Counting curves and their rational points

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Chapter 2

Polynomiality of the double ramification cycle

This chapter has already appeared as a preprint [Spe24].

Abstract. Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a sequence with sum $k(2g-2+n)$. The double ramification cycle $\mathrm{DR}_g(A) \in \mathrm{CH}^g(\overline{\mathcal{M}}_{g,n})$ is the virtual class of the locus of curves $(C, p_1, \ldots, p_n)$ where the line bundle $(\omega_C^{\mathsf{log}})^{-k}\left(\sum a_i p_i\right)$ is trivial. Although there has long been a formula for $\mathrm{DR}_g(A)$ [JPPZ17], the exact dependence on A was unknown for a long time, though it was conjectured to be polynomial in A. A proof was announced in [JPPZ17], and Pixton gave a proof incorporating ideas of Zagier in [Pix23]. Here we present an alternative proof of the polynomiality of the double ramification cycle.

2.1 Introduction

Let $\mathcal{M}_{g,n}$ be the moduli space of smooth curves $(C/S, p_1, \ldots, p_n)$ of genus g with n distinct markings. Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a sequence with sum 0. Then the *double ramification locus* is

$$\mathrm{DRL}_g(A) = \left\{(C/S, p_1, \ldots, p_n) : \mathcal{O}_C\left(\sum a_i p_i\right) \text{ is fiberwise trivial}\right\} \subset \mathcal{M}_{g,n}.$$

Equivalently, this is the locus of curves C with a rational function $f : C \to \mathbb{P}^1$ with specified ramification profile above 0 and ∞.

In 2001 Eliashberg asked the question of how to compactify this substack, and how to compute the compactification. Compactifying the substack was first done in [GV03], using stable maps to $\mathbb{P}^1$ modulo the $\mathbb{G}_m$ action. They defined the *double ramification cycle*

$$\mathrm{DR}_g(A) \in \mathrm{CH}^g(\overline{\mathcal{M}}_{g,n})$$

as the virtual class of this compactification. There are other equivalent definitions, using birational geometry of $\overline{\mathcal{M}}_{g,n}$ [Hol21] or logarithmic geometry [MW20]. These latter definitions also generalise to the *twisted double ramification cycle*

$$\mathrm{DR}_g(A)$$

for a sequence of integers $A \in \mathbb{Z}^n$ summing to $k(2g-2+n)$ for some $k \in \mathbb{Z}$. This double ramification cycle is the virtual class of the compactification of the locus

$$\left\{(C/S, p_1, \ldots, p_n) : (\omega_C^{\mathsf{log}})^{-k}\left(\sum a_i p_i\right) \text{ is fiberwise trivial}\right\}.$$

The double ramification cycle has connections with ordinary Gromov–Witten invariants through the localisation formula [JPPZ17, RK23]. It also has ties with the world of PDE's [Bur15, BR21] and to gauge theory [FTT16]. It has also been used to find relations in the tautological ring [CJ18].

Pixton conjectured a formula for the double ramification cycle, and in 2014 this was proven in [JPPZ17] for the case $k = 0$, thereby answering the second part of Eliashberg's question. They define for every integer $r \in \mathbb{Z}_{\geq 1}$ an explicit class

$$P_g^r(A) \in \mathrm{CH}^g(\overline{\mathcal{M}}_{g,n}),$$

in terms of decorated strata multiplied by certain numbers associated to the combinatorics of the graph and the integers $a_1, \ldots, a_n$ and r.

They prove that $P_g^r(A)$ is polynomial in r for r large enough. Then they define $P_g^0(A)$ to be the constant term of this polynomial expression, and they prove the equality

$$\mathrm{DR}_g(A) = P_g^0(A).$$

In [BHP$^+$20] this result was extended to a formula for the twisted double ramification cycle.

These two results were major breakthroughs, but a very simple questions remained open: what is the behaviour of $\mathrm{DR}_g(A)$ in terms of A? In [JPPZ17] the double ramification cycle was conjectured to be polynomial in A, but this is a deceptively difficult question, as the formula itself has no obvious polynomial dependency on A. A proof was announced in [JPPZ17]. A proof incorporating ideas of Zagier was made public in 2023 [Pix23]. In this paper, we present a different proof of the same theorem.

Theorem A (Theorem 2.5.1)**.** *Fix g, n. The cycle $\mathrm{DR}_g(a_1, \ldots, a_n)$ is a polynomial in $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, where we require that $(2g-2+n) \mid \sum a_i$.*

We use different techniques to prove this statement. Our methods are explicit and can be recursively used to give a polynomial expression for the double ramification cycle.

In Section 2.2 we recall the combinatorics necessary to state Pixton's formula. In Section 2.3 we give a brief recap of Pixton's formula. In Section 2.4 we prove the main technical result, a certain polynomiality statement for sums of weightings on graphs. In Section 2.5 we finally give the proof of Theorem A.

2.2 Definitions

Definition 2.2.1. We denote by $G_{g,n}$ the set of graphs

$$\Gamma = (V, H, L = (\ell_i)_{i=1}^n, r : H \to V, i : H \to H, g : V \to \mathbb{N})$$

where

1. V is the set of vertices;
2. H is the set of half edges;
3. i is an involution on H.
4. r assigns to every half edge the vertex it is incident to.
5. $L = (\ell_i)_{i=1}^n \subset H$ is a list of the legs, the fixed points of the involution i;
6. (V, H) is a connected graph;
7. $g(v)$ is the genus of vertex v;
8. for every vertex v we have $2g(v) - 2 + n(v) > 0$ where $n(v)$ is the number of half edges incident to v;
9. the genus $\sum_{v \in V} g(v) + h^1(\Gamma)$ of the graph is g.

For such a graph, we denote by E the set of edges, i.e. the set of pairs $\{h, h'\}$ of half edges with $h = i(h')$ and $h \neq h'$.

Then $G_{g,n}$ parametrises the strata of $\overline{\mathcal{M}}_{g,n}$. Every graph $\Gamma \in G_{g,n}$ corresponds to a moduli space $\overline{\mathcal{M}}_\Gamma = \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}$ together with a glueing map $\zeta_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$.

Definition 2.2.2. We fix a vector $A \in \mathbb{Z}^n$ with sum $k(2g-2+n)$. A *weighting for* A on a graph $\Gamma \in G_{g,n}$ is a map $w : H(\Gamma) \to \mathbb{Z}$ satisfying the following three conditions.

1. For $i = 1, \ldots, n$ we have $w(\ell_i) = a_i$.
2. For $h \in H \setminus L$ we have $w(h) + w(i(h)) = 0$.
3. For every vertex v we have $\sum_{h:\, r(h)=v} w(h) = k(2g(v) - 2 + n(v))$.

For $r \in \mathbb{N}_{\geq 1}$, an *weighting for* A modulo r is a map $w : H(\Gamma) \to \{0, \ldots, r-1\} \subset \mathbb{Z}$ satisfying these three conditions modulo r. We denote $W^\Gamma_{A,r}$ for the finite set of weightings for A mod r.

We fix g, n, and we fix a graph $\Gamma \in G_{g,n}$. Let $Q \in \mathbb{Z}[x_h : h \in H \setminus L]$ be a polynomial.

Definition 2.2.3. For $A = (a_1, \ldots, a_n)$ with $\sum a_i = k(2g - 2 + n)$, consider the sum

$$S^\Gamma_{A,r}(Q) = r^{-h_1(\Gamma)} \sum_{w\in W^\Gamma_{A,r}} Q(w(h)). \tag{2.2.0.1}$$

In [JPPZ20, Appendix A] they prove the following property of these sums.

Proposition 2.2.4 ([JPPZ17, Proposition 3”])**.** *The sum* $S^\Gamma_{A,r}(Q)$ *are eventually polynomial in* r*.*

This allows them to define the following quantity.

Definition 2.2.5. We denote the constant term of the polynomial expression for $S^\Gamma_{A,r}(Q)$ for large r by

$$S^\Gamma_{A,0}(Q).$$

2.3 Pixton’s formula for the double ramification cycle

Fix $g, n \in \mathbb{N}, k \in \mathbb{Z}$ with $2g - 2 + n > 0$. Fix a sequence $A \in \mathbb{Z}^n$ with sum $k(2g - 2 + n)$. In this section we will present Pixton’s formula for the double ramification cycle $\mathrm{DR}_g(A) \in \mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$. The first version of this, valid for $k = 0$, was [JPPZ17, Theorem 1]. We will present the version from [BHP+20, Theorem 7].

For $r \in \mathbb{N}_{\geq 1}$, we define the term $P_r^\Gamma \in \mathrm{CH}(\overline{\mathcal{M}}_\Gamma)$ as follows.

$$P_r^\Gamma = r^{-h^1(\Gamma)} \sum_{w \in W_{A,r}^\Gamma} \prod_{e=\{h,h'\} \in E(\Gamma)} \frac{1 - \exp\left(\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}}.$$

We remark that the factor

$$\frac{1 - \exp\left(\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}}$$

is well-defined, as the denominator formally divides the numerator.

For every choice of monomial in the factors $(\psi_h + \psi_h')$ for edges $\{h, h'\}$, the corresponding coefficient of P_r^Γ is of the form $S_{A,r}^\Gamma(Q)$ for some polynomial Q where $S_{A,r}^\Gamma(Q)$ is as defined in Definition 2.2.3. Thus P_r^Γ is eventually polynomial per Proposition 2.2.4. We define

$$P_0^\Gamma \in \mathrm{CH}(\overline{\mathcal{M}}_\Gamma)$$

to be the element obtained by taking the constant term of the polynomial expression.

Then by [BHP+20, Theorem 7] we have the formula

$$\mathrm{DR}_g(A) = \left[\exp\left(-\frac{1}{2}(k^2\kappa_1 - \sum_i a_i^2\psi_i)\right) \sum_{\Gamma \in G_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\zeta_{\Gamma,*}P_0^\Gamma\right]^g, \tag{2.3.0.1}$$

where $[\cdot]^g$ means the codimension g term.

2.4 Sums over weightings

We fix g, n as in the introduction, and we fix a graph $\Gamma \in G_{g,n}$. Write $H \setminus L = \{h_1, \ldots, h_m\}$. Let $Q \in \mathbb{Z}[x_h : h \in H \setminus L]$ be a polynomial.

Fix $A \in \mathbb{Z}^n$ with $\sum a_i = 0$. We recall the sum

$$S_{A,r}^\Gamma(Q) = r^{-h_1(\Gamma)} \sum_{w \in W_{A,r}^\Gamma} Q(w(h))$$

from (2.2.0.1), and the constant term of the polynomial expression for the sum for large r

$$S_{A,0}^\Gamma(Q).$$

The main theorem in this section is the following polynomiality statement.

Theorem 2.4.1. *For every polynomial* $Q \in \mathbb{Z}[x_h : h \in H \setminus L]$ *the function*

$$\{A \in \mathbb{Z}^n : \sum a_i = 0\} \to \mathbb{Q}$$
$$A \mapsto S^{\Gamma}_{A,0}(Q)$$

is polynomial in A.

We will prove this by induction on the number of edges of Γ. In Section 2.4.1 we treat the case where Γ has a separating edge. In Section 2.4.2 we prove a preliminary result in the case where Γ has a non-separating edge. In Section 2.4.3 we put everything together and finish the proof of Theorem 2.4.1. We will then prove Corollary 2.4.7, generalising Theorem 2.4.1 to the domain $\{A \in \mathbb{Z}^n : (2g-2+n) \mid \sum a_i\}$.

We first make some general observations and definitions.

Remark 2.4.2. We will use the notation $v \bmod r$ to denote the unique element in $\{0, \dots, r-1\}$ that is modulo r equivalent to v. We use the notation $v \equiv w \pmod r$ to denote that n and m are equivalent modulo r.

Lemma 2.4.3. *Fix* A. *Then for* r *large enough and coprime to a fixed integer, we have*

$$S^{\Gamma}_{A,0}(Q) \equiv S^{\Gamma}_{A,r}(Q) \pmod r.$$

Proof. We know $S^{\Gamma}_{A,r}(Q)$ is for r large enough a polynomial in r, with rational coefficients. If the product of the denominators is coprime to r, this means that $S^{\Gamma}_{A,r}(Q) \equiv S^{\Gamma}_{A,0}(Q) \pmod r$. □

Definition 2.4.4. Let $e = \{h_1, h_2\}$ be an edge of Γ. We let Γ_e denote the graph Γ with the edge e removed, and with two legs ℓ_{n+1}, ℓ_{n+2} added at the roots of the half-edges $r(h_1), r(h_2)$.

2.4.1 Separating edge case

In this section we will prove Theorem 2.4.1 in the case that Γ has a separating edge.

Proposition 2.4.5. *Assume* Γ *has a separating edge. Then Theorem 2.4.1 holds, assuming that it holds for graphs with fewer edges than* Γ.

Proof. By linearity we can assume Q is a monomial, and we write

$$Q = \prod_{h \in H \setminus L} x_h^{c_h}.$$

Let $e = \{h_1, h_2\}$ be a separating edge of Γ.

Then Γ_e is a disjoint union of two graphs, denoted $\Gamma_1 \sqcup \Gamma_2$. Let A_1 denote the subsequence of A consisting of elements whose corresponding vertex lies in Γ_1, and similarly for A_2. We denote the sums $\sum_{a \in A_i} a$ by s_i for $i \in \{1, 2\}$. For $i \in \{1, 2\}$ let $Q_i \in \mathbb{Z}[x_h \colon h \in H_i \setminus L_i]$ denote the monomial Q with all half edges not in $H_i \setminus L_i$ set to 1, and let c_i denote c_{h_i}.

Note that then $W^{\Gamma}_{A,r}$ splits as $W^{\Gamma_1}_{(A_1,-s_1),r} \times W^{\Gamma_2}_{(A_2,-s_2),r}$. Denote these summands by W_1 and W_2 respectively. Then we find the formula

$$\begin{aligned}
S^{\Gamma}_{A,r}(Q) &= r^{-h_1(\Gamma)} \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} Q_1(w_1)(-s_1 \bmod r)^{c_1}(-s_2 \bmod r)^{c_2} Q_2(w_2) \\
&= \left(r^{-h_1(\Gamma_1)} \sum_{w_1 \in W_1} Q_1(w_1) \right) \cdot (-s_1 \bmod r)^{c_1} \cdot \\
&\qquad\qquad (-s_2 \bmod r)^{c_2} \cdot \left(r^{-h_1(\Gamma_2)} \sum_{w_2 \in W_2} Q_2(w_2) \right) \\
&= S^{\Gamma_1}_{(A_1,-s_1),r}(Q_1) \cdot (-s_1 \bmod r)^{c_1}(-s_2 \bmod r)^{c_2} \cdot S^{\Gamma_2}_{(A_2,-s_2),r}(Q_2)
\end{aligned}$$

By our assumption that Theorem 2.4.1 holds for graphs with fewer edges, we know the first and last factors are modulo r equal to $S^{\Gamma_1}_{(A_i,-s_i),0}(Q_i)$ with $i = 1$ and 2 respectively (for r large enough and coprime to a fixed integer). In total we see that for such r we have the following equalities

$$\begin{aligned}
S^{\Gamma}_{A,0}(Q) &\equiv S^{\Gamma}_{A,r}(Q) && \pmod r \\
&= S^{\Gamma_1}_{(A_1,-s_1),r}(Q_1) \cdot (-s_1 \bmod r)^{c_1}(-s_2 \bmod r)^{c_2} \cdot S^{\Gamma_2}_{(A_2,-s_2),r}(Q_2) \\
&\equiv S^{\Gamma_1}_{(A_1,-s_1),0}(Q_1) \cdot (-s_1)^{c_1}(-s_2)^{c_2} \cdot S^{\Gamma_2}_{(A_2,-s_2),0}(Q_2) && \pmod r.
\end{aligned}$$

By our assumption that Theorem 2.4.1 holds for graphs with fewer edges than Γ the term

$$S^{\Gamma_1}_{(A_1,-s_1),0}(Q_1) \cdot (-s_1)^{c_1}(-s_2)^{c_2} \cdot S^{\Gamma_2}_{(A_2,-s_2),0}(Q_2)$$

is polynomial in A. As it agrees with $S^\Gamma_{A,0}$ modulo an infinite number of integers, we have

$$S^\Gamma_{A,0}(Q) = S^{\Gamma_1}_{(A_1,-s_1),0}(Q_1) \cdot (s_1 s_2)^{c_1} \cdot S^{\Gamma_2}_{(A_2,-s_2),0}(Q_2) \tag{2.4.1.1}$$

and in particular, $S^\Gamma_{A,0}(Q)$ is polynomial in A. □

2.4.2 Non-separating case

In this section we will treat the case where Γ has a non-separating edge.

Proposition 2.4.6. *Let $e = \{h_1, h_2\}$ be a non-separating edge, and denote $v_i = r(h_i)$ the root of h_i for $i \in \{1,2\}$. Assume for $i \in \{1,2\}$ that there is a leg ℓ_i with insertions a_i and adjacent to v_i. Let $A \in \mathbb{Z}^n$ be a vector with total sum zero. For $a \in \mathbb{Z}$, let A_a be the vector $(a_1 - a, a_2 + a, a_3, a_4, \ldots, a_n)$. Assume that Theorem 2.4.1 holds for graphs with fewer edges than Γ.*

Then there exists a polynomial $\Psi_{\Gamma,e,Q}(x_1, \ldots, x_n, y, z_1, \ldots, z_m)$ and polynomials $R_1, \ldots, R_m \in \mathbb{Z}[x_h : h \in H \setminus L]$ such that

$$S^\Gamma_{A_a,0}(Q) = \Psi_{\Gamma,e,Q}(a_1, \ldots, a_n, a, S^\Gamma_{A,0}(R_1), \ldots, S^\Gamma_{A,0}(R_m)).$$

Proof. By linearity we can assume Q is a monomial, and we write

$$Q = \prod_{h \in H \setminus L} x_h^{c_h}.$$

For $i \in \{1,2\}$ we let c_i denote c_{h_i}.

We first prove the case where $a \geq 0$. We fix an r, and let W_a denote $W^\Gamma_{A_a,r}$. We will first rewrite $S^\Gamma_{A_a,r}(Q)$. Note there is a bijection

$$\begin{aligned} \varphi_a \colon W_0 &\to W_a \\ w &\mapsto \left(h_i \mapsto \begin{cases} (w(h_1) + a) \bmod r & \text{if } i = 1 \\ (w(h_2) - a) \bmod r & \text{if } i = 2 \\ w(h_i) & \text{if } i > 2 \end{cases} \right). \end{aligned}$$

Let $Q_0(w)$ denote the monomial Q with x_{h_1} and x_{h_2} substituted by 1. We then have the formula

$$\begin{aligned} S^\Gamma_{A_a,r}(Q) &= r^{-h_1(\Gamma)} \sum_{w \in W_0} Q(\varphi_a(w)) \\ &= r^{-h_1(\Gamma)} \sum_{w \in W_0} ((w(h_1) + a) \bmod r)^{c_1} ((w(h_2) - a) \bmod r)^{c_2} Q_0(w). \end{aligned}$$

Now we will rewrite the factor $((w(h_1)+a) \bmod r)^{c_1}((w(h_2)-a) \bmod r)^{c_2}$, using some facts about $m \bmod r$ for integers m.

For any $m \not\equiv 0 \pmod r$ we have $(r-m) \bmod r = r - (m \bmod r)$ and hence

$$(m \bmod r)^{c_1}((r-m) \bmod r)^{c_2} = (m \bmod r)^{c_1}(r-(m \bmod r))^{c_2}.$$

For $m \equiv 0 \pmod r$ we have

$$(m \bmod r)^{c_1}((r-m) \bmod r)^{c_2} = \\ (m \bmod r)^{c_1}(r-(m \bmod r))^{c_2} - (1-\delta_{c_2,0})0^{c_1}r^{c_2}$$

where δ is the Kronecker delta. (This term $(1-\delta_{c_2,0})0^{c_1}r^{c_2}$ is usually 0, but can be non-zero in the case $c_1 = 0 < c_2$). Also, $m \bmod r = m - r\lfloor\frac{m}{r}\rfloor$ is a polynomial in m and $r\lfloor\frac{m}{r}\rfloor$.

Now we will use this for $m = w(h_1)+a$. All in all, the factor $((w(h_1)+a) \bmod r)^{c_1}((w(h_2)-a) \bmod r)^{c_2}$ can be rewritten as

$$p\left(w(h_1), a, r, r\lfloor\frac{w(h_1)+a}{r}\rfloor\right) - \delta_{w(h_1)+a \bmod r, 0}\cdot(1-\delta_{c_2,0})\cdot 0^{c_1}r^{c_2}$$

for the polynomial

$$p(x_0,x_1,x_2,x_3) = (x_0+x_1-x_3)^{c_1}\cdot(x_2-x_0-x_1+x_3)^{c_2}.$$

Now we assume that $r > a$. Then we have $0 \le w(h_1)+a < 2r$ and so

$$\lfloor\frac{w(h_1)+a}{r}\rfloor = \begin{cases} 0 & \text{if } r-w(h_1) > a \\ 1 & \text{if } r-w(h_1) \le a \end{cases}. \tag{2.4.2.1}$$

Define $p_0(x_0,x_1,x_2) = p(x_0,x_1,x_2,0)$ and $p_1(x_0,x_1,x_2,x_3) = \frac{p-p_0}{x_3}$. Then we can write

$$p = p_0(x_0,x_1,x_2) + x_3p_1(x_0,x_1,x_2,x_3).$$

Write

$$S_1 := r^{-h_1(\Gamma)}\sum_{w\in W_0} p_0(w(h_1),a,r)Q_0(w)$$

and

$$S_2 := r^{-h_1(\Gamma_e)}\sum_{w\in W_0}\lfloor\frac{w(h_1)+a}{r}\rfloor p_1(w(h_1),a,r,r\lfloor\frac{w(h_1)+a}{r}\rfloor)Q_0(w)$$

and

$$S_3 := r^{-h_1(\Gamma)} \sum_{w \in W_0 \colon w(h_1) = -a \pmod r} (1 - \delta_{c_2,0}) \cdot 0^{c_1} r^{c_2} Q_0(w).$$

so that $S^{\Gamma}_{A_a,r}(Q) = S_1 + S_2 - S_3$. Here Γ_e is the graph defined in Definition 2.4.4 we have used that $r^{-h_1(\Gamma)} \cdot r = r^{-h_1(\Gamma_e)}$. We will show that each of S_1, S_2, S_3 is modulo r of the required form.

Write $p_0(x_0, x_1, x_2) = \sum_{i=0}^{d} p_{0,i}(x_0) x_1^i + x_2 p_2(x_0, x_1, x_2)$ for some polynomials $p_{0,i}, p_2$. Then we get $S_1 \equiv \sum_{i=0}^{d} a^i S^{\Gamma}_{A,r}(p_{0,i}(w(h_1)) \cdot Q_0) \pmod r$. Hence for r large enough and coprime to a finite set of integers S_1 is modulo r equivalent to a polynomial in a and terms $S^{\Gamma}_{A,0}(R)$ for some polynomials R, and in particular modulo r it is of the required form.

By (2.4.2.1) we can rewrite the sum S_2 as

$$\begin{aligned} S_2 &= r^{-h_1(\Gamma_e)} \sum_{w \in W_0 \colon 1 \le r - w(h_1) \le a} p_1(w(h_1), a, r, r) Q_0(w) \\ &= \sum_{j=1}^{a} \left(p_1(r-j, a, r, r) r^{-h_1(\Gamma_e)} \sum_{w \in W^{\Gamma_e}_{(A, r-j, j-r), r}} Q_0(w) \right) \\ &= \sum_{j=1}^{a} p_1(r-j, a, r, r) S^{\Gamma_e}_{(A,-j,j),r}(Q_0) \end{aligned}$$

Taking r large enough with respect to a and coprime to a finite set of integers, for every j the factor $S^{\Gamma_e}_{(A,-j,j),r}(Q_0)$ is modulo r equivalent to $S^{\Gamma_e}_{(A,-j,j),0}(Q_0)$. By our induction hypothesis applied to the graph Γ_e, the term $q(A,j) := S^{\Gamma_e}_{(A,-j,j),0}(Q_0)$ is a polynomial in A, j. Then $S_2 \equiv \sum_{j=1}^{a} p_1(-j, a, 0, 0) q(A, j) \pmod r$, and $\sum_{j=1}^{a} p_1(-j, a, 0, 0) q(A, j)$ is a polynomial in A and a for $a \ge 0$.

Finally, S_3 is 0 unless $c_1 = 0 < c_2$, so assume $c_1 = 0 < c_2$. Then

$$S_3 = r^{c_2 - 1} S^{\Gamma_e}_{(A,-a,a),r}(Q_0).$$

Modulo r this is either 0, or $S^{\Gamma_e}_{(A,-a,a),0}(Q_0)$ which by the induction hypothesis is polynomial in A, a.

In total $S^{\Gamma}_{A_a,r}(Q)$ is modulo r equivalent to a fixed polynomial $\Psi_{\Gamma,e,Q}$ in A, a and in terms $S^{\Gamma}_{A,0}(R_i)$ for polynomials $R_1, \ldots, R_m$, for every r large enough and coprime to a fixed integer. This means that for $a \geq 0$ we find

$$S^{\Gamma}_{A_a,0}(Q) = \Psi_{\Gamma,e,Q}(a_1, \ldots, a_n, a, S^{\Gamma}_{A,0}(R_1), \ldots, S^{\Gamma}_{A,0}(R_m)).$$

This finishes the proof for $a \geq 0$.

Now we apply this to the edge with the opposite orientation and to A_a with a positive. We find that for $a, b \geq 0$ we have that $S^{\Gamma}_{A_{a-b},0}(Q)$ is a polynomial in A, a, b and terms $S^{\Gamma}_{A_a,0}(R)$ for polynomials R. These latter are polynomials in $A, a, S^{\Gamma}_{A,0}(R')$ for polynomials R', so in total

$$S^{\Gamma}_{A_{a-b},0}(Q)$$

is for $a, b \geq 0$ a polynomial in A, a, b and terms $S^{\Gamma}_{A,0}(R)$. As it is also a function of $A, a-b$ and terms $S^{\Gamma}_{A,0}(R)$, it is a polynomial in $A, a-b$ and terms $S^{\Gamma}_{A,0}(R)$ and we find that

$$S^{\Gamma}_{A_a,0}(Q) = \Psi_{\Gamma,e,Q}(a_1, \ldots, a_n, a, S^{\Gamma}_{A,0}(R_1), \ldots, S^{\Gamma}_{A,0}(R_m))$$

holds for any $a \in \mathbb{Z}$. □

2.4.3 Proof of Theorem 2.4.1

Now we have assembled the ingredients to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. We will prove this by induction on the number of edges of Γ. For Γ a graph without edges this is immediately clear. Now we assume Theorem 2.4.1 holds for graphs with fewer edges than Γ.

If Γ has a separating edge, it follows immediately from Proposition 2.4.5. We now assume there are no separating edges.

We can assume there is exactly one leg at every vertex. We number the vertices 1 through n, in accordance with the attached leg. Let T be a spanning tree of Γ. Without loss of generality, our numbering is such that for every $k = 1, \ldots, n$ the vertices $\{1, \ldots, k\}$ form a subtree T_k of T. We denote

$$P_k := \{A \in \mathbb{Z}^n \colon \sum_{i=1}^{k} a_i = 0 \wedge a_{k+1} = \cdots = a_n = 0\}.$$

Then we will prove with induction on k that for every polynomial Q the map $A \mapsto S^{\Gamma}_{A,0}(Q)$ is polynomial when restricted to P_k.

We start with a subtree with one vertex. By the total sum zero condition we have $P_1 = \{0\}$, and then for every polynomial Q the map $P_k \to \mathbb{Z}, A \mapsto S^{\Gamma}_{A,0}(Q)$ is a constant map, and in particular polynomial.

Now assume we have proven it for k and want to prove it for $k+1$. Let $j \leq k$ be the vertex that in T is adjacent to vertex $k+1$. Note that $P_{k+1} = P_k + \{a(e_k - e_j) \colon a \in \mathbb{Z}\}$ where e_i is the i'th basis vector in $\mathbb{Z}^n$. The induction step follows from Proposition 2.4.6. As $P_n = \{A \in \mathbb{Z}^n \colon \sum_i a_i = 0\}$ we are done. □

We immediately find the following corollary.

Corollary 2.4.7. *The function*

$$\{A \in \mathbb{Z}^n : (2g-2+n) \mid \sum a_i\} \to \mathbb{Q}$$
$$A \mapsto S^{\Gamma}_{A,0}(Q)$$

is polynomial in A.

Proof. We will prove this by reducing to the $k = 0$ case.

Let $\Gamma' \in G_{g,n+\#V}$ be the graph Γ with one leg added at every vertex. Let A' denote the vector $(a_1, \ldots, a_n, (-k(2g(v)-2+n(v)))_{v\in v})$. We see that the sum of the elements in A' is 0. Note that $S^{\Gamma}_{A,r}(Q) = S^{\Gamma'}_{A',r}(Q)$. The theorem follows by applying Theorem 2.4.1 to $S^{\Gamma'}_{A',r}(Q)$. □

2.5 Polynomiality of the DR cycle

In this section we will prove polynomiality of the double ramification cycle. With Theorem 2.4.1 and Pixton's formula (2.3.0.1), we conclude with the following theorem.

Theorem 2.5.1. *Fix g, n. The cycle $\mathrm{DR}_g(a_1, \ldots, a_n) \in \mathrm{CH}^g(M_{g,n})$ is a polynomial in $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, where we require that $\sum a_i$ is divisible by $(2g-2+n)$.*

Proof. By the formula (2.3.0.1), we have that $\mathrm{DR}_g(a)$ is a polynomial in a finite set of decorated strata, the a_i, and in terms $S^{\Gamma}_{A,0}(Q)$. The result follows from Corollary 2.4.7. □