Counting curves and their rational points

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Stellingen

behorende bij het proefschrift
"COUNTING CURVES AND THEIR RATIONAL POINTS"

(i) There is a log stack $\mathbb{M}_{g,n}$ of *log pointed curves* with underlying stack $\overline{\mathcal{M}}_{g,n}$ that admits logarithmic gluing maps

$$\mathbb{M}_{g_1,n_1+1} \times \mathbb{M}_{g_2,n_2+1} \to \mathbb{M}_{g_1+g_2,n_1+n_2}$$
$$\mathbb{M}_{g-1,n+2} \to \mathbb{M}_{g,n}$$

(Theorem F)

(ii) The logarithmic double ramification cycles form a partial log cohomological field theory, that is, they satisfy a certain relation with respect to the logarithmic gluing maps. (Theorem G)

(iii) For X a nice log smooth scheme, we get a log stack

$$\mathbb{M}_{g,n}(X, \beta)$$

of log stable maps to X, roughly consisting of pairs $(C, f : C \to X)$ where C is a pointed curve, with β encoding the curve class and tangency conditions, that admits gluing maps. This can be used to calculate the log Gromov–Witten invariants of X. These spaces admit logarithmic gluing maps, and the log Gromov–Witten invariants satisfy recursive relations with respect to these gluing maps.

(iv) For $a \in \mathbb{Z}^n$ with $\sum_i a_i = 0$, let $\mathrm{DR}_g(a)$ denote the double ramification cycle in $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$, the virtual class of the locus of points $(C, p_1, \ldots, p_n)$ where $\mathcal{O}_C(\sum_i a_i p_i)$ is trivial. The function

$$\left\{ a \in \mathbb{Z}^n : \sum_i a_i = 0 \right\} \mapsto \mathrm{DR}_g(a_1, \ldots, a_n)$$

is polynomial. (Theorem A, alternative proof of [Pix23])

(v) Studying $\overline{\mathcal{M}}_{g,n}$ as a logarithmic stack instead of an algebraic stack greatly simplifies the theory, and many of the classical constructions are shadows of simpler, better behaved logarithmic constructions.

(vi) For X a log scheme there is a short exact sequence

$$\mathcal{O}_X^* \to \mathsf{M}_X^{\mathsf{gp}} \to \overline{\mathsf{M}}_X^{\mathsf{gp}},$$

which one can interpret as a short exact sequence

algebraic geometry → logarithmic geometry → tropical geometry.

This exact sequence is not split, which makes the theory more complicated, but also much richer. In this sense logarithmic geometry is more than the sum of its parts.

(vii) The stack $B(\mathbb{Z}/2\mathbb{Z})$ is a cone stack, but not an Artin fan. This is the reason why Artin fans as currently defined are not functorial.

(viii) For the problem of finding rational points on curves of genus at least 2, the geometric Chabauty method outperforms the cohomological Chabauty method, in the sense that the geometric Chabauty method finds less false positives (Theorem 5.5.1, Theorem H).

(ix) Consider S the set of finite multisets of integers that are at least 2. Consider the operations on S where for $s \in S$ we can either replace $a, b \in s$ by a^b, or we can replace an element $n = a^b \in S$ with a, b if $a, b \in \mathbb{Z}_{\geq 2}$. We say $m \in \mathbb{Z}_{\geq 2}$ is *creative* if for every integer $n \in \mathbb{Z}_{\geq 2}$ we can obtain a multiset containing n when starting with the multiset $\{m\}$. Creative numbers exist, and they are exactly the powers a^b with $a, b \in \mathbb{Z}_{\geq 2}$ not both prime and $a^b \neq 16$.

(x) Animals have certain inalienable rights, derived from their sentience, as in the case of humans. Symmetrically, we have a duty, to human and non-human animals, not to arbitrarily kill, torture, use or abuse.

(xi) Animal agriculture is not a food producer, but a food waster. It is the single leading cause of deforestation, while humanity is in strong need of reforestation. Switching to a plant-based diet is the most meaningful action the world can take.

(xii) Bouldering and board games are like mathematics, in the sense that they require deep understanding of a closed domain with a fixed set of rules, on both a detail-oriented and an intuitive level. For all three, one needs calculated moves, pattern recognition, and the ability to intuit potential outcomes.

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Leiden, 12 juni 2024