Motivic invariants of character stacks

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Chapter 1

Stacks

Algebraic stacks were first introduced by Deligne and Mumford to study the moduli space of curves [DM69], and later their definition was generalized by Artin [Art74]. Roughly speaking, an algebraic stack can be thought of as a generalization of a scheme. If we view a scheme as a functor of points, its points form a set, whereas for an algebraic stack they form a groupoid. In other words, the points of a stack are allowed to have automorphisms. The notion of a stack is not specific to algebraic geometry, that is, stacks can also be defined in the context of manifolds, analytic spaces, topological spaces, or, in general, for any site, that is, category with a Grothendieck topology.

The goal of this chapter is to give a concise overview of (algebraic) stacks, with a focus on quotient stacks, which should be sufficient to understand the later chapters. For the curious reader who wishes to read more in-depth expositions of (algebraic) stacks, we refer to [Fan01, Beh14, Ols16, LM00, Stacks], in order from introductory and intuitive to detailed and rigorous.

1.1 Groupoids

Crucial to the subject of stacks is the concept of a groupoid, that is, a category in which every morphism is an isomorphism.

Definition 1.1.1. A groupoid is *finite* if it has finitely many morphisms. A groupoid is *finitely generated* if there exists a finite collection of morphisms, called *generators*, such that every morphism of the groupoid can be written as a composite of generators and inverses of generators. In particular, any finite or finitely generated groupoid has finitely many objects, because every object has at least an identity morphism. A groupoid is *essentially finite* if it is equivalent to a finite groupoid, and similarly, a groupoid is *essentially finitely generated* if it is equivalent to a finitely generated groupoid.

Denote by **Grpd** the 2-category of groupoids, whose objects are groupoids, 1-morphisms are functors, and 2-morphisms are natural transformations. Similarly, denote by **FinGrpd** and **FGGrpd** the full sub-2-categories of essentially finite groupoids and essentially finitely generated groupoids, respectively.

Definition 1.1.2. Let G be a group acting on a set X. The *action groupoid*, denoted $[X/G]$, is the groupoid whose objects are elements of X, morphisms $x \to y$ are given by elements $g \in G$ such that $y = g \cdot x$. Composition of $g\colon x \to y$ and $h\colon y \to z$ is given by $hg\colon x \to z$.

Definition 1.1.3. Let Γ be an essentially finite groupoid. The *groupoid cardinality* of Γ is defined as

$$|\Gamma| = \sum_{[x]\in\Gamma/\sim} \frac{1}{|\operatorname{Aut}(x)|} \in \mathbb{Q}$$

where $\Gamma/\sim$ denotes the set of isomorphism classes of Γ.

Example 1.1.4. Let G be a finite group acting on a finite set X. Then the groupoid cardinality of the action groupoid $\Gamma = [X/G]$ is $|X|/|G|$. Indeed, from the orbit-stabilizer theorem it follows that

$$|\Gamma| = \sum_{[x]\in[X/G]/\sim} \frac{1}{|\operatorname{Aut}(x)|} = \sum_{x\in X} \frac{1}{|Gx|}\frac{1}{|\operatorname{Aut}(x)|} = \sum_{x\in X}\frac{1}{|G|} = \frac{|X|}{|G|}.$$

Definition 1.1.5. Let $f\colon B \to A$ and $g\colon C \to A$ be morphisms of groupoids. The *fiber product* of B and C over A, denoted $B \times_A C$, is the groupoid whose objects are triples (x, y, α) with x an object of B, y an object of C and $\alpha\colon f(x) \to g(y)$ a morphism in A. A morphism from (x', y', α') to (x, y, α) is given by a pair of morphisms $(\beta\colon x' \to x, \gamma\colon y' \to y)$ such that $g(\gamma) \circ \alpha' = \alpha \circ f(\beta)$.

Note that the diagram

$$\begin{array}{ccc} B \times_A C & \xrightarrow{\pi_C} & C \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{f} & A \end{array}$$

with π_B and π_C the obvious projections, does not strictly commute whenever there are non-trivial morphisms $\alpha\colon f(x) \to g(y)$. However, there is a natural isomorphism $f \circ \pi_B \Rightarrow g \circ \pi_C$, whose component at (x, y, α) is given by α. This is the correct notion of commutativity for 2-categories, and we say this diagram *2-commutes.*

This also shows what the correct universal property of the fiber product is. For every groupoid D with morphisms $i\colon D \to B$ and $j\colon D \to C$ such that $f \circ i$ is naturally isomorphic to $g \circ j$, there exists, up to a unique natural isomorphism, a unique morphism $h\colon D \to B \times_A C$ and natural isomorphisms $\pi_B \circ h \cong i$ and $\pi_C \circ h \cong j$. One can easily verify that the above definition of the fiber product for groupoids satisfies this universal property.

Definition 1.1.6. Let $f\colon A \to B$ be a functor between groupoids, and let b be an object of B. The *fiber* of f over b is the groupoid

$$f^{-1}(b) = A \times_B \{b\}$$

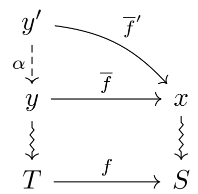
where $\{b\}$ is the groupoid with a single object b and one (identity) morphism, and $\{b\} \to B$ the natural map.

1.2 Categories fibered in groupoids

Throughout the following sections, let $\mathfrak{S}$ be a site, that is, a category equipped with a Grothendieck topology.

Definition 1.2.1. A *category over* $\mathfrak{S}$ is a category $\mathfrak{X}$ with a functor $p\colon \mathfrak{X} \to \mathfrak{S}$. An object x of $\mathfrak{X}$ is said to *lie over* an object S of $\mathfrak{S}$, or x is said to be a *lift* of S, if $p(x) = S$, and similarly for morphisms. If S is an object of $\mathfrak{S}$, the *fiber* of $\mathfrak{X}$ over S, denoted $\mathfrak{X}_S$, is the subcategory of $\mathfrak{X}$ of objects over S and morphisms over id_S. A morphism of categories over $\mathfrak{S}$ is a functor that respects the functor to $\mathfrak{S}$. If $p\colon \mathfrak{X} \to \mathfrak{S}$ and $q\colon \mathfrak{Y} \to \mathfrak{S}$ are categories over $\mathfrak{S}$, and f and g morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$, then a *2-morphism* $f \to g$ is a natural transformation $\mu\colon f \Rightarrow g$ such that all components $\mu_x\colon f(x) \to g(x)$ lie over $\mathrm{id}_{p(x)}$. The categories over $\mathfrak{S}$ form a 2-category. An *isomorphism* of categories over $\mathfrak{S}$ is a morphism which is an equivalence of categories.

Definition 1.2.2. A category $\mathfrak{X}$ over $\mathfrak{S}$ is called a *category fibered in groupoids* over $\mathfrak{S}$ if for any morphism $f\colon T \to S$ in $\mathfrak{S}$ and object x lying over S, there exists a lift $\overline{f}\colon y \to x$ of f which is unique up to unique isomorphism. That is, for any other lift $\overline{f}'\colon y' \to x$ of f, there exists a unique isomorphism $\alpha\colon y' \to y$ such that $\overline{f}' = \overline{f} \circ \alpha$.



As a motivation for the terminology, consider the following lemma.

Lemma 1.2.3. *Let $\mathfrak{X}$ be a category fibered in groupoids over $\mathfrak{S}$. Then every morphism $\varphi\colon y \to x$ of $\mathfrak{X}$ that lies over an isomorphism $f\colon T \to S$ of $\mathfrak{S}$, is an isomorphism as well. In particular, for every object S of $\mathfrak{S}$ the fiber $\mathfrak{X}_S$ is groupoid.*

Proof. Write g for the inverse of f, and choose a lift $\overline{g}\colon z \to y$ of g. As $\varphi \circ \overline{g}\colon z \to x$ lies over $f \circ g = \mathrm{id}_S$, it is a lift of id_S with target x. Since id_x is so as well, there exists a (unique) isomorphism $\alpha\colon z \to x$ such that $\varphi \circ \overline{g} = \alpha$. Now $\psi = \overline{g} \circ \alpha^{-1}$ is a right inverse of φ which lies over g. Repeating the argument, replacing φ by ψ, one shows ψ also has a right inverse, which must be φ. □

In particular, every 2-morphism between morphisms of categories over $\mathfrak{S}$ is automatically an isomorphism.

Example 1.2.4. Any object X of $\mathfrak{S}$ can be regarded as a category fibered in groupoids $p\colon \mathfrak{X} \to \mathfrak{S}$ where $\mathfrak{X}$ is the slice category $\mathfrak{S}/X$ and p simply forgets the morphism to X. Indeed, for any $f\colon T \to S$ in $\mathfrak{S}$ and $x\colon S \to X$ in $\mathfrak{X}$, there is a unique lift of f, given by $T \xrightarrow{x \circ f} X$. Hence, we can think of a category fibered in groupoids (and as we shall see later, a stack) $\mathfrak{X}$ over $\mathfrak{S}$ as a generalization of an object of $\mathfrak{S}$, and the fibers $\mathfrak{X}_S$ can be interpreted as the groupoid of S-points of $\mathfrak{X}$.

For convenience, we usually assume that for every morphism $f\colon T \to S$ in $\mathfrak{S}$ and object x over S, we have chosen a lift $f^*x \to x$ of f with target x. Depending on the context, this can be done either by direct construction, or by using a suitable version of the axiom of choice. Note that it is not required that $g^*(f^*x)$ equals $(f \circ g)^*x$, but the two are naturally isomorphic. While such a choice of lifts is not necessary, it makes it easier to write down the definition of a stack. We refer to the object f^*x as the *pullback* of x along f. When the morphism $f\colon T \to S$ is clear from context, we will also write $x|_T$ for f^*x.

Remark 1.2.5. Let $\alpha\colon x' \to x$ be a morphism in the fiber over some object S of $\mathfrak{S}$ (in particular, α is an isomorphism). Given a morphism $f\colon T \to S$, there exists a unique isomorphism $f^*\alpha\colon f^*x' \to f^*x$ such that the diagram

$$\begin{array}{ccc} f^*x' & \longrightarrow & x' \\ {\scriptstyle f^*\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ f^*x & \longrightarrow & x \end{array}$$

commutes. Namely, $f^*x' \to x' \to x$ is also a lift of f with target x. When the morphism f is clear from context, we will also write $\alpha|_T$ for $f^*\alpha$.

Notation 1.2.6. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two categories fibered in groupoids over $\mathfrak{S}$. Note that the morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$ form a category, which we denote by $\mathfrak{Y}(\mathfrak{X})$, where morphisms between morphisms are given by 2-morphisms. Moreover, since every 2-morphism is an isomorphism, this category is a groupoid. When $\mathfrak{X} = \mathfrak{S}/X$ for an object X of $\mathfrak{S}$, as in Example 1.2.4, this groupoid is in fact equivalent to the fiber $\mathfrak{Y}_X$.

Definition 1.2.7. Let $f\colon \mathfrak{X} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of categories fibered in groupoids over $\mathfrak{S}$. The *fiber product* of $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{Z}$ is the following category fibered in groupoids. Its objects over S are triples (x, y, α) with x an object of $\mathfrak{X}_S$, y an object of $\mathfrak{Y}_S$ and $\alpha\colon f(x) \to g(y)$ an isomorphism in the fiber $\mathfrak{Z}_S$. Given a morphism $f\colon S' \to S$, a morphism from (x', y', α') to (x, y, α) over f is given by a pair of morphisms $(\beta\colon x' \to x, \gamma\colon y' \to y)$ over f such that $g(\gamma) \circ \alpha' = \alpha \circ f(\beta)$. The induced diagram

$$\begin{array}{ccc} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} & \xrightarrow{\pi_{\mathfrak{Y}}} & \mathfrak{Y} \\ {\scriptstyle \pi_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle g} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Z} \end{array}$$

with $\pi_{\mathfrak{X}}$ and $\pi_{\mathfrak{Y}}$ the projections, need not strictly commute, but it *2-commutes.* That is, the two composites $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \to \mathfrak{Z}$ are related by a natural 2-morphism. Observe the similarity with Definition 1.1.5, the fiber product for groupoids.

1.3 Descent data and stacks

Informally speaking, a stack is a category fibered in groupoids where objects can be glued uniquely from local data. Let $\{S_i \to S\}$ be a covering of an object S of $\mathfrak{S}$, and let x be an object over S. Denote by x_i the pullback of x to S_i, and by S_{ij} the intersection $S_i \times_S S_j$, and similarly for S_{ijk}. The object x cannot be reconstructed solely from the x_i, also the induced isomorphisms $\alpha_{ij}\colon x_i|_{S_{ij}} \to x_j|_{S_{ij}}$, which satisfy the cocycle condition on S_{ijk}, are needed. In a stack, we want to be able to glue the x_i on the intersections via the α_{ij}. This motivates the following definition.

Definition 1.3.1. Let $\mathfrak{X}$ be a category fibered in groupoids over $\mathfrak{S}$. A *descent datum* for $\mathfrak{X}$ over an object S of $\mathfrak{S}$ is given by

(i) a covering $\{S_i \to S\}$,

(ii) for every i a lift x_i of S_i in $\mathfrak{X}$,

(iii) for every i and j an isomorphism $\alpha_{ij}\colon x_i|_{S_{ij}} \to x_j|_{S_{ij}}$ in $\mathfrak{X}_{S_{ij}}$, satisfying the cocycle condition $\alpha_{ik}|_{S_{ijk}} = \alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}}$ in $\mathfrak{X}_{S_{ijk}}$.

Such a descent datum is called *effective* if there exists a lift x of S in $\mathfrak{X}$ together with isomorphisms $\alpha_i\colon x|_{S_i} \to x_i$ in $\mathfrak{X}_{S_i}$ such that $\alpha_{ij} = \alpha_j|_{S_{ij}} \circ \alpha_i|_{S_{ij}}^{-1}$ in $\mathfrak{X}_{S_{ij}}$. In this case, one says that the x_i over S_i *descend* to x over S.

Furthermore, in a stack, we want such a gluing to be unique (up to unique isomorphism). That is, for any other gluing (x', α_i') there should be a unique isomorphism $\beta\colon x' \to x$ such that $\alpha_i' = \alpha_i \circ \beta|_{S_i}$ over S_i. To have this property, we will require that isomorphisms in fibers can be reconstructed uniquely from local data. This idea is expressed in the following definition.

Definition 1.3.2. Let $\mathfrak{X}$ be a category fibered in groupoids over $\mathfrak{S}$. We say that *isomorphisms are a sheaf* for $\mathfrak{X}$ if, for any object S of $\mathfrak{S}$, any objects x and y in $\mathfrak{X}_S$, every covering $\{S_i \to S\}$ of S, and every collection of isomorphisms $\alpha_i\colon x|_{S_i} \to y|_{S_i}$ in $\mathfrak{X}_{S_i}$ such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$, there exists a unique isomorphism $\alpha\colon x \to y$ such that $\alpha_i = \alpha|_{S_i}$.

Remark 1.3.3. Alternatively, the above definition can be expressed as follows. For any two objects x and y in $\mathfrak{X}$ lying over an object S in $\mathfrak{S}$, one can define a presheaf

$$\operatorname{Isom}(x, y)\colon (\mathfrak{S}/S)^{\mathrm{op}} \to \mathbf{Set}$$

on the slice category $\mathfrak{S}/S$, by assigning to $f\colon T \to S$ the set $\operatorname{Hom}_{\mathfrak{X}_T}(f^*x, f^*y)$ of isomorphisms from f^*x to f^*y in $\mathfrak{X}_T$, and to a morphism $g\colon T' \to T$ from $f'\colon T' \to S$ to $f\colon T \to S$ the map that is given by pullback along g, that is,

$$\operatorname{Hom}_{\mathfrak{X}_T}(f^*x, f^*y) \to \operatorname{Hom}_{\mathfrak{X}_{T'}}(g^*f^*x, g^*f^*y) \cong \operatorname{Hom}_{\mathfrak{X}_{T'}}((f')^*x, (f')^*y),$$

where the latter isomorphism is induced by the natural isomorphisms $g^*f^*x \cong (f')^*x$ and $g^*f^*y \cong (f')^*y$. Now, saying that isomorphisms are a sheaf for $\mathfrak{X}$ is equivalent to saying that $\operatorname{Isom}(x, y)$ is a sheaf for all x, y and S. Note that, while it looks as if $\operatorname{Isom}(x, y)$ depends on the choice of f^*x and f^*y, any other choice would yield a presheaf that is naturally isomorphic.

Definition 1.3.4. A *stack* over $\mathfrak{S}$ is a category fibered in groupoids $\mathfrak{X}$ over $\mathfrak{S}$ such that every descent datum for $\mathfrak{X}$ is effective and isomorphisms are a sheaf for $\mathfrak{X}$. A morphism of stacks over $\mathfrak{S}$ is simply a morphism of categories over $\mathfrak{S}$, and similarly for 2-morphisms and isomorphisms. Fiber products of stacks can be computed as fiber products of categories over groupoids.

Remark 1.3.5. As in Example 1.2.4, any object X of $\mathfrak{S}$ can be considered a category fibered in groupoids over $\mathfrak{S}$ as the slice category $\mathfrak{S}/X$, where $\mathfrak{S}/X \to \mathfrak{S}$ forgets the morphism to X. Unfortunately, this does not always give a stack, it depends on the topology on $\mathfrak{S}$. However, for most of the examples of interest it will give a stack and will be easy to prove, e.g. for schemes, manifolds, analytic spaces, topological spaces, etc. with the usual topologies [Fan01].

Definition 1.3.6. A stack $\mathfrak{X}$ over $\mathfrak{S}$ is *representable* if it is isomorphic to the stack $\mathfrak{S}/X$ for some object X of $\mathfrak{S}$. A morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$ is *representable* if, for every morphism $S \to \mathfrak{X}$ with S in $\mathfrak{S}$, the fiber product $S \times_{\mathfrak{Y}} \mathfrak{X}$ is representable.

Intuitively, this says that a morphism of stacks is representable if all of its fibers are representable.

From now on, we will simply write X for the category $\mathfrak{S}/X$ as well.

1.4 Algebraic stacks

An algebraic stack, over a fixed base scheme S, is a special type of stack over the site $\mathfrak{S} = \mathbf{Sch}_S$, where $\mathfrak{S}$ is usually equipped with the étale or fppf topology. To give a precise definition, one needs the notion of an *algebraic space*. Informally speaking, whereas a scheme is locally an affine scheme in the Zariski topology, an algebraic space is locally an affine scheme in the étale topology. For an in-depth treatment on algebraic spaces, see [LM00, Ols16, Stacks]. For our purposes, it suffices to think of an algebraic space as a geometric object slightly more general than a scheme, and to know that, just as for schemes, any algebraic space over S can naturally be considered as a category fibered in groupoids over $\mathfrak{S}$ (recall from Remark 1.3.5 that any scheme X over S can be identified with the slice category $\mathfrak{S}/X$). A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ is said to be *representable by algebraic spaces* if for every scheme T and morphism $T \to \mathfrak{Y}$, the fiber product $T \times_{\mathfrak{Y}} \mathfrak{X}$ is representable by an algebraic space.

Before giving the definition of an algebraic stack, we first need to introduce some properties of representable morphisms.

Definition 1.4.1. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of categories fibered in groupoids over $\mathfrak{S}$ which is representable by algebraic spaces. Let P be a property of morphisms of algebraic spaces which is stable under base change and fppf-local on the base, such as being *smooth*, *étale*, *unramified*, *flat*, *surjective*, *(quasi-)separated*, *affine*, *proper*, *(locally) of finite type*, *(locally) of finite presentation*, or *an (open or closed) immersion*. Then f is said to have the property P if for every scheme T over S and $T \to \mathfrak{Y}$ the base change $T \times_{\mathfrak{Y}} \mathfrak{X} \to T$ has the property P.

Definition 1.4.2. A stack $\mathfrak{X}$ over $\mathfrak{S}$ is an *Artin stack* (resp. *Deligne–Mumford stack*) if the diagonal $\Delta_{\mathfrak{X}/S}\colon \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable by algebraic spaces and there exists a smooth (resp. étale) and surjective morphism $X \to \mathfrak{X}$ for some scheme X. Such a morphism $X \to \mathfrak{X}$ is called a *presentation* of $\mathfrak{X}$.

An *algebraic stack* over $\mathfrak{S}$ will simply be an Artin stack over $\mathfrak{S}$.

Remark 1.4.3. Note that, in Definition 1.4.2, $\Delta_{\mathfrak{X}/S}$ being representable automatically implies the morphism $X \to \mathfrak{X}$ is representable, so that it makes sense to talk about this morphism being surjective, smooth or étale. Indeed, for every scheme T and morphism $T \to \mathfrak{X}$, we have that $T \times_{\mathfrak{X}} X \cong \mathfrak{X} \times_{\mathfrak{X} \times_S \mathfrak{X}} (X \times_S T)$ is representable by an algebraic space.

What follows now is a list of definitions of properties for algebraic stacks and for morphisms thereof. The general philosophy is that the common properties for schemes or algebraic spaces (and morphisms thereof) translate directly to the setting of algebraic stacks by making use of some kind of representability and the way these properties behave (e.g. often they are local on the source or target in some topology). We adopt the definitions as used by [Stacks], as indicated in the definitions. This list is by far not complete, but should cover all properties that are needed in the later chapters. For a more elaborate discussion on these properties, we refer to [Beh14, LM00, Ols16, Stacks].

Definition 1.4.4 ([Stacks, Tag 04YF])**.** Let P be a property of schemes which is local in the smooth topology, such as being *reduced*, *locally noetherian*, *normal* or *regular*. An algebraic stack $\mathfrak{X}$ is said to have P if there exists a smooth surjective morphism $X \to \mathfrak{X}$ with X a scheme having property P.

Definition 1.4.5 ([Stacks, Tag 04YC, Tag 050U])**.** An algebraic stack $\mathfrak{X}$ is *quasi-compact* if there exists a smooth and surjective morphism $X \to \mathfrak{X}$ with X a quasi-compact scheme.

A morphism of algebraic stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ is *quasi-compact* if for every quasi-compact algebraic stack $\mathfrak{Z}$ and morphism $\mathfrak{Z} \to \mathfrak{Y}$, the fiber product $\mathfrak{Z} \times_{\mathfrak{Y}} \mathfrak{X}$ is quasi-compact.

Definition 1.4.6 ([Stacks, Tag 0CHQ, Tag 0CHU, Tag 04YL])**.** Let P be any of the properties of being *affine*, *finite*, or *an (open or closed) immersion*. Then a morphism of algebraic stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ is said to have the property P if it is representable and has property P in the sense of Definition 1.4.1.

Definition 1.4.7 ([Stacks, Tag 06FM, Tag 0CIF])**.** Let P be a property of morphisms of algebraic spaces which is local on the source and target in the smooth (resp. étale) topology, such as being *locally of finite type*, *locally of finite presentation*, *flat*, or *smooth* (resp. or *unramified* or *étale*). Then a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of algebraic stacks is said to have the property P if there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f'} & V \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

with U and V algebraic spaces such that the vertical morphisms are smooth, $U \to \mathfrak{X} \times_{\mathfrak{Y}} V$ is smooth (resp. étale) and f' has the property P.

Definition 1.4.8 ([Stacks, Tag 04YW, Tag 06FS, Tag 06Q2])**.** A morphism of algebraic stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ is

- *separated* if the diagonal is proper in the sense of Definition 1.4.1,
- *quasi-separated* if the diagonal is quasi-compact and quasi-separated in the sense of Definition 1.4.1,
- of *finite type* if it is locally of finite type and quasi-compact,
- of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

1.5 Quotient stacks

A rich source of examples of algebraic stacks is given by quotients of schemes by group actions. For example, many moduli spaces are constructed in this way: one first describes a scheme X overparametrizing the objects of interest, and then describes an equivalence relation on the objects via the action of a group G on X. The moduli space should then be the quotient of X by G.

To get an intuition for what this quotient should look like, imagine a group G acting on some kind of geometric object X (e.g. a manifold or topological space). If the group action is sufficiently nice (i.e., free), the quotient $X \to X/G$ is expected to be a G-torsor, also known as a principal G-bundle. In particular, the pullback of X along any map $T \to X$ will be a G-torsor over T, and the projection $X \times_{X/G} T \to X$ will be G-equivariant. Moreover, any G-torsor $P \to T$ with an equivariant map to X conversely induces a map from T to the quotient X/G. This motivates the following definition.

Definition 1.5.1. Let G be a smooth group scheme acting on a scheme X over S. The *quotient stack* of X by G, denoted $[X/G]$, is the category over $\mathfrak{S}$ whose objects over T are diagrams

$$\begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \big\downarrow{\scriptstyle p} & & \\ T & & \end{array}$$

where $P \xrightarrow{p} T$ is a G-torsor and $P \xrightarrow{\phi} X$ is a G-equivariant morphism. The morphisms from $T' \xleftarrow{p'} P' \xrightarrow{\phi'} X$ to $T \xleftarrow{p} P \xrightarrow{\phi} X$ over $f\colon T' \to T$ are G-equivariant morphisms $\alpha\colon P' \to P$ such that $p \circ \alpha = f \circ p'$ and $\phi \circ \alpha = \phi'$. Since G-torsors can be glued from local data, it is easy to verify that $[X/G]$ is indeed a stack over $\mathfrak{S}$. There is a natural quotient map $\pi\colon X \to [X/G]$ corresponding to the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle \pi_X}\big\downarrow & & \\ X & & \end{array}$$

More generally, one can replace the scheme X by an algebraic stack $\mathfrak{X}$ to define the quotient stack $[\mathfrak{X}/G]$.

Example 1.5.2. The quotient stack $[S/G]$ corresponding to the trivial action of G on S is also known as the *classifying stack* of G and is denoted $\mathrm{B}G$.

Remark 1.5.3. For any morphism $T \xrightarrow{f} [X/G]$, the corresponding G-torsor over T with G-equivariant map to X can be recovered via pullback along π, as depicted in the following diagram.

$$\begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \pi} \\ T & \xrightarrow{f} & [X/G] \end{array}$$

Indeed, by definition of the fiber product, the objects of $T \times_{[X/G]} X$ over T' are triples of morphisms $(f\colon T' \to T, h\colon T' \to X, \alpha\colon G \times T' \to P \times_T T')$ with α being G-equivariant such that

$$\begin{array}{ccccc} & & G \times T' & & \\ & \swarrow & & \searrow^{\sigma \circ (\mathrm{id}_G \times h)} & \\ T' & & \big\downarrow{\scriptstyle \alpha} & & X \\ & \nwarrow & & \nearrow_{\phi \circ \pi_P} & \\ & & P \times_T T' & & \end{array}$$

commutes. Since α is G-equivariant, it must be of the form $\alpha(g, t') = (g \cdot \beta(t'), t')$ for $\beta\colon T' \to P$ given by $\beta(t') = \pi_P(\alpha(1, t'))$. But then $f = p \circ \beta$, $h = \phi \circ \beta$ and α can all be expressed in terms of β. Hence, $(T \times_{[X/G]} X)(T') = P(T')$ and this provides a canonical isomorphism $T \times_{[X/G]} X \cong P$.

Remark 1.5.4. The above remark shows that the quotient stack $[X/G]$ is an Artin stack with presentation $\pi\colon X \to [X/G]$. Indeed, the morphism π is smooth and surjective since $P \xrightarrow{p} T$ is smooth and surjective, as G was assumed to be smooth. Similarly, if G is a finite group, one shows that the quotient stack $[X/G]$

is a Deligne–Mumford stack. For representability of the diagonal, see [Ols16, Example 8.1.12].

Remark 1.5.5. The quotient stack $[X/G]$ indeed satisfies the quotient property, that is, for any G-invariant morphism $f\colon X \to Y$ there is an induced morphism $\overline{f}\colon [X/G] \to Y$. Indeed, for any diagram $T \leftarrow P \xrightarrow{\phi} X$, the composite $f \circ \phi$ is G-invariant, so there is an induced morphism $T \to Y$, which defines a T-point of Y.

Remark 1.5.6. The quotient stack construction is functorial in the following sense. Let G and H be smooth group schemes acting on schemes X and Y, respectively, over S. Suppose $f\colon X \to Y$ is a morphism of schemes over S, and $\varphi\colon G \to H$ a morphism of group schemes over S, such that $f(g \cdot x) = \varphi(g) \cdot f(x)$. Then there is an induced morphism of quotient stacks $\overline{f}\colon [X/G] \to [Y/H]$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ [X/G] & \xrightarrow{\overline{f}} & [Y/H] \end{array}$$

2-commutes. The morphism $\overline{f}$ is given by sending a diagram $T \xleftarrow{p} P \xrightarrow{\phi} X$ to the diagram $T \xleftarrow{p \circ \pi_P} H \times_G P \xrightarrow{\psi} Y$ where $\psi(h, p) = h \cdot f(\phi(p))$. The commutativity of the diagram follows from the natural isomorphism $H \times_G (G \times T) \cong H \times T$ of H-torsors over T.

Lemma 1.5.7. *Let G and H be smooth group schemes acting on schemes X and Y, respectively, over S. Then there is a natural isomorphism $[X/G] \times [Y/H] \cong [X \times Y/G \times H]$ given by sending a pair of diagrams $T \leftarrow P \xrightarrow{\phi} X$ and $T \leftarrow Q \xrightarrow{\psi} Y$ to the diagram $T \leftarrow P \times Q \xrightarrow{\phi \times \psi} X \times Y$.*

Proof. The described map is clearly functorial. Conversely, for any $(G \times H)$-torsor R over T, the natural isomorphism $R \cong R/H \times_T R/G$ of $(G \times H)$-torsors over T yields an inverse. □

Lemma 1.5.8. *If G acts freely on X, then $[X/G]$ is representable by an algebraic space.*

Proof. To prove this, we will use the characterization of an algebraic space as an algebraic stack whose objects all have trivial automorphism groups [Stacks, Tag 03YR]. Any point of $[X/G]$ over any T corresponds to a G-torsor over T with an equivariant morphism $\phi\colon P \to X$. An automorphism of this point is an automorphism $\alpha\colon P \to P$ over T such that $\phi \circ \alpha = \phi$. Étale-locally, $P \cong G \times T$, and ϕ is determined by its restriction $\phi' \colon S \times T \to X$ along the unit $e : S \to G$. Furthermore, $\alpha \colon G \times T \to G \times T$ is given by multiplication by some element $g \in G$. Now $g \cdot \phi'(t) = \phi'(t)$ for all $t \in T$, and since G acts freely on X we have $g = 1$, that is, $\alpha = \mathrm{id}_P$. Since this holds étale-locally, we also have $\alpha = \mathrm{id}_P$ globally, and thus this automorphism is trivial. □

Remark 1.5.9. As is reflected in the notation, the quotient stack can be thought of as a geometric analogue of the action groupoid. However, in general we have

$$[X/G](T) \not\simeq [X(T)/G(T)].$$

For example, the classifying stack BG of $G = \mathbb{Z}/2\mathbb{Z}$ has up to isomorphism precisely two $\mathbb{F}_q$-points: the trivial G-torsor $\mathbb{F}_q \to (\mathbb{F}_q)^2$ and the non-trivial G-torsor $\mathbb{F}_q \to \mathbb{F}_{q^2}$ whose G-action is given by the Frobenius automorphism, both having an automorphism group of $\mathbb{Z}/2\mathbb{Z}$. On the other side, the action groupoid has only one object with automorphism group $\mathbb{Z}/2\mathbb{Z}$.

In special cases, this discrepancy can be resolved.

Proposition 1.5.10. *Let G be an algebraic group acting on a scheme X over a field k. If (i) k is separably closed, or (ii) k is finite and G is connected, then there is an equivalence of groupoids*

$$[X/G](k) \simeq [X(k)/G(k)].$$

Proof. In both cases, any G-torsor over $\operatorname{Spec} k$ is trivial. For *(i)* because $\operatorname{Spec} k$ does not have a non-trivial étale cover, and for *(ii)* by Lang's theorem [Lan56]. Hence, the objects of the groupoid $[X/G](k)$ are G-equivariant morphisms $G \xrightarrow{\phi} X$, which are completely determined by the value $\phi(1) \in X(k)$, and morphisms $\phi \to \phi'$ are given by an element $g \in G(k)$ such that $\phi(1) = \phi'(g) = g \cdot \phi'(1)$. But this is precisely (equivalent to) $[X(k)/G(k)]$. □

1.6 Stabilizers

Definition 1.6.1. Let $\mathfrak{X}$ be an algebraic stack over $\mathfrak{S}$, and $x \colon \operatorname{Spec} K \to \mathfrak{X}$ a K-point of $\mathfrak{X}$ for some field K. The *stabilizer* of x is the fiber product

$$\mathrm{Stab}_{\mathfrak{X}}(x) = \operatorname{Spec} K \times_{\mathfrak{X}} \operatorname{Spec} K$$

as a group scheme (or more precisely, group algebraic space) over K. Indeed, for any $T \to \operatorname{Spec} K$, the T-points of $\mathrm{Stab}_{\mathfrak{X}}(x)$ can be identified with the automorphism group in $\mathfrak{X}$ of the T-point $T \to \operatorname{Spec} K \xrightarrow{x} \mathfrak{X}$. We say $\mathfrak{X}$ *has affine stabilizers* if $\mathrm{Stab}_{\mathfrak{X}}(x)$ is an affine group scheme for every x. We say $\mathfrak{X}$ *has finite stabilizers* if $\mathrm{Stab}_{\mathfrak{X}}(x)$ is a finite group scheme for every x.

Lemma 1.6.2. *Let G be a smooth group scheme acting on a scheme X. The quotient stack $\mathfrak{X} = [X/G]$ has affine stabilizers (resp. finite stabilizers) if G is affine (resp. finite).*

Proof. A point $x\colon \operatorname{Spec} K \to \mathfrak{X}$ corresponds to a G-torsor $P \xrightarrow{p} \operatorname{Spec} K$ with a G-equivariant map $P \xrightarrow{\phi} X$. As P is étale-locally trivial, we have $P \times_{\operatorname{Spec} K} \operatorname{Spec} L \cong G \times \operatorname{Spec} L$ for some finite separable field extension L/K. The G-equivariant morphism $G \times \operatorname{Spec} L \xrightarrow{\psi} X$ induced by ϕ corresponds to a point $x' = \psi(1) \in X(L)$. Now consider the base change $\operatorname{Stab}_{\mathfrak{X}}(x) \times_{\operatorname{Spec} K} \operatorname{Spec} L$. Its T-points are given by G-equivariant isomorphisms $\alpha\colon G \times T \to G \times T$ over T such that $\psi \circ \alpha = \psi$. Hence, we obtain a fiber product:

$$\begin{array}{ccc} \operatorname{Stab}_{\mathfrak{X}}(x) \times_{\operatorname{Spec} K} \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} L \\ \downarrow & & \downarrow{\scriptstyle x'} \\ G \times \operatorname{Spec} L & \xrightarrow{\quad\psi\quad} & X \end{array}$$

This shows that $\operatorname{Stab}_{\mathfrak{X}}(x) \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is a subgroup of $G \times \operatorname{Spec} L$, which is, just like $G \times \operatorname{Spec} L$, affine (resp. finite). Since being affine (resp. finite) is local in the étale topology [Stacks, Tag 02L5, Tag 02LA], it follows that $\operatorname{Stab}_{\mathfrak{X}}(x)$ is also affine (resp. finite). □

Lemma 1.6.3. *Let $f\colon \mathfrak{X} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be morphisms between algebraic stacks with affine (resp. finite) stabilizers. Then the fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ also has affine (resp. finite) stabilizers.*

Proof. Pick any point $(x, y, \alpha) \in (\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y})(K)$. An automorphism of (x, y, α) consists of morphisms $\beta\colon x \to x$ and $\gamma\colon y \to y$ such that $\alpha \circ f(\beta) = g(\gamma) \circ \alpha$. That is, the automorphism group of (x, y, α) is precisely the stabilizer of α for the action of $\operatorname{Aut}_{\mathfrak{X}}(x) \times \operatorname{Aut}_{\mathfrak{Y}}(y)$ on $\operatorname{Hom}_{\mathfrak{Z}}(f(x), g(y))$, given by $(\beta, \gamma) \cdot \alpha = g(\gamma) \circ \alpha \circ f(\beta)$. Also note that $\operatorname{Hom}_{\mathfrak{Z}}(f(x), g(y)) \cong \operatorname{Aut}_{\mathfrak{Z}}(z)$ for any object z of $\mathfrak{Z}$ isomorphic to $f(x) \cong g(y)$. This reasoning shows that the stabilizer of (x, y, α) can be identified as the fiber product in the following cartesian square

$$\begin{array}{ccc} \operatorname{Stab}_{\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}}(x, y, \alpha) & \longrightarrow & \operatorname{Spec} K \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ \operatorname{Stab}_{\mathfrak{X}}(x) \times \operatorname{Stab}_{\mathfrak{Y}}(y) & \longrightarrow & \operatorname{Stab}_{\mathfrak{Z}}(z) \end{array}$$

where z is again any object of $\mathfrak{Z}$ isomorphic to $f(x) \cong g(y)$. By assumption, all of $\operatorname{Stab}_{\mathfrak{X}}(x)$, $\operatorname{Stab}_{\mathfrak{Y}}(y)$ and $\operatorname{Stab}_{\mathfrak{Z}}(z)$ are affine (resp. finite), and therefore, the fiber product $\operatorname{Stab}_{\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}}(x, y, \alpha)$ is also affine (resp. finite). □

Definition 1.6.4. Let $\mathfrak{S}$ be an algebraic stack with affine stabilizers. Let $\mathbf{Stck}_{\mathfrak{S}}$ be the full subcategory of algebraic stacks of finite type over $\mathfrak{S}$ with affine stabilizers. By Lemma 1.6.3, this category is closed under pullbacks.

The algebraic stacks that appear in this thesis all have affine stabilizers. The following proposition shows that we can think of such algebraic stacks, at least locally, as quotient stacks of quasi-projective schemes by linear groups.

Proposition 1.6.5 ([Kre99, Proposition 3.5.9])**.** *Let $\mathfrak{X}$ be a reduced Artin stack of finite type over a field with affine stabilizers. Then $\mathfrak{X}$ admits a stratification by quotient stacks $[X_i/\mathrm{GL}_{n_i}]$ where X_i is a quasi-projective scheme.*