Chapter 4

Topological Quantum Field Theories

The aim of this chapter is to study motivic invariants of character stacks associated to closed manifolds. One of the first approaches in this direction was by Hausel and Rodriguez-Villegas [HR08], whose idea was to study the G-representation variety by counting the number of points over finite fields $\mathbb{F}_q$. They could express these counts in terms of the representation theory of the finite groups $G(\mathbb{F}_q)$, and moreover, determine from these counts the E-polynomial of the G-representation variety. We call this approach the *arithmetic method.*

A few years later, Logares, Muñoz and Newstead [LMN13] initiated the *geometric method*, a geometric approach to compute the same invariant, making use of clever stratifications of the G-representation variety. González-Prieto, Logares and Muñoz [GLM20] showed that the geometric method can be phrased in terms of a *Topological Quantum Field Theory (TQFT).*

TQFTs, originating from physics, describe the topological aspects of a quantum field theory. Atiyah [Ati88] was the first to mathematically axiomatize the notion of a TQFT, defining a TQFT as a monoidal functor from the category of bordisms to the category of vector spaces. The idea that TQFTs can be used to compute invariants of geometric objects is not a new idea. For instance, Witten, in his seminal paper [Wit89], constructed a TQFT that computes the Jones polynomial of knots.

In this chapter, we will describe both the arithmetic and geometric method, and show how they can be unified using the framework of TQFTs. Specifically, we will show that both methods can be formulated as TQFTs, and that these TQFTs can be related through natural transformations.

4.1 Monoidal categories

Central to the theory of TQFTs is the notion of a *monoidal category*. Monoidal categories were defined by Mac Lane [Mac63] under the name 'bicategory', and by Bénabou [Bén63] under the name 'categories with multiplication'.

Definition 4.1.1. A *monoidal category* is a category $\mathcal{C}$ with a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *tensor product*, an object 1 in $\mathcal{C}$, called the *unit object*, and natural isomorphisms

$$\begin{array}{ccc} \alpha \colon - \otimes (- \otimes -) \Rightarrow (- \otimes -) \otimes - & \lambda \colon 1 \otimes - \Rightarrow \mathrm{id}_{\mathcal{C}} & \rho \colon - \otimes 1 \Rightarrow \mathrm{id}_{\mathcal{C}} \\ \text{(the } \textit{associator}\text{)} & \text{(the } \textit{left unitor}\text{)} & \text{(the } \textit{right unitor}\text{)} \end{array}$$

such that the triangle

$$\begin{array}{ccc} (X \otimes 1) \otimes Y & \xrightarrow{\alpha_{X,1,Y}} & X \otimes (1 \otimes Y) \\ & {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} & \\ & X \otimes Y & \end{array}$$

and the pentagon

$$\begin{array}{ccccc} & & (X \otimes Y) \otimes (Z \otimes W) & & \\ & \nearrow^{\alpha_{X \otimes Y, Z, W}} & & \searrow^{\alpha_{X, Y, Z \otimes W}} & \\ ((X \otimes Y) \otimes Z) \otimes W & & & & X \otimes (Y \otimes (Z \otimes W)) \\ \downarrow {\scriptstyle \alpha_{X,Y,Z} \otimes \mathrm{id}_W} & & & & \uparrow {\scriptstyle \mathrm{id}_X \otimes \alpha_{Y,Z,W}} \\ (X \otimes (Y \otimes Z)) \otimes W & & \xrightarrow{\alpha_{X, Y \otimes Z, W}} & & X \otimes ((Y \otimes Z) \otimes W) \end{array}$$

commute for all objects X, Y, Z and W in $\mathcal{C}$. A *symmetric monoidal category* is a monoidal category $\mathcal{C}$ together with natural isomorphisms

$$\tau_{X,Y} \colon X \otimes Y \to Y \otimes X$$

such that

$$\tau_{Y,X} \circ \tau_{X,Y} = \mathrm{id}_{X \otimes Y}$$

and the diagrams

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{\tau_{X, Y \otimes Z}} & (Y \otimes Z) \otimes X \\ {\scriptstyle \tau_{X,Y} \otimes \mathrm{id}_Z} \downarrow & & & & \downarrow {\scriptstyle \alpha_{Y,Z,X}} \\ (Y \otimes X) \otimes Z & \xrightarrow{\alpha_{Y,X,Z}} & Y \otimes (X \otimes Z) Y & \xrightarrow{\mathrm{id}_Y \otimes \tau_{X,Z}} & Y \otimes (Z \otimes X) \end{array}$$

and

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\alpha^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\tau_{X \otimes Y, Z}} & Z \otimes (X \otimes Y) \\ {\scriptstyle \mathrm{id}_X \otimes \tau_{Y,Z}} \downarrow & & & & \downarrow {\scriptstyle \alpha^{-1}_{Z,X,Y}} \\ X \otimes (Y \otimes Z) & \xrightarrow{\alpha^{-1}_{X,Z,Y}} & (X \otimes Z) \otimes Y & \xrightarrow{\tau_{X,Z} \otimes \mathrm{id}_Y} & (Z \otimes X) \otimes Y \end{array}$$

commute for all X, Y and Z in $\mathcal{C}$.

Example 4.1.2. A basic example of a monoidal category is the category **Set** with tensor product $\times$ and unit object $\{1\}$. More generally, any category $\mathcal{C}$ with finite products can naturally be promoted to a monoidal category with tensor product $\times$ and unit object a terminal object. Such a monoidal category is called a *cartesian monoidal category*. Dually, a category with finite coproducts can be promoted to a monoidal category with tensor product $\sqcup$ and unit object an initial object, which is called a *cocartesian monoidal category*. For example, **Set** with $\sqcup$ and $\varnothing$, or the category of R-algebras $\mathbf{Alg}_R$ with $\otimes_R$ and R, for a commutative ring R. Another typical example of a monoidal category is the category of R-modules $\mathbf{Mod}_R$ with tensor product $\otimes_R$ and unit object R. All of these examples are naturally also symmetric monoidal categories.

Definition 4.1.3. A *monoidal functor* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ between monoidal categories together with a natural isomorphism

$$\mu\colon F(-) \otimes_{\mathcal{D}} F(-) \Rightarrow F(- \otimes_{\mathcal{C}} -)$$

and an isomorphism $\varepsilon\colon 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$, such that the diagrams

$$\begin{array}{ccc}
(F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) & \xrightarrow{\alpha^{\mathcal{D}}_{F(X),F(Y),F(Z)}} & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \\
\downarrow{\scriptstyle \mu_{X,Y} \otimes \mathrm{id}_{F(Z)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(X)} \otimes \mu_{Y,Z}} \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) & & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \\
\downarrow{\scriptstyle \mu_{X \otimes_{\mathcal{C}} Y, Z}} & & \downarrow{\scriptstyle \mu_{X, Y \otimes_{\mathcal{C}} Z}} \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{F(\alpha^{\mathcal{C}}_{X,Y,Z})} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z))
\end{array}$$

$$\begin{array}{ccc}
1_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) & \xrightarrow{\varepsilon \otimes \mathrm{id}_{F(X)}} & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(X) \\
\downarrow{\scriptstyle \lambda^{\mathcal{D}}_{F(X)}} & & \downarrow{\scriptstyle \mu_{1_{\mathcal{C}},X}} \\
F(X) & \xleftarrow[F(\lambda^{\mathcal{C}}_X)]{} & F(1_{\mathcal{C}} \otimes_{\mathcal{C}} X)
\end{array}
\qquad
\begin{array}{ccc}
F(X) \otimes_{\mathcal{D}} 1_{\mathcal{D}} & \xrightarrow{\mathrm{id}_{F(X)} \otimes \varepsilon} & F(X) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \\
\downarrow{\scriptstyle \rho^{\mathcal{D}}_{F(X)}} & & \downarrow{\scriptstyle \mu_{X,1_{\mathcal{C}}}} \\
F(X) & \xleftarrow[F(\rho^{\mathcal{C}}_X)]{} & F(X \otimes_{\mathcal{C}} 1_{\mathcal{C}})
\end{array}$$

commute for all objects X, Y and Z in $\mathcal{C}$. If μ is only a natural transformation, and ε only a morphism, then such a functor is called a *lax monoidal functor*. A (lax) monoidal functor between symmetric monoidal categories is said to be *symmetric* if it respects the symmetric structure, that is, the diagram

$$\begin{array}{ccc}
F(X) \otimes_{\mathcal{D}} F(Y) & \xrightarrow{\tau^{\mathcal{D}}_{F(X),F(Y)}} & F(Y) \otimes_{\mathcal{D}} F(X) \\
\downarrow{\scriptstyle \mu_{X,Y}} & & \downarrow{\scriptstyle \mu_{Y,X}} \\
F(X \otimes_{\mathcal{C}} Y) & \xrightarrow{F(\tau^{\mathcal{C}}_{X,Y})} & F(Y \otimes_{\mathcal{C}} X)
\end{array}$$

commutes for all X and Y in $\mathcal{C}$.

4.2 Bordisms

The monoidal category that is central in the theory of TQFTs is the *category of bordisms*. By convention, we consider all manifolds to be smooth.

Definition 4.2.1. Let $n \geq 1$. Given two closed $(n-1)$-dimensional manifolds M_1 and M_2, a *bordism* from M_1 to M_2 is a compact n-dimensional manifold W with boundary ∂W together with inclusions

$$M_2 \xrightarrow{i_2} W \xleftarrow{i_1} M_1$$

such that $\partial W = i_1(M_1) \sqcup i_2(M_2)$.

Definition 4.2.2. The *category of n-bordisms*, denoted $\mathbf{Bord}_n$, is the category defined as follows.

- Its objects are closed $(n-1)$-dimensional manifolds.
- A morphism $M_1 \to M_2$ is an equivalence class of bordisms from M_1 to M_2, where two such bordisms W and W' are called equivalent if there is a diffeomorphism $f \colon W \to W'$ such that the diagram

$$\begin{array}{ccccc} & \overset{i_2}{\nearrow} & W & \overset{i_1}{\nwarrow} & \\ M_2 & & \downarrow f & & M_1 \\ & \underset{i'_2}{\searrow} & W' & \underset{i'_1}{\swarrow} & \end{array} \tag{4.1}$$

  commutes. We will also refer to such equivalences classes as *bordisms*, with the understanding that it is only up to diffeomorphism.
- The composite of morphisms $W \colon M_1 \to M_2$ and $W' \colon M_2 \to M_3$ is given by $W \sqcup_{M_2} W' \colon M_1 \to M_3$. While this operation is not well-defined on bordisms (there can be multiple manifold structures on $W \sqcup_{M_2} W'$ such that the inclusions of W and W' are smooth), such a structure is unique up to diffeomorphism, making it a well-defined operation on equivalence classes of bordisms [Mil65].
- For any closed $(n-1)$-dimensional manifold M, the identity on M is given by (the equivalence class of) the cylinder $M \times [0,1]$, with the inclusions $M \times \{0\} \to M \times [0,1] \leftarrow M \times \{1\}$.

The category $\mathbf{Bord}_n$ naturally carries the structure of a symmetric monoidal category, whose tensor product is the disjoint union operator and whose unital object is the empty manifold $\varnothing$.

Definition 4.2.3. Let R be a commutative ring. An *n-dimensional Topological Quantum Field Theory (TQFT)* over R is a monoidal functor

$$Z\colon \mathbf{Bord}_n \to \mathbf{Mod}_R$$

where $\mathbf{Mod}_R$ is monoidal with tensor product $\otimes_R$ and unit object R. If such a functor is only lax monoidal it is called a *lax n-dimensional TQFT*, and similarly if it is symmetric.

Interestingly, observe that $Z(\varnothing)$ is by definition naturally isomorphic to R for any TQFT $Z\colon \mathbf{Bord}_n \to \mathbf{Mod}_R$. Hence, any closed n-dimensional manifold W, viewed as a bordism $W\colon \varnothing \to \varnothing$, induces a morphism $Z(W)\colon R \to R$ that is multiplication by $Z(W)(1) \in R$. The element $Z(W)(1)$ is an invariant associated to W, that is, it is the same for all W' diffeomorphic to W.

Definition 4.2.4. Let χ be an R-valued invariant of closed n-dimensional manifolds. An n-dimensional TQFT Z is said to *quantize* χ if $Z(W)(1) = \chi(W)$ for all closed n-dimensional manifolds W.

There are many variations on the category of bordisms, by equipping the manifolds with extra data. One common is to equip them with an orientation.

Definition 4.2.5. Let $i\colon M \to \partial W$ be an embedding of a closed oriented $(n-1)$-dimensional manifold M into the boundary of a compact oriented n-dimensional manifold W. Then i is said to be an *in-boundary* (resp. *out-boundary*) if for all $x \in M$, positively oriented bases $v_1, \ldots, v_{n-1}$ for $T_x M$, and $w \in T_{i(x)}W$ pointing inwards (resp. outwards) compared to W, the basis $di_x(v_1), \ldots, di_x(v_{n-1}), w$ for $T_{i(x)}W$ is positively oriented.

Given two closed oriented $(n-1)$-dimensional manifolds M_1 and M_2, an *oriented bordism* from M_1 to M_2 is a bordism

$$M_2 \xrightarrow{i_2} W \xleftarrow{i_1} M_1$$

with an orientation on W such that i_1 an *in-boundary* and i_2 is an *out-boundary.*

The *category of oriented n-bordisms*, denoted $\mathbf{Bord}_n^{\mathrm{or}}$, is the category whose objects are closed oriented $(n-1)$-dimensional manifolds, and morphisms are equivalence classes of oriented bordisms, with composition given as for $\mathbf{Bord}_n$.

Finally, an *n-dimensional oriented TQFT* over R is a monoidal functor

$$Z\colon \mathbf{Bord}_n^{\mathrm{or}} \to \mathbf{Mod}_R.$$

While any TQFT induces an oriented TQFT, simply by forgetting the orientation, not every oriented TQFT can be extended to a TQFT, as will be shown in Section 4.4.

Remark 4.2.6. Although in Definition 4.2.2 the category of bordisms $\mathbf{Bord}_n$ was defined as a 1-category, it can naturally be promoted to a 2-category: the objects still being closed $(n-1)$-dimensional manifolds, a 1-morphism being a bordism (rather than an equivalence class of bordisms), and a 2-morphism between bordisms being an equivalence as in (4.1). Now, if we view $\mathbf{Bord}_n$ as a 2-category, it is only natural to promote $\mathbf{Mod}_R$ to a 2-category as well and require a TQFT to be a 2-functor. We will promote $\mathbf{Mod}_R$ in the trivial way, where the only 2-morphisms are identity morphisms. Note that, in essence, this does not change the definition of a TQFT, since two bordisms which are equivalent must be sent to the same R-linear map. Therefore, in the context of TQFTs as in Definition 4.2.3, it does not matter whether we view $\mathbf{Bord}_n$ as a 2-category or simply as a 1-category.

For the correct notions of monoidal categories and monoidal functors in the context of 2-categories, see [KV94, BN96].

4.3 Physical interpretation

As mentioned, the notion of a TQFT originates from physics, and was first mathematically axiomatized by Atiyah [Ati88]. While not strictly necessary, we believe it is helpful to discuss the physical interpretation of these objects for a better intuition of the remainder of this chapter.

A TQFT describes a quantum mechanical system, specifically a quantum field theory. Space, at some point in time, is represented by a closed manifold, that is, an object of $\mathbf{Bord}_n$. A morphism in this category, a bordism connecting such manifolds, represents a part of spacetime, where the extra dimension corresponds to the dimension of time.

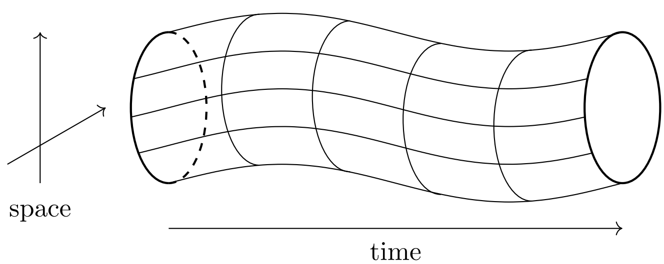


Figure 4.1: A bordism connecting two boundaries, representing a part of spacetime connecting space at two points in time.

For simplicity, let us take $R = \mathbb{C}$. Then a TQFT assigns to a space M a complex vector space $\mathcal{H} = Z(M)$ and to a spacetime $W\colon M_1 \to M_2$ a linear map $Z(W)\colon \mathcal{H}_1 \to \mathcal{H}_2$. We can think of the vector space $\mathcal{H}$ as the Hilbert space associated to M, that is, the vector space of all quantum states on this space. The linear map $Z(W)$ describes the time-evolution of the system.

What makes a TQFT '*topological*', is that the system it describes has no actual dynamics. That is, the Hamiltonian of the system is zero, and only topological effects come into play. For instance, the cylinder $M \times [0, 1]$, being topologically trivial, induces the identity on $\mathcal{H}$, and hence does not change the state of the system.

It is not uncommon for Hilbert spaces to be infinite-dimensional, and in this case the tensor product of Hilbert spaces is not simply the tensor product of the underlying vector spaces: it should be completed. For this reason, the Hilbert space $\mathcal{H}$ associated to a disjoint union $M_1 \sqcup M_2$ is not necessarily expected to be equal to the tensor product (as vector spaces) of the Hilbert spaces associated to M_1 and M_2, but at least there should be a natural morphism $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2 \to \mathcal{H}$. Although this might be an indication that the category of vector spaces is not quite the correct target for a TQFT, we will take it as motivation for the definition of a lax TQFT.

A common way to construct a TQFT is as the composite of two functors, a *field theory* and a *quantization functor*. This corresponds to describing a classical field theory, followed by a quantization procedure. The field theory $\mathcal{F}$ assigns to a manifold M a *phase space* $\mathcal{F}(M)$: a geometric object parametrizing all possible classical states, or field configuration, of the system on M. For simplicity, we can think of such a state or field as a vector bundle or a local system on M. Note that, given a bordism $W\colon M_1 \to M_2$, a field over W can be restricted to a field over any of the boundaries. In particular, we obtain the following diagram.

$$\mathcal{F}(M_1) \xleftarrow{\;i_1^*\;} \mathcal{F}(W) \xrightarrow{\;i_2^*\;} \mathcal{F}(M_2)$$

Such a diagram is known as a *correspondence* from $\mathcal{F}(M_1)$ to $\mathcal{F}(M_2)$. The field theory $\mathcal{F}$ should therefore be a functor from $\mathbf{Bord}_n$ to the category of correspondences, whose objects are some kind of geometric objects and whose morphisms are correspondences between them. Now, let us consider what happens to a composite of bordisms. Two bordisms $W\colon M_1 \to M_2$ and $W' : M_2 \to M_3$ induce the following diagram.

$$\begin{array}{ccccccc} & & & \mathcal{F}(W \sqcup_{M_2} W') & & & \\ & & \swarrow & & \searrow & & \\ & \mathcal{F}(W) & & & & \mathcal{F}(W') & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \mathcal{F}(M_1) & & & \mathcal{F}(M_2) & & & \mathcal{F}(M_3) \end{array}$$

A field over $W \sqcup_{M_2} W'$ is essentially the same as a field over W and a field over W' that agree over M_2, so the middle square will be cartesian. This is precisely how the composition of correspondences is defined, so $\mathcal{F}$ will indeed be a functor.

Next, let us consider the quantization functor $\mathcal{Q}$. This functor assigns to the phase space $\mathcal{F}(M)$ a complex vector space, its Hilbert space, whose vectors represent the *quantum states* of the system on M. For simplicity, we can think of a quantum state as a complex-valued function (a *wave function*) on $\mathcal{F}(M)$, which describes a distribution or superposition of classical states. Furthermore, on correspondences, $\mathcal{Q}$ is commonly given by a 'pull-push' construction. Given a quantum state $\psi_1 \in \mathcal{Q}(\mathcal{F}(M_1))$, that is, a complex-valued function on $\mathcal{F}(M_1)$, one can pull back ψ_1 along i_1^* to obtain $\Psi = \psi_1 \circ i_1^* \in \mathcal{Q}(\mathcal{F}(W))$, a complex-valued function on $\mathcal{F}(W)$. Next, one can push forward Ψ along i_2^*, by integrating along fibers, to obtain $\psi_2 \in \mathcal{Q}(\mathcal{F}(M_2))$ given by

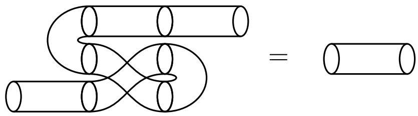
$$\psi_2(y) = \int_{(i_2^*)^{-1}(y)} \Psi(x)\, dx,$$

provided such an integral exists. The resulting map $\mathcal{Q}(\mathcal{F}(M_1)) \to \mathcal{Q}(\mathcal{F}(M_2))$ corresponds, roughly speaking, to a *(Feynman) path integral*. That is, the amplitude corresponding to state y is determined by considering all possible paths to state y (points on $\mathcal{F}(W)$ over y), and their amplitudes are added.

More generally, one could replace the above integral by a weighted integral with weight $e^{iS(x)}$, where S is a function on $\mathcal{F}(W)$ called the *action*. For TQFTs there is no such weighting, since only the topology of the bordisms is considered, and no other extra data. However, equipping the bordisms with extra data can result in QFTs with non-trivial actions. For example, one obtains a conformal field theory by equipping the bordisms with a conformal structure.

4.4 Low-dimensional TQFTs

Let us discuss some properties of an oriented TQFT $Z \colon \mathbf{Bord}_n^{\mathrm{or}} \to \mathbf{Mod}_R$. Given a closed oriented $(n-1)$-dimensional manifold M, denote by $\overline{M}$ the same manifold but with opposite orientation, and by $U_M : \varnothing \to M \sqcup \overline{M}$ and $U_M^\dagger : M \sqcup \overline{M} \to \varnothing$ be the bordisms with underlying manifold $M \times [0, 1]$. Note that the map $Z(U_M)\colon R \to Z(M) \otimes_R Z(\overline{M})$ is completely determined by the element $Z(U_M)(1) = \sum_{i=1}^m v_i \otimes \overline{v}_i$ with $v_i \in Z(M)$ and $\overline{v}_i \in Z(\overline{M})$. From the equality $(U_M^\dagger \sqcup \mathrm{id}_M) \circ (\mathrm{id}_M \sqcup \tau_{M,\overline{M}}) \circ (\mathrm{id}_M \sqcup U_M) = \mathrm{id}_M$, depicted pictorially as



it follows that

$$v = \sum_{i=1}^m Z(U_M^\dagger)(v \otimes \overline{v}_i)\, v_i$$

for all $v \in Z(M)$. In particular, $Z(M)$ is generated by $v_1, \dots, v_m$. Similarly, from the equality $(U_M^\dagger \sqcup \mathrm{id}_{\overline{M}}) \circ (\tau_{\overline{M},M} \sqcup \mathrm{id}_{\overline{M}}) \circ (\mathrm{id}_{\overline{M}} \sqcup U_M) = \mathrm{id}_{\overline{M}}$ it follows that

$$\overline{v} = \sum_{i=1}^m Z(U_M^\dagger)(v_i \otimes \overline{v})\, \overline{v}_i$$

for all $\overline{v} \in Z(\overline{M})$, so $Z(\overline{M})$ is generated by $\overline{v}_1, \dots, \overline{v}_m$. This shows that $Z(M)$ is a *dualizable object* with dual $Z(\overline{M})$, unit $Z(U_M)$ and counit $Z(U_M^\dagger)$.

Remark 4.4.1. If Z is only a lax TQFT, the image of $Z(U_M)$ need not necessarily lie in the tensor product $Z(M) \otimes Z(\overline{M})$, and consequently, the module $Z(M)$ need not be finitely generated. This will be the case for the TQFT constructed in Section 4.7.

In the category of R-modules, the dualizable objects are precisely the finitely generated projective modules [PS14]. In dimension $n = 1$, this completely characterizes the TQFT.

Proposition 4.4.2. *Let R be a commutative ring. There is an equivalence of categories*

$$\mathbf{1\text{-}TQFT}_R^{\mathrm{or}} \simeq \mathbf{FGProjMod}_R$$

between the category of 1-dimensional oriented TQFTs over R and the category of finitely generated projective R-modules, which assigns to a TQFT Z the R-module $Z(p)$, where p is the point with orientation $+1$.

Proof. As shown above, an oriented 1-TQFT Z over R determines a dualizable (that is, finitely generated projective) R-module $M = Z(p)$. From a morphism between such TQFTs (a natural transformation) we obtain a morphism between the corresponding modules. This gives a functor $\mathbf{1\text{-}TQFT}_R^{\mathrm{or}} \to \mathbf{FGProjMod}_R$.

Conversely, let M be a dualizable R-module. The objects of $\mathbf{Bord}_1^{\mathrm{or}}$ are finite disjoint unions of p and $\overline{p}$. As shown above, $Z(\overline{p})$ is dual to $Z(p)$, so by monoidality, specifying $Z(p) = M$ determines Z on objects, with $Z(\overline{p}) = \mathrm{Hom}_R(M, R)$. The only connected bordisms in $\mathbf{Bord}_1^{\mathrm{or}}$ are

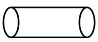
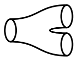
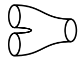
$$\mathrm{id}_p \colon p \to p, \quad \mathrm{id}_{\overline{p}} \colon \overline{p} \to \overline{p}$$

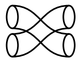
$$U_p \colon \varnothing \to p \sqcup \overline{p}, \quad U_p^\dagger \colon p \sqcup \overline{p} \to \varnothing$$

and $S^1 = U_p^\dagger \circ U_p \colon \varnothing \to \varnothing$, all of whose image under Z is canonically determined by the unit and counit of the dualizable module M. This construction, being natural in M, defines a functor $\mathbf{FGProjMod}_R \to \mathbf{1\text{-}TQFT}_R^{\mathrm{or}}$.

These functors are easily seen to be pseudo-inverses of each other, establishing the equivalence of categories. ☐

A similar characterization of oriented TQFTs can be given in dimension $n = 2$. The objects of $\mathbf{Bord}_2^{\mathrm{or}}$ are disjoint unions of S^1, the circle, where we fix an orientation of S^1. Using the classification of oriented surfaces, one can show the morphisms in $\mathbf{Bord}_2^{\mathrm{or}}$ are 'generated' by the following bordisms:

 $: S^1 \to S^1$,  $: S^1 \sqcup S^1 \to S^1$,  $: S^1 \to S^1 \sqcup S^1$,

 $: S^1 \to \varnothing$,  $: \varnothing \to S^1$ and  $: S^1 \sqcup S^1 \to S^1 \sqcup S^1$. (4.2)

That is, any bordism in $\mathbf{Bord}_2^{\mathrm{or}}$ is isomorphic to a composite of disjoint unions of these bordisms [Koc04, Proposition 1.4.13]. Hence, we expect 2-dimensional oriented TQFTs to correspond to dualizable modules with some extra algebraic structure. For $R = k$ a field, the correct algebraic structure turns out to be that of a *Frobenius algebra*.

Definition 4.4.3. A *Frobenius algebra* over a field k is an algebra A over k, whose multiplication and unit we denote by $\mu \colon A \otimes_k A \to A$ and $\eta \colon k \to A$, equipped with a bilinear form $\beta \colon A \otimes_k A \to k$, which is

- associative, that is, $\beta(\mu(a \otimes b) \otimes c) = \beta(a \otimes \mu(b \otimes c))$ for all $a, b, c \in A$,
- non-degenerate, that is, there exists a k-linear map $\gamma \colon k \to A \otimes_k A$ such that $(\beta \otimes \mathrm{id}_A)(a \otimes \gamma(1)) = a = (\mathrm{id}_A \otimes \beta)(\gamma(1) \otimes a)$ for all $a \in A$.

Remark 4.4.4. For any Frobenius algebra A, writing $\gamma(1) = \sum_i a_i \otimes b_i$ for some $a_i, b_i \in A$, we find that $a = (\mathrm{id}_A \otimes \beta)(\sum_i a_i \otimes b_i \otimes a) = \sum_i a_i \beta(b_i \otimes a)$ for all $a \in A$. In particular, A is finite-dimensional and generated by the a_i. This equality also shows that β is non-degenerate in the usual sense: if $\beta(b \otimes a) = 0$ for all $b \in A$, then $a = 0$. Similarly, one shows non-degeneracy in the other argument. Furthermore, this implies γ must be unique. Namely, if γ and γ' both satisfy the condition, then write $\gamma(1)-\gamma'(1)=\sum_i a_i\otimes b_i$ with $a_i, b_i\in A$ and the a_i are linearly independent. Since $0=(\mathrm{id}_A\otimes\beta)(\sum_i a_i\otimes b_i\otimes a)=\sum_i a_i\,\beta(b_i\otimes a)$ for all $a\in A$, it follows that $b_i=0$ for all i, so $\gamma(1)=\gamma'(1)$.

A Frobenius algebra naturally carries the structure of a k-coalgebra, see [Koc04, Section 2.3], where the comultiplication δ and counit ε are given by

$$\delta=(\mu\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\gamma)\quad\text{and}\quad\varepsilon=\beta\circ(\mathrm{id}_A\otimes\eta).\tag{4.3}$$

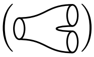
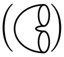
A *morphism of Frobenius algebras* is a morphism of k-algebras which is also a morphism of k-coalgebras.

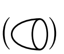
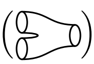
The following theorem makes a precise correspondence between 2-dimensional oriented TQFTs and Frobenius algebras. It was initially proved by Dijkgraaf [Dij89], and later reproved in more detail by others, such as [Abr96, Koc04].

Theorem 4.4.5. *Let k be a field. There is an equivalence of categories*

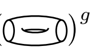
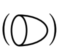
$$\mathbf{2\text{-}TQFT}_k^{\mathrm{or}}\simeq\mathbf{CFrobAlg}_k$$

between the category of 2-dimensional oriented TQFTs over k and the category of commutative Frobenius algebras over k, which assigns to a TQFT Z the Frobenius algebra $A=Z(S^1)$ and

$Z\big($  $\big)=\eta$, $Z\big($  $\big)=\mu$, $Z\big($  $\big)=\beta$,

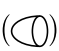
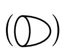
$Z\big($  $\big)=\varepsilon$, $Z\big($  $\big)=\delta$, $Z\big($  $\big)=\gamma$.

Example 4.4.6. Let S be a finite set and let $A=k^S$ be the k-algebra of k-valued functions on S, where multiplication is given pointwise. Then A admits the structure of a Frobenius algebra with $\beta(f\otimes g)(s)=\sum_{s\in S}f(s)g(s)$ and $\gamma(1)=\sum_{s\in S}\mathbb{1}_s\otimes\mathbb{1}_s$, where $\mathbb{1}_s$ denotes the indicator function. From (4.3), we find that the coalgebra structure is given by $\varepsilon(f)=\sum_{s\in S}f(s)$ and $\delta(f)=\sum_{s\in S}f(s)\mathbb{1}_s\otimes\mathbb{1}_s$. For any closed surface Σ_g of genus g, we have

$Z(\Sigma_g)(1)=Z\big($  $\big)\circ Z\big($  $\big)^g\circ Z\big($  $\big)=(\varepsilon\circ(\mu\circ\delta)^g\circ\eta)(1)=|S|.$

Therefore, $Z(M)(1)=|S|^{\pi_0(M)}$ for any general closed oriented surface M, that is, Z quantizes the number of connected components of M.

Example 4.4.7. The complex numbers $A=\mathbb{C}$ are a Frobenius algebra over $k=\mathbb{R}$ with $\beta(z_1\otimes z_2)=\mathrm{Re}(z_1z_2)$ and $\gamma(1)=1\otimes1-i\otimes i$. One quickly finds that $\delta(z)=z\otimes1-iz\otimes i$ and $\varepsilon(z)=\mathrm{Re}(z)$. The corresponding TQFT yields

$Z(\Sigma_g)(1)=Z\big($  $\big)\circ Z\big($  $\big)^g\circ Z\big($  $\big)=(\varepsilon\circ(\mu\circ\delta)^g\circ\eta)(1)=2^g.$

In this sense, this TQFT quantizes the genus of the surface.

Remark 4.4.8. Let us make a note about the difference between oriented and unoriented TQFTs. Clearly, via the forgetful functor $\mathbf{Bord}_n^{\mathrm{or}} \to \mathbf{Bord}_n$, which forgets the orientation, any TQFT induces an oriented TQFT. In this sense, an unoriented TQFT can be seen as an oriented TQFT with extra structure. However, not every oriented TQFT arises in such a way. This follows from Proposition 4.4.2 and the fact that not every finitely generated projective module (that is, dualizable module) is isomorphic to its dual.

4.5 Representation ring as TQFT

Let G be a finite group, and denote by $A = R_{\mathbb{C}}(G)$ the representation ring of G, that is, the complex algebra generated by $\mathbb{C}$-valued class functions on G. Of great importance in the representation theory of G is the inner product that is defined on A, which we will denote by $\beta\colon A \otimes_{\mathbb{C}} A \to \mathbb{C}$, and which is given by

$$\beta(a \otimes b) = \frac{1}{|G|} \sum_{g \in G} a(g) b(g^{-1}) \quad \text{for } a, b \in A.$$

A lesser known but equally important operation on A is the convolution operation $\mu\colon A \otimes_{\mathbb{C}} A \to A$ on A, which is given by

$$\mu(a \otimes b)(g) = \sum_{h \in G} a(h) b(h^{-1} g) \quad \text{for } a, b \in A,$$

and is related to the inner product via $\beta(a \otimes b) = \mu(a \otimes b)(1)$ for $a, b \in A$. The unit $\eta\colon \mathbb{C} \to A$ with respect to μ is given by $\eta(1)(1) = 1$ and $\eta(1)(g) = 0$ for $g \neq 1$. Alternatively, η can be expressed as

$$\eta(1) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(1) \chi, \tag{4.4}$$

where $\hat{G}$ denotes the set of irreducible complex characters of G. These operations give $R_{\mathbb{C}}(G)$ the structure of a commutative Frobenius algebra over $\mathbb{C}$.

Proposition 4.5.1. *The representation ring $R_{\mathbb{C}}(G)$ is a commutative Frobenius algebra over $\mathbb{C}$ with multiplication μ and bilinear form β.*

Proof. First note that μ is associative as

$$\begin{aligned} \mu(a \otimes \mu(b \otimes c))(g) &= \sum_{h_1, h_2 \in G} a(h_1) b(h_2) c(h_2^{-1} h_1^{-1} g) \\ &= \sum_{h_1', h_2' \in G} a(h_2') b(h_2'^{-1} h_1') c(h_1'^{-1} g) = \mu(\mu(a \otimes b) \otimes c)(g) \end{aligned}$$

for all $a, b, c \in A$ and $g \in G$, where $h_1' = h_1 h_2$ and $h_2' = h_1$, and μ is commutative as

$$\mu(a \otimes b)(g) = \sum_{h \in G} a(h) b(h^{-1} g) = \sum_{h' \in G} b(h') a(h'^{-1} g) = \mu(b \otimes a)(g)$$

for all $a, b \in A$ and $g \in G$, where $h' = h^{-1} g$. Furthermore, β is associative as

$$\begin{aligned} \beta(\mu(a \otimes b) \otimes c) &= \frac{1}{|G|} \sum_{g, h \in G} a(h) b(h^{-1} g) c(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g, h \in G} a(g) b(g^{-1} h) c(h^{-1}) = \beta(a \otimes \mu(b \otimes c)) \end{aligned}$$

for all $a, b, c \in A$. Finally, β is non-degenerate as $\gamma \colon \mathbb{C} \to A \otimes_{\mathbb{C}} A$ given by $\gamma(1) = \sum_{\chi \in \hat{G}} \chi \otimes \chi$ satisfies

$$(\beta \otimes \mathrm{id}_A)(a \otimes \gamma(1)) = \sum_{\chi \in \hat{G}} \beta(a \otimes \chi) \chi = a$$

by the first orthogonality theorem [Ser77, Theorem 3], and by the same argument $(\mathrm{id}_A \otimes \beta)(\gamma(1) \otimes a) = a$, for all $a \in A$. □

Remark 4.5.2. The copairing $\gamma \colon \mathbb{C} \to A \otimes_{\mathbb{C}} A$, or rather $\gamma(1)$, can be seen as an inner product on the conjugacy classes of G. As a function $G \times G \to \mathbb{C}$, it is given by

$$\gamma(1)(g_1, g_2) = |\{h \in G \mid h g_1 h^{-1} = g_2\}|.$$

Under the equivalence of Theorem 4.4.5, the representation ring $A = R_{\mathbb{C}}(G)$ corresponds to a 2-dimensional oriented TQFT

$$Z_G \colon \mathbf{Bord}_2^{\mathrm{or}} \to \mathbf{Vect}_{\mathbb{C}}.$$

From (4.3), we find that the comultiplication $\delta \colon A \to A \otimes_{\mathbb{C}} A$ and counit $\varepsilon \colon A \to \mathbb{C}$ on A are given by

$$\delta(a) = \sum_{\chi \in \hat{G}} \mu(a \otimes \chi) \otimes \chi \quad \text{and} \quad \varepsilon(a) = \frac{1}{|G|} a(1). \tag{4.5}$$

The convolution of irreducible characters $\chi, \chi' \in \hat{G}$ is well known [Isa76, Theorem 2.13] to be given by

$$\mu(\chi \otimes \chi') = \begin{cases} \frac{|G|}{\chi(1)} \chi & \text{if } \chi = \chi', \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

This implies that, for any irreducible character $\chi \in \hat{G}$,

Z_G () $(\chi) = (\mu \circ \delta)(\chi) = \frac{|G|^2}{\chi(1)^2}\chi.$ (4.7)

In other words, the irreducible characters of G form a basis of eigenvectors for the map Z_G (). The following theorem describes the invariant that this TQFT quantizes.

Theorem 4.5.3. *The TQFT Z_G quantizes the groupoid cardinality $|\mathfrak{X}_G(\Sigma_g)| = |R_G(\Sigma_g)|/|G|$. In particular,*

$$Z_G(\Sigma_g)(1) = \sum_{\chi \in \hat{G}} \left(\frac{|G|}{\chi(1)}\right)^{2g-2} = |\mathfrak{X}_G(\Sigma_g)|.$$

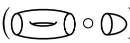
Proof. The first equality follows from (4.7) and the expressions for η and ε. For the second equality, let $f\colon G \to \mathbb{C}$ be the class function given by $f(g) = |\{(A,B) \in G^2 \mid [A,B] = g\}|$. From the explicit presentation of $R_G(\Sigma_g)$,

$$R_G(\Sigma_g) = \left\{ (A_1, B_1, \dots, A_g, B_g) \in G^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = 1 \right\},$$

and the definition of the convolution operator on $R_{\mathbb{C}}(G)$, it is clear that

$$|R_G(\Sigma_g)| = (\underbrace{f * \cdots * f}_{g \text{ times}})(1),$$

where $f * f = \mu(f \otimes f)$. Therefore, it suffices to show that f is equal to

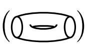
Z_G () $(1) = (\mu \circ \delta \circ \eta)(1) = \sum_{\chi \in \hat{G}} \frac{|G|}{\chi(1)}\chi,$

or equivalently, that $\beta(f \otimes \chi) = |G|/\chi(1)$ for every irreducible complex character χ of G. Note that $\beta(f \otimes \chi)$ is equal to

$$\frac{1}{|G|}\sum_{g \in G} f(g)\chi(g^{-1}) = \frac{1}{|G|}\sum_{A,B \in G} \chi([A,B]^{-1}) = \frac{1}{|G|}\sum_{A,B \in G} \chi(BAB^{-1}A^{-1}).$$

Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation with character χ. Schur's lemma implies that, for any $A \in G$, the operator $T_A = \sum_{B \in G} \rho(BAB^{-1})$ is a scalar multiple of the identity, that is, $T_A = \mathrm{tr}(T_A)/\chi(1) = |G|\chi(A)/\chi(1)$. Hence, it follows that

$$\beta(f \otimes \chi) = \frac{1}{|G|}\sum_{A,B \in G} \mathrm{tr}(T_A A^{-1}) = \frac{1}{|G|}\sum_{A \in G} \frac{|G|}{\chi(1)}\chi(A)\chi(A^{-1}) = \frac{|G|}{\chi(1)}. \quad \square$$

Example 4.5.4. When G is abelian, all irreducible representations of G are of one-dimensional, so Z_G () is simply multiplication by $|G|^2$. Therefore, $|R_G(\Sigma_g)| = |G|^{2g}$ as expected.

4.6 Arithmetic method

Let us elaborate on the arithmetic method from [HR08]. Given a complex algebraic group G, typically a linear algebraic group such as GL_n or SL_n, the goal of this method is to compute the E-polynomial of the G-representation variety $R_G(\Sigma_g)$. It tries to accomplish this using the following theorem, which is a consequence of [HR08, Theorem 6.1.2].

Theorem 4.6.1 (Katz' theorem)**.** *Let X be a complex variety with a spreading-out $\tilde{X}$ over a finitely generated $\mathbb{Z}$-algebra $R \subseteq \mathbb{C}$. If there exists a polynomial $P \in \mathbb{Z}[q]$ such that $|(\tilde{X} \times_R \mathbb{F}_q)(\mathbb{F}_q)| = P(q)$ for all ring morphisms $R \to \mathbb{F}_q$, then the E-polynomial of X is given by $P(uv) \in \mathbb{Z}[u, v]$.*

Most common linear algebraic groups G, such as GL_n and SL_n, can be defined over $\mathbb{Z}$, which determine a natural spreading-out of $R_G(\Sigma_g)$ over $R = \mathbb{Z}$. In this case, we find that

$$\begin{aligned} |R_G(\Sigma_g)(\mathbb{F}_q)| &= |\operatorname{Hom}(\pi_1(\Sigma_g, *), G(\mathbb{F}_q))| \\ &= |G(\mathbb{F}_q)|\, |\mathfrak{X}_{G(\mathbb{F}_q)}(\Sigma_g)| \\ &= |G(\mathbb{F}_q)| Z_{G(\mathbb{F}_q)}(\Sigma_g)(1). \end{aligned}$$

Hence, to compute this point count, we can apply Theorem 4.5.3. This reduces the problem to studying the representation theory of the finite groups $G(\mathbb{F}_q)$, or more specifically, to studying the dimensions of their irreducible representations. This was originally done for the groups $G = \mathrm{GL}_n$ in [HR08], and later also for the groups $G = \mathrm{SL}_n$ in [Mer15].

However, note that it is not clear at all why these point counts should be polynomial in q. It will be, by Theorem 4.5.3, when $|G(\mathbb{F}_q)|$ is polynomial in q and the irreducible representations χ of $G(\mathbb{F}_q)$ come in families in which both $\chi(1)$ and the size of the family is polynomial in q. This is the case in [HR08, Mer15], but fails already when G is not a linear algebraic group, such as an elliptic curve.

An amazing result by [BK22] shows that the above quantities are polynomial in q when G is a connected split reductive group, using Lusztig's Jordan decomposition to describe the irreducible representations of $G(\mathbb{F}_q)$. More precisely, these quantities are polynomial in q after fixing a congruence condition $q \equiv i \bmod d$, where d is an integer depending on the root datum of G, and i can be any integer. By appropriate choice of finitely generated $\mathbb{Z}$-algebra R, one can enforce the congruence condition $q \equiv 1 \bmod d$, implying that the E-polynomial of the character stack $\mathfrak{X}_G(\Sigma_g)$ is polynomial in uv [BK22, Corollary 4].

4.7 Character stack TQFT

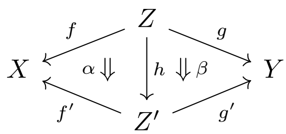
In this section, we will construct a lax TQFT quantizing the virtual class of the G-character stack in the Grothendieck ring of stacks. The construction of this TQFT is an adaptation of the work of González-Prieto, Logares and Muñoz [GLM20], the main differences being that we will not fix a set of basepoints on our manifolds, and that we focus on the character stack rather than the representation variety.

Fix a base scheme S and let G be a linear algebraic group over S. Like described in Section 4.3, the TQFT will be constructed as the composite of two functors, a field theory $\mathcal{F}_G$ and a quantization functor $\mathcal{Q}$,

$$\mathbf{Bord}_n \xrightarrow{\mathcal{F}_G} \mathrm{Corr}(\mathbf{Stck}_S) \xrightarrow{\mathcal{Q}} \mathrm{K}_0(\mathbf{Stck}_S)\text{-}\mathbf{Mod}$$

where the category $\mathrm{Corr}(\mathbf{Stck}_S)$ is defined as follows. Recall from Definition 1.6.4 that $\mathbf{Stck}_S$ is the 2-category of algebraic stacks of finite type over S with affine stabilizers, which has pullbacks by Lemma 1.6.3.

Definition 4.7.1. Let $\mathcal{C}$ be a 2-category with pullbacks. The *category of correspondences* over $\mathcal{C}$ is the 2-category, denoted by $\mathrm{Corr}(\mathcal{C})$, defined as follows. Its objects are the objects of $\mathcal{C}$. A 1-morphism from X to Y is a *correspondence*, that is, a diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathcal{C}$. A 2-morphism between correspondences $X \xleftarrow{f} Z \xrightarrow{g} Y$ and $X \xleftarrow{f'} Z' \xrightarrow{g'} Y$ is an isomorphism $h \colon Z \to Z'$ in $\mathcal{C}$ together with 2-isomorphisms $\alpha \colon f \Rightarrow f' \circ h$ and $\beta \colon g \Rightarrow g' \circ h$.



The composition of correspondences $X \xleftarrow{f} Y \xrightarrow{g} X'$ and $X' \xleftarrow{f'} Y' \xrightarrow{g'} X''$ is given by $X \xleftarrow{f \circ \pi_Y} Y \times_{X'} Y' \xrightarrow{g \circ \pi_{Y'}} X''$. If $\mathcal{C}$ is monoidal, then so is $\mathrm{Corr}(\mathcal{C})$.

Remark 4.7.2. Correspondences over $\mathcal{C}$ can be viewed as an extension of morphisms in $\mathcal{C}$, since any morphism $f \colon X \to Y$ in $\mathcal{C}$ can be seen as the correspondence $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$.

Let us start with the field theory. For this we want to consider $\mathbf{Bord}_n$ as a 2-category, as in Remark 4.2.6.

Definition 4.7.3. Let $\mathcal{F}_G \colon \mathbf{Bord}_n \to \mathrm{Corr}(\mathbf{Stck}_S)$ be the 2-functor that assigns to a closed manifold M the character stack $\mathfrak{X}_G(M)$, to a bordism $W \colon M_1 \to M_2$ the correspondence

$$\mathfrak{X}_G(M) \leftarrow \mathfrak{X}_G(W) \to \mathfrak{X}_G(M')$$

induced by the inclusions $M_i \to W$, and finally to an equivalence of bordisms $f\colon W \to W'$ the diagram

$$\begin{array}{ccccc} & & \mathfrak{X}_G(W) & & \\ & \swarrow & & \searrow & \\ \mathfrak{X}_G(M_1) & & \downarrow \wr & & \mathfrak{X}_G(M_2) \\ & \nwarrow & & \nearrow & \\ & & \mathfrak{X}_G(W') & & \end{array}$$

where the vertical isomorphism is induced by f.

Proposition 4.7.4. *$\mathcal{F}_G$ defines a symmetric monoidal functor.*

Proof. As $M \times [0,1]$ is homotopy equivalent to M, it follows that $\mathfrak{X}_G(M \times [0,1]) \cong \mathfrak{X}_G(M)$ for any closed manifold M, so $\mathcal{F}_G$ preserves identity morphisms. For any two bordisms $W\colon M_1 \to M_2$ and $W'\colon M_2 \to M_3$, the diagram

$$\begin{array}{ccccccccc} & & & & \Pi(W' \circ W) & & & & \\ & & & \nearrow & & \nwarrow & & & \\ & & \Pi(W) & & & & \Pi(W') & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \Pi(M_1) & & & & \Pi(M_2) & & & & \Pi(M_3) \end{array}$$

naturally commutes, and the square is a pushout square by the Seifert–van Kampen theorem for fundamental groupoids [Bro67]. Lemma 2.3.6 implies that the resulting square on G-character stacks is a cartesian square, which shows that $\mathcal{F}_G$ is functorial. The same lemma also shows $\mathfrak{X}_G(M_1 \sqcup M_2)$ is naturally isomorphic to $\mathfrak{X}_G(M_1) \times \mathfrak{X}_G(M_2)$, and this isomorphism clearly respects the symmetric monoidal structure, that is, $\mathcal{F}_G$ is symmetric monoidal. □

Next, we define the quantization functor. As in Remark 4.2.6, we view the category $\mathrm{K}_0(\mathbf{Stck}_S)$-**Mod** as a 2-category in the trivial way, and define $\mathcal{Q}$ as a 2-functor. Equivalently, one can think of $\mathcal{Q}$ as a 1-functor after identifying isomorphic 1-morphisms in $\mathrm{Corr}(\mathbf{Stck}_S)$.

Definition 4.7.5. Let $\mathcal{Q}\colon \mathrm{Corr}(\mathbf{Stck}_S) \to \mathrm{K}_0(\mathbf{Stck}_S)$-**Mod** be the 2-functor that assigns to an object $\mathfrak{X}$ the $\mathrm{K}_0(\mathbf{Stck}_S)$-module

$$\mathcal{Q}(\mathfrak{X}) = \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}})$$

and to a correspondence $\mathfrak{X} \xleftarrow{f} \mathfrak{Z} \xrightarrow{g} \mathfrak{Y}$ the morphism

$$\mathcal{Q}(\mathfrak{X} \xleftarrow{f} \mathfrak{Z} \xrightarrow{g} \mathfrak{Y}) = g_! \circ f^*\colon \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}}) \to \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}})$$

with f^* and $g_!$ as in Section 3.2. Note that two correspondences connected by a 2-morphism are indeed assigned to the same $\mathrm{K}_0(\mathbf{Stck}_S)$-module morphism.

Proposition 4.7.6. *$\mathcal{Q}$ is a symmetric lax monoidal functor.*

Proof. For any object $\mathfrak{X}$ of $\mathbf{Stck}_S$, it is immediate from the definition that $\mathcal{Q}(\mathrm{id}_{\mathfrak{X}}) = \mathrm{id}_{\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}})}$. Consider a composite of correspondences

$$\begin{array}{ccccccc} & & & \mathfrak{Z} & & & \\ & & \overset{h}{\swarrow} & & \overset{j}{\searrow} & & \\ & \mathfrak{Y}_1 & & & & \mathfrak{Y}_2 & \\ \overset{f_1}{\swarrow} & & \overset{g_1}{\searrow} & & \overset{f_2}{\swarrow} & & \overset{g_2}{\searrow} \\ \mathfrak{X}_1 & & & \mathfrak{X}_2 & & & \mathfrak{X}_3 \end{array}$$

for which the square is a 2-cartesian square. To show $\mathcal{Q}$ respects composition, it suffices to show $f_2^* \circ (g_1)_! = j_! \circ h^*$ as morphisms $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}_1}) \to \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}_2})$. This is true for formal reasons: for any $\mathfrak{U} \to \mathfrak{Y}_1$, the diagram

$$\begin{array}{ccccc} \mathfrak{U} \times_{\mathfrak{Y}_1} \mathfrak{Z} & \longrightarrow & \mathfrak{Z} & \xrightarrow{j} & \mathfrak{Y}_2 \\ \downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f_2} \\ \mathfrak{U} & \longrightarrow & \mathfrak{Y}_1 & \xrightarrow{g_1} & \mathfrak{X}_2 \end{array}$$

is a 2-cartesian rectangle as both squares are 2-cartesian squares. Therefore, $(f_2^* \circ (g_1)_!)([\mathfrak{U} \to \mathfrak{Y}_1]) = [\mathfrak{U} \times_{\mathfrak{Y}_1} \mathfrak{Z}] = (j_! \circ h^*)([\mathfrak{U} \to \mathfrak{Y}_1])$. The fact that $\mathcal{Q}$ is lax monoidal and symmetric follows from the natural morphism

$$\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}}) \otimes_{\mathrm{K}_0(\mathbf{Stck}_S)} \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}}) \to \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X} \times_S \mathfrak{Y}}). \qquad \square$$

Remark 4.7.7. Note that $\mathcal{Q}$ is only lax monoidal, and not monoidal, since the above morphism is not necessarily an isomorphism, see Example 3.2.10.

Finally, let us show that the TQFT obtained through the composition of $\mathcal{F}_G$ and $\mathcal{Q}$ indeed quantizes the virtual class of the G-character stack.

Theorem 4.7.8. *There exists a lax TQFT*

$$Z_G \colon \mathbf{Bord}_n \to \mathrm{K}_0(\mathbf{Stck}_S)\text{-}\mathbf{Mod}$$

given by the composite $Z_G = \mathcal{Q} \circ \mathcal{F}_G$, quantizing the virtual class of the G-character stack. That is, $[\mathfrak{X}_G(W)] = Z_G(W)(1)$ in $\mathrm{K}_0(\mathbf{Stck}_S)$ for any closed manifold W.

Proof. For any closed manifold W, viewed as a bordism from and to $\varnothing$, the corresponding field theory $\mathcal{F}_G(W)$ is given by the correspondence

$$S \xleftarrow{\;t\;} \mathfrak{X}_G(W) \xrightarrow{\;t\;} S$$

where t is the terminal morphism. Applying the quantization functor $\mathcal{Q}$, it follows that

$$Z_G(W)(1) = t_! t^*(1) = [\mathfrak{X}_G(W)] \in \mathrm{K}_0(\mathbf{Stck}_S). \qquad \square$$

4.8 Field theory of surfaces

The goal of this section is to make explicit the field theories $\mathcal{F}_G(W)$ corresponding to various bordisms W in dimension $n = 2$. We focus in particular on the generators (4.2), as any 2-dimensional oriented bordism can be built from these through composition and taking disjoint unions.

Example 4.8.1. The inclusions $S^1 \to S^1 \times [0,1]$ of the in- and out-boundary into the cylinder induce an equivalence of groupoids $\Pi(S^1) \simeq \Pi(S^1 \times [0,1])$. Therefore, the field theory $\mathcal{F}_G(S^1 \times [0,1])$ is given by the identity on $\mathcal{F}_G(S^1)$, as expected.

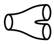
Proposition 4.8.2. *The field theory of the bordism*  *from* $\varnothing$ *to* S^1 *is given by*

$$S \longleftarrow \mathrm{B}G \xrightarrow{\;e\;} [G/G] \ ,$$

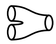
where e *is induced by the unit of* G*, and* G *acts on itself by conjugation. Similarly, the field theory of the bordism*  *from* S^1 *to* $\varnothing$ *is given by*

$$[G/G] \xleftarrow{\;e\;} \mathrm{B}G \longrightarrow S \ .$$

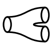
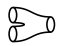
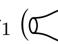
Proof. Since the fundamental group of the disk is trivial, its G-character stack is given by $\mathfrak{X}_G(D^1) = \mathrm{B}G$. The inclusion of S^1 into the disk induces the trivial homomorphism $\pi_1(S^1,*) = \mathbb{Z} \to 1 = \pi_1(D^1,*)$ between fundamental groups, and consequently the corresponding map $\mathrm{B}G \to [G/G]$ is given by the inclusion of the identity. □

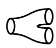
Proposition 4.8.3. *The field theory of the bordism*  *from* $S^1 \sqcup S^1$ *to* S^1 *is given by*

$$[G/G]^2 \xleftarrow{\;\pi_1 \times \pi_2\;} [G^2/G] \xrightarrow{\;m\;} [G/G]$$

where $\pi_1, \pi_2 \colon [G^2/G] \to [G/G]$ *are induced by the projections, and* m *by multiplication on* G*. Similarly, the field theory of the bordism*  *from* S^1 *to* $S^1 \sqcup S^1$ *is given by*

$$[G/G] \xleftarrow{\;m\;} [G^2/G] \xrightarrow{\;\pi_1 \times \pi_2\;} [G/G]^2 \ .$$

Proof. We will compute the field theory for , and the field theory for  can be computed completely analogous. Choose a basepoint x on the out-boundary, basepoints y and z on the in-boundary, a path γ_1 from x to y, a path γ_2 from x to z, and let α and β be generators of the fundamental group π_1 ($, x) \cong F_2$ as depicted in the figure below.

Under the inclusion of the in-boundary $S^1 \sqcup S^1$ into , the generators of $\pi_1(S^1, y) \cong \mathbb{Z}$ and $\pi_1(S^1, z) \cong \mathbb{Z}$ are sent to $\gamma_1 \alpha \gamma_1^{-1}$ and $\gamma_2 \beta \gamma_2^{-1}$, respectively.

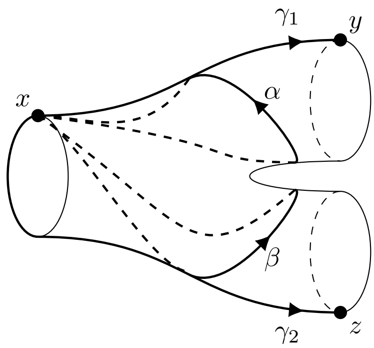
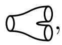


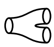
Figure 4.2: The pair of pants as a bordism from $S^1 \sqcup S^1$ to S^1. A basepoint x on the out-boundary is chosen, and basepoints y and z on the in-boundary. Also are chosen paths γ_1 from x to y and γ_2 from x to z, and two generators α and β of the fundamental group at x.

This determines the map $[G^2/G] \to [G/G]^2$ as claimed. Under the inclusion of the out-boundary S^1 into , the generator of $\pi_1(S^1, x) \cong \mathbb{Z}$ is sent to the loop $\alpha\beta$, which determines the map $[G^2/G] \to [G/G]$ as claimed. $\square$

Proposition 4.8.4. *The field theory of the bordism*  *from S^1 to S^1 is given by*

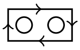
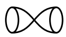
$$[G/G] \xleftarrow{\pi_1} [G^3/G] \xrightarrow{\theta} [G/G]$$

where π_1 is induced by the first projection $G^3 \to G$, and θ is induced by $G^3 \to G$ given by $(C, A, B) \mapsto C[A, B]$.

Proof. Since  is equal to the composite  $\circ$ , it suffices to compute the composite of the correspondences as given by Proposition 4.8.3. Hence, let us describe the fiber product $\mathfrak{X} = [G^2/G] \times_{[G/G]^2} [G^2/G]$. By definition of fiber products of stacks, the objects of $\mathfrak{X}$ over T are tuples $(P, Q, g_1, h_1, g_2, h_2, \alpha, \beta)$, where P and Q are G-torsors over T with G-equivariant morphisms $P \xrightarrow{(g_1,h_1)} G^2$ and $Q \xrightarrow{(g_2,h_2)} G^2$, and $\alpha, \beta \colon P \to Q$ are morphisms of G-torsors such that $g_1 = g_2 \circ \alpha$ and $h_1 = h_2 \circ \beta$. A morphism from $(P', Q', g_1', h_1', g_2', h_2', \alpha', \beta')$ to $(P, Q, g_1, h_1, g_2, h_2, \alpha, \beta)$ is a pair of morphisms of G-torsors $(\gamma_1 : P' \to P, \gamma_2 : Q' \to Q)$ such that $g_i' = g_i \circ \gamma_i$ and $h_i' = h_i \circ \gamma_i$ for $i = 1, 2$ and $\alpha = \gamma_2 \circ \alpha' \circ \gamma_1^{-1}$ and $\beta = \gamma_2 \circ \beta' \circ \gamma_1^{-1}$. Note that every object is isomorphic one with $P = Q$ and $\beta = \mathrm{id}_P$. Therefore, we can equivalently describe this category as the category whose objects over T are tuples (P, C, A, B), where P is a G-torsor over T and $C = m \circ (g_1, h_1)$, $A = h_1^{-1}$ and $B = \alpha$, and a morphism $(P', C', A', B') \to (P, C, A, B)$ is a morphism $\gamma \colon P' \to P$ such that $C' = C \circ \gamma$, $A' = A \circ \gamma$ and

$B' = \gamma^{-1} \circ B \circ \gamma$. With this description it is clear that $\mathfrak{X} \cong [G^3/G]$. Unfolding the definitions, the morphism $\mathfrak{X} \to [G/G]$ corresponding to the in-boundary is indeed given by $(C, A, B) \mapsto C$. The morphism $\mathfrak{X} \to [G/G]$ corresponding to the out-boundary is given by $(C, A, B) \mapsto B^{-1}CABA^{-1}$, which is naturally isomorphic to $C[A, B]$. □

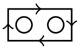
Besides orientable bordisms, there are also non-orientable bordisms, which our field theory $\mathcal{F}_G$ allows. Of interest to us are the bordisms

$: S^1 \to S^1$ and $: S^1 \to S^1$ (4.8)

corresponding to the projective plane with two punctures and the cylinder which reverses the orientation of S^1, respectively. The field theory $\mathcal{F}_G$ () is easily seen to be the correspondence

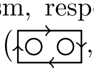
$$[G/G] \xleftarrow{\ i\ } [G/G] \xrightarrow{\ \mathrm{id}\ } [G/G]$$

where i is induced by the inversion $g \mapsto g^{-1}$. The field theory of the punctured projective plane is described by the following proposition.

Proposition 4.8.5. *The field theory of the bordism*  *from S^1 to S^1 is given by*

$$[G/G] \xleftarrow{\ \pi_1\ } [G^2/G] \xrightarrow{\ v\ } [G/G]$$

where v is given by $(B, A) \mapsto BA^2$.

Proof. Choose basepoints x and y on the in- and out-boundary of the bordism, respectively, and let α and β be generators of the fundamental group π_1 ($, x) \cong F_2$ as depicted in the figure below.

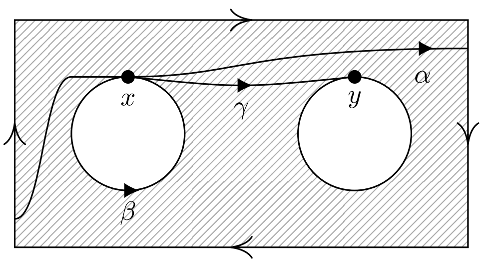
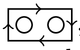


Figure 4.3: The projective plane with two punctures as a bordism from S^1 to S^1. Basepoints x and y are chosen on in- and out-boundary, respectively. Also, generators α and β of the fundamental group at x are chosen, and path γ connecting x and y.

Under the inclusion of the in-boundary S_1 into , the generator of $\pi_1(S^1, x) \cong \mathbb{Z}$ is sent to β, and under the inclusion of the out-boundary S^1, the generator of $\pi_1(S^1, y) \cong \mathbb{Z}$ is sent to $\gamma\beta\alpha^2\gamma^{-1}$. These determine the maps $[G^2/G] \to [G/G]$ as claimed. □

4.9 Arithmetic TQFT

In this section we will construct a higher-dimensional analogue of the TQFT of Section 4.5. To be precise, for any finite group G, we will construct an *arithmetic TQFT*

$$Z_G^{\#} \colon \mathbf{Bord}_n \to \mathbf{Vect}_{\mathbb{C}}$$

which will agree (up to natural isomorphism) with the TQFT of Section 4.5. Whereas the construction of the TQFT of Section 4.5 is very ad-hoc, in terms of specific operations on the representation ring $R_{\mathbb{C}}(G)$, the construction of $Z_G^{\#}$ will be very much like that of the character stack TQFT: as the composite of a field theory and a quantization functor.

Field theory and quantization

Fix a finite group G.

Definition 4.9.1. The *arithmetic field theory* is the 2-functor

$$\mathcal{F}_G^{\#} \colon \mathbf{Bord}_n \to \mathrm{Corr}(\mathbf{FinGrpd})$$

which assigns to a closed $(n-1)$-dimensional manifold M the G-character groupoid

$$\mathcal{F}_G^{\#}(M) = \mathfrak{X}_G(M)$$

and to a bordism $W \colon M_1 \to M_2$ the correspondence

$$\mathcal{F}_G^{\#}(W) = \left(\mathfrak{X}_G(M_1) \xleftarrow{i_1} \mathfrak{X}_G(W) \xrightarrow{i_2} \mathfrak{X}_G(M_2) \right).$$

Proposition 4.9.2. *$\mathcal{F}_G^{\#}$ is a symmetric monoidal functor.*

Proof. The proof is completely analogous to that of Proposition 4.7.4, where $\mathfrak{X}_G(-)$ is also sends finite colimits in **FGGrpd** to limits in **FinGrpd**. □

Definition 4.9.3. Given a groupoid A, denote by $\mathbb{C}^A$ the complex vector space of complex-valued functions on the objects of A which are invariant under isomorphism.

This construction admits some functoriality. Given a functor $f\colon A \to B$ between groupoids, we can pull back functions via

$$f^*\colon \mathbb{C}^B \to \mathbb{C}^A, \quad \varphi \mapsto \varphi \circ f.$$

Pullback is functorial in the sense that $(g \circ f)^* = f^* \circ g^*$ for functors $f\colon A \to B$ and $g\colon B \to C$. Moreover, if $\mu\colon f \Rightarrow g$ is a natural transformation between functors $f, g\colon A \to B$, then $f^* = g^*$. In particular, if A and B are equivalent groupoids, then $\mathbb{C}^A$ and $\mathbb{C}^B$ are naturally isomorphic.

Furthermore, if $f\colon A \to B$ is a functor between essentially finite groupoids, we define pushforward along f as

$$f_!\colon \mathbb{C}^A \to \mathbb{C}^B, \quad \varphi \mapsto \left(b \mapsto \sum_{[(a,\beta)] \in f^{-1}(b)/\sim} \frac{\varphi(a)}{|\operatorname{Aut}(a,\beta)|} \right)$$

where $f^{-1}(b)$ denotes the fiber product $A \times_B \{b\}$ as in Definition 1.1.6. It is an easy exercise to show that pushforward is also functorial in the sense that $(g \circ f)_! = g_! \circ f_!$ for functors $f\colon A \to B$ and $g\colon B \to C$.

Example 4.9.4. For any groupoid A, let $f\colon A \to \{*\}$ be the final morphism and let $\varphi \in \mathbb{C}^A$ be the constant function $\varphi(a) = 1$. Then $(f_!\varphi)(*) = |A|$ is the groupoid cardinality of A.

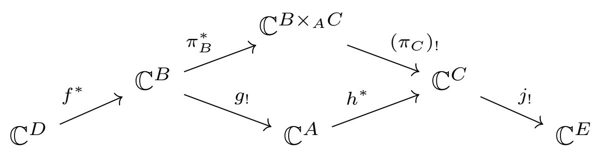
Definition 4.9.5. The *arithmetic quantization functor* is the functor

$$\mathcal{Q}^\#\colon \operatorname{Corr}(\mathbf{FinGrpd}) \to \mathbf{Vect}_\mathbb{C}$$

which assigns to a groupoid A the vector space $\mathbb{C}^A$, and which assigns to a correspondence of groupoids $A \xleftarrow{f} B \xrightarrow{g} C$ the morphism $g_! \circ f^*\colon \mathbb{C}^A \to \mathbb{C}^C$. Note that two correspondences connected by a 2-morphism are indeed assigned to the same linear map.

Lemma 4.9.6. *$\mathcal{Q}^\#$ is a symmetric monoidal functor.*

Proof. Let $D \xleftarrow{f} B \xrightarrow{g} A$ and $A \xleftarrow{h} C \xrightarrow{i} E$ be correspondences of essentially finite groupoids. The relevant diagram in $\mathbf{Vect}_\mathbb{C}$ is given by



where $\pi_B\colon B \times_A C \to B$ and $\pi_C\colon B \times_A C \to C$ are the projections. To show $\mathcal{Q}^\#$ respects composition, it suffices to show that $h^* \circ g_! = (\pi_C)_! \circ \pi_B^*$.

First note that, for any $x \in C$, the groupoids $\pi_C^{-1}(x) = (B \times_A C) \times_C \{x\}$ and $g^{-1}(h(x)) = B \times_A \{h(x)\}$ are equivalent. Explicitly, an object of $\pi_C^{-1}(x)$ is a tuple (b, c, α, γ) with $(b, c, \alpha) \in B \times_A C$ and $\gamma \colon c \to x$ a morphism in C. A morphism $(b', c', \alpha', \gamma') \to (b, c, \alpha, \gamma)$ is given by a tuple of morphisms $(\beta \colon b' \to b, \zeta \colon c' \to c)$ such that $\alpha \circ g(\beta) = h(\zeta) \circ \alpha'$ and $\gamma \circ \zeta = \gamma'$. By appropriate choice of ζ, this is equivalent to the groupoid whose objects are (b, α) with $b \in B$ and $\alpha \colon g(b) \to h(x)$ and morphisms $(b', \alpha') \to (b, \alpha)$ are morphisms $\beta \colon b' \to b$ such that $\alpha' \circ g(\beta) = \alpha$. But this is precisely $g^{-1}(h(x))$.

Now, for any $\varphi \in \mathbb{C}^B$ and any $c \in C$, it follows that

$$
\begin{aligned}
((\pi_C)_! \pi_B^* \varphi)(c) &= \sum_{[(b,c,\alpha,\gamma)] \in \pi_C^{-1}(c)/\sim} \frac{\varphi(b)}{|\operatorname{Aut}(b, c, \alpha, \gamma)|} \\
&= \sum_{[(b,\alpha)] \in g^{-1}(h(c))/\sim} \frac{\varphi(b)}{|\operatorname{Aut}(b, \alpha)|} = (h^* g_! \varphi)(c). \qquad \square
\end{aligned}
$$

Definition 4.9.7. The *arithmetic TQFT* $Z_G^{\#} \colon \mathbf{Bord}_n \to \mathbf{Vect}_{\mathbb{C}}$ is the composite $\mathcal{Q}^{\#} \circ \mathcal{F}_G^{\#}$.

Proposition 4.9.8. *The arithmetic TQFT quantizes the groupoid cardinality of the G-character groupoid, that is,*

$$Z_G^{\#}(W)(1) = |\mathfrak{X}_G(W)|$$

for any closed n-dimensional manifold W, seen as a bordism $\varnothing \to \varnothing$.

Proof. The field theory $\mathcal{F}_G^{\#}(W)$ is given by

$$\{*\} \xleftarrow{\;t\;} \mathfrak{X}_G(W) \xrightarrow{\;t\;} \{*\}$$

where t is the final morphism. Applying the quantization functor $\mathcal{Q}^{\#}$, we find

$$Z_G^{\#}(W)(1) = t_! t^*(1) = \sum_{[x] \in t^{-1}(*)/\sim} \frac{1}{|\operatorname{Aut}(x)|} = |\mathfrak{X}_G(W)|,$$

using that $t^{-1}(*) = \mathfrak{X}_G(W)$. $\square$

Remark 4.9.9. The arithmetic TQFT can be seen as a special case of the Dijkgraaf–Witten TQFT [DW90] with $\alpha = 0 \in H^n(\mathrm{B}G, \mathbb{R}/\mathbb{Z})$. This TQFT is also known as *finite gauge theory*, since the gauge group G is finite.

Comparison with the representation ring

Let us return to case $n = 2$. For a finite group G, we have $\mathfrak{X}_G(S^1) = [G/G]$ where G acts on itself by conjugation. In particular, $Z_G^{\#}(S^1)$ is the complex vector space of complex-valued functions on G which are invariant under conjugation. But this is precisely the underlying vector space of the representation ring $R_{\mathbb{C}}(G)$, that is, there is a canonical isomorphism

$$Z_G^{\#}(S^1) = \mathbb{C}^{[G/G]} \cong R_{\mathbb{C}}(G) = Z_G(S^1). \tag{4.9}$$

Proposition 4.9.10. *Let G be a finite group. For $n = 2$, there is a natural isomorphism*

$$Z_G^{\#} \cong Z_G$$

as functors $\mathbf{Bord}_2^{\mathrm{or}} \to \mathbf{Vect}_{\mathbb{C}}$ from the arithmetic TQFT to the TQFT of Section 4.5.

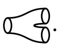
Proof. Since both $Z_G^{\#}$ and Z_G are monoidal functors, the isomorphism (4.9) naturally extends to isomorphisms $Z_G^{\#}\left(\sqcup_{i=1}^m S^1\right) \cong Z_G\left(\sqcup_{i=1}^m S^1\right)$ for all $m \geq 0$. As the category $\mathbf{Bord}_2^{\mathrm{or}}$ of 2-dimensional oriented bordisms is generated by the bordisms (4.2), it suffices to verify the naturality of the isomorphisms for these generators only.

- Case $W =$ . The field theory $\mathcal{F}_G^{\#}(W)$ is given by

  $$\{*\} \xleftarrow{\ t\ } [\{*\}/G] \xrightarrow{\ e\ } [G/G]$$

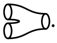
  where t is the terminal morphism and e is the inclusion of the unit of G. Hence, the morphism $Z_G^{\#}(W) : \mathbb{C} \to \mathbb{C}^{[G/G]}$ sends 1 to $e_! t^* 1$, which is precisely the indicator function on the unit of G and corresponds, under the isomorphism (4.9), to the unit $\eta(1)$ of $R_{\mathbb{C}}(G)$.
- Case $W =$ . Similarly, the field theory $\mathcal{F}_G^{\#}(W)$ is given by

  $$[G/G] \xleftarrow{\ e\ } [\{*\}/G] \xrightarrow{\ t\ } \{*\}$$

  so the morphism $Z_G^{\#}(W) : \mathbb{C}^{[G/G]} \to \mathbb{C}$ is given by $f \mapsto t_! e^* f = \frac{1}{|G|} f(1)$, which corresponds, under the isomorphism (4.9), to the counit ε of $R_{\mathbb{C}}(G)$.
- Case $W =$ . The field theory $\mathcal{F}_G^{\#}(W)$ is given by

  $$[G/G]^2 \xleftarrow{\ \pi_1 \times \pi_2\ } [G^2/G] \xrightarrow{\ m\ } [G/G]$$

  where m is multiplication on G. Hence, the morphism $Z_G^{\#}(W)\colon R_{\mathbb{C}}(G) \otimes_{\mathbb{C}} R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G)$ maps $f_1 \otimes f_2 \mapsto m_!(\pi_1 \times \pi_2)^*(f_1 \otimes f_2)$ which is precisely $\mu(f_1 \otimes f_2)$.

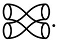
- Case $W =$ . Similarly, the field theory $\mathcal{F}_G^{\#}(W)$ is given by

$$[G/G] \xleftarrow{\;\;m\;\;} [G^2/G] \xrightarrow{\pi_1 \times \pi_2} [G/G]^2$$

so the morphism $Z_G^{\#}(W)\colon R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G) \otimes_{\mathbb{C}} R_{\mathbb{C}}(G)$ is given by $f \mapsto (\pi_1 \times \pi_2)_! m^* f$. Note that, for any $g_1, g_2 \in G$, the groupoid $(\pi_1 \times \pi_2)^{-1}(g_1, g_2)$ is equivalent to G as a set. Hence, for any irreducible character χ, we find

$$Z_G^{\#}(W)(\chi)(g_1, g_2) = \sum_{h \in G} \chi(g_1 h g_2 h^{-1}) = |G|\chi(g)\chi(h)/\chi(1),$$

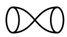
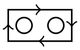
where the last equality is shown as in the proof of Theorem 4.5.3. Therefore, using (4.5) and (4.6), we find that $Z_G^{\#}(W)(\chi)$ is precisely $\delta(\chi)$.

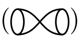
- Case $W =$ . The field theory $\mathcal{F}_G^{\#}(W)$ is given by

$$[G/G]^2 \xleftarrow{\;\;\mathrm{id}\;\;} [G/G]^2 \xrightarrow{\;\;t\;\;} [G/G]^2$$

where t switches the two copies of G. Clearly, $Z_G^{\#}(W)$ is given by $f_1 \otimes f_2 \mapsto f_2 \otimes f_1$, which is precisely τ. □

Non-orientable surfaces

Recall that the TQFT of Section 4.5 was given by the Frobenius algebra structure on the representation ring $R_{\mathbb{C}}(G)$. However, $Z_G^{\#}$ is also defined for non-orientable bordisms, so we obtain additional operations on the representation ring. In particular, let us consider the orientation-reversing cylinder : $S^1 \to S^1$ and the projective plane : $S^1 \to S^1$ as in (4.8).

The field theory $\mathcal{F}_G^{\#}$ () is easily seen to be

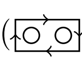
$$[G/G] \xleftarrow{\;\;i\;\;} [G/G] \xrightarrow{\;\;\mathrm{id}\;\;} [G/G]$$

where i is induced by the inversion $g \mapsto g^{-1}$. Hence, it follows that

$Z_G^{\#}$ () $: R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G), \quad f \mapsto i^* f = (g \mapsto f(g^{-1})) = \overline{f}$

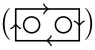
is complex conjugation of class functions.

Regarding the projective plane, we have the following lemma.

Lemma 4.9.11. *The map* $\nu \coloneqq Z_G^{\#}$ () $: R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G)$ *is given by*

$$\nu(\chi) = \varepsilon_\chi \frac{|G|}{\chi(1)} \chi$$

for any irreducible character $\chi \in \hat{G}$, *where* $\varepsilon_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$ *is known as the* Frobenius–Schur indicator *of* χ [FS06].

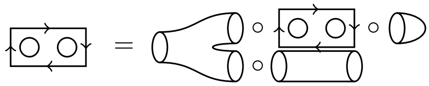
Proof. Analogous to Proposition 4.8.5, the field theory $\mathcal{F}_G^{\#}$ () is

$$[G/G] \xleftarrow{\pi_1} [G^2/G] \xrightarrow{v} [G/G]$$

where v is induced by $(B, A) \mapsto BA^2$. Applying $\mathcal{Q}^{\#}$ we find that

$$\nu(f) = v_! \pi_1^* f = \left(g \mapsto \sum_{BA^2 = g} f(B) = \sum_{h \in G} f(gh^2) \right).$$

Note that we have an equality of bordisms



which shows that $\nu(f) = \mu((\nu \circ \eta)(1) \otimes f)$ for all $f \in R_{\mathbb{C}}(G)$. We compute

$$(\nu \circ \eta)(1)(g) = \sum_{h \in G} \eta(1)(gh^2) = |\{h \in G \mid h^2 = g^{-1}\}|$$

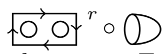
and thus, for any $\chi \in \hat{G}$, we find

$$\beta((\nu \circ \eta)(1) \otimes \chi) = \frac{1}{|G|} \sum_{g \in G} (\nu \circ \eta)(1)(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_{h \in G} \chi(h^2) = \varepsilon_\chi$$

from which we obtain that

$$(\nu \circ \eta)(1) = \sum_{\chi \in \hat{G}} \varepsilon_\chi \chi.$$

Finally, using (4.6) we conclude that $\nu(\chi) = \mu((\nu \circ \eta)(1) \otimes \chi) = \varepsilon_\chi \frac{|G|}{\chi(1)} \chi$. $\square$

This expression can be used to compute the groupoid cardinality $|\mathfrak{X}_G(N_r)|$ of the G-character groupoid of the non-orientable closed surface N_r of demigenus r, that is, the connected sum of r non-projective planes. The decomposition $N_r =$  $\circ$  yields the following proposition. Note that this formula was already known to Frobenius and Schur in [FS06, (9), p.197].

Proposition 4.9.12. *Let N_r be the closed non-orientable surface of demigenus r, that is, the surface obtained as the connected sum of r projective planes. Then*

$$Z_G^{\#}(N_r)(1) = |\mathfrak{X}_G(N_r)| = \sum_{\chi \in \hat{G}} \varepsilon_\chi^r \left(\frac{|G|}{\chi(1)} \right)^{r-2}. \qquad \square$$

4.10 Comparison of TQFTs

Let us summarize the various TQFTs constructed so far. Fix a base scheme S, a linear algebraic group G over S, a finite field $\mathbb{F}_q$, and an $\mathbb{F}_q$-rational point $x\colon \operatorname{Spec}\mathbb{F}_q \to S$ of S. The functors defined in the previous sections, i.e., the field theory and quantization functors, fit nicely together in the following (not necessarily commutative!) diagram. The dashed arrow, completing the diagram, will be defined in this section.

$$\begin{array}{ccccc} & & \mathrm{Corr}(\mathbf{Stck}_S) & \xrightarrow{\mathcal{Q}} & \mathrm{K}_0(\mathbf{Stck}_S)\text{-}\mathbf{Mod} \\ & \overset{\mathcal{F}_G}{\nearrow} & \Big\downarrow {\scriptstyle (-)(\mathbb{F}_q)} & & \Big\uparrow {\scriptstyle \mu_S^*} \\ \mathbf{Bord}_n & & & & \\ & \underset{\mathcal{F}^{\#}_{G(\mathbb{F}_q)}}{\searrow} & \mathrm{Corr}(\mathbf{FinGrpd}) & \xrightarrow[\mathcal{Q}^{\#}]{} & \mathbf{Vect}_{\mathbb{C}} \end{array}$$

Recall that for any algebraic stack $\mathfrak{X}$ over S, the groupoid $\mathfrak{X}(\mathbb{F}_q)$ is the groupoid of $\mathbb{F}_q$-points $\operatorname{Spec}\mathbb{F}_q \to \mathfrak{X}$ whose composition to S is equal to the fixed point $x\colon \operatorname{Spec}\mathbb{F}_q \to S$. In particular, $S(\mathbb{F}_q) = \{x\}$.

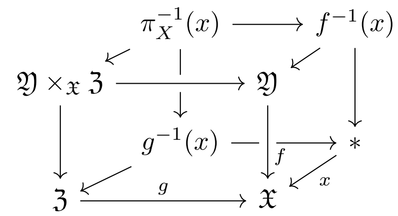
We can see the TQFT of G-character stacks, $Z_G = \mathcal{Q} \circ \mathcal{F}_G$, in the top row, and the arithmetic TQFT, $Z^{\#}_{G(\mathbb{F}_q)} = \mathcal{Q}^{\#} \circ \mathcal{F}^{\#}_{G(\mathbb{F}_q)}$, in the bottom row. Note that one can interpolate between the two: using the field theory $\mathcal{F}_G$, then taking the $\mathbb{F}_q$-rational points, and finally applying the arithmetic quantization functor $\mathcal{Q}^{\#}$, we obtain yet another TQFT given by the composite $\tilde{Z}_G = \mathcal{Q}^{\#} \circ (-)(\mathbb{F}_q) \circ \mathcal{F}_G$.

It turns out all three TQFTs all quantize different invariants. Of course, Z_G quantizes a different type of invariant (an element in the Grothendieck ring of stacks) while $Z^{\#}_G$ and $\tilde{Z}_G$ quantize a complex number. Nevertheless, in this section we will relate these TQFTs through natural transformations. More precisely, there will be a natural transformation in the square on the right in the diagram, and, if G is connected, a natural isomorphism in the triangle on the left. The functor μ_S^* will be defined in order to relate the targets of the geometric and arithmetic TQFT.

Definition 4.10.1. For any object $\mathfrak{X}$ of $\mathbf{Stck}_S$, define

$$\mu_{\mathfrak{X}}\colon \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}}) \to \mathbb{C}^{\mathfrak{X}(\mathbb{F}_q)}, \quad [\mathfrak{Y} \xrightarrow{f} \mathfrak{X}] \mapsto \big(x \mapsto |f^{-1}(x)|\big)$$

where the groupoid cardinality of $f^{-1}(x) = \mathfrak{Y}(\mathbb{F}_q) \times_{\mathfrak{X}(\mathbb{F}_q)} \{x\}$ was taken. This map is easily seen to be a morphism of rings, where multiplicativity follows from the following diagram, in which all squares are cartesian:



In particular, the morphism $\mu_S \colon \mathrm{K}_0(\mathbf{Stck}_S) \to \mathbb{C}^{S(\mathbb{F}_q)} = \mathbb{C}$ induces the functor

$$\mu_S^* \colon \mathbf{Vect}_{\mathbb{C}} \to \mathrm{K}_0(\mathbf{Stck}_S)\text{-}\mathbf{Mod}$$

given by restriction of scalars.

Proposition 4.10.2. *The maps $\mu_{\mathfrak{X}}$ define a natural transformation*

$$\mu \colon \mathcal{Q} \Rightarrow \mu_S^* \circ \mathcal{Q}^{\#} \circ (-)(\mathbb{F}_q).$$

In particular, this induces a natural transformation of TQFTs

$$Z_G \Rightarrow \mu_S^* \circ \mathcal{Q}^{\#} \circ (-)(\mathbb{F}_q) \circ \mathcal{F}_G.$$

Proof. For any correspondence $\mathfrak{X} \xleftarrow{f} \mathfrak{Z} \xrightarrow{h} \mathfrak{Y}$ in $\mathbf{Stck}_S$, the relevant diagram of $\mathrm{K}_0(\mathbf{Stck}_S)$-modules is:

$$\begin{array}{ccccc}
\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}}) & \xrightarrow{f^*} & \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Z}}) & \xrightarrow{g_!} & \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}}) \\
\downarrow{\scriptstyle \mu_{\mathfrak{X}}} & & \downarrow{\scriptstyle \mu_{\mathfrak{Z}}} & & \downarrow{\scriptstyle \mu_{\mathfrak{Y}}} \\
\mathbb{C}^{\mathfrak{X}(\mathbb{F}_q)} & \xrightarrow{f^*} & \mathbb{C}^{\mathfrak{Z}(\mathbb{F}_q)} & \xrightarrow{g_!} & \mathbb{C}^{\mathfrak{Y}(\mathbb{F}_q)}
\end{array}$$

Let us show that the first square commutes. For any stack $\mathfrak{U} \xrightarrow{h} \mathfrak{X}$ and point $z \in \mathfrak{Z}(\mathbb{F}_q)$, we have

$$\mu_{\mathfrak{Z}}(f^*[\mathfrak{U}])(z) = \sum_{[(u,z,\alpha)] \in (\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{Z})(\mathbb{F}_q)/\sim} \frac{|\operatorname{Aut}(z)|}{|\operatorname{Aut}(u,z,\alpha)|}.$$

As in the proof of Lemma 1.6.3, the group $\operatorname{Aut}(u) \times \operatorname{Aut}(z)$ acts naturally on the set $\operatorname{Hom}_{\mathfrak{X}(\mathbb{F}_q)}(h(u), f(z))$, and the stabilizer of any α in this set is precisely $\operatorname{Aut}(u,z,\alpha)$. Hence, it follows from the orbit-stabilizer theorem that

$$\begin{aligned}
\mu_{\mathfrak{Z}}(f^*[\mathfrak{U}])(z) &= \sum_{\substack{[u] \in \mathfrak{U}(\mathbb{F}_q)/\sim \\ \alpha \colon h(u) \xrightarrow{\sim} f(z)}} \frac{1}{|\operatorname{Aut}(u)|} \\
&= \sum_{[(u,\alpha)] \in h^{-1}(f(z))/\sim} \frac{1}{|\operatorname{Aut}(u)|} \\
&= |h^{-1}(f(z))| = (f^* \mu_{\mathfrak{X}}([\mathfrak{U}]))(z).
\end{aligned}$$

Next, let us show that the second square commutes. For any stack $\mathfrak{U} \xrightarrow{h} \mathfrak{Z}$ and point $y \in \mathfrak{Y}(\mathbb{F}_q)$, we have

$$(g_! \mu_{\mathfrak{Z}}[\mathfrak{U}])(y) = \sum_{[(z,\alpha)] \in g^{-1}(y)/\sim} \frac{|h^{-1}(z)|}{|\operatorname{Aut}(z)|} = |(g \circ h)^{-1}(y)| = (\mu_{\mathfrak{Y}} g_! [\mathfrak{U}])(y). \qquad \square$$

Proposition 4.10.3. *If G is connected, there is a natural isomorphism*

$$(-)(\mathbb{F}_q) \circ \mathcal{F}_G \cong \mathcal{F}^{\#}_{G(\mathbb{F}_q)}.$$

In particular, this induces a natural isomorphism of TQFTs

$$\mathcal{Q}^{\#} \circ (-)(\mathbb{F}_q) \circ \mathcal{F}_G \cong Z^{\#}_{G(\mathbb{F}_q)}.$$

Proof. Proposition 1.5.10 implies that $\mathfrak{X}_G(M)(\mathbb{F}_q)$ is naturally isomorphic to $\mathfrak{X}_{G(\mathbb{F}_q)}(M)$ for any compact manifold M. The statement now follows directly from the definitions of the field theories. $\square$

Remark 4.10.4. For non-connected G, there need not even be a natural transformation $(-)(\mathbb{F}_q) \circ \mathcal{F}_G \Rightarrow \mathcal{F}^{\#}_{G(\mathbb{F}_q)}$. Consider the 2-sphere S^2 as a bordism $\varnothing \to \varnothing$. Since the G-character stack of S^2 is $\mathrm{B}G$, one has $\tilde{Z}_G(S^2)(1) = |\mathrm{B}G(\mathbb{F}_q)|$, whereas $Z^{\#}_{G(\mathbb{F}_q)}(S^2)(1) = |R_G(S^2)(\mathbb{F}_q)|/|G(\mathbb{F}_q)| = |1|/|G(\mathbb{F}_q)|$. Already for $G = \mathbb{Z}/2\mathbb{Z}$, these quantities are different, see Remark 1.5.9.

Corollary 4.10.5. *Suppose G is connected. Then there is a natural transformation between the geometric and arithmetic TQFT*

$$Z_G \Rightarrow \mu_S^* \circ Z^{\#}_{G(\mathbb{F}_q)}. \qquad \square$$

Unfolding the definitions in dimension $n = 2$, we obtain the following theorem, relating the geometric method to the arithmetic method.

Theorem 4.10.6. *Suppose G is connected. Denote by $I \in \mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ the class of $[S/G] \to [G/G]$ induced by the unit of G. If the $\mathrm{K}_0(\mathbf{Stck}_S)$-module $\mathcal{V} = \langle Z_G$ ($)^g$ (I) for $g \in \mathbb{Z}_{\geq 0} \rangle$ is finitely generated, then:*

(i) The sums of equidimensional irreducible complex characters form a basis for the subspace $\mu_{[G/G]}(\mathcal{V}) \subseteq \mathbb{C}^{[G/G](\mathbb{F}_q)}$.

(ii) The dimensions of the irreducible complex characters of $G(\mathbb{F}_q)$ are precisely given by

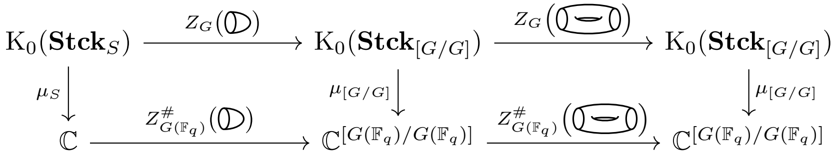
$$d_i = \frac{|G(\mathbb{F}_q)|}{\sqrt{\lambda_i}}$$

for $\lambda_i \in \mathbb{Z}$ the eigenvalues of $\mu_S(A)$, where A is any matrix representing the linear map Z_G () with respect to a generating set of $\mathcal{V}$.

(iii) Write $\mu_{[G/G]}(I) = \sum_i v_i$, where v_i are eigenvectors of $\mu_S(A)$ corresponding to the eigenvalues λ_i. Then each v_i is a scalar multiple of the sum of equidimensional characters, or more precisely,

$$v_i = \frac{d_i}{|G(\mathbb{F}_q)|} \sum_{\substack{\chi \in \hat{G} \text{ s.t.} \\ \chi(1) = d_i}} \chi.$$

Proof. By Corollary 4.10.5, we have the following commutative diagram.



The square on the right shows that

$$\mu_{[G/G]}(\mathcal{V}) = \langle Z^{\#}_{G(\mathbb{F}_q)}(\cdot)\,(\mu_{[G/G]}(I)) \text{ for } g \in \mathbb{Z}_{\geq 0} \rangle,$$

and the square on the left shows that $\mu_{[G/G]}(I) \in \mathbb{C}^{[G/G](\mathbb{F}_q)} \cong R_{\mathbb{C}}(G(\mathbb{F}_q))$ corresponds to the unit

$$\eta(1) = \frac{1}{|G(\mathbb{F}_q)|} \sum_{\chi \in \hat{G}} \chi(1)\,\chi = \sum_{d \geq 0} w_d \quad \text{with} \quad w_d = \frac{d}{|G(\mathbb{F}_q)|} \sum_{\substack{\chi \in \hat{G} \text{ s.t.} \\ \chi(1) = d}} \chi. \qquad (*)$$

Clearly, the w_d are linearly independent, and moreover, by (4.7), they are eigenvectors of $Z^{\#}_{G(\mathbb{F}_q)}(\cdot)$ with eigenvalues $|G(\mathbb{F}_q)|^2/d^2$. Since the eigenvalues are distinct, the w_d form a basis for $\mu_{[G/G]}(\mathcal{V})$, proving *(i)*.

For *(ii)*, as the matrix $\mu_S(A)$ represents $Z^{\#}_{G(\mathbb{F}_q)}(\cdot)$, its eigenvalues are precisely given by $|G(\mathbb{F}_q)|^2/d^2$. Finally, *(iii)* follows from $(*)$. ☐

4.11 $\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$-character stacks

Let us illustrate how the arithmetic TQFT and the character stack TQFT are related, and how they differ, by means of an example. Throughout this section, we consider the group $G = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{G}_m$ via $x \mapsto x^{-1}$, over any field k of characteristic not equal to 2, or more generally, over the finitely generated algebra $R = \mathbb{Z}[\frac{1}{2}]$.

Arithmetic method. Following the arithmetic method, we consider the representation theory of the finite groups $G(\mathbb{F}_q) = \mathbb{F}_q^\times \rtimes \mathbb{Z}/2\mathbb{Z}$ with q is odd. The character table of $G(\mathbb{F}_q)$ can easily be computed, e.g. using [Ser77, Proposition 25]. Fixing any generator $x \in \mathbb{F}_q^\times$, the character table of $G(\mathbb{F}_q)$ is given by

	$\{1\}$	$\{-1\}$	$\{x^\ell, x^{-\ell}\}$	$(\mathbb{F}_q^\times)^2\sigma$	$(\mathbb{F}_q^\times)^2 x\sigma$
$\rho_{\varepsilon,\delta}$	1	$\varepsilon^{\frac{q-1}{2}}$	ε^ℓ	δ	$\varepsilon\delta$
τ_k	2	$2(-1)^k$	$\zeta_{q-1}^{k\ell} + \zeta_{q-1}^{-k\ell}$	0	0

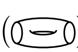
where $1 \le k, \ell \le \frac{q-3}{2}$ and $\varepsilon, \delta = \pm 1$, and σ is the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$. Summing characters of the same dimension, the character table reduces to

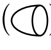
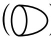
	$\{1\}$	$\{-1\}$	$\{x^\ell, x^{-\ell}\}$	$(\mathbb{F}_q^\times)^2\sigma$	$(\mathbb{F}_q^\times)^2 x\sigma$
$v_1 = \sum_{\varepsilon,\delta} \rho_{\varepsilon,\delta}$	4	$4\alpha_{(q-1)/2}$	$4\alpha_\ell$	0	0
$v_2 = \sum_k \tau_k$	$q-3$	$-2\alpha_{(q-1)/2}$	$-2\alpha_\ell$	0	0

where $\alpha_\ell = 1$ for ℓ even and $\alpha_\ell = 0$ for ℓ odd. Alternatively, this table can be expressed as

	$\{1\}$	$\{t \in G(\mathbb{F}_q) \mid t \neq 1 \text{ a square}\}$	$\{t \in G(\mathbb{F}_q) \mid t \text{ not a square}\}$
v_1	4	4	0
v_2	$q-3$	-2	0

Now, from (4.4), (4.5) and (4.7) follows that the TQFT $Z^{\#}_{G(\mathbb{F}_q)}$ is, with respect to the basis v_1, v_2, given by

$Z^{\#}_{G(\mathbb{F}_q)}$ () $= |G(\mathbb{F}_q)|^2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix},$

$Z^{\#}_{G(\mathbb{F}_q)}$ () $= \frac{1}{|G(\mathbb{F}_q)|} \begin{pmatrix} 4 & q-3 \end{pmatrix},$ $\quad Z^{\#}_{G(\mathbb{F}_q)}$ () $= \frac{1}{|G(\mathbb{F}_q)|} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$

where, of course, $|G(\mathbb{F}_q)| = 2(q-1)$. Therefore, the number of points of the $G(\mathbb{F}_q)$-representation varieties are given by

$$|R_{G(\mathbb{F}_q)}(\Sigma_g)| = (q-3)(q-1)^{2g-1} + 2^{2g+1}(q-1)^{2g-1}.$$

Applying Theorem 4.6.1 (with $R = \mathbb{Z}[\frac{1}{2}]$), we obtain the E-polynomial

$$e(R_{G \times_R \mathbb{C}}(\Sigma_g)) = (uv-3)(uv-1)^{2g-1} + 2^{2g+1}(uv-1)^{2g-1}.$$

Geometric method. Following the geometric method, the goal is to compute the $\mathrm{K}_0(\mathbf{Stck}_k)$-module morphism Z_G (). Since $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ is not finitely generated as $\mathrm{K}_0(\mathbf{Stck}_k)$-module, it is impossible to compute this map in full, so instead we restrict to a finitely generated submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ which will be invariant under this map.

Note that, via the natural map $[G/G] \to \mathrm{B}G$, we can view $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ as a $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$-module. Moreover, from Proposition 4.8.4 it is not hard to see that Z_G () promotes to a morphism of $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$-modules.

Denote by $I \in \mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ the class of the inclusion $[\{1\}/G] \to [G/G]$ of the identity, and denote by $S \in \mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ the class of the morphism $[\mathbb{G}_m/G] \to [G/G]$ induced by the squaring map $x \mapsto x^2$. Furthermore, we will make use of the following classes in $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$. Denote by A, B and C the classes $[\mathbb{G}_m/G], [\mathbb{G}_m\sigma/G]$ and $[(\mathbb{Z}/2\mathbb{Z})/G]$, respectively, where $\mathbb{G}_m$, $\mathbb{G}_m\sigma$ (recall that σ denotes the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$) and $\mathbb{Z}/2\mathbb{Z}$ are viewed as subvarieties of G on which G acts by conjugation.

Proposition 4.11.1. *The $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$-submodule $\langle I, S\rangle \subseteq \mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ is invariant under Z_G (), and*

Z_G () $(I) = A^2 \cdot I + 3B \cdot S,$

Z_G () $(S) = (A+B)^2 \cdot S.$

Proof. The image of I is the virtual class of the morphism $[G^2/G] \to [G/G]$ induced by the commutator $[-,-]\colon G^2 \to G$. Stratifying G by $\mathbb{G}_m$ and $\mathbb{G}_m\sigma$, we find

$$[x,y] = 1, \qquad [x, y\sigma] = x^2,$$
$$[x\sigma, y] = y^{-2}, \qquad [x\sigma, y\sigma] = x^2 y^{-2}.$$

The first stratum contributes $A^2 \cdot I$. The second and third stratum both contribute $B \cdot S$. After a change of variables $x' = x^2 y^{-2}$ and $y' = y$, we find that the fourth stratum contributes $B \cdot S$ as well.

Next, the image of S is the virtual class of the morphism $[(\mathbb{G}_m \times G^2)/G] \to [G/G]$ induced by

$$\mathbb{G}_m \times G^2 \to G, \quad (z, a, b) \mapsto z^2[a,b].$$

Stratifying G as above, this morphism is given by

$$z^2[x,y] = z^2, \qquad z^2[x, y\sigma] = x^2 z^2,$$
$$z^2[x\sigma, y] = y^{-2} z^2, \qquad z^2[x\sigma, y\sigma] = x^2 y^{-1} z^2.$$

The first stratum contributes $A^2 \cdot S$, the second and third stratum contribute $AB \cdot S$ each, and the fourth stratum contributes $B^2 \cdot S$. ☐

In order to repeatedly apply $Z_G\left(\ \right)$, we must understand how the scalars A, B and C behave under multiplication.

Lemma 4.11.2. *In* $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$*, the following relations hold:*

(i) $A^2 = (\mathbb{L}+2)A - (\mathbb{L}-2)C - (\mathbb{L}+1)$

(ii) $B^2 = AB$

(iii) $C^2 = 2C$

(iv) $AC = (\mathbb{L}-1)C$

Proof. *(ii)* and *(iii)* follow from the G-equivariant isomorphisms

$$\begin{aligned} \mathbb{G}_m\sigma \times \mathbb{G}_m\sigma &\to \mathbb{G}_m\sigma \times \mathbb{G}_m, & (x\sigma, y\sigma) &\mapsto (x\sigma, \tfrac{y}{x}\sigma) \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} &\to \{\pm 1\} \times \mathbb{Z}/2\mathbb{Z}, & (a, b) &\mapsto (ab, b) \end{aligned}$$

where G acts trivially on $\{\pm 1\}$. For *(i)*, the action of G on $\mathbb{G}_m$ by conjugation can be extended to $\mathbb{P}^1_k$, so that $A = [\mathbb{P}^1_k/G] - C$. After a change of variables on $\mathbb{P}^1_k$, the action of G can be described by $\sigma \cdot (x : y) = (-x : y)$. Note that this change of variables uses the assumption that 2 is invertible. Now, $[\mathbb{P}^1_k/G] = [\mathbb{A}^1_k/G] + 1$ where G acts on $\mathbb{A}^1_k$ by $\sigma \cdot x = -x$, and thus $A = [\mathbb{A}^1_k/G] + 1 - C$. One sees, similar to *(ii)* and *(iii)*, that $[\mathbb{A}^1_k/G]^2 = \mathbb{L}[\mathbb{A}^1_k/G]$ and $[\mathbb{A}^1_k/G]C = \mathbb{L}C$. It follows that

$$\begin{aligned} A^2 &= ([\mathbb{A}^1_k/G] + 1 - C)^2 \\ &= (\mathbb{L}+2)[\mathbb{A}^1_k/G] - 2\mathbb{L}C + 1 \\ &= (\mathbb{L}+2)A - (\mathbb{L}-2)C - (\mathbb{L}+1). \end{aligned}$$

Finally, *(iv)* follows as $AC = ([\mathbb{A}^1_k/G] + 1 - C)C = (\mathbb{L}-1)C$. □

The above lemma, in combination with Proposition 4.11.1, allows us to obtain the images under $Z_G\left(\ \right)$ of the elements

$$I, \quad A \cdot I, \quad C \cdot I, \quad B \cdot S, \quad AB \cdot S, \quad BC \cdot S.$$

Moreover, it follows that the $\mathrm{K}_0(\mathbf{Stck}_k)$-submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ generated by these elements is invariant under $Z_G\left(\ \right)$. In terms of these generators, the map $Z_G\left(\ \right)$ is represented by the following matrix.

$$\begin{bmatrix} -\mathbb{L}-1 & -\mathbb{L}^2-3\mathbb{L}-2 & 0 & 0 & 0 & 0 \\ \mathbb{L}+2 & \mathbb{L}^2+3\mathbb{L}+3 & 0 & 0 & 0 & 0 \\ 2-\mathbb{L} & -2\mathbb{L}^2+3\mathbb{L}+2 & \mathbb{L}^2-2\mathbb{L}+1 & 0 & 0 & 0 \\ 3 & 0 & 0 & -4\mathbb{L}-4 & -4\mathbb{L}^2-12\mathbb{L}-8 & 0 \\ 0 & 3 & 0 & 4\mathbb{L}+8 & 4\mathbb{L}^2+12\mathbb{L}+12 & 0 \\ 0 & 0 & 3 & 8-4\mathbb{L} & -8\mathbb{L}^2+12\mathbb{L}+8 & 4\mathbb{L}^2-8\mathbb{L}+4 \end{bmatrix}$$

One diagonalizes this matrix with eigenvalues

$$1, \quad 4, \quad (\mathbb{L}-1)^2, \quad (\mathbb{L}+1)^2, \quad 4(\mathbb{L}-1)^2, \quad 4(\mathbb{L}+1)^2,$$

and eigenvectors

$$\begin{bmatrix} \mathbb{L}+1 \\ -1 \\ -1 \\ -\mathbb{L}-1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbb{L}+1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ (\mathbb{L}-1)^2 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2(\mathbb{L}+1)^2 \\ -2(\mathbb{L}+1)^2 \\ (\mathbb{L}-2)(\mathbb{L}+1)^2 \\ -2 \\ 2 \\ 2-\mathbb{L} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -2 \\ \mathbb{L}-2 \end{bmatrix},$$

respectively. From the decomposition $\Sigma_g =$  $\circ$  $^g \circ$ , we can now compute the virtual class $[\mathfrak{X}_G(\Sigma_g)]$ in $\mathrm{K}_0(\mathbf{Stck}_{BG})$. Using that

Z_G () $(1) = I$, $\quad Z_G$ () $(I) = 1 \quad$ and $\quad Z_G$ () $(S) = 2$

we obtain the following theorem.

Theorem 4.11.3. *Let $G = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ over a field of characteristic not equal to 2. The virtual class of the G-character stack $\mathfrak{X}_G(\Sigma_g)$ in $\mathrm{K}_0(\mathbf{Stck}_{BG})$ equals*

$$\begin{aligned} [\mathfrak{X}_G(\Sigma_g)] = {} & \mathbb{L}^{-1}(\mathbb{L}+1-(\mathbb{L}+1)^{2g}) \\ & + \mathbb{L}^{-1}((\mathbb{L}+1)^{2g}-1)A \\ & + 2\mathbb{L}^{-1}(4^g-1)(\mathbb{L}-(\mathbb{L}+1)^{2g-2}+1)B \\ & + \tfrac{1}{2}\mathbb{L}^{-1}(\mathbb{L}(\mathbb{L}-1)^{2g}-(\mathbb{L}-2)(\mathbb{L}+1)^{2g}-2)C \\ & + 2\mathbb{L}^{-1}(4^g-1)((\mathbb{L}+1)^{2g-2}-1)AB \\ & + \mathbb{L}^{-1}(4^g-1)(\mathbb{L}(\mathbb{L}-1)^{2g-2}-(\mathbb{L}-2)(\mathbb{L}+1)^{2g-2}-2)BC. \qquad \square \end{aligned}$$

Finally, in order to obtain the virtual class of the character stack in $\mathrm{K}_0(\mathbf{Stck}_k)$, we simply need to compute the images of $1, A, B, C, AB$ and BC under the morphism

$$c_! \colon \mathrm{K}_0(\mathbf{Stck}_{BG}) \to \mathrm{K}_0(\mathbf{Stck}_k).$$

Lemma 4.11.4. *In $\mathrm{K}_0(\mathbf{Stck}_k)$, the following equalities hold:*

(i) $c_!(1) = \mathbb{L}/(\mathbb{L}^2-1)$

(ii) $c_!(A) = 1$

(iii) $c_!(B) = 1$

(iv) $c_!(C) = (\mathbb{L}-1)^{-1}$

(v) $c_!(AB) = \mathbb{L}$

(vi) $c_!(BC) = 1$

Proof. *(i)* View G as the subgroup of GL_2 generated by $\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. From the GL_2-torsor $[\mathrm{GL}_2/G] \to \mathrm{B}G$ follows that $[\mathrm{B}G] = [\mathrm{GL}_2/G]/[\mathrm{GL}_2]$. Writing $\mathrm{GL}_2 = \operatorname{Spec} k[a, b, c, d, (ad - bc)^{-1}]$, we can identify

$$[\mathrm{GL}_2/G] = \operatorname{Spec} k[ab, cd, ad + bc, (ad - bc)^{-2}] \\ = \operatorname{Spec} k[x, y, z, (z^2 - 4xy)^{-1}]$$

whose virtual class is easily seen to be $\mathbb{L}^2(\mathbb{L} - 1)$. Hence, we obtain $c_!(1) = \mathbb{L}^2(\mathbb{L} - 1)/[\mathrm{GL}_2] = \mathbb{L}/(\mathbb{L}^2 - 1)$.

(ii) From the GL_2-torsor $\mathrm{GL}_2 \times_G \mathbb{G}_m \to [\mathbb{G}_m/G]$, it follows that $[\mathbb{G}_m/G] = [\mathbb{G}_m \times_G \mathrm{GL}_2]/[\mathrm{GL}_2]$, where we can identify $[\mathbb{G}_m \times_G \mathrm{GL}_2]$ with

$$\operatorname{Spec} k[ab, cd, ad + bc, x + x^{-1}, (ad - bc)(x - x^{-1}), (ad - bc)^{-2}] \\ = \operatorname{Spec} k[u, v, w, t, s, (w^2 - 4uv)^{-1}]/(s^2 - (t^2 - 4)(w^2 - 4uv))$$

whose virtual class can be computed as $\mathbb{L}(\mathbb{L} - 1)^2(\mathbb{L} + 1)$. Hence, we obtain $c_!(A) = \mathbb{L}(\mathbb{L} - 1)^2(\mathbb{L} + 1)/[\mathrm{GL}_2] = 1$.

(iii) This case is analogous to *(ii)*.

(iv) As $[(\mathbb{Z}/2\mathbb{Z})/G] = \mathrm{B}\mathbb{G}_m$ and $\mathbb{G}_m$ is special, we find $[(\mathbb{Z}/2\mathbb{Z})/G] = (\mathbb{L} - 1)^{-1}$.

(v) Note that $[\mathbb{G}_m\sigma \times \mathbb{G}_m/G] \cong \mathrm{B}(\mathbb{Z}/2\mathbb{Z}) \times [\mathbb{G}_m/\langle\sigma\rangle]$. Since $[\mathrm{B}(\mathbb{Z}/2\mathbb{Z})] = 1$ and $[\mathbb{G}_m/\langle\sigma\rangle] = \mathbb{L}$, we find that $[\mathbb{G}_m\sigma \times \mathbb{G}_m/G] = \mathbb{L}$.

(vi) Finally, $[(\mathbb{G}_m\sigma \times (\mathbb{Z}/2\mathbb{Z}))/G] = [\mathrm{B}(\mathbb{Z}/2\mathbb{Z})] = 1$. □

Corollary 4.11.5. *Let $G = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ over a field k of characteristic not equal to 2. The virtual class of the G-character stack $\mathfrak{X}_G(\Sigma_g)$ in $\mathrm{K}_0(\mathbf{Stck}_k)$ is given by*

$$[\mathfrak{X}_G(\Sigma_g)] = \frac{(\mathbb{L} - 1)^{2g-2}\left(2^{2g+1} + \mathbb{L} - 3\right)}{2} + \frac{(\mathbb{L} + 1)^{2g-2}\left(2^{2g+1} + \mathbb{L} - 1\right)}{2}. \quad \square$$

Indeed, note that $[\mathfrak{X}_G(\Sigma_g)] \neq [R_G(\Sigma_g)]/[G]$ in $\mathrm{K}_0(\mathbf{Stck}_k)$, reflecting the fact that G is not connected.

4.12 Representation variety TQFT

While the construction of the TQFT of Theorem 4.7.8 is quite elegant, using it to explicitly compute the virtual class of character stacks can be rather hard. When M is a connected closed manifold and G a special algebraic group, the virtual class of the G-character stack $\mathfrak{X}_G(M)$ can also be computed as

$[\mathfrak{X}_G(M)] = [R_G(M)]/[G]$ by Proposition 3.5.5. Hence, if there were to exist a (lax) TQFT that quantizes the virtual class of the G-representation variety $R_G(M)$, this would lead to a more practical approach, as more stratifications will be allowed for in computations: stratifications on $R_G(M)$, as opposed to stratifications on $\mathfrak{X}_G(M)$, need not be G-equivariant with respect to the action of G by conjugation. Such a TQFT was proposed by [GLM20], making use of *pointed bordisms* instead of bordisms, that is, bordisms equipped with a choice of basepoints on their boundaries. These basepoints are used to keep track of any non-trivial loops that arise when bordisms are composed. The downside of this TQFT is that it does not quite quantize the virtual class of $R_G(M)$, but rather $[G]^n[R_G(M)]$, where n is the number of basepoints on M. Without these basepoints, no such TQFT exists.

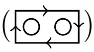
Nevertheless, we can define the following morphisms, which will effectively compute the virtual class of the representation variety. However, we stress that these maps do *not* come from a TQFT.

Definition 4.12.1. Let G be a linear algebraic group over a field k. Define the following $\mathrm{K}_0(\mathbf{Var}_k)$-module morphisms,

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{Var}_G), \quad 1 \mapsto \left[\{1\} \to G\right]$

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_k), \quad \begin{bmatrix} X \\ \downarrow f \\ G \end{bmatrix} \mapsto \left[f^{-1}(1)\right]$

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_G), \quad \begin{bmatrix} X \\ \downarrow f \\ G \end{bmatrix} \mapsto \begin{bmatrix} X \times G^2 & (x, A, B) \\ \downarrow & \downarrow \\ G & f(x)[A, B] \end{bmatrix}$

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_G), \quad \begin{bmatrix} X \\ \downarrow f \\ G \end{bmatrix} \mapsto \begin{bmatrix} X \times G & (x, A) \\ \downarrow & \downarrow \\ G & f(x)A^2 \end{bmatrix}$

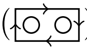
where $\{1\} \to G$ is the inclusion of the unit, and $[A, B] = ABA^{-1}B^{-1}$ denotes the group commutator. Furthermore, we define the $\mathrm{K}_0(\mathbf{Var}_k)$-module morphism

Z_G^{rep} () $: \begin{array}{rcl} \mathrm{K}_0(\mathbf{Var}_G) \otimes \mathrm{K}_0(\mathbf{Var}_G) & \to & \mathrm{K}_0(\mathbf{Var}_G) \\ \left[X \xrightarrow{f} G\right] \otimes \left[Y \xrightarrow{g} G\right] & \mapsto & \left[X \times Y \xrightarrow{m \circ (f \times g)} G\right] \end{array}$

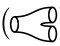
where $m \colon G \times G \to G$ denotes the multiplication map.

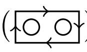
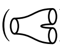
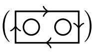
From the explicit presentations of the G-representation varieties of the orientable and non-orientable surfaces, $R_G(\Sigma_g)$ and $R_G(N_r)$, as in Example 2.1.4, it follows that their virtual classes can be computed through the following formulas.

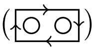
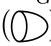
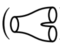
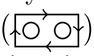
$[R_G(\Sigma_g)] = Z_G^{\mathrm{rep}}$ () $\circ\, Z_G^{\mathrm{rep}}$ ()^g $\circ\, Z_G^{\mathrm{rep}}$ () (1) (4.10)

$[R_G(N_r)] = Z_G^{\mathrm{rep}}$ () $\circ\, Z_G^{\mathrm{rep}}$ ()^g $\circ\, Z_G^{\mathrm{rep}}$ () (1) (4.11)

Moreover, it follows immediately from the expressions in Definition 4.12.1 that the following relations hold, for all $X \in \mathrm{K}_0(\mathbf{Var}_G)$:

Z_G^{rep} () $(X) = Z_G^{\mathrm{rep}}$ () $\big(X \otimes \big(Z_G^{\mathrm{rep}}$ () $\circ\, Z_G^{\mathrm{rep}}$ ()$\big)(1)\big)$ (4.12)

Z_G^{rep} () $(X) = Z_G^{\mathrm{rep}}$ () $\big(X \otimes \big(Z_G^{\mathrm{rep}}$ () $\circ\, Z_G^{\mathrm{rep}}$ ()$\big)(1)\big)$ (4.13)

Let us explain why the above equations are useful. They show that, in order to compute Z_G^{rep} () and Z_G^{rep} (), it suffices to understand only the image of Z_G^{rep} () (1) under these maps, and to compute Z_G^{rep} (). However, this latter is only 'linear' in the two inputs, whereas the original maps Z_G^{rep} () and Z_G^{rep} () are 'quadratic' in their inputs. This will result in significant simplifications in the concrete computations of the following chapters.