# Motivic invariants of character stacks

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# Introduction

The theory of representations of groups is a rich and fascinating subject in mathematics. For certain classes of groups, the representation theory is fairly well understood. For example, for finite groups, the representation theory is largely described by their character table, and for connected compact Lie groups, the representation theory is given by the theorem of the highest weight. However, the representation theory of finitely generated groups, lying somewhere in between, is not so easily described. For a finitely generated group $\Gamma$, the set of $n$-dimensional representations $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$, denoted

$$\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C})),$$

defines a complex variety, called the *representation variety* of $\Gamma$. Recall that two representations $\rho, \rho' \colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ are isomorphic if $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ for some $g \in \mathrm{GL}_n(\mathbb{C})$ and all $\gamma \in \Gamma$. In other words, the group $\mathrm{GL}_n(\mathbb{C})$ acts by conjugation on the representation variety $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))$, and the quotient $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$, known as the *character variety* of $\Gamma$, can be thought of as a geometric analogue of the character table. A subtle point is that it is not completely clear how to take this quotient. Using geometric invariant theory [Mum65], one arrives at the classical definition of the character variety. Another possibility is to enter the realm of algebraic stacks, to arrive at the quotient stack

$$[\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})],$$

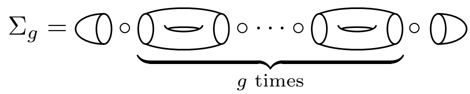
known as the *character stack* of $\Gamma$, for which the character variety is a coarse moduli space. More generally, one may replace $\mathbb{C}$ by any field $k$, and $\mathrm{GL}_n$ by any algebraic group $G$ over $k$. As an example, when $\Gamma = \mathbb{Z}$, a representation from $\Gamma$ into $G$ is simply the choice of an element of $G$, so the representation variety is isomorphic to $G$, and the character variety, or character stack, is the appropriate quotient of $G$ by the action of $G$ by conjugation on itself. In general, the geometry of these spaces can be quite complicated and is a wide field of study. The goal of this thesis is to provide a better understanding of the geometry of these spaces.

Many finitely generated groups arise as the fundamental group $\Gamma = \pi_1(M, *)$ of a connected compact manifold $M$ with a basepoint $*$. In this case, representations of $\Gamma$ into $G$ correspond to $G$-local systems on $M$, and isomorphic representations correspond to isomorphic local systems [Sza09, Corollary 2.6.2]. In this sense, the character variety (or stack) of $\Gamma$ can be seen as the moduli space of $G$-local systems on $M$, and is in the literature also known as the *Betti moduli space* of $M$. In the particular case that $M$ is the underlying manifold of a complex smooth projective curve $C$, this space appears in the geometric Langlands program [BD96, BN18] and plays a major role in non-abelian Hodge theory [Cor88, Don87, Sim91, Sim94], where it is strongly related to a moduli space of Higgs bundles on $C$ and a moduli space of flat connections on $C$. The study of these moduli spaces motivated the $P = W$ conjecture [CHM12], which was recently proved [MS22, Hau+22]. The main focus of this thesis will be the case where $M$ has dimension 2. Such manifolds $M$ are either orientable and classified by their genus, or non-orientable and classified by their demigenus.

The geometry of the representation variety (and of its quotients) can be studied in many ways, for instance by computing their invariants. When $k$ is a finite field, one could count the number of $k$-rational points, and when $k = \mathbb{C}$, one could compute the singular cohomology of the analytification. In this thesis, we focus on invariants $\chi$ that are *additive* and *multiplicative* in the sense that $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ and $\chi(X \times Y) = \chi(X)\chi(Y)$ whenever $X$ and $Y$ are varieties over $k$ and $Z \subseteq X$ a closed subvariety. We call these *motivic invariants*, and they include the point count when $k$ is finite, and the Euler characteristic of the analytification when $k = \mathbb{C}$. Another such invariant for $k = \mathbb{C}$, which is central in this thesis, is the *$E$-polynomial*, a refinement of the Euler characteristic. The $E$-polynomial of a complex variety is a polynomial in two variables whose coefficients reflect the mixed Hodge structure on its cohomology. In this thesis we discuss various such invariants, and develop tools for computing them. In particular, we focus on the universal such invariant, called the *virtual class*, which takes values in the *Grothendieck ring of varieties*.

The computation of motivic invariants for representation varieties of orientable surfaces started with Hausel and Rodriguez-Villegas [HR08], who studied the representation variety by counting the number of points over finite fields $\mathbb{F}_q$. They could express these counts in terms of the representation theory of the finite groups $G(\mathbb{F}_q)$, and moreover, infer from these counts the $E$-polynomial of the representation variety. This approach, which we will call the *arithmetic method*, has led many to study the $E$-polynomials of character varieties for various $\Gamma$ and $G$ [HLR11, Mer15, Let15, MR15, Cam17, BH17, LR22, BK22].

A few years later, Logares, Muñoz and Newstead [LMN13] initiated the *geometric* *method*: a geometric approach to compute the $E$-polynomial of the representation variety, making use of its additive and multiplicative property and clever stratifications. González-Prieto, Logares and Muñoz [GLM20] showed that the geometric method can be phrased in terms of a *Topological Quantum Field Theory (TQFT)*, a concept originating from physics. In particular, an orientable surface of genus $g$ can be considered as a composite of manifolds with boundaries, known as *bordisms*, as follows:




In short, a TQFT (over some commutative ring $R$) assigns to every boundary (possibly empty) an $R$-module, and to every bordism between boundaries a linear map between the corresponding modules, such that composition of bordisms corresponds to composition of the linear maps. In other words, a TQFT is a certain functor from the category of bordisms to the category of $R$-modules. Now, the idea of the geometric method is that the $E$-polynomial of the representation variety corresponds to the image of $\Sigma_g$ under the TQFT, and so the computation of this $E$-polynomial can be broken down into a simpler computation for each bordism. It was shown later [Gon20] that the same construction can be used to compute the virtual class of the representation variety in the Grothendieck ring of varieties.

Both the arithmetic and geometric method are discussed in detail in Chapter 4. One of the main results of this chapter, which is based on [GHV23], is that the two methods can be unified into a single framework. In particular, we show how the arithmetic method can be translated into the language of TQFTs, and moreover, we show that the TQFTs, for the geometric and arithmetic method, are related through natural transformations. As a consequence, we describe how parts of the character tables of the finite groups $G(\mathbb{F}_q)$, specifically the dimensions of the irreducible representations of $G(\mathbb{F}_q)$, are related to the eigenvalues of the image of the bordism  under the TQFT corresponding to the geometric method.

Another aim of this thesis, besides giving theoretical descriptions, is to apply the above methods to explicitly compute invariants of the representation varieties and character stacks of surfaces, for certain algebraic groups $G$. In Chapter 5 we focus on the group $G = \mathrm{SL}_2$, generalizing the results of [LMN13, MM16, LR22] where the $E$-polynomials of the representation varieties were computed. Lifting these computations from $E$-polynomials to the Grothendieck ring of varieties introduces many subtle problems that have to be dealt with. In Chapter 6, based on [HV22, Vog24], we concentrate on the groups of $n \times n$ upper triangular matrices and unipotent upper triangular matrices. By means of computer-assisted calculations, we compute the virtual classes of the character stacks of $\Sigma_g$ for $n \leq 5$ through the geometric method, and their $E$-polynomials for $n \leq 10$ through the arithmetic method.

Finally, in Chapter 7, we turn our attention to the representation varieties and character stacks of the free groups $F_n$ and free abelian groups $\mathbb{Z}^n$. These spaces, parametrizing tuples (resp. commuting tuples) of elements of $G$, have also been widely studied [Bai07, AC07, FL11, PS13, FL14, RS19, FS21]. When considering the homology of these spaces, an interesting phenomenon emerges: as shown in [RS21], the homology groups of these spaces (and many variations thereof) stabilize as $n$ tends to infinity, in a well-defined sense due to [CF13] known as *representation stability*. In Chapter 7, we will combine the notion of representation stability with that of *motivic stability* [VW15] to define an analogous notion of *motivic representation stability* for stability in the Grothendieck ring of varieties. As an application, we will show that the character stacks of $F_n$ and $\mathbb{Z}^n$ stabilize in this sense for the linear groups $G = \mathrm{GL}_r$.

These explicit applications and computations have led to a number of new computational techniques. For instance, the study of equivariant motivic invariants, in Section 3.6, describes how motivic invariants, in particular the virtual class, behave with respect to finite group actions. The results in this section are crucial to the computations for the $\mathrm{SL}_2$-character stacks, and to the definition of motivic representation stability. Other new computational techniques include the introduction of *algebraic representatives*, in Section 6.1, and the development of an algorithm for computing virtual classes, in Section 3.4. Without these techniques, the computations for the character stacks for upper triangular matrices of high rank would not have been possible.