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The parabolic Anderson model on Galton-Watson trees

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Appendix of Part II

APPENDIX B

Appendix: Chapter 4

§B.1 Largest eigenvalue

We recall the Rayleigh-Ritz formula for the principal eigenvalue of the Anderson Hamiltonian. For $\Lambda \subset V$ and $q: V \rightarrow [-\infty, \infty)$, let $\lambda_\Lambda(q; G)$ denote the largest eigenvalue of the operator $\Delta_G + q$ in Λ with Dirichlet boundary conditions on $V \setminus \Lambda$, i.e.,

$$\lambda_\Lambda(q; G) := \sup \{ \langle (\Delta_G + q)\phi, \phi \rangle_{\ell^2(V)} : \phi \in \mathbb{R}^V, \text{supp } \phi \subset \Lambda, \|\phi\|_{\ell^2(V)} = 1 \}. \quad (\text{B.1})$$

Lemma B.1.1. *[Spectral bounds]*

(1) For any $\Gamma \subset \Lambda \subset V$,

$$\max_{z \in \Gamma} q(z) - D_{\bar{z}} \leq \lambda_\Gamma(q; G) \leq \lambda_\Lambda(q; G) \leq \max_{z \in \Lambda} q(z) \quad (\text{B.2})$$

with $\bar{z} = \arg \max_{z \in \Gamma} q(z)$ and $D_{\bar{z}}$ the degree of $\bar{z}$.

- (2) The eigenfunction corresponding to $\lambda_\Lambda(q; G)$ can be taken to be non-negative.
- (3) If q is real-valued and $\Gamma \subsetneq \Lambda$ is finite and connected in G , then the second inequality in (B.2) is strict and the eigenfunction corresponding to $\lambda_\Lambda(q; G)$ is strictly positive.

Proof. Write

$$\begin{aligned} \langle (\Delta_G + q)\phi, \phi \rangle_{\ell^2(V)} &= \sum_{x \in \Lambda} [(\Delta_G \phi)(x) + q(x)\phi(x)] \phi(x) \\ &= \sum_{x \in \Lambda} \sum_{\substack{y \in \Lambda: \\ \{x, y\} \in E_\Lambda}} [\phi(y) - \phi(x)] \phi(x) + \sum_{x \in \Lambda} q(x)\phi(x)^2 \\ &= -\frac{1}{2} \sum_{\substack{x, y \in \Lambda: \\ \{x, y\} \in E_\Lambda}} [\phi(x) - \phi(y)]^2 + \sum_{x \in \Lambda} q(x)\phi(x)^2, \end{aligned} \quad (\text{B.3})$$

where the first sum in the last line runs over all ordered pairs (x, y) with $(x, y) \neq (y, x)$, which gives rise to the factor $\frac{1}{2}$. The upper bound in (B.2) follows from the estimate

$$\langle (\Delta_G + q)\phi, \phi \rangle \leq \sum_{x \in \Lambda} q(x)\phi(x)^2 \leq \max_{z \in \Lambda} q(z) \sum_{x \in \Lambda} \phi(x)^2 = \max_{z \in \Lambda} q(z). \quad (\text{B.4})$$

To get the lower bound in (B.2), we use the fact that λ_Λ is non-decreasing in q . Hence, replacing $q(z)$ by $-\infty$ for every $z \neq \bar{z}$ and taking as test function $\phi = \bar{\phi} = \delta_{\bar{z}}$, we get from (B.3) that

$$\begin{aligned} \lambda_\Lambda(q; G) &\geq -\frac{1}{2} \sum_{\substack{x, y \in \Lambda: \\ \{x, y\} \in E_\Lambda}} [\bar{\phi}(x) - \bar{\phi}(y)]^2 + \sum_{x \in \Lambda} q(x) \bar{\phi}(x)^2 \\ &= -\frac{1}{2} \sum_{\substack{y \in \Lambda: \\ \{\bar{z}, y\} \in E_\Lambda}} 1 + q(\bar{z}) = -D_{\bar{z}} + \max_{z \in \Lambda} q(z), \end{aligned} \quad (\text{B.5})$$

which settles the claim in (1). The claims in (2) and (3) are standard. $\square$

Inside $\mathcal{GW}$, fix a finite connected subset $\Lambda \subset V$, and let H_Λ denote the Anderson Hamiltonian in Λ with zero Dirichlet boundary conditions on $\Lambda^c = V \setminus \Lambda$ (i.e., the restriction of the operator $H_G = \Delta_G + \xi$ to the class of functions supported on Λ). For $y \in \Lambda$, let u_Λ^y be the solution of

$$\begin{aligned} \partial_t u(x, t) &= (H_\Lambda u)(x, t), & x \in \Lambda, t > 0, \\ u(x, 0) &= \delta_y(x), & x \in \Lambda, \end{aligned} \quad (\text{B.6})$$

and set $U_\Lambda^y(t) := \sum_{x \in \Lambda} u_\Lambda^y(x, t)$. The solution admits the Feynman-Kac representation

$$u_\Lambda^y(x, t) = \mathbb{E}_y \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}_{\{\tau_{\Lambda^c} > t, X_t = x\}} \right], \quad (\text{B.7})$$

where τ_{Λ^c} is the hitting time of Λ^c . It also admits the spectral representation

$$u_\Lambda^y(x, t) = \sum_{k=1}^{|\Lambda|} e^{t\lambda_\Lambda^k} \phi_\Lambda^k(y) \phi_\Lambda^k(x), \quad (\text{B.8})$$

where $\lambda_\Lambda^1 \geq \lambda_\Lambda^2 \geq \dots \geq \lambda_\Lambda^{|\Lambda|}$ and $\phi_\Lambda^1, \phi_\Lambda^2, \dots, \phi_\Lambda^{|\Lambda|}$ are, respectively, the eigenvalues and the corresponding orthonormal eigenfunctions of H_Λ . These two representations may be exploited to obtain bounds for one in terms of the other, as shown by the following lemma.

Lemma B.1.2. *[Bounds on the solution] For any $y \in \Lambda$ and any $t > 0$,*

$$\begin{aligned} e^{t\lambda_\Lambda^1} \phi_\Lambda^1(y)^2 &\leq \mathbb{E}_y \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{\tau_{\Lambda^c} > t, X_t = y\}} \right] \\ &\leq \mathbb{E}_y \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{\tau_{\Lambda^c} > t\}} \right] \leq e^{t\lambda_\Lambda^1} |\Lambda|^{1/2}. \end{aligned} \quad (\text{B.9})$$

Proof. The first and third inequalities follow from (B.7)–(B.8) after a suitable application of Parseval's identity. The second inequality is elementary. $\square$