Counting curves and their rational points

Spelier, P.

Chapter 1

Introduction

In this thesis, we consider two different topics related to algebraic curves. The first topic is counting curves and intersection theory on moduli spaces of curves. The thesis takes a logarithmic geometry approach, focussing on log moduli spaces and their intersection theoretical properties. The second topic is finding rational points on curves, using a general principle called Chabauty's method.

The six main chapters of this thesis are the six papers [Spe24, Spe22, HS23, HS22a, DRHS23, BDRHS24].

In Section 1.1 we discuss the general motivation for studying moduli spaces of curves and their intersection theory by relating it to curve counting. In Section 1.2 we shortly discuss how logarithmic geometry enters the picture. In Sections 1.1.1, 1.2.1 and 1.2.2 we introduce Chapters 2, 3 and 4.

In Section 1.3 we introduce the problem of finding rational points on curves, Chabauty's method, and some of its many variations. In Sections 1.3.1, 1.3.2 and 1.3.3 we introduce Chapters 5, 6 and 7.

1.1 Moduli spaces of curves and intersection theory

Enumerative geometry is the study of computing numbers of solutions to geometric problems. The main example is given by Gromov–Witten invariants,

counts of curves in varieties satisfying certain conditions. Classically, some examples are:

- there is exactly one line through two general points in the plane;
- there is exactly one conic through five general points in the plane;
- there are exactly twelve nodal cubics through eight general points in the plane.

Currently, these invariants are viewed through the lens of *intersection theory* on *moduli spaces of curves*.

The moduli space of smooth curves[1] $\mathcal{M}_{g,n}$ parametrises smooth curves of genus g with n marked points. Its compactification $\overline{\mathcal{M}}_{g,n}$ parametrises curves of genus g with n marked points with at worst nodal singularities and finite automorphism groups. The crucial insight is that these spaces, parametrising geometric objects, are themselves geometric objects, namely Deligne-Mumford stacks (orbifolds). The intersection theory on $\overline{\mathcal{M}}_{g,n}$ is encoded in the Chow ring $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$, consisting of sums of closed substacks up to rational equivalence, with an intersection product. A condition on curves can now be thought of as a closed substack of $\overline{\mathcal{M}}_{g,n}$, and intersection theory will tell us the (virtual) cardinality of the intersection of the conditions.

This way of thinking has paid off. One of the first general results was the count of genus 0 degree d curves in $\mathbb{P}^2$ passing through $3d-1$ general points, generalising the counts mentioned above [KM94, Kon95]. Currently, complicated generalisations of these problems are being studied, for example generalising the target variety, or counting higher genus curves.

In Chapter 2, summarised in Section 1.1.1, we study the *double ramification cycle*, a cycle $\mathrm{DR}_g(A) \in \mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$, dependent on a sequence $A \in \mathbb{Z}^n$ with sum 0, that is related to Gromov–Witten invariants and can be seen as a count of curves in the stack $[\mathbb{P}^1/\mathbb{G}_m]$. In [JPPZ17] they announced a proof of polynomiality of $\mathrm{DR}_g(A)$ in A, and in [Pix23] gave a proof. We give an alternative proof of the fact that $\mathrm{DR}_g(A)$ is polynomial in A.

In Chapter 3 and Chapter 4 we take a logarithmic geometry approach to moduli spaces. Logarithmic geometry is a key language for combining geometry and combinatorics, and is in general very useful for breaking problems into smaller pieces by degenerating varieties to non-smooth but simpler varieties. In Section 1.2 we introduce log geometry and give some motivations. In Sec-

[1] In this section, a curve C/S is a proper, flat scheme with reduced and connected fibers of dimension 1.

tions 1.2.1 and 1.2.2 we give introductions to Chapters 3 and 4 respectively.

1.1.1 Chapter 2: Polynomiality of the double ramification cycle

Let $A \in \mathbb{Z}^n$ be a sequence with sum 0. Then the *double ramification locus* is the substack

$$\mathrm{DRL}_g(A) = \left\{(C/S, p_1, \ldots, p_n) : \mathcal{O}_C\left(\sum a_i p_i\right) \text{ is fiberwise trivial}\right\} \subset \mathcal{M}_{g,n}.$$

Equivalently, this is the locus of curves C that admit a rational map to a $\mathbb{P}^1$ with ramification profile over 0 and ∞ given by the positive and negative entries of the a_i respectively. This can be compactified to a cycle in $\mathrm{CH}^g(\overline{\mathcal{M}}_{g,n})$ in several equivalent ways. In 2003 Graber and Vakil [GV03] defined the *double ramification cycle* $\mathrm{DR}_g(A)$ as the the virtual class of the space of stable maps to $\mathbb{P}^1$ with specified tangency conditions modulo the $\mathbb{G}_m$-action. Since then there have also been constructions using birational geometry of $\overline{\mathcal{M}}_{g,n}$ [HKP18] or logarithmic geometry [MW20]. Using either of these latter constructions, one can also define the *twisted double ramification cycle*

$$\mathrm{DR}_g(A)$$

for a vector $A \in \mathbb{Z}^n$ with sum $k(2g-2+n)$ for some integer k. This is the virtual class of the compactification of the locus

$$\left\{(C/S, p_1, \ldots, p_n) : (\omega_C^{\log})^{-k}\left(\sum a_i p_i\right) \text{ is fiberwise trivial}\right\}.$$

For a long time, it was an open question on how to actually compute the (twisted) double ramification cycle. In 2016 [JPPZ17] answered this by giving a formula for $\mathrm{DR}_g(A)$ for $k = 0$, known as Pixton's formula. Later this was generalised to general k [BHP$^+$20].

One question about the double ramification cycle has remained open for quite some time: whether $\mathrm{DR}_g(A)$ is polynomial in A. The formula itself does not seem polynomial in A, but instead depends on the combinatorics of A and the strata of $\overline{\mathcal{M}}_{g,n}$. In [JPPZ17] they announced a proof of this statement, and in 2023 Pixton gave a proof incorporating ideas of Zagier of the polynomiality of $\mathrm{DR}_g(A)$ [Pix23]. We present an alternative proof of the following statement.

Theorem 1 (Theorem A)**.** *Fix g, n. The cycle $\mathrm{DR}(a_1, \ldots, a_n) \in \mathrm{CH}^g(M_{g,n})$ is a polynomial in $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, where we require that $(2g-2+n) \mid \sum a_i$.*

1.2 Logarithmic moduli spaces

Since the computation of the genus 0 Gromov–Witten invariants of $\mathbb{P}^2$ [KM94, Kon95], the goal has shifted to computing higher genus invariants of more complicated varieties. One of the main tools we have is *degeneration*. When degenerating a space, the curves inside degenerate as well, and its Gromov–Witten invariants can degenerate into *log Gromov–Witten invariants*, counts of curves that also take into account tangency conditions [GS13, ACGS20b, RK23]. This calls for a language where we can both degenerate varieties and curves, and deal with the complicated combinatorics and geometry that appears. *Logarithmic geometry*, or log geometry, is an extension of algebraic geometry particularly equipped at dealing with degenerations. A log scheme X is a scheme $\underline{X}$ with some extra structure. See for example [Kat89, Tem22] for introductions to log schemes, or [Ogu18] for a reference work.

Example 1.2.1. Any scheme X with a normal crossings boundary divisor D has a natural log structure.

A log curve C/S is roughly a log smooth map of log schemes such that the underlying map of schemes is a curve. See [Kat96] for a more in depth study of this terms.

Every nodal curve has a canonical log curve structure on it. The log stack parametrising stable log curves is simply $\overline{\mathcal{M}}_{g,n}$, with log structure given by the normal crossings boundary divisor $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. Per definition, all log curves are all log smooth. This log smoothness causes all log curves to behave similarly to smooth curves. And correspondingly, as nodal curves are shadows of log curves, many classical constructions on $\overline{\mathcal{M}}_{g,n}$ turn out to be shadows of easier, better behaved log constructions. Constructing log analogues has been fruitful even in order to prove purely algebraic results [HPS19, HMOP23, RK23] (see e.g. [FJP23, KRZB16] for some examples where classical results are proven by generalising classical constructions to *tropical geometry*, the combinatorial part of log geometry). In Chapter 3 and Chapter 4 we treat two log constructions, relating to enriched structures and gluing maps respectively.

1.2.1 Chapter 3: The moduli stack of enriched structures and a logarithmic compactification

There is interest in understanding how line bundles can degenerate when a smooth curve degenerates to a nodal curve. This has been studied extensively in the context of limits of so-called *linear series*, subspaces of the space of

global sections of a line bundle on a smooth curve (see for example [GH80, HM82, FJP23], or the introduction of [Oss19]). Eisenbud and Harris initiated this area by studying how these linear series degenerate if the smooth curve degenerates into two curves meeting in a single node [EH86]. Currently, an important tool in the study of limit linear series is the following notion of an enriched structure on a curve, defined by Mainò.

Definition 1.2.2 (First definition of [Mai98])**.** Let $C/\operatorname{Spec} k$ with $k = \overline{k}$ be a stable curve with irreducible components $C_1, \dots, C_n$. An *enriched structure* on C is a collection of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ on C such that there exists a regular smoothing $\mathfrak{C}/\operatorname{Spec} k[[t]]$ with $\mathcal{L}_i \cong O_{\mathfrak{C}}(C_i)|_C$. An *enriched curve* is a stable curve together with an enriched structure.

Mainò's work on this has been important for Osserman's work on limit linear series [Oss16, Oss19], and in other problems degenerating structures on smooth curves to a nodal curve [EM02, ES07].

However, Definition 1.2.2 does not readily generalise to curves over a base scheme, as the combinatorial notion of irreducible components break down. In [AP14] Abreu and Pacini study a tropical version of Definition 1.2.2, with the goal to find a well-behaved moduli space whose geometric points correspond to enriched curves, and a well-behaved modular compactification.

There is an intricate combinatorial definition of a stack in [BH19] whose geometric points corresponds to enriched curves, but some questions remained open, namely whether their definition has a modular smooth compactification, and whether it is an open inside a blowup of the moduli space $\mathfrak{M}$ of prestable curves. Log geometry is perfect for generalising combinatorial definitions, and it turns out that with logarithmic tools, we can easily find a moduli space whose geometric points correspond to enriched curves.

We do this by defining when a log curve is *rich* (Definition 3.1.1). This defines a substack of the stack of log curves, whose underlying algebraic stack does not embed into the stack of curves.

Theorem 2 (Theorem B,Theorem C)**.** *There is a log substack* $\operatorname{Min}(\boldsymbol{RLC})$ *of* $\mathfrak{M}$ *consisting of* rich *log curves whose fiber over a geometric point* $C/\operatorname{Spec} k$ *naturally corresponds to the set of enriched structures on* C*. It has a modular compactification* $\operatorname{Min}(\boldsymbol{WRLC})$ *consisting of* weakly rich *log curves, which is a smooth blowup*[2] *of* $\mathfrak{M}$*.*

[2]In fact, it is a *log* blowup of $\mathfrak{M}$. A log blowups of a log stack with divisorial log structure is an iterated blowup in boundary strata. Log blowups have many special properties and play an important role in log geometry

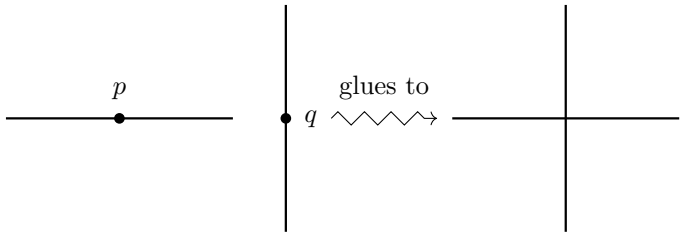

Figure 1.1: The gluing of two lines.

This affirmatively answers the two open questions from [BH19].

We also generalise the notion of richness to *r-richness* for $r \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, in order to study the universal Néron-model-admitting stack $\widetilde{\mathcal{M}}$. This stack roughly parametrises curves C/S that are smooth over a dense open $U \subset S$ and that admit a proper separated Néron model J/S of the Jacobian $\mathfrak{Jac}(C|_U)/U$. This universal Néron-model-admitting stack is not quasi-compact, but it does admit a stratification with nice stacks, as per the following theorem.

Theorem 3 (Theorem D,Theorem E)**.** *For $r \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ there is a smooth log substack* $\mathrm{Min}(\boldsymbol{RLC}_r)$ *of* $\mathfrak{M}$ *consisting of* r-rich *log curves that give rise to a moduli stratification*

$$\widetilde{\mathcal{M}} = \mathrm{Min}(\boldsymbol{RLC}_\infty) = \bigcup_{r \in \mathbb{Z}_{\geq 1}} \mathrm{Min}(\boldsymbol{RLC}_r).$$

For $r \in \mathbb{Z}_{\geq 1}$ the space $\mathrm{Min}(\boldsymbol{RLC}_r)$ *has a modular compactification*

$$\mathrm{Min}(\boldsymbol{WRLC}_r)$$

consisting of weakly r-rich *log curves, which is a blowup of* $\mathfrak{M}$.

1.2.2 Chapter 4: Logarithmic cohomological field theories

A crucial notion in the classical study of $\overline{\mathcal{M}}_{g,n}$ is the *gluing maps*.

Definition 1.2.3. Let C/S be a (possibly disconnected) curve with two distinct points $p, q : S \to C$. We let C^{gl} denote the curve with p and q glued together, or more formally the quotient C under the equivalence relation $p \sim q$.

For an example, see Figure 1.1. This construction induces gluing maps

$$\begin{aligned}\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} &\to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},\\ \overline{\mathcal{M}}_{g-1,n+2} &\to \overline{\mathcal{M}}_{g,n}.\end{aligned} \tag{1.2.2.1}$$

These gluing maps automatically land in the boundary divisor of nodal curves. They help us immensely in understanding $\overline{\mathcal{M}}_{g,n}$ recursively. For example, they are vital in the construction of the tautological ring $\mathsf{R}^*(\overline{\mathcal{M}}_{g,n})$, a subring of $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$ containing many geometrically natural classes [Mum83], and for computing many interesting classes, such as certain Gromov–Witten invariants, in $\mathsf{R}^*(\overline{\mathcal{M}}_{g,n})$.

A particularly nice phenomenon they capture is that of *(partial) cohomological field theories* or *CohFT*. These are collections of cycle classes satisfying some compatibilities with the gluing maps. For example, Gromov–Witten invariants of a fixed variety form a CohFT. The notion of a (partial) CohFT, a collection of classes satisfying some compatibility with respect to gluing maps, has been very fruitful. A *semisimple CohFT* is by the Givental-Teleman classification always of a specific form, and the Givental-Teleman classification has been used to calculate many intersection classes, giving rise to many formulas, and to the famous *Pixton relations* in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ [PPZ15]. An important example is that of the double ramification cycles explained in Section 1.1.1, that were proven to be a partial CohFT in [BR21].

However, the gluing maps (1.2.2.1) do not respect the logarithmic structure on $\overline{\mathcal{M}}_{g,n}$, in that they do not lift to maps of log stacks. Fundamentally, the issue is that the log structure on a log curve remembers the *length* of every node. When creating a new node with the gluing map, there is often no valid (or at least, no functorial) choice for the length of the new node.

In joint work with David Holmes, we have solved these issues, by keeping track of the lengths of markings.

Theorem 4 (Theorem F, Holmes–S.)**.** *There is a log stack $\mathbb{M}_{g,n}$ with underlying stack $\overline{\mathcal{M}}_{g,n}$ that admits logarithmic gluing maps*

$$\begin{aligned}\mathbb{M}_{g_1,n_1+1} \times \mathbb{M}_{g_2,n_2+1} &\to \mathbb{M}_{g_1+g_2,n_1+n_2}\\ \mathbb{M}_{g-1,n+2} &\to \mathbb{M}_{g,n}\end{aligned}$$

We use this to define (partial) log CohFTs, and prove the following generalisation of the result of [BR21].

Theorem 5 (Theorem G, Holmes–S.)**.** *The logarithmic double ramification cycles form a partial logarithmic cohomological field theory.*

These results open up a new research line, asking what theorems and definitions about CohFTs we can generalise to the logarithmic case, and how it can help in calculating logarithmic classes. It also opens up a new and simpler approach to *evaluation maps*, and to logarithmic Gromov–Witten invariants. This is subject of future work.

1.3 Chabauty's method

An ancient mathematical problem is that of solving diophantine equations. Given a multivariate polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ with rational coefficients, one can study the rational zeros $\{x \in \mathbb{Q}^n : f(x) = 0\}$. From an algebro-geometric perspective, this corresponds to *rational points* on the algebraic variety given by the zeroes of f inside $\mathbb{C}^n$. Much has been said about this for all kinds of varieties, but we will focus on the case of algebraic curves over $\mathbb{Q}$, and how to explicitly find their rational points.

Algebraic curves are stratified by their genus $g \in \mathbb{Z}_{\geq 0}$. While for C a curve of genus 0 or 1 it is in practice possible to find the rational points[3], for C a curve of genus at least 2 it is much more difficult.

It was conjectured in 1922 by Mordell [Mor22b] that if $g \geq 2$ then the set of rational points $C(\mathbb{Q})$ is always finite. In 1983, this was finally proven by Faltings [Fal83a]. However, Faltings' theorem is not *effective*. That is, given any specific curve $C/\mathbb{Q}$, it does not help with determining $C(\mathbb{Q})$. Guessing all rational points on a curve is easy, as a brute force search usually gives a few small points, with heuristics suggesting that one does not need to seek any further. But finding effective algorithms for provably determining $C(\mathbb{Q})$ is a difficult task. The current state-of-the-art methods are based on Chabauty's partial proof of Faltings' theorem in 1941 [Cha41], which we will explain now.

Let $C/\mathbb{Q}$ be a curve of genus $g \geq 2$, and let J be its Jacobian. Assume C has a rational point b, and use this to construct an Abel–Jacobi embedding $C \to J$.

[3] In theory this is a very difficult problem for $g = 1$, related to finiteness of the Tate-Shafarevich group of an elliptic curve. In practice, [Cre97] gives many tricks to rapidly compute the rational points of an elliptic curve, and the LMFDB database [LMF23] contains the rational points of 3824372 elliptic curves.

Let p be a prime. Then we can form the commutative diagram of embeddings

$$\begin{array}{ccc} C(\mathbb{Q}) & \longrightarrow & J(\mathbb{Q}) \\ \downarrow & & \downarrow \\ C(\mathbb{Q}_p) & \longrightarrow & J(\mathbb{Q}_p) \end{array} \tag{1.3.0.1}$$

Let r be the rank of the Mordell–Weil group $J(\mathbb{Q})$. Then if $r < g$, the closure $\overline{J(\mathbb{Q})}$ of $J(\mathbb{Q})$ inside $J(\mathbb{Q}_p)$ is of p-adic dimension less than g, while $J(\mathbb{Q}_p)$ is of dimension g. Hence we expect the intersection $\overline{J(\mathbb{Q})} \cap C(\mathbb{Q}_p)$ to be finite. By construction the intersection contains $C(\mathbb{Q})$, which then would also be finite. Indeed, Chabauty proved the intersection is always finite for $r < g$ [Cha41].

The first known way to make this effective was developed by Coleman [Col85a], in a method now known as Chabauty–Coleman. He defined the Coleman integral $\int_P^Q \omega$ for $P, Q \in C(\mathbb{Q}_p)$ and $\omega \in H^0(C_{\mathbb{Q}_p}, \Omega_{C_{\mathbb{Q}_p}})$, which naturally extends to a pairing $\int : J(\mathbb{Q}_p) \times H^0(C_{\mathbb{Q}_p}, \Omega_{C_{\mathbb{Q}_p}}) \to \mathbb{Q}_p$. Locally, integrals can be determined by expanding ω as a power series and integrating formally.

Then one can define the vanishing differentials $V \subset H^0(C_{\mathbb{Q}_p}, \Omega_{C_{\mathbb{Q}_p}})$ to be the differentials orthogonal to $J(\mathbb{Q})$ under this pairing. As $r < g$, this has dimension at least $g - r \geq 1$. Then the rational points P of C satisfy $\int_b^P \omega = 0$ for $\omega \in V$. This gives at least $g - r$ equations that rational points satisfy, and by locally expanding these equations as power series, one can use Hensel's lemma to find all of the (finitely many) p-adic solutions. One can then recognise the rational points as a subset, and only finitely many other points remain. Often (and conjecturally always) one can use the Mordell–Weil sieve [BS10, Sik15] to show that these remaining points do not come from rational points.

There are currently many different variations on this theme, and the name of the game is to find explicit algorithms that work in ever more cases and produce ever smaller sets of p-adic points.

The modern perspective on Chabauty's method and Chabauty–Coleman is that the map $C \to J$ induces the abelianisation of the fundamental group of C. Replacing this abelian quotient by bigger, non-abelian quotients one can hopefully find effective finiteness proofs even when the assumption $r < g$ does not hold. This perspective is worked out by Kim, and is known as Chabauty–Kim [Kim09]. He produces an infinite sequence of subsets of $C(\mathbb{Q}_p)$

$$C(\mathbb{Q}_p) \supseteq C(\mathbb{Q}_p)_1 \supseteq C(\mathbb{Q}_p)_2 \supseteq \cdots \supseteq C(\mathbb{Q}).$$

Here $C(\mathbb{Q}_p)_1$ is the same set as produced by Chabauty–Coleman. The sequence is conjectured to be eventually finite for any curve, and is even conjectured to be eventually equal to $C(\mathbb{Q})$ for any curve. However, computing these higher Chabauty sets $C(\mathbb{Q}_p)_n$ for $n \geq 2$ is extremely difficult. For a slightly bigger set $C(\mathbb{Q}_p)_{\mathrm{Coh}} \supseteq C(\mathbb{Q}_p)_2$ this method, called (cohomological) quadratic Chabauty, has been made explicit [BD18, BD21], which has famously led to finding all rational points on the "cursed curve" $X_{\mathrm{s}}(13)$ [BDM$^+$19].

Let ρ be the rank of the Néron–Severi group $\mathrm{NS}(J)$. This is always at least 1. Cohomological quadratic Chabauty theoretically produces a finite set whenever $r < g+\rho-1$, but has only been made algorithmic in the case where $r = g$, the map $J(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to J(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism, and $\rho \geq 2$.

Recently, Edixhoven and Lido developed a geometric version of quadratic Chabauty, called geometric quadratic Chabauty [EL21]. They work with a $\mathbb{G}_m^{\rho-1}$-bundle T/J, and embed C into T. After spreading out over $\mathbb{Z}$, the $\mathbb{Z}$-points of T are in bijection with $J(\mathbb{Q}) \times \mathbb{G}_m(\mathbb{Z})^{\rho-1} = J(\mathbb{Q}) \times \{\pm 1\}^{\rho-1}$. As the dimension of T is $g+\rho-1$, one expects $C(\mathbb{Q})_{\mathrm{Geo}} \coloneqq \overline{T(\mathbb{Z})} \cap C(\mathbb{Z}_p)$ to be finite if $r < g+\rho-1$, and indeed this holds. Geometric quadratic Chabauty was conjectured to be comparable to cohomological quadratic Chabauty.

Jointly with Sachi Hashimoto we first proved a comparison theorem between Chabauty–Coleman and a geometric variant, called geometric linear Chabauty, and then jointly with Juanita Duque-Rosero and Sachi Hashimoto we proved a comparison theorem between geometric quadratic Chabauty and cohomological quadratic Chabauty [HS22a, DRHS23]. These papers are reproduced in Chapter 5 and Chapter 6, and the main theorems are summarised in Section 1.3.1 and Section 1.3.2.

In Section 1.3.3 we give a short introduction for Chapter 7 on computing *local heights*, an important ingredient of quadratic Chabauty.

1.3.1 Chapter 5: A geometric linear Chabauty comparison theorem

In my master's thesis I developed and implemented a geometric version of Chabauty–Coleman, called geometric linear Chabauty, which can also be interpreted as a linear case of geometric quadratic Chabauty. In [HS22a], joint work with Sachi Hashimoto, we proved a comparison result. Assume $r < g$, and let $C(\mathbb{Q}_p)_{\mathrm{GLC}}$ be the set of p-adic points obtained by geometric linear Chabauty, and $C(\mathbb{Q}_p)_{\mathrm{CC}}$ the set obtained by Chabauty–Coleman. Then we showed the following theorem.

Theorem 6 (Theorem 5.5.1,Hashimoto–S.)**.** *Geometric linear Chabauty outperforms Chabauty–Coleman, in that in the case $r < g$ we have an inclusion*

$$C(\mathbb{Q}_p)_{\mathrm{CC}} \supseteq C(\mathbb{Q}_p)_{\mathrm{GLC}} \supseteq C(\mathbb{Q}).$$

In addition, we give an explicit characterisation of the difference $C(\mathbb{Q}_p)_{\mathrm{CC}} \setminus C(\mathbb{Q}_p)_{\mathrm{GLC}}$.

1.3.2 Chapter 6: Geometric quadratic Chabauty and p-adic heights

In this chapter we prove the following comparison theorem.

Theorem 7 (Theorem H,Duque-Rosero–Hashimoto–S.)**.** *Geometric quadratic Chabauty outperforms cohomological quadratic Chabauty, in that in the case where $r = g$, the map $J(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to J(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism, and $\rho \geq 2$ we have an inclusion*

$$C(\mathbb{Q}_p)_{\mathrm{Coh}} \supseteq C(\mathbb{Q}_p)_{\mathrm{Geo}} \supseteq C(\mathbb{Q}).$$

In addition, we give an explicit characterisation of the difference $C(\mathbb{Q}_p)_{\mathrm{Coh}} \setminus C(\mathbb{Q}_p)_{\mathrm{Geo}}$.

1.3.3 Chapter 7: Local heights on hyperelliptic curves and quadratic Chabauty

Let C be a curve with $r = g, \rho \geq 2$ and for which the map $J(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to J(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism. Front and center in cohomological quadratic Chabauty on C is the *Néron–Tate height*, a quadratic function

$$h : J(\mathbb{Q}) \to \mathbb{Q}_p$$

defined with respect to some trace 0 element Z of $\mathrm{End}(J)$. We first explain how exactly h is used in cohomological quadratic Chabauty. This function splits as a sum of functions

$$h = h_p + \sum_{\ell \neq p} h_\ell$$

where ℓ ranges over primes distinct from p, and for any prime q the function h_q is the *local height at* q and $h_q : J(\mathbb{Q}_q) \to \mathbb{Q}_p$. The function h_p is an analytic

function, while for $\ell \neq p$ the function h_ℓ only takes finitely many values. For all primes of good reduction, h_ℓ is even identically zero.

By the assumption that the map $J(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to J(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism one can naturally extend the quadratic function h to a quadratic function $h : J(\mathbb{Q}_p) \to \mathbb{Q}_p$. Then the function $\rho = h - h_p : J(\mathbb{Q}_p) \to \mathbb{Q}_p$ lands in the finite set

$$S = \left\{ \sum_{\ell \neq p} q_\ell : \forall \ell, q_\ell \in \operatorname{im} h_\ell \right\}$$

when evaluated on the rational points. And hence this is also true for the pullback of ρ to the curve C. By finding the zeroes in C of $\rho - c$ for $c \in S$, one hence finds all rational points of C.

Computing h_p has been implemented in [BDM$^+$] and in other places. But computing S is deceptively difficult. Accordingly, quadratic Chabauty has almost exclusively been used for curves where all local heights h_ℓ for $\ell \neq p$ happen to vanish.

Joint with Alex Betts, Juanita Duque-Rosero and Sachi Hashimoto we have given an algorithm to compute these local heights h_ℓ for hyperelliptic curves and odd primes ℓ. This allows quadratic Chabauty to be applied to curves whose rational points could not be computed because of non-trivial local heights. We present the first example of quadratic Chabauty applied to a curve with non-trivial local heights contributions from two bad primes, as per the following theorem.

Theorem 8 (Theorem I, Betts–Duque-Rosero–Hashimoto–S.)**.** *The curve*

$$y^2 = x^6 + 18/5x^4 + 6/5x^3 + 9/5x^2 + 6/5x + 1/5$$

has 10 *rational points.*