{7-3s}+$
$9q^{-18s}(q-1)^{3-7s}+9q^{-19s}(q-1)^{5-5s}+8q^{-16s}(q-1)^{1-9s}+8q^{-s}(q-1)^{9-s}+7q^{-2s}(q-$
$1)^{9-s}+7q^{-12s}(q-1)^{8-2s}+7q^{-s}(q-1)^{1-9s}+6q^{-3s}(q-1)^{9-s}+5q^{-4s}(q-1)^{9-s}+5q^{-19s}(q-$
$1)^{6-4s}+4q^{-5s}(q-1)^{9-s}+4q^{-17s}(q-1)^{7-3s}+4q^{-19s}(q-1)^{4-6s}+3q^{-6s}(q-1)^{9-s}+$

$3q^{-13s}(q-1)^{8-2s}+2q^{-7s}(q-1)^{9-s}+q^{-8s}(q-1)^{9-s}+q^{-14s}(q-1)^{8-2s}+q^{-18s}(q-1)^{7-3s}+q^{-20s}(q-1)^{6-4s}+q^{-20s}(q-1)^{5-5s}+126(q-1)^{6-4s}+126(q-1)^{5-5s}+84(q-1)^{7-3s}+84(q-1)^{4-6s}+36(q-1)^{8-2s}+36(q-1)^{3-7s}+9(q-1)^{9-s}+9(q-1)^{2-8s}+(q-1)^{1-9s}+(q-1)^{10}$ □

The computation times[2] for the representation zeta functions $\zeta_{\mathbb{U}_n}(s)$ are given by

n	2	3	4	5	6	7	8	9	10
time	0.01s	0.08s	0.25s	0.77s	2.41s	8.18s	27.43s	1m44s	6m52s

and those for the representation zeta functions $\zeta_{\mathbb{T}_n}(s)$ are given by

n	2	3	4	5	6	7	8	9	10
time	0.03s	0.15s	0.51s	1.81s	6.32s	26.16s	1m53s	6m39s	55m46s

6.4 Arithmetic-geometric correspondence

In this section, we give some more insight into the correspondence between the arithmetic and geometric method. In one direction, Theorem 4.10.6 shows how information on the geometric side can be translated to the arithmetic side. More precisely, the eigenvalues and eigenvectors of the geometric TQFT Z_G partially describe the character tables of the finite groups $G(\mathbb{F}_q)$. Let us illustrate how, in the other direction, the arithmetic information provides geometric insight into the Z_G. In particular, we will show how the representation theory of the groups $\mathbb{U}_n(\mathbb{F}_q)$ of unipotent upper triangular matrices over $\mathbb{F}_q$ can be used to simplify the corresponding geometric TQFT Z_G. This yields a new smaller set of generators for this TQFT motivated by the arithmetic side. More precisely, we obtain this new generating set by canonically lifting the sums of equidimensional characters to the Grothendieck ring of varieties. These generators will be given by virtual classes of locally closed subvarieties of G.

Unipotent 3×3 matrices. Consider the group $\mathbb{U}_3(\mathbb{F}_q)$ of unipotent upper triangular matrices of rank 3 over a finite field $\mathbb{F}_q$,

$$\mathbb{U}_3(\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{F}_q \right\}.$$

The irreducible complex characters of $\mathbb{U}_3(\mathbb{F}_q)$ are of dimension 1 or q. Denote the set of 1-dimensional characters by X_1 and of the q-dimensional characters by

[2] As performed on an Intel®Xeon®CPU E5-4640 0 @ 2.40GHz.

X_q. Summing the 1-dimensional characters, we find that $v_1 = \sum_{\chi \in X_1} \chi$ is given by

$$v_1 \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{cases} q & \text{if } x = z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Summing the q-dimensional characters, we find that $v_2 = \sum_{\chi \in X_q} \chi$ is given by

$$v_2 \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{cases} -q & \text{if } x = z = 0 \text{ and } y \neq 0, \\ q(q-1) & \text{if } x = y = z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, since the eigenvectors v_1 and v_2 are 'polynomial in q'-valued on locally closed subsets of $G = \mathbb{U}_3$, we can naturally lift these eigenvectors along the morphism $\mu_{[G/G]} \colon \mathrm{K}_0(\mathbf{Stck}_{[G/G]}) \to \mathbb{C}^{[G/G](\mathbb{F}_q)} = R_{\mathbb{C}}(G(\mathbb{F}_q))$ of Definition 4.10.1, replacing q by $\mathbb{L}$ to obtain

$$\mathbb{L}[\{x = z = 0\}] \quad \text{and} \quad -\mathbb{L}[\{x = z = 0,\ y \neq 0\}] + \mathbb{L}(\mathbb{L} - 1)[\{x = y = z = 0\}]$$

respectively, expressed in terms of the virtual classes of G-equivariant subvarieties of $\mathbb{U}_3$. Indeed, from the computations of Section 6.2, whose results can be found in Appendix A, it can be seen that these elements are eigenvectors of Z_G, with eigenvalues $\mathbb{L}^6$ and $\mathbb{L}^4$, respectively. In fact, the submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ generated by these two eigenvectors contains Z_G () (1) and is invariant under Z_G(), and is therefore a simplification of the $M = 5$ generators as used in Section 6.2.

Unipotent 4×4 matrices. Consider the group $\mathbb{U}_4(\mathbb{F}_q)$ of unipotent upper triangular matrices of rank 4 over a finite field $\mathbb{F}_q$,

$$\mathbb{U}_4(\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} : a, b, c, d, e, f \in \mathbb{F}_q \right\}.$$

This group has three families of irreducible complex characters: the 1-dimensional characters X_1, the q-dimensional characters X_q and the q^2-dimensional characters X_{q^2}. Summing equidimensional characters, we find

$$\sum_{\chi \in X_1} \chi \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{cases} q^2 & \text{if } a = d = f = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\sum_{\chi \in X_q} \chi \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{cases} q^4 & \text{if } a = b = d = e = f = 0, \\ -q^2 & \text{if } a = d = f \\ 0 & \text{otherwise}, \end{cases}$$

$$\sum_{\chi \in X_{q^2}} \chi \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{cases} q^3(q-1) & \text{if } a = b = c = d = e = f = 0, \\ -q^3 & \text{if } a = b = d = e = f = 0 \text{ and } c \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

We lift these to elements in $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$, for $G = \mathbb{U}_4$, given by

$$\begin{aligned} &\mathbb{L}^2[\{a = d = f = 0\}], \\ &\mathbb{L}^4[\{a = b = d = e = f = 0\}] - \mathbb{L}^2[\{a = d = f = 0\}], \text{ and} \\ &\mathbb{L}^3(\mathbb{L} - 1)[\{a = b = c = d = e = f = 0\}] - \mathbb{L}^3[\{a = b = d = e = f = 0, \ c \neq 0\}]. \end{aligned}$$

Again, the computations of Section 6.2, whose results can be found in Appendix A, show that these elements are eigenvectors of Z_G, with eigenvalues $\mathbb{L}^{12}$, $\mathbb{L}^{10}$ and $\mathbb{L}^8$, respectively. These three elements to generate a submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ which can replace the one from Section 6.2 with $M = 16$ generators, a significant simplification.

Remark 6.4.1. During the algorithmic computations of Section 6.3, it is, in principle, possible to keep track of the irreducible characters of $\mathbb{U}_n(\mathbb{F}_q)$ and $\mathbb{T}_n(\mathbb{F}_q)$ for $6 \leq n \leq 10$. Then, as in the above examples, the sums of equidimensional characters can be lifted to elements in $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ for $G = \mathbb{U}_n$ or $G = \mathbb{T}_n$, respectively. While we have not attempted this, these lifts would generate a submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ which, one could hopefully show, is invariant under Z_G (). This would provide a way to extend the geometric method to groups of upper triangular matrices of rank ≥ 6, even though there are infinitely many conjugacy classes.