Appendix of Part II

APPENDIX B

Appendix: Chapter 4

§B.1 Largest eigenvalue

We recall the Rayleigh-Ritz formula for the principal eigenvalue of the Anderson Hamiltonian. For $\Lambda \subset V$ and $q\colon V \to [-\infty, \infty)$, let $\lambda_\Lambda(q; G)$ denote the largest eigenvalue of the operator $\Delta_G + q$ in Λ with Dirichlet boundary conditions on $V \setminus \Lambda$, i.e.,

$$\lambda_\Lambda(q; G) := \sup\left\{\langle(\Delta_G + q)\phi, \phi\rangle_{\ell^2(V)}\colon\ \phi \in \mathbb{R}^V,\ \operatorname{supp}\phi \subset \Lambda,\ \|\phi\|_{\ell^2(V)} = 1\right\}. \tag{B.1}$$

Lemma B.1.1. *[Spectral bounds]*

(1) *For any* $\Gamma \subset \Lambda \subset V$,

$$\max_{z\in\Gamma} q(z) - D_{\bar z} \le \lambda_\Gamma(q; G) \le \lambda_\Lambda(q; G) \le \max_{z\in\Lambda} q(z) \tag{B.2}$$

with $\bar z = \arg\max_{z\in\Gamma} q(z)$ *and* $D_{\bar z}$ *the degree of* $\bar z$.

(2) *The eigenfunction corresponding to* $\lambda_\Lambda(q; G)$ *can be taken to be non-negative.*

(3) *If* q *is real-valued and* $\Gamma \subsetneq \Lambda$ *is finite and connected in* G*, then the second inequality in* (B.2) *is strict and the eigenfunction corresponding to* $\lambda_\Lambda(q; G)$ *is strictly positive.*

Proof. Write

$$\begin{aligned}\langle(\Delta_G + q)\phi, \phi\rangle_{\ell^2(V)} &= \sum_{x\in\Lambda} \left[(\Delta_G\phi)(x) + q(x)\phi(x)\right]\phi(x)\\ &= \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda:\\ \{x,y\}\in E_\Lambda}} [\phi(y) - \phi(x)]\phi(x) + \sum_{x\in\Lambda} q(x)\phi(x)^2\\ &= -\tfrac{1}{2} \sum_{\substack{x,y\in\Lambda:\\ \{x,y\}\in E_\Lambda}} [\phi(x) - \phi(y)]^2 + \sum_{x\in\Lambda} q(x)\phi(x)^2,\end{aligned} \tag{B.3}$$

where the first sum in the last line runs over all ordered pairs (x, y) with $(x, y) \neq (y, x)$, which gives rise to the factor $\frac{1}{2}$. The upper bound in (B.2) follows from the estimate

$$\langle(\Delta_G + q)\phi, \phi\rangle \le \sum_{x\in\Lambda} q(x)\phi(x)^2 \le \max_{z\in\Lambda} q(z) \sum_{x\in\Lambda} \phi(x)^2 = \max_{z\in\Lambda} q(z). \tag{B.4}$$

To get the lower bound in (B.2), we use the fact that λ_Λ is non-decreasing in q. Hence, replacing $q(z)$ by $-\infty$ for every $z \neq \bar{z}$ and taking as test function $\phi = \bar{\phi} = \delta_{\bar{z}}$, we get from (B.3) that

$$\begin{aligned} \lambda_\Lambda(q;G) &\geq -\tfrac{1}{2} \sum_{\substack{x,y\in\Lambda:\\ \{x,y\}\in E_\Lambda}} \left[\bar{\phi}(x) - \bar{\phi}(y)\right]^2 + \sum_{x\in\Lambda} q(x)\bar{\phi}(x)^2 \\ &= -\tfrac{1}{2} \sum_{\substack{y\in\Lambda:\\ \{\bar{z},y\}\in E_\Lambda}} 1 + q(\bar{z}) = -D_{\bar{z}} + \max_{z\in\Lambda} q(z), \end{aligned} \tag{B.5}$$

which settles the claim in (1). The claims in (2) and (3) are standard. □

Inside $\mathcal{GW}$, fix a finite connected subset $\Lambda \subset V$, and let H_Λ denote the Anderson Hamiltonian in Λ with zero Dirichlet boundary conditions on $\Lambda^c = V \backslash \Lambda$ (i.e., the restriction of the operator $H_G = \Delta_G + \xi$ to the class of functions supported on Λ). For $y \in \Lambda$, let u^y_Λ be the solution of

$$\begin{aligned} \partial_t u(x,t) &= (H_\Lambda u)(x,t), & x \in \Lambda,\, t > 0, \\ u(x,0) &= \delta_y(x), & x \in \Lambda, \end{aligned} \tag{B.6}$$

and set $U^y_\Lambda(t) := \sum_{x\in\Lambda} u^y_\Lambda(x,t)$. The solution admits the Feynman-Kac representation

$$u^y_\Lambda(x,t) = \mathbb{E}_y\left[\exp\left\{\int_0^t \xi(X_s)\mathrm{d}s\right\} \mathbb{1}\{\tau_{\Lambda^c} > t, X_t = x\}\right], \tag{B.7}$$

where τ_{Λ^c} is the hitting time of Λ^c. It also admits the spectral representation

$$u^y_\Lambda(x,t) = \sum_{k=1}^{|\Lambda|} \mathrm{e}^{t\lambda^k_\Lambda} \phi^k_\Lambda(y)\phi^k_\Lambda(x), \tag{B.8}$$

where $\lambda^1_\Lambda \geq \lambda^2_\Lambda \geq \cdots \geq \lambda^{|\Lambda|}_\Lambda$ and $\phi^1_\Lambda, \phi^2_\Lambda, \ldots, \phi^{|\Lambda|}_\Lambda$ are, respectively, the eigenvalues and the corresponding orthonormal eigenfunctions of H_Λ. These two representations may be exploited to obtain bounds for one in terms of the other, as shown by the following lemma.

Lemma B.1.2. *[Bounds on the solution] For any $y \in \Lambda$ and any $t > 0$,*

$$\begin{aligned} \mathrm{e}^{t\lambda^1_\Lambda}\phi^1_\Lambda(y)^2 &\leq \mathbb{E}_y\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbb{1}_{\{\tau_{\Lambda^c} > t, X_t = y\}}\right] \\ &\leq \mathbb{E}_y\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbb{1}_{\{\tau_{\Lambda^c} > t\}}\right] \leq \mathrm{e}^{t\lambda^1_\Lambda}|\Lambda|^{1/2}. \end{aligned} \tag{B.9}$$

Proof. The first and third inequalities follow from (B.7)–(B.8) after a suitable application of Parseval's identity. The second inequality is elementary. □

Appendix B