Motivic invariants of character stacks

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Chapter 3

Motivic invariants

When studying a geometric object, say a compact manifold X, one can try to understand X by means of its invariants. One of the simplest invariants is the *Euler characteristic* of X, a topological invariant, which is an integer $\chi(X) \in \mathbb{Z}$ given by the alternating sum of its Betti numbers

$$\chi(X) = \sum_{k \geq 0} (-1)^k \dim_{\mathbb{C}} H^k(X; \mathbb{C}).$$

There are many ways in which the Euler characteristic can be refined. For instance, when X is (the analytification of) a smooth projective complex variety, the cohomology groups $H^k(X; \mathbb{C})$ admit a Hodge structure by the Hodge decomposition theorem [PS08, Theorem 1.8]. The *Hodge polynomial* of X,

$$P_{\text{Hodge}}(X) = \sum_{p,q \geq 0} \dim_{\mathbb{C}} H^{p,q}(X)\, u^p v^q \in \mathbb{Z}[u, v] \tag{3.1}$$

specializes to the Euler characteristic for $u = v = -1$. One may replace $H^k(X; \mathbb{C})$ by the compactly supported cohomology groups $H^k_c(X; \mathbb{C})$ in order to extend the Euler characteristic to non-compact X. Analogously, as explained in Section 3.1, by work of Deligne [Del71b, Del74] the Hodge polynomial can be extended to an invariant for all complex varieties, possibly non-smooth and non-projective, called the *E-polynomial* $e(X) \in \mathbb{Z}[u, v]$, also known as the *Hodge–Deligne polynomial* or *Serre polynomial.* This invariant is additive and multiplicative in the sense that $e(X) = e(Z) + e(X \setminus Z)$ and $e(X \times Y) = e(X)\, e(Y)$ for all complex varieties X and Y and closed subvarieties $Z \subseteq X$.

The goal of this chapter is to discuss various such invariants, and to give tools for computing them. Our main focus will be on the invariant that takes values in the *Grothendieck ring of varieties*, defined in Section 3.2, which is universal among all additive and multiplicative invariants.

3.1 Mixed Hodge structures

Let X be a complex variety. It was shown by Deligne [Del71a, Del71b] that the singular cohomology groups with compact support $H^k_c(X;\mathbb{Q})$ naturally admit the structure of a mixed Hodge structure. Let us recall the definition of a (mixed) Hodge structure.

Definition 3.1.1. A *Hodge structure of weight* $k \in \mathbb{Z}$ is a pair $(H, F^\bullet H)$ consisting of a finite-dimensional rational vector space H and a decreasing filtration $F^\bullet H$ on $H_\mathbb{C} = H \otimes_\mathbb{Q} \mathbb{C}$,

$$H_\mathbb{C} \supseteq \cdots \supseteq F^p H \supseteq F^{p+1} H \supseteq \cdots \supseteq 0,$$

such that $H_\mathbb{C} = F^p \oplus \overline{F^q}$ for $p+q = k+1$. A morphism of Hodge structures of the same weight $(H, F^\bullet H) \to (H', F^\bullet H')$ is a linear map $f \colon H \to H'$ which preserves the filtration, that is, $f_\mathbb{C}(F^p H) \subseteq F^p H'$ for all p. A *mixed Hodge structure* is a triple $(H, W_\bullet H, F^\bullet H)$ consisting of a finite-dimensional rational vector space H, an increasing filtration $W_\bullet H$ on H, called the *weight filtration*,

$$0 \subseteq \cdots \subseteq W_k H \subseteq W_{k+1} H \subseteq \cdots \subseteq H,$$

and a decreasing filtration $F^\bullet H$ on $H_\mathbb{C}$,

$$H_\mathbb{C} \supseteq \cdots \supseteq F^p H \supseteq F^{p+1} H \supseteq \cdots \supseteq 0,$$

such that the induced filtration of $F^\bullet H$ on the graded pieces $(\operatorname{Gr}^W_k H) \otimes_\mathbb{Q} \mathbb{C} = (W_k H / W_{k+1} H) \otimes_\mathbb{Q} \mathbb{C}$ are Hodge structures of weight k. A morphism of mixed Hodge structures is a linear map which preserves both the increasing and decreasing filtration. The categories of Hodge structures and of mixed Hodge structures are denoted by **HS** and **MHS**, respectively.

Now, more precisely, Deligne showed that the cohomology groups $H^k_c(X;\mathbb{Q})$ and their complexification $H^k_c(X;\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} = H^k_c(X;\mathbb{C})$ can naturally be equipped with weight filtrations $W_\bullet$ and decreasing filtrations $F^\bullet$, respectively, such that the triples $H^k_c(X) = (H^k_c(X;\mathbb{Q}), W_\bullet, F^\bullet)$ are mixed Hodge structures. Moreover, the construction is functorial in X, agrees with the usual Hodge decomposition when X is smooth and projective, and is compatible with various classical exact sequences in cohomology. For the explicit construction and more details, we refer to [Del71b, Del74, PS08].

There is an exact functor from the category of mixed Hodge structures to the category of finite-dimensional bigraded complex vector spaces [Del71b, Theorem 1.2.10]:

$$\operatorname{Gr}^*_F \operatorname{Gr}^W_* : \mathbf{MHS} \to (\mathbf{Vect}^{\mathbb{Z}\times\mathbb{Z}}_\mathbb{C})_{\text{fin}}, \quad H \mapsto \bigoplus_{p,q\in\mathbb{Z}} \operatorname{Gr}^p_F \operatorname{Gr}^W_{p+q} H_\mathbb{C} \tag{3.2}$$

In the case of a mixed Hodge structure on $H^k_c(X;\mathbb{Q})$, we denote its bigraded pieces by

$$H^{k;p,q}_c(X) = \operatorname{Gr}^p_F \operatorname{Gr}^W_{p+q} H^k_c(X;\mathbb{C}).$$

In fact, $H^{k;p,q}_c(X)$ is non-zero only if $p, q \geq 0$. The dimensions of these vector spaces can be collected as the coefficients of a polynomial. This way we obtain the following definition, as first introduced in [DK86].

Definition 3.1.2. Let X be a complex variety. The *E-polynomial* of X (also known as the *Hodge–Deligne polynomial* or the *Serre polynomial*) is the polynomial $e(X) \in \mathbb{Z}[u, v]$ given by

$$e(X) = \sum_{k,p,q\in\mathbb{Z}} (-1)^k \dim_{\mathbb{C}} H^{k;p,q}_c(X)\, u^p v^q.$$

In particular, when X is smooth and projective, the E-polynomial $e(X)$ coincides with the Hodge polynomial (3.1), up to the change of signs induced by $u \mapsto -u$ and $v \mapsto -v$.

Amazingly, the E-polynomial is additive and multiplicative, in the sense that

$$e(X) = e(Z) + e(X \setminus Z) \quad \text{and} \quad e(X \times Y) = e(X)\, e(Y) \tag{3.3}$$

for complex varieties X and Y, and $Z \subseteq X$ a closed subvariety. These properties follow from the long exact sequence

$$\cdots \to H^k_c(X \setminus Z;\mathbb{C}) \to H^k_c(X;\mathbb{C}) \to H^k_c(Z;\mathbb{C}) \to H^{k+1}_c(X \setminus Z;\mathbb{C}) \to \cdots \tag{3.4}$$

of mixed Hodge structures [PS08, p.138], and the Künneth formula [Del74, Proposition 8.2.10], respectively, together with the fact that (3.2) is exact.

3.2 Grothendieck ring of varieties

As seen in the previous section, the E-polynomial is an additive and multiplicative invariant (3.3). In this section, we will define the *Grothendieck ring of varieties*: the ring in which the universal invariant, among all additive and multiplicative invariants, takes values. This means, in particular, that when computing the E-polynomial of some complex variety using only these properties, one might as well compute the invariant in the Grothendieck ring of varieties, to obtain a more refined invariant. One of the advantages to the Grothendieck ring of varieties is that, as opposed to other invariants, it can be defined for varieties over any field k, and also more generally in the relative setting for varieties over a base variety S.

The Grothendieck ring of varieties $\mathrm{K}_0(\mathbf{Var}_k)$ was originally introduced in a letter from Grothendieck to Serre [CS01, 16 Aug. 1964], and came with a hypothetical morphism

$$\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{M}(k)) \tag{3.5}$$

to the 'Grothendieck group of the abelian category of motives'. For this reason, we refer to these invariants as *motivic invariants*. To gain some understanding about this morphism, we first introduce the Grothendieck group of an abelian or triangulated category.

Definition 3.2.1. The *Grothendieck group* of an abelian category $\mathcal{A}$, denoted $\mathrm{K}_0(\mathcal{A})$, is the free abelian group on isomorphism classes $[A]$ of objects A of $\mathcal{A}$, modulo the relations

$$[B] = [A] + [C]$$

for all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$. Similarly, the *Grothendieck group* of a triangulated category $\mathcal{A}$, also denoted $\mathrm{K}_0(\mathcal{A})$, is the free abelian group on isomorphism classes $[A]$ of objects A of $\mathcal{A}$, modulo the relations

$$[B] = [A] + [C]$$

for all distinguished triangles $A \to B \to C \to A[1]$ in $\mathcal{A}$. When $\mathcal{A}$ is a tensor triangulated category, the tensor product $\otimes$ induces the structure of a commutative ring on $\mathrm{K}_0(\mathcal{A})$ given on generators by

$$[A][B] = [A \otimes B].$$

Remark 3.2.2. The Grothendieck group of an abelian category $\mathcal{A}$ is naturally isomorphic to that of its derived category $D^b(\mathcal{A})$ as triangulated category. In particular, the functor $\mathcal{A} \to D^b(\mathcal{A})$, which assigns to any object A the complex with A concentrated in degree 0, induces a morphism $\mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(D^b(\mathcal{A}))$. It is an easy exercise in homological algebra to show that an inverse is given by $[A^\bullet] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [H^i(A^\bullet)]$.

Even though the category of varieties is neither abelian nor triangulated (not even additive), the Grothendieck ring of varieties is defined similarly, where exact sequences are replaced by closed immersions with open complements. For many invariants, these notions can be related through a long exact sequence such as (3.4).

Definition 3.2.3. Let S be a variety over a field k. The *Grothendieck ring of varieties over S*, denoted $\mathrm{K}_0(\mathbf{Var}_S)$, is the free abelian group on isomorphism classes $[X]$ of varieties X over S, modulo the relations

$$[X] = [Z] + [X \setminus Z]$$

for all closed immersions $Z \to X$ of varieties over S. It admits the structure of a commutative ring, where multiplication is given on generators by

$$[X][Y] = [(X \times_S Y)^{\mathrm{red}}].$$

In particular, the classes $[\varnothing]$ and $[S]$ are the zero and unit of this ring, respectively. For any variety X over S, the element $[X]$ in $\mathrm{K}_0(\mathbf{Var}_S)$ is also known as the *virtual class* of X.

Remark 3.2.4. Although the Grothendieck ring is generated by isomorphism classes of varieties X, one could allow for X to be non-reduced without affecting the ring. Indeed, $X^{\mathrm{red}} \subseteq X$ is a closed subscheme with complement $\varnothing$, so that $[X^{\mathrm{red}}] = [X]$. Also, in this case, one can define multiplication simply by $[X][Y] = [X \times_S Y]$. Similarly, we can omit the condition that X be separated since any scheme X of finite type over S can be partitioned into finitely many separated subschemes $X_1, \dots, X_n$, so that $[X] = [X_1] + \cdots + [X_n]$. However, we cannot permit any X which is not quasi-compact over S. For example, if $X = \bigsqcup_{\mathbb{Z}} S$ and $Z = S$, then $X \setminus Z \cong X$ which would imply $1 = [Z] = 0$, collapsing the ring to the trivial ring. Indeed, Grothendieck originally defined his ring allowing for isomorphism classes of all schemes X of finite type over S [CS01, 16 Aug. 1964].

Notation 3.2.5. To distinguish between virtual classes over different bases, we sometimes write $[X]_S$ to emphasize the virtual class lives in $\mathrm{K}_0(\mathbf{Var}_S)$. When the base S is clear from context, or when $S = \operatorname{Spec} k$, we simply write $[X]$.

Definition 3.2.6. The virtual class of the affine line $\mathbb{A}^1_S$ over S in $\mathrm{K}_0(\mathbf{Var}_S)$ is called *Lefschetz class* and is denoted by $\mathbb{L}$.

Example 3.2.7.
- The virtual class of affine n-space is $[\mathbb{A}^n_S]_S = \mathbb{L}^n$ for any $n \geq 0$.
- Since $\mathbb{P}^n_S \setminus \mathbb{P}^{n-1}_S \cong \mathbb{A}^n_S$, it follows by induction on n that $[\mathbb{P}^n_S]_S = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1$ for all $n \geq 0$.

Example 3.2.8. The following invariants are additive and multiplicative, and hence factor through the Grothendieck ring of varieties.

- For $S = \operatorname{Spec} \mathbb{C}$, the E-polynomial (see Definition 3.1.2) factors through $\mathrm{K}_0(\mathbf{Var}_{\mathbb{C}})$, which gives a ring morphism

$$e \colon \mathrm{K}_0(\mathbf{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v], \quad [X] \mapsto e(X). \tag{3.6}$$

- For any point $\operatorname{Spec} \mathbb{F}_q \to S$, one can count $\mathbb{F}_q$-rational points

$$\#_{\mathbb{F}_q} \colon \operatorname{Ob}(\mathbf{Var}_S) \to \mathbb{Z}, \quad X \mapsto |X(\mathbb{F}_q)|.$$

  This map, being additive and multiplicative, factors through $\mathrm{K}_0(\mathbf{Var}_S)$.

- Let $S = \operatorname{Spec} k$ for a field k with $\operatorname{char}(k) = 0$. Then there is a ring morphism

  $$\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q}))$$

  to the Grothendieck group of the $\mathbb{Q}$-linearization of Voevodsky's triangulated category of effective geometric motives [BD07, Appendix A]. This morphism sends the virtual class $[X]$ of a variety X over k to the class $[M^c_{\mathrm{gm}}(X)]$ of its motive with compact support. Viewing $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ as a substitute for the derived category of the 'abelian category of motives', this map would be the morphism (3.5) that Grothendieck had in mind in his letter.
- Again, let $S = \operatorname{Spec} k$ for a field k with $\operatorname{char}(k) = 0$. Then there is a ring morphism

  $$\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{CHMot}_k)$$

  to the Grothendieck group of the category of Chow motives over k with rational coefficients [GN02, (5.5)].

Let us describe some formal and functorial properties of the Grothendieck ring of varieties. Given a morphism $f \colon X \to Y$ of varieties over S, there is an induced ring morphism

$$f^* \colon \mathrm{K}_0(\mathbf{Var}_Y) \to \mathrm{K}_0(\mathbf{Var}_X), \quad [W]_Y \mapsto [W \times_Y X]_X.$$

Indeed, this map is well-defined since, for any variety W over Y and closed subvariety $Z \subseteq W$, we have $[W \times_Y X]_X = [Z \times_Y X]_X + [(W \setminus Z) \times_Y X]_X$. Similarly, f^* respects multiplication as $(W \times_Y W') \times_Y X \cong (W \times_Y X) \times_X (W' \times_Y X)$ for any varieties W and W' over Y. The morphism f^* turns $\mathrm{K}_0(\mathbf{Var}_X)$ into a $\mathrm{K}_0(\mathbf{Var}_Y)$-algebra, and in particular $\mathrm{K}_0(\mathbf{Var}_X)$ is a $\mathrm{K}_0(\mathbf{Var}_S)$-algebra for every variety X over S. Moreover, f^* is a morphism of $\mathrm{K}_0(\mathbf{Var}_S)$-algebras.

Similarly, the morphism $f \colon X \to Y$ induces a map

$$f_! \colon \mathrm{K}_0(\mathbf{Var}_X) \to \mathrm{K}_0(\mathbf{Var}_Y), \quad [W]_X \mapsto [W]_Y$$

which is a morphism of $\mathrm{K}_0(\mathbf{Var}_S)$-modules. However, note that $f_!$ is generally not a morphism of rings.

Remark 3.2.9. The maps f^* and $f_!$ can more generally be seen as functors

$$\mathbf{Var}_X \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathbf{Var}_Y$$

given by pulling back along f and post-composing with f, respectively, forming an adjoint pair $f_! \dashv f^*$. Indeed, for any varieties U over X and V over Y there is a natural bijection

$$\mathrm{Hom}_Y(U, V) \cong \mathrm{Hom}_X(U, V \times_Y X).$$

Example 3.2.10. Let X and Y be varieties over S. There is a natural morphism of $\mathrm{K}_0(\mathbf{Var}_S)$-algebras

$$\mathrm{K}_0(\mathbf{Var}_X) \otimes_{\mathrm{K}_0(\mathbf{Var}_S)} \mathrm{K}_0(\mathbf{Var}_Y) \to \mathrm{K}_0(\mathbf{Var}_{X\times_S Y})$$

given, on generators, by $[U]_X \otimes [V]_Y \mapsto [U \times_S V]_{X\times_S Y}$ for all varieties U over X and V over Y. This map is generally not surjective. For example, let $X = Y = \mathbb{A}^1_k$ over $S = \operatorname{Spec} k$ for a finite field $k = \mathbb{F}_q$. Consider the class $[\Delta_{\mathbb{A}^1_k}]$ of the diagonal in $X \times Y = \mathbb{A}^2_k$, and suppose $[\Delta_{\mathbb{A}^1_k}]$ is equal to the image of $\sum_{i=1}^n u_i \otimes v_i$ under this map for some $u_i, v_i \in \mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^1_k})$. Note that every $\mathbb{F}_{q^m}$-rational point $x \in \mathbb{A}^1_k(\mathbb{F}_{q^m})$ induces a ring morphism

$$\#_x \colon \mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^1_k}) \to \mathbb{Z}, \quad [U \xrightarrow{f} \mathbb{A}^1_k] \mapsto |f^{-1}(x)|$$

counting the number of $\mathbb{F}_{q^m}$-rational points in the fiber over x. The same construction works for $\mathbb{A}^2_k$, and together they induce the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^1_k}) \otimes_{\mathrm{K}_0(\mathbf{Var}_k)} \mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^1_k}) & \longrightarrow & \mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^2_k}) \\
\downarrow & & \downarrow \\
\left(\prod_{x\in\mathbb{A}^1_k(\mathbb{F}_{q^m})} \mathbb{Z}\right) \otimes \left(\prod_{x\in\mathbb{A}^1_k(\mathbb{F}_{q^m})} \mathbb{Z}\right) & & \prod_{x\in\mathbb{A}^2_k(\mathbb{F}_{q^m})} \mathbb{Z} \\
\| & & \| \\
\mathbb{Z}^{q^m} \otimes \mathbb{Z}^{q^m} & \xrightarrow{\sim} & \mathbb{Z}^{q^m\times q^m}
\end{array}$$

Now, the image of $[\Delta_{\mathbb{A}^1_k}]$ in $\mathbb{Z}^{q^m\times q^m}$ corresponds to the $q^m \times q^m$ identity matrix, which has rank q^m, while the image of $\sum_{i=1}^n u_i \otimes v_i$ has rank at most n. This yields a contradiction for sufficiently large m.

3.3 Stratifications and fibrations

Let S be a variety over a field k.

Definition 3.3.1. Let X be a variety over S. A *stratification* of X is a collection of disjoint locally closed subvarieties $\{X_i\}_{i\in I}$ of X such that $X = \bigcup_{i\in I} X_i$.

Lemma 3.3.2 ([Bri12, Lemma 2.2])**.** *Let X be a variety over S and $\{X_i\}_{i\in I}$ a stratification of X. Then only finitely many of the X_i are non-empty and $[X] = \sum_{i\in I}[X_i]$ in $\mathrm{K}_0(\mathbf{Var}_S)$.*

Proof. Proof by induction on the dimension of X. If $\dim X = 0$, then X is a finite set of points, and the result is clear. Now assume that $\dim X > 0$ and that the result holds for all varieties of dimension less than $\dim X$.

We prove the result for X by induction on the number of irreducible components of X. If this number is 1, that is, X is irreducible, then some $U = X_i$ contains the generic point of X and is therefore open. The complement $Z = X \setminus U$ is of smaller dimension than X and is stratified by the other X_i. Since $[X] = [Z]+[U]$, the result follows from the induction hypothesis (on $\dim X$).

Now suppose that X is reducible. Take an irreducible component and remove the intersections with the other irreducible components, which gives an irreducible open subset $U \subseteq X$. The complement $Z = X \setminus U$ is a closed subvariety with fewer irreducible components than X. Note that $\{Z \cap X_i\}_{i\in I}$ and $\{U \cap X_i\}_{i\in I}$ are stratifications of Z and U, respectively, so only finitely many strata are non-empty, and we have

$$[U] = \sum_{i\in I}[U \cap X_i] \quad \text{and} \quad [Z] = \sum_{i\in I}[Z \cap X_i].$$

This follows, if $\dim Z$ (resp. $\dim U$) is less than $\dim X$, from the induction hypothesis on the dimension, or, if $\dim Z$ (resp. $\dim U$) is equal to $\dim X$, from the induction hypothesis on the number of irreducible components. Finally, $[X_i] = [U \cap X_i] + [Z \cap X_i]$ implies that $[X] = [U] + [Z] = \sum_{i\in I}[X_i]$. □

Lemma 3.3.3. *Let $f\colon Y \to X$ be a fiber bundle of varieties over S with fiber F which is locally trivial in the Zariski topology. That is, there exists an open cover $Y = \cup_{i\in I} U_i$ such that $f^{-1}(U_i)$ is isomorphic to $F \times U_i$ over U_i for each $i \in I$. Then $[Y]_S = [F] \cdot [X]_S$ in $\mathrm{K}_0(\mathbf{Var}_S)$.*

Proof. From the given open cover, we construct a stratification of Y as follows. Let $Z_0 = Y$ and inductively construct Z_{j+1} for $j \geq 0$: if $Z_j \neq \varnothing$, there exists some $i \in I$ such that $Z_j \cap U_i \neq \varnothing$, and set $Z_{j+1} = Z_j \setminus (Z_j \cap U_i)$. As Y is noetherian, this results in a finite descending chain of closed sets

$$Y = Z_0 \supsetneq Z_1 \supsetneq \ldots \supsetneq Z_n \supsetneq Z_{n+1} = \varnothing$$

and the locally closed sets $Y_j = Z_j \setminus Z_{j+1}$ for $j = 0, 1, \ldots, n$ form a stratification of Y. Moreover, since $Y_j \subseteq U_i$ for some i by construction, f is trivial over each Y_j, that is, $f^{-1}(Y_j) \cong F \times Y_j$. Using Lemma 3.3.2 we conclude

$$[X]_S = \sum_{j=0}^{n} \left[f^{-1}(Y_j)\right]_S = \sum_{j=0}^{n}[F] \cdot [Y_j]_S = [F] \cdot [Y]_S. \qquad \square$$

Example 3.3.4. For any $n \geq 0$, the natural morphism $\mathbb{A}_S^{n+1} \setminus \{0\} \to \mathbb{P}_S^n$ is a fiber bundle with fiber $\mathbb{G}_m$ which is locally trivial in the Zariski topology. Indeed, we have

$$[\mathbb{G}_m] \cdot [\mathbb{P}_S^n]_S = (\mathbb{L} - 1)(\mathbb{L}^n + \cdots + \mathbb{L} + 1) = \mathbb{L}^{n+1} - 1 = [\mathbb{A}_S^{n+1} \setminus \{0\}]_S.$$

Example 3.3.5. Consider the general linear group GL_n of rank n over a field k. The morphism $\mathrm{GL}_n \to \operatorname{Spec} k$ factors as

$$\mathrm{GL}_n = Y_n \to Y_{n-1} \to \cdots \to Y_0 = \operatorname{Spec} k$$

where $Y_m \subseteq \mathrm{Mat}_{n\times m}$ denotes the locally closed subvariety of m-linearly independent vectors over k, and the morphisms are given by forgetting the last vector. Now, for any $m = 1, \ldots, n$, the variety Y_m can be regarded as the open complement in $Y_{m-1} \times \mathbb{A}^n_k$ of the closed subvariety $Y_{m-1} \times \mathbb{A}^{m-1}_k$. In particular, $[Y_m] = (\mathbb{L}^n - \mathbb{L}^{m-1})[Y_{m-1}]$. Therefore, by induction on m, we obtain

$$[\mathrm{GL}_n] = \prod_{m=1}^{n} (\mathbb{L}^n - \mathbb{L}^{m-1}).$$

Proposition 3.3.6. *Let S be a variety with stratification $\{S_i\}_{i\in I}$ and write $f_i \colon S_i \to S$ for the immersion of S_i into S. Then the map*

$$\mathrm{K}_0(\mathbf{Var}_S) \to \bigoplus_{i\in I} \mathrm{K}_0(\mathbf{Var}_{S_i}), \quad X \mapsto (f_i^* X)_{i\in I}$$

is an isomorphism of $\mathrm{K}_0(\mathbf{Var}_k)$-algebras.

Proof. Every f_i^* is a morphism of $\mathrm{K}_0(\mathbf{Var}_k)$-algebras, so this map is as well. Its inverse is given by $(X_i)_{i\in I} \mapsto \sum_{i\in I} (f_i)_! X_i$, which is well-defined because only finitely many S_i are non-empty by Lemma 3.3.2. Indeed, it is a right inverse to the given map because

$$f_i^*(f_j)_! = \begin{cases} \mathrm{id}_{\mathrm{K}_0(\mathbf{Var}_{S_i})} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is a left inverse because any variety T over S is stratified by $\{T \times_S S_i\}_{i\in I}$, so that

$$[T]_S = \sum_{i\in I} [T \times_S S_i]_S = \sum_{i\in I} (f_i)_! f_i^* [T]_S. \qquad \square$$

Notation 3.3.7. For any $X \in \mathrm{K}_0(\mathbf{Var}_S)$, we will write $X|_{S_i} \in \mathrm{K}_0(\mathbf{Var}_{S_i})$ for the components of the image of X under this isomorphism.

Inclusion-exclusion matrix

Let X be a variety over S with stratification $\{X_i\}_{i\in I}$ and let Y be a variety over X. The goal of this subsection is to show that, in order to compute the virtual classes $[Y \times_X X_i]$ in $\mathrm{K}_0(\mathbf{Var}_S)$ for all i, it is sufficient to compute the virtual classes $[Y \times_X \overline{X}_i]$ for all i instead, where $\overline{X}_i$ denotes the Zariski closure of X_i in X, making use of an inclusion-exclusion principle.

Example 3.3.8. Suppose X is stratified by a closed subvariety $X_0 \subseteq X$ and its open complement $X_1 = X \backslash X_0$. If we were to compute $[X_0]$ and $[X_1]$ in $\mathrm{K}_0(\mathbf{Var}_S)$, computing the latter would likely result in the computation $[X]-[X_0]$, so that the result of the computation of $[X_0]$ can be reused. Therefore, instead of computing $[X_0]$ and $[X_1]$, one can compute $[\overline{X}_0] = [X_0]$ and $[\overline{X}_1] = [X]$, from which formally follows that $[X_1] = [\overline{X}_1] - [\overline{X}_0]$.

Lemma 3.3.9. *Let X be a variety over S with stratification $\{X_i \neq \varnothing\}_{i \in I}$. Then $\overline{X}_i = \overline{X}_j$ if and only if $i = j$.*

Proof. For each $i \in I$, write $X_i = Z_i \cap U_i$ for some closed $Z_i \subseteq X$ and open $U_i \subseteq X$. Without loss of generality, we may assume $Z_i = \overline{X}_i$. Now, if $\overline{X}_i = \overline{X}_j$ for some $i, j \in I$, then both X_i and X_j are open and dense in $\overline{X}_i = \overline{X}_j \neq \varnothing$, so they must intersect. But this contradicts the assumption that X_i and X_j are disjoint, since they are part of the stratification. □

Definition 3.3.10. Let X be a variety over S with a finite stratification $\{X_i \neq \varnothing\}_{i \in I}$. Put a partial order on I where $i \leq j$ if and only if $\overline{X}_i \subseteq \overline{X}_j$. Reflexivity and transitivity are clear, and anti-symmetry follows from the above lemma. The virtual classes $[X_i]$ and $[\overline{X}_i]$ in $\mathrm{K}_0(\mathbf{Var}_S)$ are now linearly related through

$$[\overline{X}_i] = \sum_{j \in I} A_{ij} [X_j]$$

where $A_{ij} = 1$ for $j \leq i$ and $A_{ij} = 0$ for $i < j$. Hence, the A_{ij} define a linear map $A \colon \mathbb{Z}^I \to \mathbb{Z}^I$ with determinant 1. The inverse $C = A^{-1}$ is called the *inclusion-exclusion matrix* of the stratification, and satisfies

$$[X_i] = \sum_{j \in I} C_{ij} [\overline{X}_j].$$

Corollary 3.3.11. *Let X be a variety over S with finite stratification $\{X_i\}_{i \in I}$ and corresponding inclusion-exclusion matrix C. Then for any variety Y over X, we have*

$$[Y \times_X X_i]_S = \sum_{j \in I} C_{ij} [Y \times_X \overline{X}_j]_S.$$ □

Special algebraic groups

Special algebraic groups were first introduced by Serre [Ser58]. In this section, we describe some basic properties of these groups, and show why they are extremely useful in the context of computing virtual classes.

Definition 3.3.12. An algebraic group G over a field k is *special* if any G-torsor in the étale topology is locally trivial in the Zariski topology.

Lemma 3.3.13. *Let G be a special algebraic group. Then for every G-torsor of varieties $P \to X$ in the étale topology over S, we have $[P]_S = [G] \cdot [X]_S$ in* $\mathrm{K}_0(\mathbf{Var}_S)$.

Proof. The G-torsor $P \to X$ is Zariski-locally trivial, so the result follows from Lemma 3.3.3. □

Example 3.3.14. In general, the equality $[P]_S = [G] \cdot [X]_S$ fails to hold when G is not special. Consider for instance the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ and the G-torsor $P = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} = X$ given by $x \mapsto x^n$. Then $[P] = \mathbb{L} - 1 \neq n(\mathbb{L} - 1) = [G][X]$ for $n \geq 2$, showing $\mathbb{Z}/n\mathbb{Z}$ is not special for $n \geq 2$.

Corollary 3.3.15 (Motivic orbit-stabilizer theorem)**.** *Let G be an algebraic group over k acting on a variety X. For any point $\xi \in X(k)$, if the stabilizer* $\mathrm{Stab}(\xi)$ *is special, then*

$$[G] = [\mathrm{Stab}(\xi)][\mathrm{Orbit}(\xi)]$$

in $\mathrm{K}_0(\mathbf{Var}_k)$.

Proof. Since the map $G \to \mathrm{Orbit}(\xi)$ given by $g \mapsto g \cdot \xi$ is a $\mathrm{Stab}(\xi)$-torsor, the result follows from Lemma 3.3.13. □

Proposition 3.3.16. *Let $1 \to N \hookrightarrow G \xrightarrow{\pi} H \to 1$ be an exact sequence of algebraic groups.*

(i) If N and H are special, then so is G.

(ii) If the sequence splits and G is special, then so is H.

(iii) If the sequence splits and G is special, then so is N.

Proof. (i) Any G-torsor $X \to S$ can be written as the composite of the N-torsor $X \to X/N$ and the H-torsor $X/N \to X/G \cong S$. As H is special, there exist opens $S_i \subseteq S$ such that $(X/N) \times_S S_i \cong H \times S_i$. Pulling back the N-torsor $X \times_S S_i \to H \times S_i$ along $S_i \xrightarrow{(1,\mathrm{id})} H \times S_i$ gives an N-torsor $Y_i \to S_i$, which is also Zariski-locally trivial as N is special. Hence, there exist opens $S_{ij} \subseteq S_i$ such that $Y_i \times_{S_i} S_{ij} \cong N \times S_{ij}$. There is now a natural morphism $G \times S_{ij} \to X \times_S S_{ij}$ of G-torsors over S_{ij}, which must be an isomorphism. Therefore, $X \to S$ is Zariski-locally trivial.

(ii) As sequence splits, there exists a section $\sigma \colon H \to G$ to π, i.e., $\pi \circ \sigma = \mathrm{id}_H$. Let $X \to S$ be an H-torsor, and consider the G-torsor $G \times_H X \coloneqq (G \times X)/H \to S$, where H acts on $G \times X$ via $h \cdot (g, x) = (g\sigma(h)^{-1}, h \cdot x)$. This G-torsor factors

through the N-torsor $G \times_H X \to X$ given by $(g, x) \mapsto \pi(g) \cdot x$. Hence, any trivialization of $G \times_H X$ induces a trivialization of X, and such a trivialization exists as G is special.

(iii) Let $X \to S$ be an N-torsor, and consider the G-torsor $G \times_N X \coloneqq (G \times X)/N \to S$, where N acts on $G \times X$ via $n \cdot (g, x) = (gn^{-1}, n \cdot x)$. As G is special, there exist opens $S_i \subseteq S$ and G-equivariant isomorphism $\varphi_i \colon (G \times_N X) \times_S S_i \to G \times S_i$. These induce N-equivariant isomorphisms $X \times_S S_i \to N \times S_i$ given by $(x, s) \mapsto (g\,\sigma(\pi(g^{-1})), s)$, where $(g, s) = \varphi_i((1, x), s)$, showing the S_i also trivialize X. □

Example 3.3.17.
- By Hilbert's Theorem 90, the general linear groups GL_n are special over any field k [Mil80, Proposition III.4.9, Lemma III.4.10].
- The exact sequence $1 \to \mathrm{SL}_n \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m \to 1$ splits, so it follows from Proposition 3.3.16 *(iii)* that SL_n is also special over any field k.
- The additive group $\mathbb{G}_a$ is special over any field k [Mil80, Proposition III.3.7].
- The projective linear group PGL_n is not special for $n \geq 2$. In fact, the PGL_n-torsors over a variety X which are not Zariski-locally trivial are classified by the Brauer group of X, which is in general non-trivial [Mil80, IV §2].

3.4 Algorithmic computations

Let k be a field. In this section, we describe, from a practical and computational point of view, various strategies for computing the virtual class of varieties in $\mathrm{K}_0(\mathbf{Var}_k)$, in terms of the classes of some simple varieties, such as $\mathbb{L} = [\mathbb{A}^1_k]$. These strategies are combined in a recursive algorithm, Algorithm 3.4.3. We remark already that the algorithm will not be a general recipe for computing virtual classes in $\mathrm{K}_0(\mathbf{Var}_k)$: it is allowed to fail. In fact, whenever the algorithm does not fail, it will return the virtual class of the given variety as a polynomial in $\mathbb{L}$. Of course, there exist varieties whose virtual class is not of this form, but it turns out that this algorithm is sufficiently general for the purposes of the later chapters.

In order to algorithmically manipulate varieties, we will encode them as follows. While not all varieties can be encoded in such a way, this should not be too much of a restriction since any variety can be stratified into varieties of this form.

Notation 3.4.1. Let $A = \{x_1, \ldots, x_n\}$ be a finite set, and let F and G be finite subsets of $k[A]$. Then we write

$$X(A, F, G)$$

for the reduced locally closed subvariety of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ given by $f = 0$ for all $f \in F$ and $g \neq 0$ for all $g \in G$.

Furthermore, we will introduce a notation for the evaluation of polynomials.

Notation 3.4.2. Given an element $x \in A$ and polynomials $f \in k[A]$ and $u \in k[A \setminus \{x\}]$, denote by $\operatorname{eval}^x_u(f)$ the evaluation of f in $x = u$. For polynomials $u, v \in k[A \setminus \{x\}]$, write $\operatorname{eval}^x_{u/v}(f)$ for the evaluation of f in $x = u/v$ multiplied by $v^{\deg_x(f)}$, so that $\operatorname{eval}^x_{u/v}(f) \in k[A \setminus \{x\}]$. Similarly, for subsets $F \subseteq k[A]$, write $\operatorname{eval}^x_u(F) = \{\operatorname{eval}^x_u(f) : f \in F\}$ and $\operatorname{eval}^x_{u/v}(F) = \{\operatorname{eval}^x_{u/v}(f) : f \in F\}$.

An implementation of this algorithm can be found at [Vog22].

Algorithm 3.4.3. Input: Finite sets A, F and G as in Notation 3.4.1.

Output: The virtual class $[X] \in \mathrm{K}_0(\mathbf{Var}_k)$ of $X = X(A, F, G)$ as a polynomial in $\mathbb{L} = [\mathbb{A}^1_k]$.

1. If F contains a non-zero constant or if $0 \in G$, then $X = \varnothing$, so return $[X] = 0$.
2. If $F = G = \varnothing$ or $A = \varnothing$, then $X = \mathbb{A}^{|A|}_k$, so return $[X] = \mathbb{L}^{|A|}$.
3. If $F, G \subseteq k[A \setminus \{x\}]$ for some $x \in A$, then $X \cong \mathbb{A}^1_k \times X'$ with $X' = X(A \setminus \{x\}, F, G)$, so return $[X] = \mathbb{L}\,[X']$.
4. If $f = u^n$ (with $n > 1$) for some $f \in F$ and $u \in k[A]$, then we replace f with u without changing X, that is, $X = X(A, (F \setminus \{f\}) \cup \{u\}, G)$. Similarly, if $g = u^n$ (with $n > 1$) for some $g \in G$ and $u \in k[A]$, then $X = X(S, F, (G \setminus \{g\}) \cup \{u\})$. Continue with this new presentation.
5. If $f \in k[x]$ for some $f \in F$ and $x \in A$, and if f factors as $f = c(x - a_1) \cdots (x - a_m)$ for some $c \in k^\times$ and $a_i \in k$, then return $[X] = \sum_{i=1}^m [X_i]$ with
$$X_i = X\left(A \setminus \{x\}, \operatorname{eval}^x_{a_i}(F \setminus \{f\}), \operatorname{eval}^x_{a_i}(G)\right).$$
6. Suppose $f = uv$ for some $f \in F$ and non-constant $u, v \in k[A]$. Then X is stratified by its closed subvariety given by $u = 0$ and its open complement given by $u \neq 0$ and $v = 0$. Hence, return $[X] = [X_1] + [X_2]$ with
$$X_1 = X(A, (F \setminus \{f\}) \cup \{u\}, G),$$
$$X_2 = X(A, (F \setminus \{f\}) \cup \{v\}, G \cup \{u\}).$$
7. Suppose $f = ux + v$ for some element $x \in A$ and polynomials $f \in F$ and $u, v \in k[A \setminus \{x\}]$, with u non-zero. Then X is stratified by its closed

subvariety given by $u = v = 0$ and its open complement given by $u \neq 0$ and $a = -v/u$. Hence, return $[X] = [X_1] + [X_2]$ with

$$X_1 = X(A, (F \setminus \{f\}) \cup \{u, v\}, G),$$
$$X_2 = X(A, \text{eval}_a(F \setminus \{f\}, -v/u), \text{eval}_a(G, -v/u) \cup \{u\}).$$

8. Suppose $\text{char}(k) \neq 2$ and $f = ux^2 + vx + w$ for some element $x \in A$ and polynomials $f \in F$ and $u, v, w \in k[A \setminus \{a\}]$ with u non-zero. Moreover, suppose that the discriminant $D = v^2 - 4uw$ is a square, that is, $D = d^2$ for some $d \in k[A \setminus \{a\}]$. Then return $[X] = [X_1] + [X_2] + [X_3] + [X_4]$, where X is stratified by the following varieties:

$$X_1 = X(A, (F \setminus \{f\}) \cup \{u, vx + w\}, G),$$
$$X_2 = X(A, \text{eval}^x_{-v/2u}(F \setminus \{f\}) \cup \{d\}, \text{eval}^x_{-v/2u}(G) \cup \{u\}),$$
$$X_3 = X(A, \text{eval}^x_{(-v-d)/2u}(F \setminus \{f\}), \text{eval}^x_{(-v-d)/2u}(G) \cup \{u, d\}),$$
$$X_4 = X(A, \text{eval}^x_{(-v+d)/2u}(F \setminus \{f\}), \text{eval}^x_{(-v+d)/2u}(G) \cup \{u, d\}).$$

9. If $G \neq \varnothing$, pick any $g \in G$, and return $[X] = [X_1] - [X_2]$ with

$$X_1 = X(A, F, G \setminus \{g\}),$$
$$X_2 = X(A, F \cup \{g\}, G).$$

10. If none of the above rules apply, fail.

Remark 3.4.4. Of course, it is possible to replace Step 10 with:

10'. If none of the above rules apply, create a new symbol for the variety $X(A, F, G)$ and return that.

However, this raises the question of what it means to 'compute the virtual class' of a variety. For the purpose of computing motivic invariants, an expression for the virtual class of a variety in terms of the classes of other varieties is only useful if the motivic invariants of those other varieties are known. As far as the applications in this thesis go, the varieties to which this algorithm will be applied all have a virtual class that is a polynomial in $\mathbb{L}$.

3.5 Grothendieck ring of stacks

In order to study motivic invariants of stacks, we would like to have an analogue of the Grothendieck ring of varieties for stacks. A number of constructions have been proposed by various authors, such as in [Joy07, Toë05, BD07]. We will follow the construction by Ekedahl as in [Eke09a, Eke09b], since its definition is closest to Definition 3.2.3. As in Definition 1.6.4, we will restrict to algebraic (Artin) stacks which are of finite type over a field with affine stabilizers.

Definition 3.5.1. Let $\mathfrak{S}$ be an algebraic stack of finite type over a field k with affine stabilizers. The *Grothendieck ring of stacks over* $\mathfrak{S}$, denoted $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{S}})$, is the free abelian group on isomorphism classes of algebraic stacks of finite type over $\mathfrak{S}$ with affine stabilizers, modulo the relations

(1) $[\mathfrak{X}] = [\mathfrak{Z}] + [\mathfrak{X} \setminus \mathfrak{Z}]$ for all closed immersions $\mathfrak{Z} \to \mathfrak{X}$ over $\mathfrak{S}$,

(2) $[\mathfrak{E}] = \mathbb{L}^n[\mathfrak{X}]$ for any vector bundle $\mathfrak{E}$ over $\mathfrak{X}$ of rank n, where $\mathbb{L} = [\mathbb{A}^1_{\mathfrak{S}}]$.

Multiplication is given on generators by $[\mathfrak{X}][\mathfrak{Y}] = [\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}]$.

Remark 3.5.2. If S is a variety over k, then the inclusion $\mathbf{Var}_S \to \mathbf{Stck}_S$ induces a ring morphism

$$\mathrm{K}_0(\mathbf{Var}_S) \to \mathrm{K}_0(\mathbf{Stck}_S).$$

In particular, any relation that holds in $\mathrm{K}_0(\mathbf{Var}_S)$ also holds in $\mathrm{K}_0(\mathbf{Stck}_S)$.

Example 3.5.3. Consider the classifying stack $\mathrm{B}\mathbb{G}_m = [\mathrm{Spec}(k)/\mathbb{G}_m]$ over a field k. The natural morphism $[\mathbb{A}^1_k/\mathbb{G}_m] \to \mathrm{B}\mathbb{G}_m$ is a vector bundle of rank one, so $[\mathbb{A}^1_k/\mathbb{G}_m] = \mathbb{L}[\mathrm{B}\mathbb{G}_m]$ in $\mathrm{K}_0(\mathbf{Stck}_k)$. On the other hand, the closed subscheme $\mathrm{B}\mathbb{G}_m \subseteq [\mathbb{A}^1_k/\mathbb{G}_m]$, given by the origin, yields the relation $[\mathbb{A}^1_k/\mathbb{G}_m] = [\mathrm{B}\mathbb{G}_m] + [\mathbb{G}_m/\mathbb{G}_m] = [\mathrm{B}\mathbb{G}_m]+1$. Therefore, $(\mathbb{L}-1)[\mathrm{B}\mathbb{G}_m] = 1$ and hence $[\mathrm{B}\mathbb{G}_m]$ is invertible with inverse $(\mathbb{L} - 1) = [\mathbb{G}_m]$.

The above example can be generalized to other algebraic groups, and more general quotient stacks. The following proposition treats the case $G = \mathrm{GL}_n$.

Proposition 3.5.4. *For any $n \geq 0$, the element $[\mathrm{GL}_n]$ in $\mathrm{K}_0(\mathbf{Stck}_k)$ is invertible, and $[\mathfrak{X}/\mathrm{GL}_n]_{\mathfrak{S}} = [\mathrm{GL}_n]^{-1} \cdot [\mathfrak{X}]_{\mathfrak{S}}$ in $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{S}})$ for any algebraic stack $\mathfrak{X}$ over $\mathfrak{S}$ in $\mathbf{Stck}_{\mathfrak{S}}$ with an action of GL_n such that the map $\mathfrak{X} \to \mathfrak{S}$ is G-invariant.*

Proof. As in Example 3.3.5, for any $0 \leq m \leq n$, let $Y_m \subseteq \mathrm{Mat}_{n \times m}$ be the subscheme of m-linearly independent vectors. The group GL_n acts naturally on each Y_m, and we construct $\mathfrak{Y}_m = [(Y_m \times \mathfrak{X})/\mathrm{GL}_n]$. Now, the quotient $\mathfrak{X} \to [\mathfrak{X}/\mathrm{GL}_n]$ factors as

$$\mathfrak{X} = \mathfrak{Y}_n \to \mathfrak{Y}_{n-1} \to \cdots \to \mathfrak{Y}_0 = [\mathfrak{X}/\mathrm{GL}_n].$$

For any $1 \leq m \leq n$, the scheme Y_m can be identified with the open complement of $Y_{m-1} \times \mathbb{A}^{m-1}_k$ inside $Y_{m-1} \times \mathbb{A}^n_k$. Hence, $[\mathfrak{Y}_m]_{\mathfrak{S}} = (\mathbb{L}^n - \mathbb{L}^{m-1})[\mathfrak{Y}_{m-1}]_{\mathfrak{S}}$ for all

$1 \leq m \leq n$, and thus $[\mathfrak{X}]_{\mathfrak{S}} = \left(\prod_{m=0}^{n-1}(\mathbb{L}^n - \mathbb{L}^m)\right)[\mathfrak{X}/\mathrm{GL}_n]_{\mathfrak{S}} = [\mathrm{GL}_n] \cdot [\mathfrak{X}/\mathrm{GL}_n]_{\mathfrak{S}}$. Specializing to the case $\mathfrak{X} = \mathfrak{S} = \operatorname{Spec} k$, we find that $[\mathrm{GL}_n]$ is invertible with inverse $[\mathrm{BGL}_n]$. Therefore, $[\mathfrak{X}/\mathrm{GL}_n]_{\mathfrak{S}} = [\mathrm{GL}_n]^{-1} \cdot [\mathfrak{X}]_{\mathfrak{S}}$. □

Proposition 3.5.5. *Let G be a special algebraic group over a field k. Then $[G]$ is invertible in $\mathrm{K}_0(\mathbf{Stck}_k)$, and for any G-torsor of $\mathfrak{P} \to \mathfrak{X}$ in $\mathbf{Stck}_{\mathfrak{S}}$, one has $[\mathfrak{X}]_{\mathfrak{S}} = [G]^{-1} \cdot [\mathfrak{P}]_{\mathfrak{S}}$ in $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{S}})$.*

Proof. Using Proposition 1.6.5, we can reduce to the case where $\mathfrak{X} = [X/\mathrm{GL}_n]$ for a quasi-projective scheme X. Now form the following cartesian diagram.

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathfrak{P} & \longrightarrow & [X/\mathrm{GL}_n] \end{array}$$

Then P is a GL_n-torsor over $\mathfrak{P}$, as described in Section 1.5, and a G-torsor over X. By Proposition 3.5.4 and Lemma 3.3.13, we have $[\mathrm{GL}_n] \cdot [\mathfrak{P}]_{\mathfrak{S}} = [P]_{\mathfrak{S}} = [G] \cdot [X]_{\mathfrak{S}} = [G][\mathrm{GL}_n] \cdot [\mathfrak{X}]_{\mathfrak{S}}$, and hence $[\mathfrak{P}]_{\mathfrak{S}} = [G] \cdot [\mathfrak{X}]_{\mathfrak{S}}$. In the special case that $\mathfrak{P} = \mathfrak{S} = \operatorname{Spec} k$ and $\mathfrak{X} = \mathrm{B}G$, we find that $[G]$ is invertible with inverse $[\mathrm{B}G]$ and thus $[\mathfrak{X}]_{\mathfrak{S}} = [G]^{-1} \cdot [\mathfrak{P}]_{\mathfrak{S}}$. □

Example 3.5.6. In general, it need not be the case that $[\mathrm{B}G] = [G]^{-1}$. For example, consider the group $G = \mu_n$ of n-th roots of unity. The morphism $\mathbb{G}_m \to \mathrm{B}\mu_n$, corresponding to the μ_n-torsor $\mathbb{G}_m \to \mathbb{G}_m$ given by $x \mapsto x^n$, is a $\mathbb{G}_m$-torsor itself, so it follows that $[\mathrm{B}\mu_n] = [\mathbb{G}_m]/[\mathbb{G}_m] = 1$. It was shown by Ekedahl that also $[\mathrm{B}S_n] = 1$ for the symmetric groups S_n for all $n \geq 0$ [Eke09b]. He also showed that there are finite groups G for which $[\mathrm{B}G] \neq 1$.

From Proposition 3.5.4 and the expression of $[\mathrm{GL}_n]$ in terms of $\mathbb{L}$, see Example 3.3.5, it follows that the elements $\mathbb{L}$ and $\mathbb{L}^n - 1$ for all $n \geq 1$ are invertible in $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{S}})$. Hence, if $\mathfrak{S} = S$ is a variety over k, there is a natural map from the localization $\mathrm{K}_0(\mathbf{Var}_S)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1} : n \geq 1]$ (where we adjoined inverses of $\mathbb{L}$ and $\mathbb{L}^n - 1$ for all $n \geq 1$) to $\mathrm{K}_0(\mathbf{Stck}_S)$. In fact, this map is an isomorphism.

Theorem 3.5.7 ([Eke09a, Theorem 1.2])**.** *The map $\mathrm{K}_0(\mathbf{Var}_S)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1} : n \geq 1] \to \mathrm{K}_0(\mathbf{Stck}_S)$ is an isomorphism of rings.*

Remark 3.5.8. The isomorphism of Theorem 3.5.7 allows us to extend any invariant $\chi \colon \mathrm{K}_0(\mathbf{Var}_S) \to R$ to $\mathrm{K}_0(\mathbf{Stck}_S)$, possibly after inverting $\chi(\mathbb{L})$ and $\chi(\mathbb{L}^n - 1)$ in R, for all $n \geq 1$, provided they are not zero-divisors in R. In particular, this extends the E-polynomial to all algebraic stacks $\mathfrak{X}$ of finite type over $\mathbb{C}$ with affine stabilizers,

$$e \colon \mathrm{K}_0(\mathbf{Stck}_{\mathbb{C}}) \to \mathbb{Z}[\![u, v]\!][u^{-1}, v^{-1}]. \tag{3.7}$$

This approach is taken for example in [Joy07, Theorem 4.10]. Alternatively, given a presentation $X \to \mathfrak{X}$, one can construct a simplicial scheme $X^\bullet$ resolving $\mathfrak{X}$, given by $X^n = X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$ ($n+1$ times). Now, Deligne's construction applies in fact to simplicial schemes [Del74], so that the cohomology groups $H_c^k(\mathfrak{X}, \mathbb{C})$ of the geometric realization of the analytification of $X^\bullet$ admit a mixed Hodge structure, which can be shown to be independent of the presentation X. The corresponding E-polynomial $e(\mathfrak{X})$ agrees with (3.7). For details, see [BD07] or [Toë05]. In the particular case of a quotient stack $\mathfrak{X} = [X/G]$ with G a connected group, one has $e(\mathfrak{X}) = e(X)/e(G)$.

3.6 Equivariant motivic invariants

Let G be a finite group acting on a complex variety X. The action of G turns the cohomology groups $H_c^k(X;\mathbb{C})$ into representations of G, by functoriality of cohomology. Moreover, the action of G, being algebraic, respects the mixed Hodge structure [PS08, FS21], so the graded pieces $H_c^{k;p,q}(X) = \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^k(X;\mathbb{C})$, see (3.2), turn into representations of G as well. From this, one constructs the *G-equivariant E-polynomial*

$$e^G(X) = \sum_{k,p,q} (-1)^k u^p v^q \otimes [H_c^{k;p,q}(X)] \in \mathbb{Z}[u,v] \otimes R_{\mathbb{C}}(G),$$

where $R_{\mathbb{C}}(G)$ denotes the representation ring of G. The G-equivariant E-polynomial is still additive and multiplicative, that is,

$$e^G(X) = e^G(Z) + e^G(X \setminus Z) \quad \text{and} \quad e^G(X \times Y) = e^G(X)\, e^G(Y)$$

for complex varieties X and Y with a G-action, and $Z \subseteq X$ a G-invariant closed subvariety [FS21]. The original E-polynomial $e(X)$ can be obtained from $e^G(X)$ via the map $\dim\colon R_{\mathbb{C}}(G) \to \mathbb{Z}$.

In this section, we investigate to which extent other invariants can be made G-equivariant, with a special focus on the virtual class in the Grothendieck ring of varieties.

Definition 3.6.1. Let G be an algebraic group over a field k, and S a variety over k. A *G-variety* over S is a variety X over S with an action of G such that $X \to S$ is G-invariant and X admits a cover by G-invariant affine opens. Denote by $\mathbf{Var}_S^G$ the category of G-varieties over S and G-equivariant morphisms over S between them. The *Grothendieck ring of G-varieties over S*, denoted $\mathrm{K}_0(\mathbf{Var}_S^G)$, is defined, analogous to Definition 3.2.3, as the free abelian group on isomorphism classes $[X]$ of G-varieties X over S modulo the relations $[X] = [Z] + [X \setminus Z]$ for all G-invariant closed subvarieties $Z \subseteq X$. Multiplication is given on generators by $[X][Y] = [X \times_S Y]$, where G acts diagonally on $X \times_S Y$.

Now, more precisely, we investigate whether an invariant $\chi\colon \mathrm{Ob}(\mathbf{Var}_k) \to R$, for some commutative ring R, can be promoted to some $\chi^G\colon \mathrm{Ob}(\mathbf{Var}_k^G) \to R \otimes R_{\mathbb{C}}(G)$ such that χ is obtained from χ^G via the map $\dim\colon R_{\mathbb{C}}(G) \to \mathbb{Z}$, while remaining additive or multiplicative. We show this is possible in many cases, such as for $R = \mathrm{K}_0(\mathbf{MHS})$ or $R = \mathrm{K}_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{C}))$. However, we also show this is not possible for $R = \mathrm{K}_0(\mathbf{Var}_k)$. Nevertheless, under certain assumptions on G, we will provide a construction which, although far from ideal, provides a new tool for computations in $\mathrm{K}_0(\mathbf{Var}_k)$.

Let us start with a positive result.

Proposition 3.6.2. *Let G be a finite group with splitting field K. Let $\mathcal{A}$ be an idempotent complete K-linear tensor category, whose unit object is denoted by K. Suppose G acts on an object X of $\mathcal{A}$. Then X decomposes in $\mathcal{A}$ as*

$$X \cong \bigoplus_{\rho} X_\rho \otimes [\rho],$$

for some objects X_ρ of $\mathcal{A}$, where ρ ranges over the irreducible representations of G, and $[\rho] \coloneqq K^{\oplus \dim \rho}$. Moreover, the isomorphism is G-equivariant when G acts trivially on X_ρ and via ρ on $[\rho]$, and the objects X_ρ are uniquely determined up to isomorphism.

Proof. Denote by $\rho_1, \dots, \rho_n$ the irreducible representations of G over K. The Artin–Wedderburn theorem gives an isomorphism $K[G] \cong \prod_{i=1}^n \mathrm{Mat}_{d_i \times d_i}(K)$, where $d_i = \dim \rho_i$, given by $g \mapsto (\rho_i(g))_{i=1}^n$ [Ser77, Proposition 10]. This decomposition corresponds to a sequence $e_1, \dots, e_n \in K[G]$ of pairwise orthogonal central idempotents such that $\sum_{i=1}^n e_i = 1$. For every $i = 1, \dots, n$, the idempotents e_i induce idempotent morphisms $X \xrightarrow{e_i} X$ which, by assumption, split as $X \xrightarrow{r_i} Y_i \xrightarrow{s_i} X$ with $s_i \circ r_i = e_i$ and $r_i \circ s_i = \mathrm{id}_{Y_i}$. Orthogonality of the idempotents implies, for $i \neq j$, that $s_i \circ r_i \circ s_j \circ r_j = 0$ and hence $r_i \circ s_j = 0$. Therefore, we have an isomorphism

$$\bigoplus_{i=1}^n Y_j \underset{\coprod_i r_i}{\overset{\coprod_i s_i}{\rightleftarrows}} X. \quad (\sim)$$

Since the e_i are central in $K[G]$, the action of G restricts to Y_i for every i.

Every idempotent e_i corresponds to a factor $\mathrm{Mat}_{d_i \times d_i}(K)$. Write E_{jk} for the $d_i \times d_i$ matrix which is zero everywhere, except at position (j, k), where the entry is one. Then $e_i = \sum_{j=1}^{d_i} e_{ij}$ for the pairwise orthogonal idempotents $e_{ij} = E_{jj}$. As above, this yields a decomposition

$$Y_i \cong \bigoplus_{j=1}^{d_i} Z_{ij}.$$

Moreover, the Z_{ij} are isomorphic for all j since E_{jk} defines an isomorphism from Z_{ik} to Z_{ij}, with inverse E_{kj}, so $Y_i \cong Z_i \otimes K^{\oplus d_i}$. Under this isomorphism, G acts trivially on Z_i and on $K^{\oplus d_i}$ via ρ_i, because of the isomorphism $K[G] \cong \prod_{i=1}^n \mathrm{Mat}_{d_i \times d_i}(K)$. Therefore, $Y_i \cong Z_i \otimes [\rho_i]$. Finally, the $X_{\rho_i} \coloneqq Z_i$ are uniquely determined up to isomorphism, as they correspond to the idempotents e_{ij}. □

Remark 3.6.3. Suppose G acts on objects X and Y in $\mathcal{A}$ as in Proposition 3.6.2. Then it follows from the uniqueness statement that

$$(X \otimes Y)_{\rho_k} \cong \bigoplus_{i,j=1}^n (X_{\rho_i} \otimes Y_{\rho_j})^{\oplus a_{ij}^k},$$

where $a_{ij}^k \in \mathbb{Z}_{\geq 0}$ are the *Clebsch–Gordan series*, given by $\rho_i \otimes \rho_j \cong \bigoplus_{k=1}^n \rho_k^{\oplus a_{ij}^k}$.

Now, let $\mathcal{A}$ be as in Proposition 3.6.2, and suppose we are given a functor $\mathcal{X} \colon \mathbf{Var}_k \to \mathcal{A}$. If $\mathcal{A}$ is an abelian or triangulated category, we obtain an invariant $\chi\colon \mathrm{Ob}(\mathbf{Var}_k) \to \mathrm{K}_0(\mathcal{A})$ which, using Proposition 3.6.2, promotes to an invariant

$$\chi^G\colon \mathrm{Ob}(\mathbf{Var}_k^G) \to \mathrm{K}_0(\mathcal{A}) \otimes R_K(G)$$

where $\mathcal{X}(X) \cong \bigoplus_\rho X_\rho \otimes [\rho]$ is sent to $\sum_\rho [X_\rho] \otimes [\rho]$. One can obtain χ from χ^G via $\dim\colon R_K(G) \to \mathbb{Z}$ as the image of $[\rho] = K^{\oplus \dim \rho}$ in $\mathrm{K}_0(\mathcal{A})$ equals $\dim \rho$.

Furthermore, if G-invariant closed subvarieties $Z \subseteq X$ induce exact sequences $0 \to \mathcal{X}(X \setminus Z) \to \mathcal{X}(X) \to \mathcal{X}(Z) \to 0$ (when $\mathcal{A}$ is abelian) or distinguished triangles $\mathcal{X}(X \setminus Z) \to \mathcal{X}(X) \to \mathcal{X}(Z) \to \mathcal{X}(X \setminus Z)[1]$ (when $\mathcal{A}$ is triangulated) with G-equivariant maps in $\mathcal{A}$, then χ^G will also be additive, that is, $\chi^G(X) = \chi^G(Z) + \chi^G(X \setminus Z)$. In this case, χ^G descends to a group morphism

$$\chi^G\colon \mathrm{K}_0(\mathbf{Var}_k^G) \to \mathrm{K}_0(\mathcal{A}) \otimes R_K(G). \tag{3.8}$$

Moreover, when $\mathcal{A}$ is tensor triangulated and there are natural isomorphisms $\mathcal{X}(X \times Y) \cong \mathcal{X}(X) \otimes \mathcal{X}(Y)$ for all G-varieties X and Y, where G acts diagonally on $X \times Y$, it follows from Remark 3.6.3 that χ^G is multiplicative, that is, $\chi^G(X \times Y) = \chi^G(X)\,\chi^G(Y)$. In this case, (3.8) is a ring morphism.

Example 3.6.4.
- Let $\mathcal{A} = D^b(\mathbf{MHS})$ be the derived category of mixed Hodge structures. The assignment of the mixed Hodge structures $H_c^k(X)$ to a complex variety X can be promoted to a functor $\mathcal{X} = R\Gamma(-, \mathbb{Q})\colon \mathbf{Var}_\mathbb{C} \to \mathcal{A}$ such that $H_c^k(X)$ is the k-th cohomology group of $\mathcal{X}(X)$ [Beĭ86]. The resulting G-equivariant invariant χ^G is additive by the G-equivariant long exact sequence (3.4). It is also multiplicative by the Künneth formula, and hence induces a ring morphism

$$\chi^G\colon \mathrm{K}_0(\mathbf{Var}_\mathbb{C}^G) \to \mathrm{K}_0(\mathbf{MHS}) \otimes R_\mathbb{C}(G).$$

- Extending the previous example, note that the exact functor $\mathrm{Gr}_F^* \mathrm{Gr}_*^W$ in (3.2) induces an exact functor $D^b(\mathbf{MHS}) \to D^b((\mathbf{Vect}_{\mathbb{C}}^{\mathbb{Z}\times\mathbb{Z}})_{\text{fin}})$, and let $\mathcal{X}$ be the composite

$$\mathbf{Var}_{\mathbb{C}} \xrightarrow{R\Gamma(-,\mathbb{Q})} D^b(\mathbf{MHS}) \xrightarrow{\mathrm{Gr}_F^* \mathrm{Gr}_*^W} D^b((\mathbf{Vect}_{\mathbb{C}}^{\mathbb{Z}\times\mathbb{Z}})_{\text{fin}}).$$

  The induced invariant χ^G is still additive and multiplicative, and hence induces ring morphism

$$\chi^G \colon \mathrm{K}_0(\mathbf{Var}_{\mathbb{C}}^G) \to \mathrm{K}_0((\mathbf{Vect}_{\mathbb{C}}^{\mathbb{Z}\times\mathbb{Z}})_{\text{fin}}) \otimes R_{\mathbb{C}}(G) \cong \mathbb{Z}[u^{\pm 1}, v^{\pm 1}] \otimes R_{\mathbb{C}}(G)$$

  which is precisely the G-equivariant E-polynomial.

- Let $\mathcal{A} = \mathrm{DM}_{\text{gm}}^{\text{eff}}(k, K)$ be the K-linearization of Voevodsky's triangulated category of effective geometric motives, with k a field of characteristic zero, and let $\mathcal{X} \colon \mathbf{Var}_k \to \mathcal{A}$ be the motive M_{gm} or the motive with compact support M_{gm}^c. The induced invariant χ^G is multiplicative, and if $\mathcal{X} = M_{\text{gm}}^c$ also additive.

Grothendieck ring of varieties

Unfortunately, the Grothendieck ring of varieties $\mathrm{K}_0(\mathbf{Var}_k)$ is not given by the Grothendieck group of an abelian (or triangulated) category $\mathcal{A}$. To get an idea of how an analogous construction could work for $\mathrm{K}_0(\mathbf{Var}_k)$, we first consider some properties of the G-equivariant E-polynomial.

Let G be a finite group and $H \subseteq G$ a subgroup. Denote by

$$\mathrm{Res}_H^G \colon R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(H) \quad \text{and} \quad \mathrm{Ind}_H^G \colon R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G)$$

the restriction and induction maps [Ser77, p.28]. Using the same symbols, we define restriction and induction for G-varieties.

Definition 3.6.5. Let G be an algebraic group over k with a subgroup $H \subseteq G$, and S a variety over k. Define the functors

$$\mathrm{Res}_H^G \colon \mathbf{Var}_S^G \to \mathbf{Var}_S^H \quad \text{and} \quad \mathrm{Ind}_H^G \colon \mathbf{Var}_S^H \to \mathbf{Var}_S^G$$

where Res_H^G restricts the action from G to H (in fact, Res_H^G is defined for any morphism $H \to G$ of algebraic groups), and $\mathrm{Ind}_H^G(Y) = (G \times Y) /\!\!/ H$, where H acts on $G \times Y$ via $h \cdot (g, y) = (gh^{-1}, h \cdot y)$ and G acts on the resulting quotient by left multiplication on the factor of G. Note that, by [PV89, Theorem 4.19], the quotient $(G \times Y) /\!\!/ H$ is a variety, even when H is non-reductive. It is easy to see that these functors descend to the Grothendieck ring of varieties

$$\mathrm{Res}_H^G \colon \mathrm{K}_0(\mathbf{Var}_S^G) \to \mathrm{K}_0(\mathbf{Var}_S^H) \quad \text{and} \quad \mathrm{Ind}_H^G \colon \mathrm{K}_0(\mathbf{Var}_S^H) \to \mathrm{K}_0(\mathbf{Var}_S^G).$$

When G and H are finite, the underlying variety of $\mathrm{Ind}_H^G(Y)$ is simply $\bigsqcup_{G/H} Y$.

Lemma 3.6.6. *Let G be a finite group and $H \subseteq G$ a subgroup.*

(i) $e^H(\mathrm{Res}_H^G(X)) = \mathrm{Res}_H^G(e^G(X))$ *for all objects X of* $\mathbf{Var}_k^G$,

(ii) $e^G(\mathrm{Ind}_H^G(Y)) = \mathrm{Ind}_H^G(e^H(Y))$ *for all objects Y of* $\mathbf{Var}_k^H$,

(iii) $e(X /\!\!/ G) = \langle T, e^G(X)\rangle$, *where $T \in R_{\mathbb{C}}(G)$ corresponds to the trivial representation, and $\langle -, - \rangle$ denotes the inner product of characters.*

Proof. *(i)* and *(ii)* directly follow from the definitions of Res_H^G and Ind_H^G for representations and varieties. *(iii)* follows from [FS21, Proposition 4.3]. □

The following example shows how these properties can be used to compute the G-equivariant E-polynomials in some simple cases.

Example 3.6.7. Consider $G = \mathbb{Z}/2\mathbb{Z}$ and denote by $T, N \in R_{\mathbb{C}}(G)$ the trivial and non-trivial character of G. For any G-variety X, we have

$$e^G(X) = \alpha \otimes T + \beta \otimes N$$

for some $\alpha, \beta \in \mathbb{Z}[u, v]$. The properties of Lemma 3.6.6 imply that $e(X /\!\!/ G) = \alpha$ and $e(X) = \mathrm{Res}_1^G(e^G(X)) = \alpha + \beta$. Therefore,

$$e^G(X) = e(X /\!\!/ G) \otimes T + (e(X) - e(X /\!\!/ G)) \otimes N.$$

Example 3.6.8. Consider $G = S_3$ and denote by $T, S, D \in R_{\mathbb{C}}(G)$ the trivial, sign and standard representation. For any G-variety X, we have

$$e^G(X) = \alpha \otimes T + \beta \otimes S + \gamma \otimes D$$

for some $\alpha, \beta, \gamma \in \mathbb{Z}[u, v]$. For $\tau = (1\ 2)$ and $\rho = (1\ 2\ 3)$ in S_3, we find

$$e(X) = \alpha + \beta + 2\gamma,\ e(X /\!\!/ \langle\tau\rangle) = \alpha + \gamma,\ e(X /\!\!/ \langle\rho\rangle) = \alpha + \beta,\ e(X /\!\!/ G) = \alpha.$$

In particular, it follows that

$$\alpha = e(X /\!\!/ G),\ \beta = e(X) - 2 \cdot e(X /\!\!/ \langle\tau\rangle) + e(X /\!\!/ G),\ \gamma = e(X /\!\!/ \langle\tau\rangle) - e(X /\!\!/ G).$$

Note that, since there are more subgroups than irreducible representations, the relation

$$e(X) - 2 \cdot e(X /\!\!/ \langle\tau\rangle) - e(X /\!\!/ \langle\rho\rangle) + 2 \cdot e(X /\!\!/ G) = 0 \tag{3.9}$$

will always hold.

Let us return to the Grothendieck ring of varieties. Given a G-variety X, we want to define $[X]^G \in \mathrm{K}_0(\mathbf{Var}_k) \otimes R_{\mathbb{C}}(G)$ such that

$$\langle T_H, \mathrm{Res}_H^G [X]^G \rangle = [X /\!\!/ H] \tag{3.10}$$

for every subgroup $H \subseteq G$. Unfortunately, here we run into trouble trying to define $[X]^G$. The following example shows that the analogue of (3.9) need not hold in $\mathrm{K}_0(\mathbf{Var}_k)$ in general.

Example 3.6.9 ([Saw22])**.** Let $G = S_3$ and let X be a complex smooth projective curve of genus $6g+1$ for some $g \geq 1$ with a free action of S_3. By the Riemann–Hurwitz formula, the quotients $X /\!\!/ \langle \tau \rangle$, $X /\!\!/ \langle \rho \rangle$ and $X /\!\!/ S_3$ have genera $3g+1, 2g+1$ and $g+1$, respectively. Hence, none of these quotients are stably birational to X or to each other. Now, the isomorphism $\mathrm{K}_0(\mathbf{Var}_{\mathbb{C}})/(\mathbb{L}) \cong \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$ by Larsen and Lunts [LL03] shows there is no $\mathbb{Z}$-linear relation between their classes in $\mathrm{K}_0(\mathbf{Var}_{\mathbb{C}})$.

It seems that having too many subgroups results in $[X]^G$ being ill-defined. A possible remedy could be to fix a set of subgroups of G. On the other hand, having too few subgroups could also be a problem, e.g. for $G = \mathbb{Z}/3\mathbb{Z}$, which has 3 irreducible representations but only 2 subgroups. For this reason, we will focus only on rational representations of G. This makes sense in analogy with the G-equivariant E-polynomial, since $H_c^k(X;\mathbb{C}) = H_c^k(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. Finally, note that the quotient $X /\!\!/ H$ only depends on the conjugacy class of H.

Definition 3.6.10. Let G be a finite group, and let $\mathcal{H}$ be a set of conjugacy classes of subgroups of G. Define the map

$$\Psi_G^{\mathcal{H}} \colon R_{\mathbb{Q}}(G) \to \bigoplus_{[H] \in \mathcal{H}} \mathbb{Z}, \quad V \mapsto \left(\langle T_H, \mathrm{Res}_H^G V \rangle \right)_{[H] \in \mathcal{H}}.$$

Lemma 3.6.11. *If $\mathcal{H}$ contains the conjugacy classes of all subgroups of G, then $\Psi_G^{\mathcal{H}}$ is injective.*

Proof. Take any $V \in R_{\mathbb{Q}}(G)$ such that $\langle T_H, \mathrm{Res}_H^G V \rangle = 0$ for all H. By Frobenius reciprocity, this is the same as $\langle \mathrm{Ind}_H^G T_H, V \rangle = 0$ for all H. Now, by [Ser77, Theorem 30], the elements $\mathrm{Ind}_H^G T_H$ generate $R_{\mathbb{Q}}(G)$, so $V = 0$. ☐

Shrinking $\mathcal{H}$ appropriately, the map $\Psi_G^{\mathcal{H}}$ will still be injective, and its image will have rank equal to $|\mathcal{H}|$. In particular, $\Psi_G^{\mathcal{H}} \otimes \mathbb{Q}$ will be an isomorphism, so further tensoring with $\mathrm{K}_0(\mathbf{Var}_k)$ shows the existence and uniqueness of an element $[X]^G \in \mathrm{K}_0(\mathbf{Var}_k) \otimes R_{\mathbb{Q}}(G) \otimes \mathbb{Q}$ satisfying (3.10) for all $[H] \in \mathcal{H}$. We end up with the following definition.

Definition 3.6.12. Let G be a finite group and $\mathcal{H}$ a set of conjugacy classes of subgroups of G such that

$$\Psi_G^{\mathcal{H}} \otimes \mathbb{Q}\colon R_{\mathbb{Q}}(G) \otimes \mathbb{Q} \to \bigoplus_{H \in \mathcal{H}} \mathbb{Q} \tag{3.11}$$

is an isomorphism. In this case we say that $\mathcal{H}$ is a *good* set of conjugacy classes of subgroups of G. Then for any G-variety X over S, the *G-virtual class* of X is the unique element $[X]^G \in \mathrm{K}_0(\mathbf{Var}_S) \otimes R_{\mathbb{Q}}(G) \otimes \mathbb{Q}$ such that

$$\langle T_H, \mathrm{Res}^G_H [X]^G \rangle = [X /\!\!/ H]$$

in $\mathrm{K}_0(\mathbf{Var}_S)$ for all $[H] \in \mathcal{H}$.

Remark 3.6.13. The G-virtual class is clearly additive, that is, $[X]^G = [Z]^G + [X \setminus Z]^G$ for all G-invariant closed subvarieties $Z \subseteq X$. Hence, it induces a group morphism

$$[-]^G\colon \mathrm{K}_0(\mathbf{Var}_S^G) \to \mathrm{K}_0(\mathbf{Var}_S) \otimes R_{\mathbb{Q}}(G) \otimes \mathbb{Q}.$$

Example 3.6.14. Let $G = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$ and let $\mathcal{H}$ be the set of conjugacy classes of all subgroups of G. Then $\Psi_G^{\mathcal{H}} \otimes \mathbb{Q}$ is injective by Lemma 3.6.11, and an isomorphism because $|\mathcal{H}|$ equals the number of divisors of n, which is equal to the rank of $R_{\mathbb{Q}}(G)$.

Example 3.6.15. Consider the symmetric group $G = S_n$ with the set $\mathcal{H} = \{S_{\lambda_1} \times \cdots \times S_{\lambda_k} : \lambda \text{ a partition of } n\}$ of *Young subgroups*. From the representation theory of S_n [FH91, Lecture 4] it can be shown that (3.11) is an isomorphism. In particular, the irreducible representations of S_n are parametrized by the partitions λ of n. Denote by V_λ the irreducible representation of S_n corresponding to such a partition λ. Now, for any $V = \sum_\lambda a_\lambda [V_\lambda] \in R_{\mathbb{Q}}(S_n)$, we find that

$$\begin{aligned}
\Psi_G^{\mathcal{H}}(V) &= \left(\left\langle T_{S_\lambda}, \mathrm{Res}^{S_n}_{S_\lambda} V \right\rangle \right)_{S_\lambda \in \mathcal{H}} \\
&= \left(\left\langle \mathrm{Ind}^{S_n}_{S_\lambda} T_{S_\lambda}, V \right\rangle \right)_{S_\lambda \in \mathcal{H}} \\
&= \left(\textstyle\sum_\mu a_\mu K_{\mu\lambda} \right)_{S_\lambda \in \mathcal{H}}
\end{aligned}$$

where $K_{\mu\lambda}$ are the *Kostka numbers*, by Young's rule [FH91, Corollary 4.39]. Since $K_{\lambda\lambda} = 1$ and $K_{\mu\lambda} = 0$ for $\mu < \lambda$ (for the lexicographical order on partitions), it follows that $\Psi_G^{\mathcal{H}}$ is invertible.

Example 3.6.16. Suppose G_1 and G_2 are finite groups with good sets of conjugacy classes of subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then

$$\mathcal{H} = \{[H_1 \times H_2] : [H_1] \in \mathcal{H}_1 \text{ and } [H_2] \in \mathcal{H}_2\}$$

is a good set of conjugacy classes of subgroups of $G_1 \times G_2$ if $R_{\mathbb{Q}}(G_1) = R_{\mathbb{C}}(G_1)$ or $R_{\mathbb{Q}}(G_2) = R_{\mathbb{C}}(G_2)$. In particular, this provides good sets of conjugacy classes of subgroups for all Young subgroups $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Even though the G-virtual class is additive, the following example shows that in general, already for $G = \mathbb{Z}/2\mathbb{Z}$, the G-virtual class is not multiplicative.

Example 3.6.17. Let $G = \mathbb{Z}/2\mathbb{Z}$ and take A and B elliptic curves over $k = \mathbb{C}$ with $[A] \neq [B]$ and $A \times A \cong B \times B$ as abelian varieties, as in [Poo02, Lemma 3]. Equip the elliptic curves A and B with the G-action of negation, $P \mapsto -P$. Now suppose that the G-virtual class is multiplicative. Then, using the notation $[X]^G = [X]_+ \otimes T + [X]_- \otimes N$ with $[X]_+ = [X /\!\!/ G]$ and $[X]_- = [X] - [X /\!\!/ G]$, we find that

$$
\begin{aligned}
[A \times A]_+ &= [A]_+^2 + [A]_-^2 = [\mathbb{P}^1_k]^2 + ([A] - [\mathbb{P}^1_k])^2 = [A]^2 + 2[\mathbb{P}^1_k]^2 - 2[A][\mathbb{P}^1_k] \\
[B \times B]_+ &= [B]_+^2 + [B]_-^2 = [\mathbb{P}^1_k]^2 + ([B] - [\mathbb{P}^1_k])^2 = [B]^2 + 2[\mathbb{P}^1_k]^2 - 2[B][\mathbb{P}^1_k]
\end{aligned}
$$

where $A /\!\!/ G \cong B /\!\!/ G \cong \mathbb{P}^1_k$. Since the isomorphism $A \times A \cong B \times B$ is G-equivariant, we have $[A \times A]_+ = [B \times B]_+$, and hence $([A] - [B])[\mathbb{P}^1_k] = 0$ in $\mathrm{K}_0(\mathbf{Var}_k)$. However, the Albanese map $\mathrm{K}_0(\mathbf{Var}_k) \to \mathbb{Z}[\mathrm{AV}_k]$, described in [Poo02, Section 4], from the Grothendieck ring of varieties to the monoid ring of abelian varieties over k sends $([A] - [B])[\mathbb{P}^1_k]$ to $[A] - [B]$, which is non-zero in $\mathbb{Z}[\mathrm{AV}_k]$. Therefore, the G-virtual class cannot be multiplicative.

Nevertheless, we present the following construction, to measure to which extent the G-virtual class is multiplicative.

Lemma 3.6.18. *Let G be a finite group and $\mathcal{H}$ a good set of conjugacy classes of subgroups of G. Let $\mathcal{V}_G^{\mathcal{H}} \subseteq \mathrm{K}_0(\mathbf{Var}_S^G)$ be the subset of elements X such that $[XY]^G = [X]^G[Y]^G$ for all $Y \in \mathrm{K}_0(\mathbf{Var}_S^G)$. Then $\mathcal{V}_G^{\mathcal{H}}$ is a $\mathrm{K}_0(\mathbf{Var}_S)$-subalgebra of $\mathrm{K}_0(\mathbf{Var}_S^G)$.*

Proof. Note that $\mathcal{V}_G^{\mathcal{H}}$ is the left radical of the $\mathrm{K}_0(\mathbf{Var}_S)$-bilinear form

$$
\begin{aligned}
\mathrm{K}_0(\mathbf{Var}_S^G) \times \mathrm{K}_0(\mathbf{Var}_S^G) &\to \mathrm{K}_0(\mathbf{Var}_S) \otimes R_{\mathbb{Q}}(G) \otimes \mathbb{Q} \\
(X, Y) &\mapsto [XY]^G - [X]^G[Y]^G
\end{aligned}
$$

and is therefore a subgroup of $\mathrm{K}_0(\mathbf{Var}_S)$. Furthermore, $\mathcal{V}_G^{\mathcal{H}}$ is closed under multiplication, because for all $X, Y \in \mathcal{V}_G^{\mathcal{H}}$ and $Z \in \mathrm{K}_0(\mathbf{Var}_S^G)$ we have

$$
[(XY)Z]^G = [X(YZ)]^G = [X]^G[YZ]^G = [X]^G[Y]^G[Z]^G = [XY]^G[Z]^G. \quad \square
$$

Theorem 3.6.19. *Let G be a finite group, $\mathcal{H}$ a good set of conjugacy classes of subgroups of G, and suppose that k is a splitting field for G.*

(i) If G acts linearly on $\mathbb{A}^1_k$, then $\mathcal{V}^{\mathcal{H}}_G$ contains $[\mathbb{A}^1_k]$.

(ii) If G acts diagonally on $\mathbb{A}^n_k$, then $\mathcal{V}^{\mathcal{H}}_G$ contains $[\mathbb{A}^n_k]$.

(iii) If G acts diagonally on $\mathbb{P}^n_k$, then $\mathcal{V}^{\mathcal{H}}_G$ contains $[\mathbb{P}^n_k]$.

(iv) Let $[H] \in \mathcal{H}$ be such that $[H' \cap gHg^{-1}] \in \mathcal{H}$ for all $[H'] \in \mathcal{H}$ and $g \in G$. Then the set G/H of cosets with the natural action of G lies in $\mathcal{V}^{\mathcal{H}}_G$.

Proof. (i) As $[\mathbb{A}^1_k /\!\!/ H] = \mathbb{L}$ for any finite group H acting linearly on $\mathbb{A}^1_k$, we have $[\mathbb{A}^1_k]^G = \mathbb{L} \otimes T$ where $T \in R_{\mathbb{Q}}(G)$ corresponds to the trivial representation. Hence, it suffices to show that $[(\mathbb{A}^1_k \times Y) /\!\!/ H]_S = \mathbb{L}[Y /\!\!/ H]_S$ for all $Y \in \mathbf{Var}^G_S$ and $[H] \in \mathcal{H}$. Take such Y and H, write $\tau \colon H \to \mathrm{GL}_1(k)$ for the representation via which H acts on $\mathbb{A}^1_k$, and let $N = \ker \tau$. Since

$$(\mathbb{A}^1_k \times Y) /\!\!/ H = (\mathbb{A}^1_k \times (Y /\!\!/ N)) /\!\!/ (H/N)$$

we may, replacing H by H/N and Y by $Y /\!\!/ N$, assume that H is a finite cyclic group, that is, $H = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$.

Also, we may assume $Y = \operatorname{Spec} R$ is affine. Write $R^H \subseteq R$ for the subring of H-invariants. Then R is finitely generated as R^H-module [Mon80, Corollary 5.9], so it can be written as

$$R = R^H\langle \sigma_{1,1}, \dots, \sigma_{1,m_1} \rangle \oplus \cdots \oplus R^H\langle \sigma_{n-1,1}, \dots, \sigma_{n-1,m_{n-1}} \rangle$$

for some $\sigma_{i,j} \in R$ such that H acts via $(a \bmod n) \cdot \sigma_{i,j} \mapsto \zeta_n^{ai} \sigma_{i,j}$, where $\zeta_n \in k$ is a primitive n-th root of unity. Note that, for any $1 \leq i \leq n-1$, we have $\sigma_{i,1}^n = r$ for some $r \in R^H$, and for any $2 \leq j \leq m_i$, we have $\sigma_{i,1}^{n-1}\sigma_{i,j} = s$ for some $s \in R^H$. But then, over the closed subvariety of Y given by $r = 0$, we have $\sigma_{i,1}^n = r = 0$, so we can omit $\sigma_{i,1}$ from the generators. Similarly, over the open complement where r is invertible (so $\sigma_{i,1}$ is invertible as well), we can remove $\sigma_{i,j} = \frac{s}{r}\sigma_{i,1}$ from the generators. Hence, after sufficiently many stratifications, we may reduce to the case that

$$R = R^H \oplus R^H\langle \sigma_d \rangle \oplus R^H\langle \sigma_{2d} \rangle \oplus \cdots \oplus R^H\langle \sigma_{n-d} \rangle$$

for some $d \geq 1$ dividing n, and some $\sigma_i \in R^\times$, such that H acts via $(a \bmod n) \cdot \sigma_i = \zeta_n^{ai} \cdot \sigma_i$. In particular, for any $1 \leq m \leq n/d$, we have $\sigma_d^m = r_m \sigma_{md}$ for some $r_m \in (R^H)^\times$, that is, $\sigma_{md} = \sigma_d^m / r_m$, and hence

$$R = R^H[\sigma]/(\sigma^{n/d} - r)$$

with $\sigma = \sigma_d$ and $r = r_{n/d}$.

Note that $\tau\colon H \to \mathrm{GL}_1(k)$ must be of the form $\tau(a \bmod n) = \zeta_n^{ac}$ for some $0 \le c \le n-1$. Stratifying $\mathbb{A}^1_k$ as $\{0\} \sqcup (\mathbb{A}^1_k \setminus \{0\})$, we find that

$$(\{0\} \times Y) /\!\!/ H = Y /\!\!/ H$$

and

$$\begin{aligned}((\mathbb{A}^1_k \setminus \{0\}) \times Y) /\!\!/ H &\cong \operatorname{Spec} R[x^{\pm 1}]^H \\ &\cong \operatorname{Spec} \left(R^H[\sigma, x^{\pm 1}]/(\sigma^{n/d} - r) \right)^H \\ &\cong \operatorname{Spec} R^H \left\langle x^i \sigma^j : (i,j) \in L \right\rangle,\end{aligned}$$

where $L = \left\{ (i,j) \in \mathbb{Z}^2 \mid ci + dj \equiv 0 \bmod n \right\}$ is a lattice. Take some $(i_0, j_0) \in L$ such that $i_0 > 0$ is minimal, and write $w = x^{i_0}\sigma^{j_0}$. Then, for any other $(i,j) \in L$, we must have $i = mi_0$ for some $m \in \mathbb{Z}$, and hence $(x^i\sigma^j)/w^m$ is an element of R^H. Therefore,

$$((\mathbb{A}^1_k \setminus \{0\}) \times Y) /\!\!/ H \cong \operatorname{Spec} R^H[w^{\pm 1}] \cong \left(\mathbb{A}^1_k \setminus \{0\}\right) \times (Y /\!\!/ H).$$

Finally, we find that

$$\begin{aligned}[(\mathbb{A}^1_k \times Y) /\!\!/ H]_S &= [(\{0\} \times Y) /\!\!/ H]_S + [((\mathbb{A}^1_k \setminus \{0\}) \times Y) /\!\!/ H]_S \\ &= [Y /\!\!/ H]_S + (\mathbb{L} - 1)[Y /\!\!/ H]_S \\ &= \mathbb{L}[Y /\!\!/ H]_S\end{aligned}$$

as desired.

Since $\mathcal{V}_G^{\mathcal{H}}$ is closed under multiplication, *(ii)* follows from *(i)*. For *(iii)*, stratify $\mathbb{P}^n_S$ as $\mathbb{A}^n_S \sqcup \mathbb{P}^{n-1}_S$, so that the result follows from *(i)* and by induction on n.

For *(iv)*, note that $[G/H]^G = 1 \otimes \operatorname{Ind}_H^G(T_H)$, where $T_H \in R_{\mathbb{Q}}(H)$ corresponds to the trivial representation. Now, for any $[H'] \in \mathcal{H}$, we can choose representatives gH for the points of the quotient $(G/H) /\!\!/ H'$. Note that the stabilizer of gH for the action of H' is $H' \cap gHg^{-1}$, and therefore

$$((G/H) \times Y) /\!\!/ H' = \bigsqcup_{[gH] \in (G/H) /\!\!/ H'} Y /\!\!/ (H' \cap gHg^{-1}).$$

Since $[H' \cap gHg^{-1}] \in \mathcal{H}$ by assumption, it follows that the coefficients of $[(G/H) \times Y]_S^G$ can be written naturally in terms of the coefficients of $[Y]_S^G$, and therefore $[G/H]$ must be contained in $\mathcal{V}_G^{\mathcal{H}}$. □

Remark 3.6.20. The condition in *(iv)* of the above theorem is trivially satisfied when $\mathcal{H}$ contains the conjugacy classes of all subgroups of G. Also, it is satisfied in the case of Example 3.6.15 with $G = S_n$. That is, the intersection of conjugates of Young subgroups is again the conjugate of a Young subgroup.

We conclude this section with two examples, both for the group $G = \mathbb{Z}/2\mathbb{Z}$. Let $T, N \in R_{\mathbb{Q}}(G)$ correspond to the trivial and non-trivial irreducible representation of G, respectively. For any G-variety X, the G-virtual class is, similar to Example 3.6.7, given by

$$[X]^{\mathbb{Z}/2\mathbb{Z}} = [X /\!\!/ (\mathbb{Z}/2\mathbb{Z})] \otimes T + ([X] - [X /\!\!/ (\mathbb{Z}/2\mathbb{Z})]) \otimes N. \tag{3.12}$$

Example 3.6.21. Let k be an algebraically closed field with $\operatorname{char}(k) \neq 2$. Consider the subvariety $M = \{A \in \mathrm{SL}_2 \mid \operatorname{tr} A \neq \pm 2\}$ of SL_2, over k, of diagonalizable non-scalar matrices. Note that we have a cartesian diagram

$$\begin{array}{ccc} \mathrm{GL}/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1\}) & \longrightarrow & M \\ \big\downarrow & & \big\downarrow \scriptstyle{\mathrm{tr}} \\ \mathbb{A}^1_k \setminus \{0, \pm 1\} & \longrightarrow & \mathbb{A}^1_k \setminus \{\pm 2\} \end{array}$$

where $D \subseteq \mathrm{GL}_2$ is the subgroup of diagonal matrices, and GL_2/D the left coset space. The bottom morphism is given by $\lambda \mapsto \lambda + \lambda^{-1}$, and the top morphism by $(P, \lambda) \mapsto P \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) P^{-1}$. The group $G = \mathbb{Z}/2\mathbb{Z}$ acts both on GL_2/D and $\mathbb{A}^1_k \setminus \{0, \pm 1\}$, via $P \mapsto P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\lambda \mapsto \lambda^{-1}$, respectively, and we can identify M with $(\mathrm{GL}_2/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1\})) /\!\!/ G$. Since $\mathbb{A}^1_k \setminus \{0, \pm 1\}$ is a projective line minus some points, its class lies in $\mathcal{V}^{\mathcal{H}}_G$ using Theorem 3.6.19, so we can compute $[M] \in \mathrm{K}_0(\mathbf{Var}_k)$ from the G-virtual classes $[\mathrm{GL}_2/D]^G$ and $[\mathbb{A}^1_k \setminus \{0, \pm 1\}]^G$. Using $(\mathbb{A}^1_k \setminus \{0, \pm 1\}) /\!\!/ G \cong \mathbb{A}^1_k \setminus \{\pm 2\}$ and (3.12), we find that

$$[\mathbb{A}^1_k \setminus \{0, \pm 1\}]^G = (\mathbb{L} - 2) \otimes T - 1 \otimes N.$$

Similarly, from $[(\mathrm{GL}_2/D) /\!\!/ G] = \mathbb{L}^2$ follows that

$$[\mathrm{GL}_2/D]^G = \mathbb{L}^2 \otimes T + \mathbb{L} \otimes N$$

and hence

$$[\mathrm{GL}_2/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1\})]^G = (\mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}) \otimes T + (2\mathbb{L}^2 - 2\mathbb{L}) \otimes N.$$

Therefore, $[M] = \mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}$.

Example 3.6.22. Consider $G = \mathbb{Z}/2\mathbb{Z}$ acting on $X = \mathbb{G}_m$ via $x \mapsto x^{-1}$, over any field k. As $[X] = \mathbb{L} - 1$ and $[X /\!\!/ G] = \mathbb{L}$, we obtain $[X]^G = \mathbb{L} \otimes T - 1 \otimes N$. Since X can be seen as a projective line minus two points, its class lies in $\mathcal{V}^{\mathcal{H}}_G$, so we find that

$$[X^n /\!\!/ G] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{L} & -1 \\ -1 & \mathbb{L} \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{(\mathbb{L} - 1)^n + (\mathbb{L} + 1)^n}{2}.$$