Motivic invariants of character stacks
Vogel, J.T.

Chapter 2

Character stacks

In this chapter we will define, and describe various properties of, character stacks, which are the main objects of study in this thesis. Roughly speaking, they are the moduli space of representations of a finitely generated group Γ into a linear algebraic group G. While Γ can be any finitely generated group, it most commonly arises as the fundamental group $\pi_1(M, *)$ of a compact manifold M. In fact, every finitely presented group arises in this way. In this case, it is well known that representations $\pi_1(M, *) \to G$ correspond to G-local systems on M [Sza09, Corollary 2.6.2]. Moreover, isomorphic local systems correspond to conjugate representations. Therefore, one is interested in the quotient of the space parametrizing *all* representations $\Gamma \to G$ (this space will be called the 'representation variety'), by the action of conjugation by G. This quotient will be the G-character stack of Γ.

2.1 Representation varieties

Fix a base scheme S. Typically, S will be $\operatorname{Spec} k$ where k is a field or a finitely generated $\mathbb{Z}$-algebra. Let G be a linear algebraic group over S, by which we understand a closed subgroup of the group scheme GL_r over S for some $r \geq 0$.

Definition 2.1.1. Let Γ be a finitely generated group. The *G-representation variety* of Γ is the scheme $R_G(\Gamma)$ over S whose functor of points is given by

$$R_G(\Gamma)(T) = \operatorname{Hom}(\Gamma, G(T)).$$

Let us explain why $R_G(\Gamma)$ is indeed representable. After choosing a presentation

$$\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid r_i(\gamma_1, \ldots, \gamma_n) = 1 \text{ for } i \in I \rangle,$$

any representation $\rho\colon \Gamma \to G(T)$ can be identified with the image of its generators, that is, the tuple $(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \in G(T)^n$. However, not all tuples in $G(T)^n$ define such a representation because of the relations r_i between the generators. Every such relation r_i, which is a word in the symbols γ_i, defines a morphism $r_i\colon G^n \to G$ given on points by $(g_1, \dots, g_n) \mapsto r_i(g_1, \dots, g_n)$, and hence a closed subscheme $X_i \subseteq G^n$ as in the pullback diagram

$$\begin{array}{ccc} X_i & \longrightarrow & G^n \\ \big\downarrow & & \big\downarrow{\scriptstyle r_i} \\ S & \xrightarrow{\ e\ } & G \end{array}$$

where e is the unit of G, a closed immersion [Stacks, Tag 047G]. Now, the intersection of all X_i over G^n realizes $R_G(\Gamma)$ as a closed subscheme of G^n. This closed subscheme corresponds to the sheaf of ideals in $\mathcal{O}_{G^n}$ that is generated by the sheaves of ideals $\mathcal{I}_i \subseteq \mathcal{O}_{G^n}$ corresponding to the X_i. Indeed, we have

$$R_G(\Gamma)(T) = \bigcap_{i \in I} \big\{ t \in G(T)^n \mid r_i(t) = 1 \big\} = \bigcap_{i \in I} X_i(T) = \Big(\bigcap_{i \in I} X_i \Big)(T).$$

Remark 2.1.2. The G-representation variety $R_G(\Gamma)$ will always be separated and of finite type over S, as it is a closed subscheme of G^n, which itself is separated and of finite type over S. Moreover, $R_G(\Gamma)$ is affine over S, as G^n is affine over S. However, the G-representation variety may be non-reduced. For example, it was shown in [LM85, (2.10.4)] that for the von Dyck group $\Gamma = \langle a, b, c \mid a^3 = b^3 = c^3 = abc = 1 \rangle \cong \mathbb{Z}^2 \rtimes S_3$ and $G = \mathrm{GL}_2$ over $S = \operatorname{Spec} \mathbb{C}$, the G-representation variety $R_G(\Gamma)$ is non-reduced.

For us, the main example of a finitely generated group Γ is the fundamental group of a compact manifold.

Proposition 2.1.3. *Let M be a connected compact manifold with a basepoint x. Then $\pi_1(M, x)$ is finitely presented.*

Proof. Every compact manifold M is homotopy equivalent to a finite CW-complex [Whi40]. Since M is connected, this finite CW-complex can be chosen to consist of a single 0-cell corresponding to x. It follows that the fundamental group of M has a presentation with a generator for every 1-cell and a relation for every 2-cell, and is therefore finitely presented. □

When M is a connected compact manifold, we will simply write $R_G(M)$ instead of $R_G(\pi_1(M, x))$ and call it the *G-representation variety* of M. Note that this scheme is, up to isomorphism, independent of the chosen basepoint x since the fundamental group $\pi_1(M, x)$ is, up to isomorphism, independent of x.

Example 2.1.4.
- The circle S^1 has fundamental group $\pi_1(S^1, *) \cong \mathbb{Z}$, from which follows that $R_G(S^1) \cong G$.

- The fundamental group of a closed orientable surface Σ_g of genus g can be presented as $\pi_1(\Sigma_g, *) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$, where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ denotes the commutator. Therefore, $R_G(\Sigma_g)$ is the closed subscheme of G^{2g} given by $\prod_{i=1}^{g} [A_i, B_i] = 1$.
- Let N_r be the connected sum of r projective planes, that is, the non-orientable closed surface of demigenus r. Its fundamental group can be presented as $\pi_1(N_r, *) = \langle a_1, \ldots, a_r \mid a_1^2 \cdots a_r^2 = 1 \rangle$. Hence, $R_G(N_r)$ is the closed subvariety of G^r given by $\prod_{i=1}^{r} A_i^2 = 1$.

While the G-representation variety $R_G(\Gamma)$ is an interesting object on its own, it cannot quite be regarded as the moduli space of representations of Γ into G. Namely, two different points of $R_G(\Gamma)$ might represent isomorphic representations, that is, representations that are related through conjugation. Formulated differently, the linear algebraic group G acts on the G-representation variety by conjugation

$$(g \cdot \rho)(\gamma) = g \rho(\gamma) g^{-1}$$

for all $g \in G(T)$ and $\gamma \in R_G(\Gamma)(T)$. In this sense, the correct moduli space should be the quotient of $R_G(\Gamma)$ by the action of G. Unfortunately, quotients are famously hard in algebraic geometry, and it is not always clear which quotient one wants to take.

One possibility is to take the *Geometric Invariant Theory (GIT)* quotient as developed by Mumford [Mum65]. Given an affine variety $X = \operatorname{Spec} R$ over a field k with an action of a linear algebraic group G over k, encoded by a ring morphism $\hat{\sigma} \colon R \to R \otimes_k \mathcal{O}_G(G)$, the (affine) GIT quotient of X by G is

$$X /\!\!/ G = \operatorname{Spec} R^G$$

where $R^G = \{ r \in R \mid \hat{\sigma}(r) = r \otimes 1 \}$ denotes the subring of invariants of R. The projection $X \to X /\!\!/ G$ corresponds to the inclusion $R^G \subseteq R$. Even though the GIT quotient can be constructed as a scheme, it was shown by Nagata that in general the resulting scheme need not be of finite type over k [Nag59]. However, he also showed that if G is reductive, the ring of invariants will be finitely generated over k [Nag64].

Definition 2.1.5. Let G be a reductive linear algebraic group over a field k, and Γ a finitely generated group. The *G-character variety* of Γ is the GIT quotient

$$X_G(\Gamma) = R_G(\Gamma) /\!\!/ G.$$

Remark 2.1.6. In the literature, the term 'G-character variety' is also used for a notion which is different, but related, to the above definition. Given a linear algebraic group $G \subseteq \mathrm{GL}_n$ over $k = \mathbb{C}$, one defines

$$\chi_G(\Gamma) = \operatorname{Spec} \mathbb{C}[\tau_\gamma \mid \gamma \in \Gamma]$$

to be the spectrum of the complex algebra generated by the functions $\tau_\gamma \colon \rho \mapsto \operatorname{tr}(\rho(\gamma))$ on $R_G(\Gamma)$. Since the functions τ_γ are invariant under the action of G, there is a canonical morphism

$$X_G(\Gamma) \to \chi_G(\Gamma).$$

While this morphism is known to be an isomorphism for various G, such as SL_n, GL_n, Sp_{2n} and O_n, see [FL11, Theorem A.1] and [Pro76], it fails to be so for other groups, such as SO_{2n} [Sik13].

Besides the GIT quotient, there are other ways to construct quotients. In the following sections we will apply the theory of quotient stacks, as encountered in Section 1.5, to take the quotient in the category of stacks, defining the *G-character stack*. One advantage to this approach is that the quotient remembers the automorphisms of the representations. Another advantage is that we do not need to assume that G is reductive.

2.2 Character groupoids

Before we properly introduce the G-character stack, we will first forget all geometry, and let G be an ordinary group. Furthermore, we will allow for a more general setup, with Γ being a groupoid, rather than a group.

Definition 2.2.1. Let G be a group. For any groupoid Γ, the *G-character groupoid* of Γ, denoted $\mathfrak{X}_G(\Gamma)$, is the groupoid whose objects are functors $\rho \colon \Gamma \to G$ (where G is seen as a groupoid with a single object), and whose morphisms $\rho_1 \to \rho_2$ are given by natural transformations $\mu \colon \rho_1 \Rightarrow \rho_2$.

The map $\mathfrak{X}_G$ can naturally be extended to a 2-functor $\mathfrak{X}_G \colon \mathbf{Grpd} \to \mathbf{Grpd}^{\mathrm{op}}$. Explicitly:

- For any functor $f \colon \Gamma' \to \Gamma$ between groupoids, let $\mathfrak{X}_G(f) \colon \mathfrak{X}_G(\Gamma) \to \mathfrak{X}_G(\Gamma')$ be the functor given by precomposition $\mathfrak{X}_G(f)(\rho) = \rho \circ f$ for any $\rho \in \mathfrak{X}_G(\Gamma)$, and $\mathfrak{X}_G(f)(\mu) = \mu f$ for any morphism $\mu \colon \rho_1 \to \rho_2$.
- For any natural transformation $\eta \colon f_1 \Rightarrow f_2$ between functors $f_1, f_2 \colon \Gamma' \to \Gamma$, let $\mathfrak{X}_G(\eta) \colon \mathfrak{X}_G(f_1) \Rightarrow \mathfrak{X}_G(f_2)$ be the natural transformation given by $(\mathfrak{X}_G(\eta)_\rho)_{x'} = \rho(\eta_{x'})$ for all $\rho \in \mathfrak{X}_G(\Gamma)$ and $x' \in \Gamma'$. Indeed, this defines a natural transfor-

mation as the square

$$\begin{array}{ccc} \rho(f_1(x')) & \xrightarrow{\rho(\eta_{x'})} & \rho(f_2(x')) \\ {\scriptstyle \rho(f_1(\gamma'))}\big\downarrow & & \big\downarrow{\scriptstyle \rho(f_2(\gamma'))} \\ \rho(f_1(y')) & \xrightarrow[\rho(\eta_{y'})]{} & \rho(f_2(y')) \end{array}$$

commutes for every $\gamma' \colon x' \to y'$ in Γ' by naturality of η, and this is natural in ρ.

Note that $\mathfrak{X}_G$ strictly preserves composition of 1-morphisms and 2-morphisms, and therefore defines a strict 2-functor.

Corollary 2.2.2. *An equivalence between groupoids Γ and Γ' naturally induces an equivalence between the G-character groupoids $\mathfrak{X}_G(\Gamma)$ and $\mathfrak{X}_G(\Gamma')$.* □

Let us apply the above corollary as follows in the case that G is a finite group. If Γ is a finitely generated groupoid, then it can easily be seen that the groupoid $\mathfrak{X}_G(\Gamma)$ is finite. But now it follows from Corollary 2.2.2 that $\mathfrak{X}_G(\Gamma)$ is essentially finite if Γ is essentially finitely generated. Therefore, for G finite, we can restrict $\mathfrak{X}_G$ to a 2-functor

$$\mathfrak{X}_G \colon \mathbf{FGGrpd} \to \mathbf{FinGrpd}^{\mathrm{op}}.$$

As before, the main example of a finitely generated groupoid Γ for us comes from a compact manifold.

Definition 2.2.3. Let M be a compact manifold. The *fundamental groupoid* of M is the groupoid $\Pi(M)$ whose objects are the points of M, and morphisms $x \to y$ are given by homotopy classes of paths from x to y.

For any smooth map of manifolds $f \colon M \to N$, there is an induced a functor $\Pi(f) \colon \Pi(M) \to \Pi(N)$. In particular, one can think of Π as a functor $\Pi \colon \mathbf{Mnfd} \to \mathbf{Grpd}$ from the category of manifolds to the category of groupoids. Moreover, Π can be promoted to a 2-functor if one considers **Mnfd** as a 2-category where 2-morphisms are given by smooth homotopies.

Note that the fundamental groupoid $\Pi(M)$ is essentially finitely generated when M is a compact manifold. Namely, choosing a basepoint $x_1, \ldots, x_n$ on each of the finitely many connected component of M, we find that $\Pi(M)$ is equivalent to $\pi_1(M, x_1) \sqcup \cdots \sqcup \pi_1(M, x_n)$, which is finitely generated by Proposition 2.1.3.

Definition 2.2.4. Let G be a group and let M be a compact manifold. The *G-character groupoid* of M, denoted $\mathfrak{X}_G(M)$, is defined as $\mathfrak{X}_G(\Pi(M))$, where $\Pi(M)$ is the fundamental groupoid of M. In particular, if G is finite, $\mathfrak{X}_G(M)$ is essentially finite.

Let us elaborate a bit more on the groupoid $\mathfrak{X}_G(M)$. Its objects $\rho\colon \Pi(M) \to G$ assign to every homotopy class of paths γ an element $\rho(\gamma)$ of G. A morphism from ρ_1 to ρ_2 is a natural transformation $\mu\colon \rho_1 \Rightarrow \rho_2$. Such a natural transformation can be thought of as a function $\mu\colon M \to G$ such that $\rho_2(\gamma) = \mu(y)\rho_1(\gamma)\mu(x)^{-1}$ for any path $\gamma\colon x \to y$ in $\Pi(M)$. Such transformations are known in physics as *local gauge transformations.*

With this characterization, the G-character groupoid can be defined in an alternative way. Let $\mathcal{G}_\Gamma = \prod_{x\in\Gamma} G$ be the *group of local gauge transformations*, which acts on the set $X = \mathrm{Hom}(\Gamma, G)$ via

$$((g_x)_{x\in\Gamma} \cdot \rho)(\gamma) = g_y \rho(\gamma) g_x^{-1}$$

for any $\rho \in X$ and $\gamma\colon x \to y$ in Γ. Now, the G-character groupoid $\mathfrak{X}_G(\Gamma)$ is equivalent to the action groupoid $[X/\mathcal{G}_\Gamma]$. This alternative description will be of crucial importance in defining the G-character stacks.

2.3 Character stacks

The G-character stack will be defined as the geometric analogue of the G-character groupoid, replacing the action groupoid by the quotient stack. Fix a base scheme S and let G be a linear algebraic group over S.

Definition 2.3.1. Let Γ be a finitely generated groupoid. The *G-representation variety* of Γ is the scheme over S whose functor of points is given by

$$R_G(\Gamma)(T) = \mathrm{Hom}(\Gamma, G(T)),$$

where $G(T)$ is seen as a groupoid with a single object. Completely analogous to the discussion below Definition 2.1.1, the G-representation variety is representable by a closed subscheme of G^n for some n.

Importantly, note that $R_G(\Gamma)$ is not well-defined up to equivalence of Γ. That is, $R_G(\Gamma)$ need not be isomorphic to $R_G(\Gamma')$ even when Γ is equivalent to Γ'. This problem will be fixed once we pass to the G-character stack.

Analogous to the previous section, for a finitely generated groupoid Γ, we define the *group of local gauge transformations* to be the group scheme

$$\mathcal{G}_\Gamma = \prod_{x\in\Gamma} G$$

which, as a finite product of linear algebraic groups, is again a linear algebraic group over S. It acts naturally on $R_G(\Gamma)$, and the action is pointwise given by

$$((g_x)_{x\in\Gamma} \cdot \rho)(\gamma) = g_y \rho(\gamma) g_x^{-1}$$

for all $(g_x)_{x\in\Gamma} \in \mathcal{G}_\Gamma(T)$ and $\rho \in R_G(\Gamma)(T)$ and $\gamma\colon x \to y$ in Γ.

Definition 2.3.2. Let Γ be a finitely generated groupoid. The *G-character stack* of Γ is the quotient stack

$$\mathfrak{X}_G(\Gamma) = [R_G(\Gamma)/\mathcal{G}_\Gamma]\,.$$

As for the G-character groupoids, we want to extend $\mathfrak{X}_G(-)$ to essentially finitely generated groupoids, and promote it to a 2-functor $\mathbf{FGGrpd} \to \mathbf{Stck}_S^{\mathrm{op}}$, where $\mathbf{Stck}_S$ is the category of algebraic stacks of finite type over S with affine stabilizers, as defined in Definition 1.6.4.

Let $f\colon \Gamma' \to \Gamma$ be a functor between finitely generated groupoids. Such a functor induces a morphism between the representation varieties, given by pullback

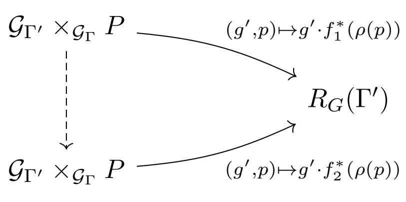
$$f^*\colon R_G(\Gamma) \to R_G(\Gamma'), \quad \rho \mapsto \rho \circ f \quad \text{for all } \rho \in R_G(\Gamma)(T),$$

and also a morphism of algebraic groups

$$\mathcal{G}_f\colon \mathcal{G}_\Gamma \to \mathcal{G}_{\Gamma'}, \quad (g_x)_{x\in\Gamma} \mapsto (g_{f(x')})_{x'\in\Gamma'}.$$

In particular, as described in Remark 1.5.6, there is an induced map on character stacks $\mathfrak{X}_G(f)\colon \mathfrak{X}_G(\Gamma) \to \mathfrak{X}_G(\Gamma')$ that sends a $\mathcal{G}_\Gamma$-torsor P to the $\mathcal{G}_{\Gamma'}$-torsor $\mathcal{G}_{\Gamma'} \times_{\mathcal{G}_\Gamma} P$. Note that this construction is functorial in f.

Next, let $\eta\colon f_1 \Rightarrow f_2$ be a natural transformation between functors $f_1, f_2\colon \Gamma' \to \Gamma$. We want to assign a 2-morphism $\mathfrak{X}_G(\eta)\colon \mathfrak{X}_G(f_1) \Rightarrow \mathfrak{X}_G(f_2)$ to this natural transformation, which amounts to, for every $\mathcal{G}_\Gamma$-torsor P over T with $\mathcal{G}_\Gamma$-equivariant map $\rho\colon P \to R_G(\Gamma)$, a morphism of $\mathcal{G}_{\Gamma'}$-torsors (as indicated by the dashed arrow) such that the diagram



commutes. Analogous to the case for G-character groupoids, this morphism is given by $(g', p) \mapsto (g'\rho(p)(\eta_{x'}), p)$. One easily sees that this map is well-defined, that is, respects the $\mathcal{G}_\Gamma$-action on both sides.

Corollary 2.3.3. *Any equivalence between finitely generated groupoids Γ and Γ' naturally induces an isomorphism between the G-character stacks $\mathfrak{X}_G(\Gamma)$ and $\mathfrak{X}_G(\Gamma')$.* □

This corollary allows us to extend the definition of the G-character stack to groupoids Γ which are only essentially finitely generated, but only up to a natural isomorphism. In particular, we obtain a 2-functor

$$\mathfrak{X}_G(-)\colon \mathbf{FGGrpd} \to \mathbf{Stck}_S^{\mathrm{op}}.$$

We are now able to define the G-character stack of a compact manifold.

Definition 2.3.4. Let M be a compact manifold (possibly with boundary). It was shown that the fundamental groupoid $\Pi(M)$ of M is essentially finitely generated, that is, is equivalent to a finitely generated groupoid Γ. The *G-character stack* of M is defined as

$$\mathfrak{X}_G(M) = \mathfrak{X}_G(\Gamma).$$

This definition is, up to isomorphism, independent of the choice of Γ by the above corollary.

Remark 2.3.5. It might be tempting to define the G-character stack of Γ, similar to the G-representation variety, as the category fibered in groupoids over $\mathfrak{S} = \mathbf{Sch}_S$ whose fiber over an object T is the G-character groupoid $\mathfrak{X}_{G(T)}(\Gamma)$. However, these groupoids are different as explained in Remark 1.5.9.

Lemma 2.3.6. *$\mathfrak{X}_G(-)$ sends finite colimits in* $\mathbf{FGGrpd}$ *to limits in* $\mathbf{Stck}_S$*.*

Proof. Let $\Gamma = \operatorname{colim}_{i \in I} \Gamma_i$ be a colimit in $\mathbf{FGGrpd}$. Up to equivalence, we can assume all Γ_i and Γ are finitely generated groupoids. A T-point of $\lim_{i \in I} \mathfrak{X}_G(\Gamma_i)$ is a collection of $\mathcal{G}_{\Gamma_i}$-torsors P_i over T with $\mathcal{G}_{\Gamma_i}$-equivariant morphisms $\rho_i\colon P_i \to R_G(\Gamma_i)$, which are compatible in the sense that there are natural isomorphisms $\mathcal{G}_{\Gamma_i} \times_{\mathcal{G}_{\Gamma_j}} P_j \cong P_i$ in $\mathfrak{X}_G(\Gamma_i)$ for every $i \to j$ in I. On the other hand, a T-point of $\mathfrak{X}_G(\Gamma)$ is a $\mathcal{G}_\Gamma$-torsor P over T with a $\mathcal{G}_\Gamma$-equivariant morphism $\rho\colon P \to R_G(\Gamma)$. Note that ρ, on T'-points, is given by

$$\rho\colon P(T') \to R_G(\Gamma)(T') = \operatorname{Hom}(\operatorname*{colim}_{i \in I} \Gamma_i, G(T')) = \lim_{i \in I} \operatorname{Hom}(\Gamma_i, G(T'))$$

so ρ is equivalently described by compatible morphisms $\rho_i\colon P \to R_G(\Gamma_i)$ which are $\mathcal{G}_{\Gamma_i}$-equivariant, where $\mathcal{G}_{\Gamma_i}$ acts on P via $\mathcal{G}_\Gamma$.

These two descriptions are related as follows. From the $\mathcal{G}_\Gamma$-torsor P, one constructs the $\mathcal{G}_{\Gamma_i}$-torsors $P_i = \mathcal{G}_{\Gamma_i} \times_{\mathcal{G}_\Gamma} P$, which are naturally compatible. Conversely, from the P_i one constructs $\lim_{i \in I} P_i$, where the limit is taken as schemes over T, which naturally comes with the structure of a $(\lim_{i \in I} \mathcal{G}_{\Gamma_i})$-torsor, and one puts $P = \mathcal{G}_\Gamma \times_{(\lim_{i \in I} \mathcal{G}_{\Gamma_i})} \lim_{i \in I} P_i$. This induces the desired isomorphism between $\lim_{i \in I} \mathfrak{X}_G(\Gamma_i)$ and $\mathfrak{X}_G(\Gamma)$. ∎