Motivic invariants of character stacks

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Chapter 5

SL_2-character stacks

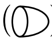
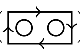
In this chapter, we apply the theory of Chapter 4 to compute the virtual classes of the G-character stacks $\mathfrak{X}_G(M)$, for M equal to both orientable and non-orientable closed surfaces, and G equal to

$$\mathrm{SL}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}.$$

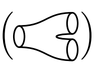
Even though $G = \mathrm{SL}_2$ is one of the simplest non-trivial groups, the resulting computations are quite intricate. Throughout this chapter, we work over an algebraically closed field k with $\mathrm{char}(k) \neq 2$.

Similar computations were performed by [LMN13, MM16] to compute the corresponding E-polynomials. While the scissor relation (3.3) is the main ingredient in these computations, they cannot simply be lifted to the Grothendieck ring of varieties. Instead, many subtle points arise and have to be dealt with, such as the study of $\mathbb{P}^1$-fibrations, equivariant motivic invariants (as in Section 3.6), and non-zero elements in the Grothendieck ring of varieties whose E-polynomial is zero.

As $G = \mathrm{SL}_2$ is a special group, the virtual class of the G-character stack $\mathfrak{X}_G(M)$ is equal to that of the G-representation variety $R_G(M)$ divided by $[\mathrm{SL}_2] = \mathbb{L}(\mathbb{L} - 1)(\mathbb{L} + 1)$. Hence, we can apply the theory of Section 4.12, allowing us to make non-equivariant stratifications. In order to use (4.10), (4.11), (4.12) and (4.13), we will compute

Z_G^{rep} () $\circ$ Z_G^{rep} () and Z_G^{rep} () $\circ$ Z_G^{rep} () (5.1)

in Section 5.2 and Section 5.3, respectively, and in Section 5.4 we compute

Z_G^{rep} (). (5.2)

It is not necessary to compute these maps in full. Rather it suffices to compute their restriction to a certain finitely generated submodule of $\mathrm{K}_0(\mathbf{Var}_G)$. The generators for this submodule are described in Section 5.1. For the computation of some of these maps, we need an extra relation in the Grothendieck ring of varieties regarding $\mathbb{P}^1$-fibrations. This will also be discussed in Section 5.1.

Finally, in Section 5.5 we collect and discuss the results.

5.1 Generators, relations and $\mathbb{P}^1$-fibrations

Let us introduce some notation. The following varieties are all considered naturally as varieties over $G = \mathrm{SL}_2$.

$$\begin{aligned}
I_+ &= \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}, \\
I_- &= \{\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}, \\
J_+ &= \{A \in G \mid A \text{ conjugate to } \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\}, \\
J_- &= \left\{A \in G \mid A \text{ conjugate to } \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)\right\}, \\
M &= \{A \in G \mid \mathrm{tr}(A) \neq \pm 2\}, \\
X_2 &= \{(A, \ell) \in M \times \mathbb{A}^1_k \mid \ell^2 = \mathrm{tr}(A) - 2\}, \\
X_{-2} &= \{(A, \ell) \in M \times \mathbb{A}^1_k \mid \ell^2 = \mathrm{tr}(A) + 2\}, \\
X_{2,-2} &= \{(A, \ell) \in M \times \mathbb{A}^1_k \mid \ell^2 = \mathrm{tr}(A)^2 - 4\}, \\
Y &= \{(A, \omega) \in M \times \mathbb{A}^1_k \setminus \{0\} \mid \mathrm{tr}(A) = \omega^2 + \omega^{-2}\},
\end{aligned} \tag{5.3}$$

By the same symbols, we will also denote their virtual class in $\mathrm{K}_0(\mathbf{Var}_G)$. These elements will be the generators of the $\mathrm{K}_0(\mathbf{Var}_k)$-submodule of $\mathrm{K}_0(\mathbf{Var}_G)$ on which (5.1) and (5.2) will be computed. A useful alternative presentation of the last five generators is as follows:

$$\begin{aligned}
M &\cong (\mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1\}) /\!\!/ S_2 \to G, && (P, \lambda) \mapsto P \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) P^{-1} \\
X_2 &\cong (\mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\}) /\!\!/ S_2 \to G, && (P, \omega) \mapsto P \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^{-2} \end{pmatrix} P^{-1} \\
X_{-2} &\cong (\mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\}) /\!\!/ S_2 \to G, && (P, \omega) \mapsto P \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix} P^{-1} \\
X_{2,-2} &\cong \mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1\} \to G, && (P, \lambda) \mapsto P \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) P^{-1} \\
Y &\cong \mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\} \to G, && (P, \omega) \mapsto P \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix} P^{-1}
\end{aligned}$$

where $D \subseteq \mathrm{GL}_2$ is the subgroup of diagonal matrices, and where S_2 acts on the left coset space GL_2/D by $P \mapsto P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and acts on the coordinates λ and ω by $\lambda \mapsto \lambda^{-1}$ and $\omega \mapsto \omega^{-1}$. The following lemma gives a better understanding of the relation between these generators.

Lemma 5.1.1. *The following relations hold in* $\mathrm{K}_0(\mathbf{Var}_G)$*:*

$$X_2^2 = 2X_2, \quad X_{-2}^2 = 2X_{-2}, \quad X_{2,-2}^2 = 2X_{2,-2}$$
$$\text{and} \quad Y = X_2X_{-2} = X_2X_{2,-2} = X_{-2}X_{2,-2}.$$

Proof. The first equality follows from

$$X_2 \times_M X_2 = \{(A, \ell_1, \ell_2) \in M \times \mathbb{A}_k^2 \mid \ell_1^2 = \mathrm{tr}(A) - 2 \text{ and } \ell_2 = \pm\ell_1\} \cong X_2 \sqcup X_2,$$

and similarly for the second and third. The final two equalities follow from the fact that, if $\ell_1^2 = \mathrm{tr}(A) - 2$ and $\ell_2^2 = \mathrm{tr}(A) + 2$, then $(\ell_1\ell_2)^2 = \mathrm{tr}(A)^2 - 4$. Finally, the fourth equality follows from the isomorphism

$$Y \xrightarrow{\sim} X_2 \times_M X_{-2} = \{(A, \ell_1, \ell_2) \in M \times \mathbb{A}_k^2 \mid \ell_1^2 = \mathrm{tr}(A) - 2 \text{ and } \ell_2^2 = \mathrm{tr}(A) + 2\}$$

which is given by $(A, \omega) \mapsto (A, \omega - \omega^{-1}, \omega + \omega^{-1})$ with inverse $(A, \ell_1, \ell_2) \mapsto (A, \frac{1}{2}(\ell_1 + \ell_2))$. □

Remark 5.1.2. The symbols X_2, X_{-2} and $X_{2,-2}$ were adopted from [Gon20] and reflect the monodromy action of these spaces as covering spaces over M. They are double covers of M, and have non-trivial monodromy for loops around $\mathrm{tr}\, A = 2$ or $\mathrm{tr}\, A = -2$ as indicated by the subscript of the symbol. More precisely, write T for the trivial representation of $\pi_1(M, *)$ and N_2 (resp. N_{-2}) for the 1-dimensional representation that sends a loop around $\mathrm{tr}\, A = 2$ (resp. $\mathrm{tr}\, A = -2$) to -1. Then the monodromy representations of X_2, X_{-2} and $X_{2,-2}$ are $T \oplus N_2$, $T \oplus N_{-2}$ and $T \oplus N_2 \otimes N_{-2}$, respectively. Since $Y \cong X_2 \times_M X_{-2}$, it follows that Y is a 4-to-1 cover of M with monodromy representation $T \oplus N_2 \oplus N_{-2} \oplus N_2 \otimes N_{-2}$. In particular, the monodromy representation of $M \sqcup M \sqcup Y$ is equal to that of $X_2 \sqcup X_{-2} \sqcup X_{2,-2}$. This is also the case for their *Hodge monodromy representation* [LMN13, (5)], and for this reason, the generator Y is not needed in the E-polynomial computations of [LMN13, MM16]. However, the analogous equality does not (necessarily) hold in $\mathrm{K}_0(\mathbf{Var}_G)$ as $M \sqcup M \sqcup Y$ is not isomorphic to $X_2 \sqcup X_{-2} \sqcup X_{2,-2}$ over M: the former has two sections over M whereas the latter has none.

Finally, when computing the images of the generators (5.3) under the maps (5.1) and (5.2), we will encounter some non-trivial $\mathbb{P}^1$-fibrations. Recall, a $\mathbb{P}^1$-*fibration* is a morphism $P \to X$ which is étale-locally of the form $X \times \mathbb{P}_k^1 \xrightarrow{\pi_X} X$, where π_X denotes the projection to X. However, as many motivic invariants $\chi\colon \mathrm{K}_0(\mathbf{Var}_k) \to R$ satisfy $\chi(P) = \chi(\mathbb{P}_k^1)\chi(X)$ for all $\mathbb{P}^1$-fibrations $P \to X$, we will impose this relation as well. This includes the point-count over finite fields, and the E-polynomial and Euler characteristic over $\mathbb{C}$ [MOV09, Lemma 2.4].

Definition 5.1.3. Let S be a variety over k. Denote by $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S)$ the quotient of $\mathrm{K}_0(\mathbf{Var}_S)$ by relations of the form

$$[P]_S = [\mathbb{P}^1_k] \cdot [X]_S \tag{5.4}$$

for all $\mathbb{P}^1$-fibrations $P \to X$ over S. Similarly, denote by $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Stck}_S)$ the quotient of $\mathrm{K}_0(\mathbf{Stck}_S)$ by the same relations. Furthermore, if G is a finite group, denote by $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S^G)$ the quotient of $\mathrm{K}_0(\mathbf{Var}_S^G)$ by the same relations, for all G-equivariant $\mathbb{P}^1$-fibrations $P \to X$ over S.

We will need the G-equivariant version when dealing with varieties of the form $X /\!\!/ G$, and we want to stratify X in a G-equivariant manner. In that case, it is important that taking the quotient with respect to G respects the relation (5.4).

Proposition 5.1.4. *Let S be variety over k, and let G be a finite group. The morphism $\mathrm{K}_0(\mathbf{Var}_S^G) \to \mathrm{K}_0(\mathbf{Var}_S)$ given by $[X]_S \mapsto [X /\!\!/ G]_S$ descends to a morphism*

$$\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S^G) \to \mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S).$$

Proof. It must be shown that for every G-equivariant $\mathbb{P}^1$-fibration $P \to X$ over S, we have $[P /\!\!/ G]_S = [\mathbb{P}^1_k] \cdot [X /\!\!/ G]_S$ in $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S)$.

If G does not act faithfully on X, then $N = \{g \in G \mid g \cdot x = x \text{ for all } x \in X\}$ is a normal subgroup of G which acts trivially on X. Since $X /\!\!/ G = X /\!\!/ (G/N)$ and $P /\!\!/ G = (P /\!\!/ N) /\!\!/ (G/N)$, we may replace G by G/N and P by $P' = P /\!\!/ N$ (still a $\mathbb{P}^1$-fibration over X) and assume that G does act faithfully on X.

Next, let $\mathcal{H}$ be a set of representatives for the conjugacy classes of subgroups of G. Stratify $X = \bigsqcup_{H \in \mathcal{H}} X_H$, where $X_H = \{x \in X \mid \mathrm{Stab}(x) \text{ is conjugate to } H\}$. Note that the action of G restricts to X_H since $\mathrm{Stab}(g \cdot x) = g \,\mathrm{Stab}(x) g^{-1}$ for all $g \in G$ and $x \in X$. Furthermore, we have $X_H /\!\!/ G = Y_H /\!\!/ N_G(H)$, where $Y_H = \{x \in X \mid \mathrm{Stab}(x) = H\}$ and $N_G(H)$ is the normalizer of H in G, and similarly $(P \times_X X_H) /\!\!/ G = (P \times_X Y_H) /\!\!/ N_G(H)$. Hence, replacing G by $N_G(H)$ and X by Y_H, we may assume $\mathrm{Stab}(x)$ is constant and normal in G. Moreover, since we could assume G to act faithfully on X, we can assume the action of G on X to be free.

After stratifying X into smooth strata, the quotient map $X \to X /\!\!/ G$ is étale [Dré04, Proposition 4.11], so from the cartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & P /\!\!/ G \\ \downarrow & & \downarrow \\ X & \longrightarrow & X /\!\!/ G \end{array}$$

it follows that $P /\!\!/ G \to X /\!\!/ G$ is a $\mathbb{P}^1$-fibration over S as well. Therefore, $[P /\!\!/ G]_S = [\mathbb{P}^1_k] \cdot [X /\!\!/ G]_S$, as desired. □

5.2 Orientable surfaces

The goal of this section is to prove the following proposition, which completely characterizes the first map of (5.1). The stratifications used are similar to those in [LMN13], but adapted to the setting of $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_G)$.

Proposition 5.2.1. *The virtual class of* $G^2 \to G$ *given by* $(A, B) \mapsto [A, B]$ *in* $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_G)$ *is equal to*

$\Big(Z_G^{\mathrm{rep}}\big($  $\big) \circ Z_G^{\mathrm{rep}}\big($  $\big)\Big)(1)$

$$\begin{aligned} &= \mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)(\mathbb{L}+4)I_+ + \mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)I_- \\ &\quad + \mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)J_+ + \mathbb{L}^2(\mathbb{L}+3)J_- + (\mathbb{L}-1)^2(\mathbb{L}+1)M \\ &\quad + 2\mathbb{L}(\mathbb{L}+1)X_2 - \mathbb{L}(\mathbb{L}+1)X_{-2} - (\mathbb{L}-1)^2 X_{2,-2} + \mathbb{L}(\mathbb{L}-2)Y. \end{aligned}$$

Proof. Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ and stratify based on the conjugacy class of $[A, B]$.

- If $[A, B] = 1$, we consider the following cases.
  - Case $A = \pm 1$. Since any B commutes with A, this stratum has a virtual class equal to $2[G]I_+ = 2\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)I_+$.
  - Case $A \in J_\pm$. Conjugate A to $\left(\begin{smallmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{smallmatrix}\right)$ to find that B must be of the form $\left(\begin{smallmatrix} \pm 1 & x \\ 0 & \pm 1 \end{smallmatrix}\right)$. Hence, we obtain $4\mathbb{L}[J_+]I_+ = 4\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)I_+$.
  - Case $A \in M$. Note that A can be conjugated to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for some $\lambda \neq 0, \pm 1$, after which B must be diagonal. Hence, this stratum can be identified with
  $$\Big(\Big\{(P, \lambda, \mu) \in \mathrm{GL}_2/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1\}) \times (\mathbb{A}^1_k \setminus \{0\})\Big\}\Big) /\!\!/ S_2$$
  where $A = P\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)P^{-1}$ and $B = P\left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{smallmatrix}\right)P^{-1}$, and where S_2 acts via $(P, \lambda, \mu) \mapsto \left(P\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \lambda^{-1}, \mu^{-1}\right)$. To compute the virtual class of this quotient, we apply Section 3.6 with the finite (cyclic) group $S_2 = \mathbb{Z}/2\mathbb{Z}$. Using notation as in (3.12), we find
  $$\begin{aligned} [\mathbb{A}^1_k \setminus \{0, \pm 1\}]^{S_2} &= (\mathbb{L}-2) \otimes T - 1 \otimes N, \\ [\mathbb{A}^1_k \setminus \{0\}]^{S_2} &= \mathbb{L} \otimes T - 1 \otimes N, \\ [\mathrm{GL}_2/D]^{S_2} &= \mathbb{L}^2 \otimes T + \mathbb{L} \otimes N. \end{aligned}$$
  Therefore, we obtain $\mathbb{L}(\mathbb{L}-2)(\mathbb{L}-1)(\mathbb{L}+1)I_+$.

Together, these cases add up to $\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)(\mathbb{L}+4)I_+$.

- Suppose $[A,B]=-1$. From the equivalent expressions $ABA^{-1}=-B$ and $B^{-1}AB=-A$ follows that $\operatorname{tr} A=\operatorname{tr} B=0$. In particular, we can conjugate A to $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, after which B must be of the form $\left(\begin{smallmatrix} 0 & y \\ -1/y & 0 \end{smallmatrix}\right)$. Hence, we obtain $\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)I_-$.

- Suppose $[A,B]\in J_+$. Conjugate $[A,B]$ to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. From $AB=\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)BA$ follows that $\operatorname{tr} B=\operatorname{tr}(ABA^{-1})=\operatorname{tr}\big(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)B\big)$ and hence $z=0$. Similarly, $\operatorname{tr} A=\operatorname{tr}\big(BAB^{-1}\big)=\operatorname{tr}\big(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^{-1}A\big)$ implies $c=0$. Now $\det A=\det B=1$ yields $d=a^{-1}$ and $w=x^{-1}$, and the only remaining equation is $y(a-a^{-1})-b(x-x^{-1})=1/ax$. Consider the following cases.

  - If $a\neq\pm1$, we can solve for $y=(1/ax+b(x-x^{-1}))/(a-a^{-1})$, and obtain $\mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)J_+$.
  - If $a=\pm1$, we must have $x\neq\pm1$ and can solve for $b=a/(1-x^2)$. We obtain $2\mathbb{L}(\mathbb{L}-3)J_+$.

  Together, these cases add up to $\mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)J_+$.

- Suppose $[A,B]\in J_-$. Conjugate $[A,B]$ to $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. From $\operatorname{tr} B=\operatorname{tr}(ABA^{-1})=\operatorname{tr}\big(\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)B\big)$ follows that $z=2(x+w)$, and from $\operatorname{tr} A=\operatorname{tr}(BAB^{-1})=\operatorname{tr}\big(\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)^{-1}A\big)$ follows that $c=-2(a+d)$. The only remaining equation is $ay-bw-bx-dw+dy=0$. Consider the following cases.

  - Case $w=-x$. From $\det B=1$ follows that $x=\pm i$. The action of conjugation by $\{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\}\cong\mathbb{G}_a$ turns this stratum into a $\mathbb{G}_a$-torsor over the stratum with $y=0$. On this stratum with $y=0$, we can solve for $d=0$, and $\det A=1$ implies $a\neq0$ and $b=1/2a$. Hence, we obtain $2\mathbb{L}(\mathbb{L}-1)J_-$.
  - Case $w\neq-x$. The action of conjugation by $\{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\}\cong\mathbb{G}_a$ turns this stratum into a $\mathbb{G}_a$-torsor over the stratum with $w=0$. On this stratum with $w=0$, it follows from $\det B=1$ that $x\neq0$ and $y=-1/2x$. We can solve for $b=-(a+d)/2x^2$. Finally, $\det A=1$ translates to $ad-(a+d)^2/x^2=1$.
    - Case $d=-a$. Solve for $a=\pm i$ to obtain $2\mathbb{L}(\mathbb{L}-1)J_-$.
    - Case $d\neq-a$. Make a substitution $x'=(a+d)/x$ to rewrite the equation as $ad-(x')^2=1$. This is easily seen to give $\mathbb{L}(\mathbb{L}^2-\mathbb{L}+4)J_-$.

  Together, these cases add up to $\mathbb{L}^2(\mathbb{L}+3)J_-$.

- Suppose $[A,B]\in M$. Diagonalizing $[A,B]$, this stratum can be expressed as

$$\left\{(P,A,B,\lambda)\in\mathrm{GL}_2/D\times G^2\times(\mathbb{A}^1_k\setminus\{0,\pm1\})\;\middle|\;[A,B]=\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\right\}/\!\!/\,S_2$$

where S_2 acts via $\lambda \mapsto \lambda^{-1}$ and $P \mapsto P\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and on A and B via conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. From $\operatorname{tr} A = \operatorname{tr}(BAB^{-1}) = \operatorname{tr}\left(\left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{smallmatrix}\right) A\right)$ follows that $d = a/\lambda$, and from $\operatorname{tr} B = \operatorname{tr}(ABA^{-1}) = \operatorname{tr}\left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B\right)$ that $w = \lambda x$. The relevant equations are now $ax + bz - \lambda(ax + cy) = 0$ and $\det A = a^2\lambda^{-1} - bc = 1$ and $\det B = \lambda x^2 - yz = 1$. Consider the following cases.

- Case $b = c = 0$. It follows that $a^2 = \lambda$, $x = 0$ and $z = -y^{-1}$. Note that S_2 acts via $a \mapsto d = a/\lambda = a^{-1}$ and $y \mapsto z = -y^{-1}$. Therefore, we obtain the following S_2-virtual classes
$$[\{y \neq 0\}]^{S_2} = \mathbb{L} \otimes T - 1 \otimes N$$
$$[\mathrm{GL}_2/D \times \{a^2 = \lambda\}]^{S_2}_M = X_{-2} \otimes T + (Y - X_{-2}) \otimes N.$$
Multiplying these and taking the quotient by S_2, we obtain $(\mathbb{L}+1)X_{-2} - Y$.
- Case $y = z = 0$. Similarly, we obtain $(\mathbb{L} + 1)X_{-2} - Y$.
- Case $b = 0$ or $c = 0$, but not both. The action of S_2 swaps b and c, so we can identify the S_2-quotient with the stratum where $b = 0$ and $c \neq 0$. The action of conjugation by $\left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\right\} \cong \mathbb{G}_m$ turns this stratum into a $\mathbb{G}_m$-torsor over the stratum with $c = 1$. On this stratum with $c = 1$, we find that $a^2 = \lambda$, $y = ax(\lambda^{-1} - 1)$ with $x \neq 0$, and $z = (\lambda x^2 - 1)/y$. Hence, we obtain $(\mathbb{L} - 1)^2 Y$.
- Case $y = 0$ or $z = 0$, but not both. Similarly, we obtain $(\mathbb{L} - 1)^2 Y$.
- In the above cases, we have counted twice the stratum given by $b = z = 0$ or $c = y = 0$, so we need to subtract it once. Note that these conditions cannot be satisfied simultaneously, and moreover, the action of S_2 swaps them. Therefore, we can identify the S_2-quotient with the stratum where $b = z = 0$ (and $c, y \neq 0$). The action of conjugation by $\left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\right\} \cong \mathbb{G}_m$ turns this stratum into a $\mathbb{G}_m$-torsor over the stratum with $c = 1$. On this stratum with $c = 1$, we find $a^2 = \lambda$ and solve for $(x, y) = \pm\left(a^{-1}, a^{-1} - a\right)$. Hence, we obtain $-2(\mathbb{L} - 1)Y$, where the minus sign signifies this stratum must be subtracted from the total.
- Case $bcyz \neq 0$. Solve for $c = (a^2/\lambda - 1)/b$ and $z = (\lambda x^2 - 1)/y$. The conditions $c, z \neq 0$ translate to $a^2 \neq \lambda$ and $x^{-2} \neq \lambda$. The remaining equation is
$$x^2 - \frac{a'(\lambda - 1)}{(\lambda + 1)} x y' + \left(1 - \frac{(a')^2}{\lambda + \lambda^{-1} + 2}\right) (y')^2 = \lambda^{-1},$$
where we made substitutions $y' = y/b$ and $a' = a(1 + \lambda^{-1})$. The condition $a^2 \neq \lambda$ translates to $(a')^2 \neq \lambda + \lambda^{-1} + 2$. This equation describes a family of conics over the plane $\{(a', \lambda) \mid (a')^2 \neq \lambda + \lambda^{-1} + 2\}$ with discriminant $D = (a' - 2)(a' + 2)$. To compute its virtual class, the idea is to complete

this family of conics to a $\mathbb{P}^1$-fibration over the (a', λ)-plane, for $D \neq 0$, so that relation (5.4) can be used. The stratum at infinity will be computed separately, and must be subtracted from the total.

Note that the variable $b \neq 0$ is independent of a', x and y', except through the action of S_2 given by $b \mapsto c = (a'\lambda/(\lambda+1)^2 - 1)/b$. Extending b to be $\mathbb{P}^1$-valued, we can regard this stratum as a $\mathbb{P}^1$-fibration minus the stratum with $b = 0$ or $b = \infty$. Note that the cases $b = 0$ and $b = \infty$ are interchanged by the action of S_2. Hence, for the sake of the computation, we can effectively act as if b is completely independent of a', x and y', with S_2-virtual class $[\{b \neq 0\}]^{S_2} = \mathbb{L} \otimes T - 1 \otimes N$.

* Case $D = 0$. Solve for $a' = \pm 2$. Suppose $a' = 2$. Then $\left(x - \frac{\lambda-1}{\lambda+1} y'\right)^2 = \lambda^{-1}$. Write $\omega = x - \frac{\lambda-1}{\lambda+1} y'$ so that $\omega^2 = \lambda^{-1}$ and note that S_2 acts via $\omega \mapsto -\omega^{-1}$. The condition $x^2 \neq \lambda^{-1}$ translates to $x \neq \pm\omega$. Substituting $x' = x/\omega$ yields $x' \neq \pm 1$ and S_2 acts via $x' \mapsto -x'$. From the S_2-virtual classes

$$\begin{aligned} [\{b \neq 0\}]^{S_2} &= \mathbb{L} \otimes T - 1 \otimes N \\ [\{x' \neq \pm 1\}]^{S_2} &= (\mathbb{L} - 1) \otimes T - 1 \otimes N \\ [\mathrm{GL}_2/D \times \{\omega^2 = \lambda^{-1}\}]^{S_2}_M &= X_2 \otimes T + (Y - X_2) \otimes N \end{aligned}$$

we obtain $\mathbb{L}(\mathbb{L}+1)X_2 - (2\mathbb{L}-1)Y$. The case $a' = -2$ is completely similar, so we double this virtual class to obtain $2\mathbb{L}(\mathbb{L}+1)X_2 - (4\mathbb{L} - 2)Y$.

* Case $D \neq 0$. Complete the family of conics to a $\mathbb{P}^1$-fibration given by

$$X^2 - \frac{a'(\lambda-1)}{(\lambda+1)} XY + \left(1 - \frac{(a')^2}{\lambda + \lambda^{-1} + 2}\right) Y^2 = \lambda^{-1} Z^2, \tag{5.5}$$

over the base $B = \mathrm{GL}_2/D \times \{(a', \lambda) \mid a' \neq \pm 2 \text{ and } (a')^2 \neq \lambda + \lambda^{-1} + 2\}$. Regarding B as the open complement of $(a')^2 = \lambda + \lambda^{-1} + 2$, we compute its S_2-virtual class as

$$\begin{aligned} [B]^{S_2}_M = (\mathbb{L} - 2)(M \otimes T + (X_{2,-2} - M) \otimes N) \\ - (X_{-2} \otimes T + (Y - X_{-2}) \otimes N). \end{aligned}$$

Multiplying by $[\mathbb{P}^1_k] = \mathbb{L} + 1$ and by $[\{b \neq 0\}]^{S_2} = \mathbb{L} \otimes T - 1 \otimes N$, and taking the quotient by S_2, we obtain

$$(\mathbb{L} - 2)(\mathbb{L}+1)^2 M - (\mathbb{L}+1)^2 X_{-2} - (\mathbb{L}-2)(\mathbb{L}+1)X_{2,-2} + (\mathbb{L}+1)Y.$$

* Now we must subtract the stratum of points at infinity, that is, the points given by $Z = 0$. Since there are no solutions with $X = 0$, we can work

with the dehomogenized coordinate Y/X. In fact, writing $u = Y/X \cdot \left(1 - \frac{(a')^2\lambda}{(\lambda+1)^2}\right)$, equation (5.5) reduces to

$$\left(2u - \frac{a'(\lambda-1)}{(\lambda+1)}\right)^2 = (a'-2)(a'+2).$$

Substituting $u' = 2u - \frac{a'(\lambda-1)}{\lambda+1}$, we find that u' is invariant under S_2, and the equation simplifies to

$$(u')^2 = (a')^2 - 4.$$

Regarding this stratum as the open complement of $(a')^2 = \lambda + \lambda^{-1} + 2$, we compute its S_2-virtual class as

$$\left[\mathrm{GL}_2/D \times \left\{\substack{(u')^2=(a')^2-4\neq 0 \\ (a')^2\neq\lambda+\lambda^{-1}+2}\right\}\right]_M^{S_2} = (\mathbb{L}-3)(M \otimes T - (X_{2,-2} - M) \otimes N) \\ - Y \otimes (T+N).$$

Multiplying by $[\{b \neq 0\}]^{S_2} = \mathbb{L} \otimes T - 1 \otimes N$ and taking the quotient by S_2, we obtain

$$-\Big((\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}-3)X_{2,-2} - (\mathbb{L}-1)Y\Big),$$

where the overall minus sign signifies this stratum should be subtracted from the total.

* Finally, we must subtract the stratum where $x^{-2} = \lambda$. In this case, we solve for $y' = 0$ or $y' = \frac{a'x(\lambda-1)(\lambda+1)}{(\lambda+1)^2-(a')^2\lambda}$. When $a' = 0$, these values coincide, so from the S_2-virtual classes

$$[\{b \neq 0\}]^{S_2} = \mathbb{L} \otimes T - 1 \otimes N$$
$$[\mathrm{GL}_2/D \times \{x^{-2} = \lambda\}]_M^{S_2} = X_{-2} \otimes T + (Y - X_{-2}) \otimes N$$

we obtain

$$-\Big((\mathbb{L}+1)X_{-2} - Y\Big).$$

When $a' \neq 0$, the values for y' are interchanged by the action of S_2. Hence, we can identify the S_2-quotient with the stratum where $y' = 0$. The condition $(a')^2 \neq \lambda + \lambda^{-1} + 2$ translates to $a' \neq \pm(x + x^{-1})$. This gives

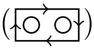
$$-\Big((\mathbb{L}-5)(\mathbb{L}-1)Y\Big).$$

Together, these cases add up to $(\mathbb{L}-1)^2(\mathbb{L}+1)M + 2\mathbb{L}(\mathbb{L}+1)X_2 - \mathbb{L}(\mathbb{L}+1)X_{-2} - (\mathbb{L}-1)^2X_{2,-2} + \mathbb{L}(\mathbb{L}-2)Y$. □

5.3 Non-orientable surfaces

Analogous to the previous section, we prove the following proposition, characterizing the second map of (5.1).

Proposition 5.3.1. *The virtual class of $G \to G$ given by $A \mapsto A^2$ in $\mathrm{K}_0(\mathbf{Var}_G)$ is equal to*

$$\left(Z_G^{\mathrm{rep}}\left(\right.\right.$$  $$\left.\right) \circ Z_G^{\mathrm{rep}}\left(\right.$$  $$\left.\left.\right)\right)(1) = 2I_+ + \mathbb{L}(\mathbb{L}+1)I_- + 2J_+ + X_{-2}.$$

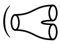
Proof. Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and stratify based on the conjugacy class of A^2.

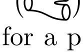
- If $A^2 = 1$, then $A = \pm 1$, so we obtain $2I_+$.
- Suppose $A^2 = -1$. If $b = 0$, then $d = a^{-1}$ with $a = \pm i$, contributing $2\mathbb{L}I_-$. If $b \neq 0$ and $c = 0$, then $d = a^{-1}$ with $a \neq \pm i$, contributing $2(\mathbb{L}-1)I_-$. If $b, c \neq 0$, then $d = -a$ and $a^2 + bc = -1$, contributing $(\mathbb{L}-2)(\mathbb{L}-1)I_-$. In total, we obtain $\mathbb{L}(\mathbb{L}+1)I_-$.
- Suppose $A^2 \in J_+$. By conjugating we can assume $A^2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. There are no solutions for $c \neq 0$, and $c = 0$ yields $a = d = b/2 = \pm 1$, so we obtain $2J_+$.
- There are no solutions with $A^2 \in J_-$.
- Suppose $A^2 \in M$. This stratum is given by

  $$\left(\mathrm{GL}_2/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\})\right) /\!\!/ S_2 \to G, \quad (P, \omega) \mapsto P \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix} P^{-1},$$

  where S_2 acts on ω via $\omega \mapsto \omega^{-1}$. Hence, this is equal to X_{-2}. □

5.4 Multiplication in SL_2

In this section, we compute the images $Z_G^{\mathrm{rep}}\left(\right.$  $\left.\right)(X \otimes Y)$ for all pairs (X, Y) of generators in (5.3), in a series of lemmas. For conciseness, we will omit some cases, but those can be obtained directly from the cases we do compute. For example, the cases with $X = I_-$ are straightforward, and the cases with $X = J_-$ and $X = X_{-2}$ can be derived from those with $X = J_+$ and X_2.

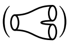
First, let us fix some notation. When computing $Z_G^{\mathrm{rep}}\left(\right.$  $\left.\right)(X \otimes Y)$ for a pair (X, Y), we write A for a point of X and $B = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ for a point of Y. When Y is of the form $(\mathrm{GL}_2/D \times \Lambda) /\!\!/ S_2$ for some S_2-variety Λ over $\mathbb{A}^1_k \setminus \{0, \pm 1\}$, we also write $B = P \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} P^{-1}$ with $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2/D$ and $\mu \in \mathbb{A}^1_k \setminus \{0, \pm 1\}$. Recall that S_2 acts on (P, μ) via $(P, \mu) \mapsto \left(P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \mu^{-1}\right)$. More specifically, when $\Lambda = \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\}$, we write $\mu = \omega^2$ with $\omega \in \Lambda$. Similarly, when X is of the form $(\mathrm{GL}_2/D \times \Lambda) /\!\!/ S_2$ for such Λ, we write $A = Q \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} Q^{-1}$ with $Q \in \mathrm{GL}_2/D$ and $\rho \in \mathbb{A}^1_k \setminus \{0, \pm 1\}$, and write $\rho = \nu^2$ when $\Lambda = \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\}$.

When dealing with the strata where $AB \in M$, we usually want to diagonalize AB. This can be done once we base change along the double cover $(\mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1\}) \to M$. We write λ for the coordinate on $\mathbb{A}^1_k \setminus \{0, \pm 1\}$. The group S_2 acts on this double cover via $(P, \lambda) \mapsto \left(P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \lambda^{-1}\right)$.

Strata often admit symmetry by the action of conjugation with some subgroup of SL_2. When this happens for the subgroups $\{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\} \cong \mathbb{G}_a$ or $\left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\right\} \cong \mathbb{G}_m$, we will speak of *$\mathbb{G}_a$-symmetry* or *$\mathbb{G}_m$-symmetry*, respectively. In these cases, such a stratum turns into a (Zariski-locally trivial) $\mathbb{G}_a$-torsor or $\mathbb{G}_m$-torsor, so to compute its virtual class it suffices to compute that of the base.

Finally, to avoid confusion between the various S_2-actions, we write S_2^λ, S_2^μ and S_2^ρ to differentiate between them.

Lemma 5.4.1.

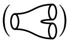


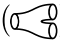
$$Z_G^{\mathrm{rep}}\left(\,\cdot\,\right)(J_+ \otimes J_+) = (\mathbb{L}+1)(\mathbb{L}-1)I_+ + (\mathbb{L}-2)J_+ + \mathbb{L}J_- + (\mathbb{L}+1)M - X_{2,-2}$$

Proof. Stratify based on the conjugacy class of the product AB.

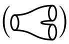
- If $AB = 1$, then $A = B^{-1}$, so we obtain $[J_+]I_+ = (\mathbb{L}+1)(\mathbb{L}-1)I_+$.
- If $AB = -1$, there are no solutions as $\operatorname{tr} A = 2 \neq -2 = -\operatorname{tr} B^{-1}$.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} w-z & x-y \\ -z & x \end{smallmatrix}\right)$. From $\operatorname{tr} A = \operatorname{tr} B = 2$ follows that $z = 0$ and (using $\det B = 1$) also $x = w = 1$. Furthermore, $y \neq 0, 1$ as $A, B \neq 1$, so we obtain $(\mathbb{L}-2)J_+$.
- If $AB \in J_-$, then conjugate to $AB = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} -w-z & x+y \\ z & -x \end{smallmatrix}\right)$. From $\operatorname{tr} A = \operatorname{tr} B = 2$ follows that $z = -4$. Fix $x = 0$ using $\mathbb{G}_a$-symmetry, and solve for $w = 2$ and $y = 1/4$. We obtain $\mathbb{L}J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda w & -\lambda y \\ -z/\lambda & x/\lambda \end{smallmatrix}\right)$. From $\operatorname{tr} B = 2$ follows that $w = 2 - x$. From $\operatorname{tr} A = 2$ and $\det A = 1$ and $\lambda \neq 1$ follows that $z \neq 0$. From $\det B = 1$ follows that $y = (xw-1)/z$. From $\operatorname{tr} A = 2$ follows that $x = \frac{2\lambda}{\lambda+1}$. Make a substituting $z' = z\frac{\lambda+1}{\lambda-1}$, and note that S_2^λ acts via $z' \mapsto 1/z'$. Now, from the S_2^λ-virtual classes
$$[\{z' \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$
$$[\mathrm{GL}_2/D \times \{\lambda \neq 0, \pm 1\}]_M^{S_2^\lambda} = M \otimes T - (X_{2,-2} - M) \otimes N$$
follows that the quotient by S_2^λ is $(\mathbb{L}+1)M - X_{2,-2}$. □

Lemma 5.4.2.

Z_G^{rep} () $(J_+ \otimes M) = \mathbb{L}(\mathbb{L}-2)(J_+ + J_-) + (\mathbb{L}-3)(\mathbb{L}+1)M + 2X_{2,-2}$

Proof. Note that Z_G^{rep} () $(X \otimes G) = [X] \cdot G$ for all $X \in \mathrm{K}_0(\mathbf{Var}_G)$. Since $G = I_+ + I_- + J_+ + J_- + M$, the result can be derived from the above lemma. □

Lemma 5.4.3.

Z_G^{rep} () $(J_+ \otimes X_2) = \mathbb{L}(\mathbb{L}-3)(J_+ + J_-) + (\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}+1)X_2 - (\mathbb{L}-3)X_{2,-2} + \mathbb{L}Y$

Proof. Stratify based on the conjugacy class of the product AB.

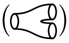
- If $AB = \pm 1$, there are no solutions.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} w-z & x-y \\ -z & x \end{smallmatrix}\right)$. From $\operatorname{tr} A = 2$ follows that $z = x + w - 2$ and from $\det B = 1$ that $y = (xw-1)/z$. Furthermore, we can solve for $w = \ell^2 - x + 2$ with $\ell \neq 0, \pm 2i$. Hence, we obtain $\mathbb{L}(\mathbb{L}-3)J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-3)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda w & -\lambda y \\ -z/\lambda & x/\lambda \end{smallmatrix}\right)$. Consider the following cases.
  - Case $y = z = 0$. There are no solutions.
  - Case $y = 0$ or $z = 0$, but not both. Since the action of S_2^λ swaps y and z, we can identify the S_2^λ-quotient with the stratum where $z = 0$. From $\operatorname{tr} A = 2$ and $\det A = 1$ follows that $x = w^{-1} = \lambda$, so in particular $\ell^2 = \lambda + \lambda^{-1} - 2$. Since $A \neq 1$, we have $y \neq 0$, so we obtain $(\mathbb{L}-1)Y$.
  - Case $yz \neq 0$. From $\operatorname{tr} A = 2$ follows that $w = (2 - x/\lambda)/\lambda$ and from $\det B = 1$ that $y = (xw-1)/z$. We substitute $z' = z\lambda/(x-\lambda)$ so that S_2^λ acts via $z' \mapsto 1/z'$. Using $\ell^2 = \operatorname{tr} B - 2$, we can solve for $x = \lambda(\ell^2\lambda + 2\lambda - 2)/(\lambda^2 - 1)$. The conditions $y \neq 0$ and $\operatorname{tr} B \neq \pm 2$ translate to $\ell^2 \neq \lambda + \lambda^{-1} - 2$ and $\ell^2 \neq 0, -4$. From the S_2^λ-virtual classes

$$[\{z' \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$

$$\left[\mathrm{GL}_2/D \times \left\{ \begin{smallmatrix} \ell^2 \neq \lambda + \lambda^{-1} - 2 \\ \ell \neq 0, \pm 2i \end{smallmatrix} \right\} \right]_M^{S_2^\lambda} = (\mathbb{L}-3)(M \otimes T + (X_{2,-2} - M) \otimes N) - (X_2 \otimes T + (Y - X_2) \otimes N)$$

we obtain $(\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}+1)X_2 - (\mathbb{L}-3)X_{2,-2} + Y$. □

Lemma 5.4.4.

Z_G^{rep} () $(J_+ \otimes X_{2,-2}) = \mathbb{L}(\mathbb{L}-3)(J_+ + J_-) + (\mathbb{L}-3)(\mathbb{L}+1)M + 2X_{2,-2}$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = \pm 1$, there are no solutions.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) B^{-1}$. We have $\gamma \neq 0$ since $\operatorname{tr} A = 2$ and $\mu \neq \pm 1$. Hence, we can fix $\gamma = 1$, $\alpha = 0$ and $\beta = 1$ by lifting P to GL_2 and using $\mathbb{G}_a$-symmetry. Now $\operatorname{tr} A = 2$ implies $\delta = -\frac{\mu-1}{\mu+1}$ with $\mu \neq 0, \pm 1$, so we obtain $\mathbb{L}(\mathbb{L}-3)J_+$.
- If $AB \in J_+$, then similarly we obtain $\mathbb{L}(\mathbb{L}-3)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix}\lambda&0\\0&\lambda^{-1}\end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix}\lambda&0\\0&\lambda^{-1}\end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. The action of S_2^λ swaps α and γ, so we can break the S_2^λ-action and consider only the stratum with $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. From $\operatorname{tr} A = 2$ follows that $\mu = \lambda$. Furthermore, we must have $\beta \neq 0$ to ensure $A \neq 1$, so we obtain $(\mathbb{L}-1)X_{2,-2}$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. The action of S_2^λ swaps β and δ, so we can identify the S_2^λ-quotient with the stratum where $\delta = 0$. Fix $\beta = \gamma = 1$ by lifting P to GL_2. From $\operatorname{tr} A = 2$ follows that $\mu = \lambda^{-1}$. Furthermore, there are no conditions on α other than $\alpha \neq 0$, so we obtain another $(\mathbb{L}-1)X_{2,-2}$.
  - Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\mu = \lambda^{\pm 1}$, and use $\operatorname{tr} A = 2$ to solve for $\beta = \alpha\frac{(\lambda-\mu)^2}{(\lambda\mu-1)^2}$. Note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

$$[\{\alpha \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$

$$\left[\mathrm{GL}_2/D \times \left\{\begin{smallmatrix}\lambda \neq 0, \pm 1\\ \mu \neq 0, \pm 1, \lambda^{\pm 1}\end{smallmatrix}\right\}\right]_M^{S_2^\lambda} = (\mathbb{L}-3)(M \otimes T + (X_{2,-2} - M) \otimes N) - X_{2,-2} \otimes (T + N)$$

  we obtain $(\mathbb{L}-3)(\mathbb{L}+1)M - 2(\mathbb{L}-2)X_{2,-2}$. □

Lemma 5.4.5.

Z_G^{rep} () $(J_+ \otimes Y) = \mathbb{L}(\mathbb{L}-5)(J_+ + J_-) + (\mathbb{L}-5)(\mathbb{L}+1)M - (\mathbb{L}-5)X_{2,-2} + (\mathbb{L}-1)Y$

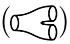
Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = \pm 1$, there are no solutions.
- If $AB \in J_+$, the computation is the same as for $A \in J_+$ and $B \in X_{2,-2}$, but with $\mu = \omega^2$ and $\omega \neq 0, \pm 1, \pm i$. Hence, we obtain $\mathbb{L}(\mathbb{L}-5)J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-5)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. Identify the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. From $\operatorname{tr} A = 2$ follows that $\omega^2 = \lambda$. Furthermore, we must have $\beta \neq 0$ to ensure $A \neq 1$, so we obtain $(\mathbb{L}-1)Y$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. Identify the S_2^λ-quotient with the stratum where $\delta = 0$. Fix $\beta = \gamma = 1$ by lifting P to GL_2. From $\operatorname{tr} A = 2$ follows that $\omega^2 = \lambda^{-1}$. Furthermore, there are no conditions on α other than $\alpha \neq 0$, so we obtain another $(\mathbb{L}-1)Y$.
  - Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\mu = \lambda^{\pm 1}$, and use $\operatorname{tr} A = 2$ to solve for $\beta = \alpha \frac{(\lambda-\mu)^2}{(\lambda\mu-1)^2}$. Note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

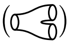
$$\begin{aligned} [\{\alpha \neq 0\}]^{S_2^\lambda} &= \mathbb{L} \otimes T - 1 \otimes N \\ \left[\mathrm{GL}_2/D \times \left\{ \begin{smallmatrix} \lambda \neq 0, \pm 1 \\ \omega^2 \neq 0, \pm 1, \lambda^{\pm 1} \end{smallmatrix} \right\}\right]_M^{S_2^\lambda} &= (\mathbb{L}-5)(M \otimes T + (X_{2,-2} - M) \otimes N) \\ &\quad - Y \otimes (T + N) \end{aligned}$$

we obtain $(\mathbb{L}-5)(\mathbb{L}+1)M - (\mathbb{L}-5)X_{2,-2} - (\mathbb{L}-1)Y$. □

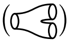
Lemma 5.4.6.

Z_G^{rep} () $(M \otimes M)$

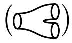
$$\begin{aligned} &= \mathbb{L}(\mathbb{L}^2 - 2\mathbb{L} - 1)(I_+ + I_-) \\ &\quad + \mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)(J_+ + J_-) \\ &\quad + (\mathbb{L}^3 - 4\mathbb{L}^2 + 3\mathbb{L} + 4)M - 4X_{2,-2} \end{aligned}$$

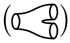
Z_G^{rep} () $(M \otimes X_2)$

$$\begin{aligned} &= \mathbb{L}(\mathbb{L}^2 - 3\mathbb{L} - 2)(I_+ + I_-) \\ &\quad + \mathbb{L}(\mathbb{L}-4)(\mathbb{L}-1)(J_+ + J_-) \\ &\quad + (\mathbb{L}^3 - 5\mathbb{L}^2 + 2\mathbb{L} + 6)M + \mathbb{L}(X_2 + X_{-2}) \\ &\quad + 2(\mathbb{L}-3)X_{2,-2} - 2\mathbb{L}Y \end{aligned}$$

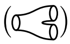
Z_G^{rep} () $(M \otimes X_{2,-2})$

$$\begin{aligned} &= \mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)(I_+ + I_-) \\ &\quad + \mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)(J_+ + J_-) \\ &\quad + (\mathbb{L}-3)(\mathbb{L}-2)(\mathbb{L}+1)M - 6X_{2,-2} \end{aligned}$$

Z_G^{rep}  $(M \otimes Y) = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)(I_+ + I_-) + \mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)(J_+ + J_-) + (\mathbb{L}-5)(\mathbb{L}-2)(\mathbb{L}+1)M + 2(\mathbb{L}-5)X_{2,-2} - 2\mathbb{L}Y$

Proof. Note that Z_G^{rep}  $(G \otimes X) = [X] \cdot G$ for all $X \in \mathrm{K}_0(\mathbf{Var}_G)$. Since $G = I_+ + I_- + J_+ + J_- + M$, the result follows from the earlier lemmas. □

Lemma 5.4.7.

Z_G^{rep}  $(X_{2,-2} \otimes X_{2,-2}) = 2\mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)(I_+ + I_-) + \mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)(J_+ + J_-) + (\mathbb{L}-3)^2(\mathbb{L}+1)M + (\mathbb{L}^2 - 4\mathbb{L} - 9)X_{2,-2}$

Proof. Stratify based on the conjugacy class of the product AB.

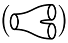
- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[X_{2,-2} \times_M X_{2,-2}]I_+ = 2[X_{2,-2}]I_+ = 2\mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)I_+$.
- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[X_{2,-2} \times_M X_{2,-2}]I_- = 2[X_{2,-2}] = 2\mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)I_-$.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. Solving for $\rho = \mu^{\pm 1} \neq 0, \pm 1$, we obtain $2\mathbb{L}(\mathbb{L}-3)J_+$.
  - Case $\gamma \neq 0$. Fix $\gamma = 1$, $\alpha = 0$ and $\beta = 1$ by lifting P to GL_2 and using $\mathbb{G}_a$-symmetry. Using $\operatorname{tr} A = \rho + \rho^{-1}$, solve for $\delta = -\frac{(\mu-\rho)(\mu\rho-1)}{\rho(\mu-1)(\mu+1)}$. Since $\mu, \rho \neq 0, \pm 1$, we obtain $\mathbb{L}(\mathbb{L}-3)^2 J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. Identify the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\mu^{-1})^{\pm 1}$. In both cases $\mu \neq 0, \pm 1, \pm\lambda$, so we obtain $2\mathbb{L}(\mathbb{L}-5)X_{2,-2}$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. Identify the S_2^λ-quotient with the stratum where $\beta = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\mu^{-1})^{\pm 1}$. In both cases $\mu \neq 0, \pm 1, \pm\lambda$, so we obtain $2(\mathbb{L}-1)(\mathbb{L}-5)X_{2,-2}$.

- Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. Note that $\alpha \neq \beta$ is automatically satisfied as there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$. Note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

$$[\{\alpha \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$
$$\left[\mathrm{GL}_2/D \times \left\{\substack{\lambda,\mu\neq 0,\pm 1 \\ \rho \neq 0,\pm 1,\lambda^{\pm 1}\mu^{\pm 1}}\right\}\right]_M^{S_2^\lambda} = (\mathbb{L}-3)^2(M \otimes T + (X_{2,-2} - M) \otimes N) - 2(\mathbb{L}-5)X_{2,-2} \otimes (T + N)$$

we obtain $(\mathbb{L}-3)^2(\mathbb{L}+1)M - (3\mathbb{L}^2 - 18\mathbb{L} + 19)X_{2,-2}$. ☐

Lemma 5.4.8.

$$Z_G^{\mathrm{rep}}\left(\right.$$  $$\left.\right)(X_{2,-2} \otimes Y) = 2\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)(I_+ + I_-) + \mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)(J_+ + J_-) + (\mathbb{L}-5)(\mathbb{L}-3)(\mathbb{L}+1)M + (\mathbb{L}-5)(\mathbb{L}+3)X_{2,-2} - 4\mathbb{L}Y$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[X_{2,-2} \times_M Y]I_+ = 2[Y]I_+ = 2\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_+$.
- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[X_{2,-2} \times_M Y]I_- = 2[Y]I_+ = 2\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_-$.
- If $AB \in J_+$, the computation is the same as for $A \in X_{2,-2}$ and $B \in X_{2,-2}$, but with $\mu = \omega^2$ and $\omega \neq 0, \pm 1, \pm i$. Hence, we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. Identify the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\mu^{-1})^{\pm 1}$. In both cases $\mu = \omega^2 \neq 0, \pm 1, \pm\lambda$, so we obtain $2\mathbb{L}(\mathbb{L}-5)X_{2,-2} - 4\mathbb{L}Y$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. Identify the S_2^λ-quotient with the stratum where $\beta = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\mu^{-1})^{\pm 1}$. In both cases $\mu = \omega^2 \neq 0, \pm 1, \pm\lambda$, so we obtain $2(\mathbb{L}-1)(\mathbb{L}-5)X_{2,-2} - 4(\mathbb{L}-1)Y$.
  - Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. Furthermore,

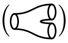
note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

$$[\{\alpha \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$

$$\left[\mathrm{GL}_2/D \times \left\{ \substack{\lambda,\omega^2,\rho\neq 0,\pm 1 \\ \rho \neq \lambda^{\pm 1}\omega^{\pm 2}} \right\}\right]_M^{S_2^\lambda} = (\mathbb{L}-5)(\mathbb{L}-3)(M \otimes T + (X_{2,-2} - M) \otimes N)$$
$$- (2(\mathbb{L}-5)X_{2,-2} - 4Y) \otimes (T + N)$$

we obtain $(\mathbb{L}-5)(\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}-5)(3\mathbb{L}-5)X_{2,-2} + 4(\mathbb{L}-1)Y$. □

Lemma 5.4.9.

Z_G^{rep} () $(Y \otimes Y) = 4\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)(I_+ + I_-)$
$$+ \mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)(J_+ + J_-)$$
$$+ (\mathbb{L}-5)^2(\mathbb{L}+1)M - (\mathbb{L}-5)^2 X_{2,-2} + 2\mathbb{L}(\mathbb{L}-9)Y$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[Y \times_M Y]I_+ = 4[Y]I_+ = 4\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_+$.
- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[Y \times_M Y]I_- = 4[Y]I_- = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_-$.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. Then $\nu^2 = \omega^{\pm 2}$, that is, $\nu = \pm\omega^{\pm 1} \neq 0, \pm 1, \pm i$. Hence, we obtain $4\mathbb{L}(\mathbb{L}-5)J_+$.
  - Case $\gamma \neq 0$. Fix $\gamma = 1$, $\alpha = 0$ and $\beta = 1$ by lifting P to GL_2 and using $\mathbb{G}_a$-symmetry. Solve for $\delta = -\frac{(\mu-\rho)(\mu\rho-1)}{\rho(\mu-1)(\mu+1)}$. Since $\omega, \nu \neq 0, \pm 1, \pm i$, we obtain $\mathbb{L}(\mathbb{L}-5)^2 J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. Identify the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$. If $\nu^2 = \lambda\omega^{-2}$, then substituting $u = \nu\omega$ yields $u^2 = \lambda$ with $\omega \neq 0, \pm 1, \pm i, \pm u, \pm iu$. The case $\nu^2 = (\lambda\omega^{-2})^{-1}$ is similar with $u = \nu/\omega$, so we obtain $2\mathbb{L}(\mathbb{L}-9)Y$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. Identify the S_2^λ-quotient with the stratum where $\beta = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$. Again, substituting $u = \nu\omega^{\pm 1}$, respectively, we obtain $2(\mathbb{L}-9)(\mathbb{L}-1)Y$.

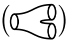
– Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. Furthermore, note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

$$[\{\alpha \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$

$$\left[\mathrm{GL}_2/D \times \left\{ \substack{\lambda,\omega^2,\nu^2 \neq 0,\pm 1 \\ \nu^2 \neq \lambda^{\pm 1}\omega^{\pm 2}} \right\}\right]_M^{S_2^\lambda} = (\mathbb{L}-5)^2(M \otimes T + (X_{2,-2} - M) \otimes N) - 2(\mathbb{L}-9)Y \otimes (T + N)$$

we obtain $(\mathbb{L}-5)^2(\mathbb{L}+1)M - (\mathbb{L}-5)^2 X_{2,-2} - 2(\mathbb{L}-9)(\mathbb{L}-1)Y$. □

Lemma 5.4.10.

Z_G^{rep} () $(X_{2,-2} \otimes X_2) = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)(I_+ + I_-)$

$$+ \mathbb{L}(\mathbb{L}-4)(\mathbb{L}-1)(J_+ + J_-)$$
$$+ (\mathbb{L}-3)^2(\mathbb{L}+1)M + (\mathbb{L}-9)X_{2,-2} - 2\mathbb{L}Y$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[X_{2,-2} \times_M X_2]I_+ = [Y]I_+ = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_+$.
- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[X_{2,-2} \times_M X_{-2}]I_- = [Y]I_+ = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_-$.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\gamma\delta = 0$. Identify the S_2^μ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. Solve for $\rho = \omega^{\pm 2}$. Hence, we obtain $2\mathbb{L}(\mathbb{L}-5)J_+$.
  - Case $\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2, and fix $\alpha = 0$ using $\mathbb{G}_a$-symmetry. Note that there are no solutions with $\rho = \omega^{\pm 2}$, and solve for $\beta = -\frac{\rho(\mu-1)(\mu+1)}{(\mu-\rho)(\mu\rho-1)}$. Since

    $$\left[\left\{ \substack{\rho,\omega^2 \neq 0,\pm 1 \\ \rho \neq \omega^{\pm 2}} \right\} /\!\!/ \, S_2^\mu\right] = (\mathbb{L}-3)^2 - (\mathbb{L}-5) = \mathbb{L}^2 - 7\mathbb{L} + 14,$$

    we obtain $\mathbb{L}(\mathbb{L}^2 - 7\mathbb{L} + 14)J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-4)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.

– Case P is (anti-)diagonal. Identify the S_2^μ-quotient with the stratum where P is diagonal. Fix $\alpha = \delta = 1$ by lifting P to GL_2 and using $\mathbb{G}_m$-symmetry. Solve for $\rho = (\lambda\omega^{-2})^{\pm 1}$, and identify the S_2^λ-quotient with the stratum where $\rho = \lambda\omega^{-2}$. From the conditions $\omega \neq 0, \pm 1, \pm i$ and $\omega^2 \neq \pm\lambda$, we obtain $(\mathbb{L} - 5)X_{2,-2} - 2Y$.

– Case P has *one* zero. Identify the S_2^μ-quotient with the stratum where $\alpha\gamma = 0$, and subsequently the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\omega^{-2})^{\pm 1}$. From the conditions $\beta \neq 0$, $\omega \neq 0, \pm 1, \pm i$ and $\omega^2 \neq \pm\lambda$, we obtain $2(\mathbb{L} - 5)(\mathbb{L} - 1)X_{2,-2} - 4(\mathbb{L} - 1)Y$.

– Case P has *no* zeros. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda - \mu\rho)(\lambda\rho - \mu)}{(\lambda\mu - \rho)(\lambda\mu\rho - 1)}$. Substituting $\alpha' = \alpha\frac{\lambda\rho - \mu}{\lambda\mu - \rho}$, we find that S_2^λ and S_2^μ act via $\alpha' \mapsto 1/\alpha'$ and $\alpha' \mapsto \alpha'$, respectively. From the $S_2^\lambda \times S_2^\mu$-virtual classes

$$[\{\alpha' \neq 0\}]^{S_2^\lambda \times S_2^\mu} = (\mathbb{L} \otimes T^\lambda - 1 \otimes N^\lambda) \otimes T^\mu$$

$$\left[\mathrm{GL}_2/D \times \left\{\begin{smallmatrix}\rho, \omega^2 \neq 0, \pm 1 \\ \rho \neq \lambda^{\pm 1}\omega^{\pm 2}\end{smallmatrix}\right\}\right]_M^{S_2^\lambda \times S_2^\mu} =$$

$$(\mathbb{L} - 3)((\mathbb{L} - 3) \otimes T^\mu - 2 \otimes N^\mu)(M \otimes T^\lambda + (X_{2,-2} - M) \otimes N^\lambda)$$
$$- ((\mathbb{L} - 5)X_{2,-2} - 2Y) \otimes (T^\lambda + N^\lambda) \otimes (T^\mu \otimes N^\mu)$$

we obtain $(\mathbb{L} - 3)^2(\mathbb{L} + 1)M - 2(\mathbb{L}^2 - 6\mathbb{L} + 7)X_{2,-2} + 2(\mathbb{L} - 1)Y$. □

Lemma 5.4.11.

$$Z_G^{\mathrm{rep}}\left(\cdot\right)(Y \otimes X_2) = 2\mathbb{L}(\mathbb{L} - 5)(\mathbb{L} + 1)(I_+ + I_-)$$
$$+ \mathbb{L}(\mathbb{L} - 5)(\mathbb{L} - 1)(J_+ + J_-)$$
$$+ (\mathbb{L} - 5)(\mathbb{L} - 3)(\mathbb{L} + 1)M$$
$$- (\mathbb{L} - 5)(\mathbb{L} - 3)X_{2,-2} + \mathbb{L}(\mathbb{L} - 9)Y$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[Y \times_M X_2]I_+ = 2[Y]I_+ = 2\mathbb{L}(\mathbb{L} - 5)(\mathbb{L} + 1)I_+$.

- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[Y \times_M X_{-2}]I_- = 2[Y]I_- = 2\mathbb{L}(\mathbb{L} - 5)(\mathbb{L} + 1)I_-$.

- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix}1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix}1 & 1 \\ 0 & 1\end{smallmatrix}\right)B^{-1}$. Consider the following cases.

– Case $\gamma\delta = 0$. Identify the S_2^μ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. Solve for $\nu^2 = \omega^{\pm 2}$, that is, $\nu = \pm\omega^{\pm 1}$. Since $\omega \neq 0, \pm 1, \pm i$, we obtain $4\mathbb{L}(\mathbb{L} - 5)J_+$.

– Case $\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2, and fix $\alpha = 0$ using $\mathbb{G}_a$-symmetry. Note that there are no solutions with $\rho = \omega^{\pm 2}$, and solve for $\beta = -\frac{\rho(\mu-1)(\mu+1)}{(\mu-\rho)(\mu\rho-1)}$. Since

$$\left[\left\{\begin{smallmatrix}\nu,\omega\neq 0,\pm 1,\pm i\\ \nu\neq\pm\omega^{\pm 1}\end{smallmatrix}\right\} /\!\!/ S_2^\mu\right] = (\mathbb{L} - 5)(\mathbb{L} - 3) - 2(\mathbb{L} - 5) = (\mathbb{L} - 5)^2,$$

we obtain $\mathbb{L}(\mathbb{L} - 5)^2 J_+$.

- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L} - 5)(\mathbb{L} - 1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right) B^{-1}$. Consider the following cases.

– Case P is (anti-)diagonal. Identify the S_2^μ-quotient with the stratum where P is diagonal. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$, and identify the S_2^λ-quotient with the stratum where $\nu^2 = \lambda\omega^{-2}$. Substitute $u = \nu\omega$ so that $u^2 = \lambda$. From the condition $\omega \neq 0, \pm 1, \pm i, \pm u, \pm iu$, we obtain $(\mathbb{L} - 9)Y$.

– Case P has *one* zero. Identify the S_2^μ-quotient with the stratum where $\alpha\gamma = 0$, and subsequently the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$. Again, substituting $u = \nu\omega^{\pm 1}$, respectively, we obtain $2(\mathbb{L} - 9)(\mathbb{L} - 1)Y$.

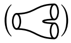
– Case P has *no* zeros. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. Substituting $\alpha' = \alpha\frac{\lambda\rho-\mu}{\lambda\mu-\rho}$, we find that S_2^λ and S_2^μ act via $\alpha' \mapsto 1/\alpha'$ and $\alpha' \mapsto \alpha'$, respectively. From the $S_2^\lambda \times S_2^\mu$-virtual classes

$$[\{\alpha' \neq 0\}]^{S_2^\lambda \times S_2^\mu} = (\mathbb{L} \otimes T^\lambda - 1 \otimes N^\lambda) \otimes T^\mu$$

$$\begin{aligned}\left[\mathrm{GL}_2/D \times \left\{\begin{smallmatrix}\nu,\omega\neq 0,\pm 1,\pm i\\ \nu^2\neq\lambda^{\pm 1}\omega^{\pm 2}\end{smallmatrix}\right\}\right]_M^{S_2^\lambda \times S_2^\mu} &= \\ (\mathbb{L} - 5)((\mathbb{L} - 3) \otimes T^\mu - 2 \otimes N^\mu)&(M \otimes T^\lambda + (X_{2,-2} - M) \otimes N^\lambda) \\ &- (\mathbb{L} - 9)Y \otimes (T^\lambda + N^\lambda) \otimes (T^\mu \otimes N^\mu)\end{aligned}$$

we obtain $(\mathbb{L}-5)(\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}-5)(\mathbb{L}-3)X_{2,-2} - (\mathbb{L}-9)(\mathbb{L}-1)Y$. $\square$

Lemma 5.4.12.

Z_G^{rep} 

$$
\begin{aligned}
(X_2 \otimes X_2) = {} & 2\mathbb{L}(\mathbb{L}^2 - 3\mathbb{L} - 2)I_+ + \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_- \\
& + \mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_+ + \mathbb{L}(\mathbb{L}-4)(\mathbb{L}-1)J_- \\
& + (\mathbb{L}-3)^2(\mathbb{L}+1)M - \mathbb{L}(\mathbb{L}-3)X_{-2} \\
& - (\mathbb{L}-3)^2 X_{2,-2} + \mathbb{L}(\mathbb{L}-6)Y
\end{aligned}
$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$. It follows that $\nu^2 = \omega^{\pm 2}$, that is, $\nu = \pm\omega^{\pm 1}$, so identify the S_2^ρ-quotient with the stratum where $\nu = \pm\omega^{-1}$. From the S_2^μ-virtual classes

$$
\begin{aligned}
[\mathrm{GL}_2/D]^{S_2^\mu} &= \mathbb{L}^2 \otimes T + \mathbb{L} \otimes N \\
\left[\left\{ \begin{smallmatrix} \omega \neq 0, \pm 1, \pm i \\ \nu \neq \pm\omega \end{smallmatrix} \right\}\right]^{S_2^\mu} &= 2((\mathbb{L}-3) \otimes T - 2 \otimes N)
\end{aligned}
$$

  we obtain $2\mathbb{L}(\mathbb{L}^2 - 3\mathbb{L} - 2)I_+$.

- If $AB = -1$, then solve for $A = -B^{-1}$. It follows that $\nu^2 = -\omega^{\pm 2}$, that is, $\nu = \pm i\omega^{\pm 1}$. Identify the S_2^ρ-quotient with the stratum where $\nu = \pm i\omega^{-1}$, and subsequently identify the S_2^μ-quotient with the stratum where $\nu = i\omega^{-1}$. We obtain $[Y]I_- = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_-$.

- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.

  - Case $\gamma\delta = 0$. Identify the S_2^μ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. It follows that $\nu^2 = \omega^{\pm 2}$, that is, $\nu = \pm\omega^{\pm 1}$, so identify the S_2^ρ-quotient with the stratum where $\nu = \pm\omega$. Since $\omega \neq 0, \pm 1, \pm i$, we obtain $2\mathbb{L}(\mathbb{L}-5)J_+$.
  - Case $\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2, and fix $\alpha = 0$ using $\mathbb{G}_a$-symmetry. Note that there are no solutions with $\rho = \omega^{\pm 2}$, and solve for $\beta = -\frac{\rho(\mu-1)(\mu+1)}{(\mu-\rho)(\mu\rho-1)}$. Since

$$
\left[\left\{ \begin{smallmatrix} \nu, \omega \neq 0, \pm 1, \pm i \\ \nu^2 \neq \omega^{\pm 2} \end{smallmatrix} \right\} /\!\!/ S_2^\mu \times S_2^\rho \right] = (\mathbb{L}-3)^2 - 2(\mathbb{L}-3) = (\mathbb{L}-5)(\mathbb{L}-3)
$$

    we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-3)J_+$.

- If $AB \in J_-$, then conjugate to $AB = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.

  - Case $\gamma\delta = 0$. Similarly to the above we obtain $2\mathbb{L}(\mathbb{L}-5)J_-$.

– Case $\gamma\delta \ne 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2, and fix $\alpha = 0$ using $\mathbb{G}_a$-symmetry. Note that there are no solutions with $\rho = -\mu^{\pm 1}$, and solve for $\beta = \frac{\rho(\mu-1)(\mu+1)}{(\mu+\rho)(\mu\rho+1)}$. Since

$$\left[\left\{ {\nu,\omega\ne 0,\pm 1,\pm i \atop \nu \ne \pm i\omega^{\pm 1}} \right\} /\!\!/ S_2^\mu \times S_2^\rho\right] = (\mathbb{L}-3)^2 - (\mathbb{L}-5) = \mathbb{L}^2 - 7\mathbb{L} + 14$$

we obtain $\mathbb{L}(\mathbb{L}^2 - 7\mathbb{L} + 14)J_-$.

- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.

  – Case P is (anti)-diagonal. Identify the S_2^μ-quotient with the stratum where P is diagonal. Fix $\alpha = \delta = 1$ by lifting P to GL_2. It follows that $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$, so identify the S_2^ρ-quotient with the stratum where $\nu^2 = \lambda\omega^{-2}$. Substitute $u = \nu\omega$ so that $u^2 = \lambda$. Since $\omega \ne 0, \pm 1, \pm i, \pm u, \pm iu$, we find

$$\left[\mathrm{GL}_2/D \times \left\{ {u^2=\lambda \atop \omega\ne 0,\pm 1,\pm i,\pm u,\pm iu} \right\}\right]_M^{S_2^\lambda} =$$
$$(X_{-2} \otimes T + (Y - X_{-2}) \otimes N)((\mathbb{L}-6) \otimes T - 3 \otimes N)$$

  so we obtain $(\mathbb{L}-3)X_{-2} - 3Y$.

  – Case P has *one* zero. Identify the S_2^μ-quotient with the stratum where $\alpha\gamma = 0$, and subsequently the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Identify the S_2^ρ-quotient with the stratum where $\nu^2 = \lambda\omega^{-2}$. Substitute $u = \nu\omega$ so that $u^2 = \lambda$ and $\omega \ne 0, \pm 1, \pm i, \pm u, \pm iu$. Furthermore, $\beta \ne 0$, so we obtain $(\mathbb{L}-9)(\mathbb{L}-1)Y$.

  – Case P has *no* zeros. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. The various S_2-actions on α are given by

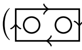
$$\alpha \overset{S_2^\lambda}{\mapsto} \alpha^{-1}, \quad \alpha \overset{S_2^\mu}{\mapsto} \beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}, \quad \alpha \overset{S_2^\rho}{\mapsto} \alpha.$$

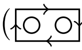
  Extending α to be $\mathbb{P}^1$-valued, we can consider this stratum as a $\mathbb{P}^1$-fibration minus the stratum where $\alpha = 0$ or $\alpha = \infty$. Since the cases $\alpha = 0$ or $\alpha = \infty$ are interchanged by S_2^λ but invariant under S_2^μ, we can effectively act as if α is invariant under S_2^μ and S_2^ρ and has S_2^λ-virtual class $[\{\alpha \ne 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$. Together with the S_2^λ-virtual class

$$\left[\mathrm{GL}_2/D \times \left\{ {\nu,\omega\ne 0,\pm 1,\pm i \atop \nu^2 \ne \lambda^{\pm 1}\omega^{\pm 2}} \right\} /\!\!/ S_2^\mu \times S_2^\rho\right]_M^{S_2^\lambda} =$$
$$(\mathbb{L}-3)^2(M \otimes T + (X_{2,-2} - M) \otimes N)$$
$$- (X_{-2} \otimes T + (Y - X_{-2}) \otimes N)((\mathbb{L}-6) \otimes T - 3 \otimes N)$$

  we obtain $(\mathbb{L}-3)^2(\mathbb{L}+1)M - (\mathbb{L}-3)(\mathbb{L}+1)X_{-2} - (\mathbb{L}-3)^2 X_{2,-2} + (4\mathbb{L}-6)Y$. □

5.5 Results

Using (4.13), Proposition 5.3.1 and the lemmas in Section 5.4, we obtain an expression for the matrix associated with Z_G^{rep} () with respect to the generators (5.3).

Z_G^{rep} () =

$$\begin{bmatrix}
2 & \mathbb{L}^2+\mathbb{L} & 2\mathbb{L}^2-2 & 0 & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} & 2\mathbb{L}^3-6\mathbb{L}^2-4\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} & 2\mathbb{L}^3-8\mathbb{L}^2-10\mathbb{L} \\
\mathbb{L}^2+\mathbb{L} & 2 & 0 & 2\mathbb{L}^2-2 & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & 2\mathbb{L}^3-6\mathbb{L}^2-4\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} & 2\mathbb{L}^3-8\mathbb{L}^2-10\mathbb{L} \\
2 & 0 & \mathbb{L}^2-\mathbb{L}-2 & 2\mathbb{L}^2 & \mathbb{L}^3-3\mathbb{L}^2 & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-\mathbb{L} & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} \\
0 & 2 & 2\mathbb{L}^2 & \mathbb{L}^2-\mathbb{L}-2 & \mathbb{L}^3-3\mathbb{L}^2 & \mathbb{L}^3-4\mathbb{L}^2-\mathbb{L} & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} \\
0 & 0 & \mathbb{L}^2-1 & \mathbb{L}^2-1 & \mathbb{L}^3-2\mathbb{L}^2-\mathbb{L}+2 & \mathbb{L}^3-3\mathbb{L}^2-\mathbb{L}+3 & \mathbb{L}^3-3\mathbb{L}^2-\mathbb{L}+3 & \mathbb{L}^3-3\mathbb{L}^2-\mathbb{L}+3 & \mathbb{L}^3-5\mathbb{L}^2-\mathbb{L}+5 \\
0 & 1 & 0 & -\mathbb{L}-1 & \mathbb{L} & -\mathbb{L}^2+\mathbb{L} & \mathbb{L}^2+\mathbb{L} & 0 & 0 \\
1 & 0 & -\mathbb{L}-1 & 0 & \mathbb{L} & \mathbb{L}^2+\mathbb{L} & -\mathbb{L}^2+\mathbb{L} & 0 & 0 \\
0 & 0 & 1-\mathbb{L} & 1-\mathbb{L} & 2\mathbb{L}-2 & -\mathbb{L}^2+4\mathbb{L}-3 & -\mathbb{L}^2+4\mathbb{L}-3 & \mathbb{L}^2+2\mathbb{L}-3 & -\mathbb{L}^2+6\mathbb{L}-5 \\
0 & 0 & \mathbb{L} & \mathbb{L} & -2\mathbb{L} & \mathbb{L}^2-4\mathbb{L} & \mathbb{L}^2-4\mathbb{L} & -2\mathbb{L} & 2\mathbb{L}^2-6\mathbb{L}
\end{bmatrix}$$

This matrix can be diagonalized with eigenvalues

$$0, \quad -\mathbb{L}(\mathbb{L}-1), \quad \mathbb{L}(\mathbb{L}-1), \quad \mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1), \quad (\mathbb{L}-1)(\mathbb{L}+1),$$
$$-\mathbb{L}(\mathbb{L}+1), \quad 2\mathbb{L}(\mathbb{L}+1), \quad 2\mathbb{L}(\mathbb{L}-1), \quad \mathbb{L}(\mathbb{L}+1)$$

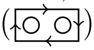
and respective eigenvectors

$$\begin{bmatrix} 0\\0\\0\\0\\2\\-1\\-1\\-1\\1 \end{bmatrix}
\begin{bmatrix} -\mathbb{L}-1\\ \mathbb{L}+1\\-1\\1\\0\\0\\0\\0\\0 \end{bmatrix}
\begin{bmatrix} -\mathbb{L}^2+4\mathbb{L}+5\\ -\mathbb{L}^2+4\mathbb{L}+5\\ 5-\mathbb{L}\\ 5-\mathbb{L}\\0\\0\\0\\0\\2 \end{bmatrix}
\begin{bmatrix} 1\\1\\1\\1\\1\\0\\0\\0\\0 \end{bmatrix}
\begin{bmatrix} \mathbb{L}\\ \mathbb{L}\\0\\0\\-1\\0\\0\\1\\0 \end{bmatrix}
\begin{bmatrix} (\mathbb{L}-1)^2\\ -(\mathbb{L}-1)^2\\ 1-\mathbb{L}\\ \mathbb{L}-1\\0\\-2\\2\\0\\0 \end{bmatrix}
\begin{bmatrix} 1-\mathbb{L}\\ \mathbb{L}-1\\1\\-1\\0\\-1\\1\\0\\0 \end{bmatrix}
\begin{bmatrix} \mathbb{L}+1\\ \mathbb{L}+1\\1\\1\\0\\0\\0\\-1\\1 \end{bmatrix}
\begin{bmatrix} -(\mathbb{L}-1)^2\\ -(\mathbb{L}-1)^2\\ \mathbb{L}-1\\ \mathbb{L}-1\\0\\-2\\-2\\0\\2 \end{bmatrix}.$$

The following theorem now follows from (4.11).

Theorem 5.5.1. *For any $r \geq 0$, the virtual class of the* SL_2*-character stack of* N_r *in* $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Stck}_k)$ *is*

$$\begin{aligned}
[\mathfrak{X}_{\mathrm{SL}_2}(N_r)] = {} & \frac{1}{4}\mathbb{L}^{r-2}(\mathbb{L}+1)^{r-2}((\mathbb{L}-1)(1+(-1)^r)-(-2)^{r+1}) \\
& + \frac{1}{4}\mathbb{L}^{r-2}(\mathbb{L}-1)^{r-2}((\mathbb{L}-1)(1+(-1)^r)+2^{r+1}-4) \\
& + (\mathbb{L}^{r-2}+1)(\mathbb{L}-1)^{r-2}(\mathbb{L}+1)^{r-2}. \qquad \square
\end{aligned}$$

Remark 5.5.2. The first eigenvector, with eigenvalue 0, corresponds to the element $2M + Y - X_2 - X_{-2} - X_{2,-2} \in \mathrm{K}_0(\mathbf{Var}_G)$. We encountered this element already in Remark 5.1.2, where it was shown to be non-zero. On the other hand, the (Hodge) monodromy representation of $M + M + Y$ agrees with that of $X_2 + X_{-2} + X_{2,-2}$, so it is not surprising to encounter this element in the kernel of Z_G^{rep} ().

Similarly, for the orientable surfaces, using (4.12), Proposition 5.2.1 and the lemmas in Section 5.4, we obtain an expression for the matrix associated with Z_G^{rep} () with respect to the same set of generators.

Z_G^{rep} () =

$$\left[\begin{smallmatrix}
\mathbb{L}^4+4\mathbb{L}^3-\mathbb{L}^2-4\mathbb{L} & \mathbb{L}^3-\mathbb{L} & \mathbb{L}^5-2\mathbb{L}^4-4\mathbb{L}^3+2\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^5+3\mathbb{L}^4-\mathbb{L}^3-3\mathbb{L}^2 & \mathbb{L}^6-2\mathbb{L}^5-4\mathbb{L}^4+3\mathbb{L}^2+2\mathbb{L} & \mathbb{L}^6-11\mathbb{L}^4-3\mathbb{L}^3+10\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-3\mathbb{L}^5-8\mathbb{L}^4+7\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-9\mathbb{L}^4-\mathbb{L}^3+8\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-\mathbb{L}^5-20\mathbb{L}^4-4\mathbb{L}^3+19\mathbb{L}^2+5\mathbb{L} \\
\mathbb{L}^3-\mathbb{L} & \mathbb{L}^4+4\mathbb{L}^3-\mathbb{L}^2-4\mathbb{L} & \mathbb{L}^5+3\mathbb{L}^4-\mathbb{L}^3-3\mathbb{L}^2 & \mathbb{L}^5-2\mathbb{L}^4-4\mathbb{L}^3+2\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-4\mathbb{L}^4+3\mathbb{L}^2+2\mathbb{L} & \mathbb{L}^6-3\mathbb{L}^5-8\mathbb{L}^4+7\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-11\mathbb{L}^4-3\mathbb{L}^3+10\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-9\mathbb{L}^4-\mathbb{L}^3+8\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-\mathbb{L}^5-20\mathbb{L}^4-4\mathbb{L}^3+19\mathbb{L}^2+5\mathbb{L} \\
\mathbb{L}^3-2\mathbb{L}^2-3\mathbb{L} & \mathbb{L}^3+3\mathbb{L}^2 & \mathbb{L}^5+\mathbb{L}^4+3\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^5-3\mathbb{L}^3-6\mathbb{L}^2 & \mathbb{L}^6-2\mathbb{L}^5-3\mathbb{L}^4+\mathbb{L}^3+3\mathbb{L}^2 & \mathbb{L}^6-3\mathbb{L}^5-4\mathbb{L}^4-3\mathbb{L}^3+9\mathbb{L}^2 & \mathbb{L}^6-3\mathbb{L}^5-\mathbb{L}^4-3\mathbb{L}^3+6\mathbb{L}^2 & \mathbb{L}^6-2\mathbb{L}^5-5\mathbb{L}^4+6\mathbb{L}^2 & \mathbb{L}^6-4\mathbb{L}^5-4\mathbb{L}^4-8\mathbb{L}^3+15\mathbb{L}^2 \\
\mathbb{L}^3+3\mathbb{L}^2 & \mathbb{L}^3-2\mathbb{L}^2-3\mathbb{L} & \mathbb{L}^5-3\mathbb{L}^3-6\mathbb{L}^2 & \mathbb{L}^5+\mathbb{L}^4+3\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-3\mathbb{L}^4+\mathbb{L}^3+3\mathbb{L}^2 & \mathbb{L}^6-3\mathbb{L}^5-\mathbb{L}^4-3\mathbb{L}^3+6\mathbb{L}^2 & \mathbb{L}^6-3\mathbb{L}^5-4\mathbb{L}^4-3\mathbb{L}^3+9\mathbb{L}^2 & \mathbb{L}^6-2\mathbb{L}^5-5\mathbb{L}^4+6\mathbb{L}^2 & \mathbb{L}^6-4\mathbb{L}^5-4\mathbb{L}^4-8\mathbb{L}^3+15\mathbb{L}^2 \\
\mathbb{L}^3-\mathbb{L}^2-\mathbb{L}+1 & \mathbb{L}^3-\mathbb{L}^2-\mathbb{L}+1 & \mathbb{L}^5-2\mathbb{L}^3+\mathbb{L} & \mathbb{L}^5-2\mathbb{L}^3+\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-2\mathbb{L}^4+2\mathbb{L}^3+3\mathbb{L}^2-2 & \mathbb{L}^6-3\mathbb{L}^5-3\mathbb{L}^4+4\mathbb{L}^3+5\mathbb{L}^2-\mathbb{L}-3 & (\mathbb{L}-1)^2(\mathbb{L}+1)\left(\mathbb{L}^3-2\mathbb{L}^2-4\mathbb{L}-3\right) & (\mathbb{L}-3)(\mathbb{L}-1)^2(\mathbb{L}+1)^3 & (\mathbb{L}-5)(\mathbb{L}-1)^2(\mathbb{L}+1)^3 \\
2\mathbb{L}^2+2\mathbb{L} & -\mathbb{L}^2-\mathbb{L} & -2\mathbb{L}^3-4\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3+2\mathbb{L}^2+\mathbb{L} & \mathbb{L}^3+\mathbb{L}^2 & 2\mathbb{L}^4+2\mathbb{L}^3 & -2\mathbb{L}^4+2\mathbb{L}^2 & 0 & 0 \\
-\mathbb{L}^2-\mathbb{L} & 2\mathbb{L}^2+2\mathbb{L} & \mathbb{L}^3+2\mathbb{L}^2+\mathbb{L} & -2\mathbb{L}^3-4\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3+\mathbb{L}^2 & -2\mathbb{L}^4+2\mathbb{L}^2 & 2\mathbb{L}^4+2\mathbb{L}^3 & 0 & 0 \\
-\mathbb{L}^2+2\mathbb{L}-1 & -\mathbb{L}^2+2\mathbb{L}-1 & -2\mathbb{L}^3+4\mathbb{L}^2-2\mathbb{L} & -2\mathbb{L}^3+4\mathbb{L}^2-2\mathbb{L} & 4\mathbb{L}^3-6\mathbb{L}^2+2 & -2\mathbb{L}^4+11\mathbb{L}^3-13\mathbb{L}^2+\mathbb{L}+3 & -2\mathbb{L}^4+11\mathbb{L}^3-13\mathbb{L}^2+\mathbb{L}+3 & \mathbb{L}^4+6\mathbb{L}^3-12\mathbb{L}^2+2\mathbb{L}+3 & -3\mathbb{L}^4+20\mathbb{L}^3-26\mathbb{L}^2+4\mathbb{L}+5 \\
\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^2-2\mathbb{L} & 2\mathbb{L}^3-2\mathbb{L}^2+2\mathbb{L} & 2\mathbb{L}^3-2\mathbb{L}^2+2\mathbb{L} & -4\mathbb{L}^3+2\mathbb{L}^2 & 2\mathbb{L}^4-11\mathbb{L}^3+7\mathbb{L}^2 & 2\mathbb{L}^4-11\mathbb{L}^3+7\mathbb{L}^2 & -6\mathbb{L}^3+6\mathbb{L}^2 & 4\mathbb{L}^4-20\mathbb{L}^3+16\mathbb{L}^2
\end{smallmatrix}\right]$$

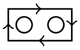
It turns out this matrix can be diagonalized using the same set of eigenvectors. The corresponding eigenvalues are

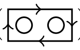
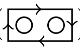
$$0,\quad \mathbb{L}^2(\mathbb{L}-1)^2,\quad \mathbb{L}^2(\mathbb{L}-1)^2,\quad \mathbb{L}^2(\mathbb{L}-1)^2(\mathbb{L}+1)^2,\quad (\mathbb{L}-1)^2(\mathbb{L}+1)^2,$$
$$\mathbb{L}^2(\mathbb{L}+1)^2,\quad 4\mathbb{L}^2(\mathbb{L}+1)^2,\quad 4\mathbb{L}^2(\mathbb{L}-1)^2,\quad \mathbb{L}^2(\mathbb{L}+1)^2.$$

The following theorem now follows from (4.10). The corresponding E-polynomials can be seen to agree with [MM16].

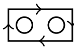
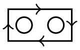
Theorem 5.5.3. *For any $g \geq 0$, the virtual class of the* SL_2*-character stack of* Σ_g *in* $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Stck}_k)$ *is*

$$\begin{aligned}[\mathfrak{X}_{\mathrm{SL}_2}(\Sigma_g)] = &\frac{1}{2}\mathbb{L}^{2g-2}(\mathbb{L}+1)^{2g-2}(2^{2g}+\mathbb{L}-1)\\ &+\frac{1}{2}\mathbb{L}^{2g-2}(\mathbb{L}-1)^{2g-2}(2^{2g}+\mathbb{L}-3)\\ &+(\mathbb{L}^{2g-2}+1)(\mathbb{L}-1)^{2g-2}(\mathbb{L}+1)^{2g-2}. \qquad \square\end{aligned}$$

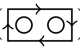
The fact that both matrices can be simultaneously diagonalized is not too surprising considering the fact that  and  commute as bordisms. Furthermore, it can be seen that

Z_G^{rep} ( $)^3 = Z_G^{\mathrm{rep}}$ ( $) \circ Z_G^{\mathrm{rep}}$ ()

which reflects the equality of bordisms

 $^3 =$  $\circ$  .

What is remarkable is that the equality

Z_G^{rep} ( $)^2 = Z_G^{\mathrm{rep}}$ ()

holds for $G = \mathrm{SL}_2$ (at least on the set of generators (5.3)), even though it does not for general G. For example, it already fails to hold for $G = \mathbb{G}_m$. Comparing Theorem 5.5.3 and Theorem 5.5.1, we find the following.

Corollary 5.5.4. $[\mathfrak{X}_{\mathrm{SL}_2}(\Sigma_g)] = [\mathfrak{X}_{\mathrm{SL}_2}(N_{2g})]$ *in* $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Stck}_k)$ *for all* $g \geq 0$. $\square$

An explanation for this relation between the orientable and non-orientable case can be given for the corresponding E-polynomials, from the point of view of the arithmetic method.

Suppose G is a linear algebraic group over a finitely generated $\mathbb{Z}$-algebra R. Comparing Theorem 4.5.3 and Proposition 4.9.12, it follows that, for any morphism $R \to \mathbb{F}_q$, the point counts $|R_G(\Sigma_g)(\mathbb{F}_q)|$ and $|R_G(N_{2g})(\mathbb{F}_q)|$ agree whenever the Frobenius–Schur indicators ε_χ of all irreducible characters χ of $G(\mathbb{F}_q)$ are equal to ± 1. That is, if all irreducible representations of $G(\mathbb{F}_q)$ are either real or pseudoreal.

Indeed, if we take $G = \mathrm{SL}_2$ and $R = \mathbb{Z}[1/2, i]$, then a map $R \to \mathbb{F}_q$ exists if and only if $q \equiv 1 \bmod 4$. For such q, any element of $\mathrm{SL}_2(\mathbb{F}_q)$ is conjugate to its inverse, and hence

$$\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$$

for all $g \in \mathrm{SL}_2(\mathbb{F}_q)$ and irreducible characters χ of $\mathrm{SL}_2(\mathbb{F}_q)$. This shows that all irreducible characters χ of $\mathrm{SL}_2(\mathbb{F}_q)$, with $q \equiv 1 \bmod 4$, are either real or pseudoreal, that is, $\varepsilon_\chi = \pm 1$, and hence

$$|R_{\mathrm{SL}_2}(\Sigma_g)(\mathbb{F}_q)| = |R_{\mathrm{SL}_2}(N_{2g})(\mathbb{F}_q)|.$$

Since these numbers are polynomial in q, it follows from Theorem 4.6.1 (Katz' theorem) that $e(R_{\mathrm{SL}_2}(\Sigma_g)) = e(R_{\mathrm{SL}_2}(N_{2g}))$, and in turn that $e(\mathfrak{X}_{\mathrm{SL}_2}(\Sigma_g)) = e(\mathfrak{X}_{\mathrm{SL}_2}(N_{2g}))$.