The parabolic Anderson model on Galton-Watson trees

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Summary

This thesis investigates the parabolic Anderson model (PAM), which is the Cauchy problem for the heat equation with random potential. The PAM is a mathematical model that describes how mass (i.e. matter or energy) flows in a medium in the presence of a random potential, which acts as a field of sources and sinks. Without the potential, diffusion causes the mass to be evenly distributed across the medium. However, the potential vastly changes this behaviour as mass tends to concentrate around the sources and deplete around the sinks. The goal of the thesis is to understand the long-term behaviour of the mass: its asymptotic growth rate, as well as how and where it concentrates.

The equation that drives the PAM is solved by a functional of a random walk that is given by the well-known Feynman-Kac representation. This representation is the starting point for the analysis of the PAM. The PAM has been extensively studied on regular lattices and is well understood there. It is known that the upper tail of the distribution of the potential fully determines the asymptotic behaviour of the mass for both the growth rate and the locations of high concentration. However, the lattice is not always a suitable model and we look for extensions to random graphs. Very little is known for general graphs and the literature is extremely sparse. The present thesis is a contribution to this developing area. Because sparse random graphs can often be approximated by trees, the natural first step is to consider the PAM on a tree. In particular, this thesis is devoted to studying the PAM on random trees with potentials having a double-exponential distribution.

The study of the PAM is naturally divided into two cases: quenched and annealed, i.e. almost surely with respect to the potential and averaged over the potential, respectively. The thesis covers both and is therefore divided into two parts: Part I is dedicated to the annealed model and Part II to the quenched model.

Part I consists of Chapters 2 and 3 and considers the annealed model. Chapter 2 investigates the PAM on a regular tree and derives the first two terms of the asymptotic growth of the mass. As was the case on the lattice, the argument requires finding asymptotically matching upper and lower bounds. The main challenge is to deal with the exponential growth of the graph size, which is not present in the lattice. A novel technique of folding random walk paths is devised and its large deviation behaviour is analysed to achieve the upper bound. Furthermore, the concentration behaviour of the mass is determined and is given in terms of a minimiser of a variational formula. The argument relies on a rearrangement inequality that overcomes the lack of translation invariance.

Chapter 3 extends Chapter 2 to a Galton-Watson tree with large periodicity – a generalisation of the regular tree. This requires carefully navigating the non-homogeneity of the Galton-Watson tree and dealing with degrees that are random. This chapter is the crucial step in understanding the PAM on the regular Galton-Watson tree, which approximates many sparse random graph models.

Part II consists of Chapters 4 and 5 and considers the quenched model. Chapter 4 investigates the PAM on a Galton-Watson tree. The asymptotic growth rate of the total mass was derived in previous work by den Hollander, König and dos Santos under the restrictive assumption that the degrees are bounded. Chapter 4 extends their analysis to trees with unbounded degrees, and identifies the weakest condition on the degree distribution under which their arguments still hold. This is done by uniformly controlling the appearance of large degrees in subtrees. The existence and uniqueness of the Feynman-Kac representation on exponentially growing graphs is also shown in this chapter.

Chapter 5 again considers the PAM on a Galton-Watson tree, and extends the previous work by den Hollander, König and dos Santos to the version of model in which the Laplacian is replaced by with the degree-normalised Laplacian. This amounts to the random walk in the Feynman-Kac representation having jump rate equal to 1 instead of the degree of the vertex it resides on. The normalisation causes the Laplacian to no longer be symmetric, which results in different spectral properties. We find that the asymptotic growth rate of the total mass is the same, while the variational formula describing the distribution of mass is different. The weakest condition required on the degree distribution is also identified.