The parabolic Anderson model on Galton-Watson trees

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PART II

PARABOLIC ANDERSON MODEL: QUENCHED

CHAPTER 4

Parabolic Anderson model on a Galton-Watson tree revisited

This chapter is based on the following paper:
F. den Hollander and D. Wang. The parabolic Anderson model on a Galton-Watson tree revisited. *J. Stat. Phys.*, 189(1):Paper 8, 2022.

Abstract

In [12] a detailed analysis was given of the large-time asymptotics of the total mass of the solution to the parabolic Anderson model on a supercritical Galton-Watson random tree with an i.i.d. random potential whose marginal distribution is double-exponential. Under the assumption that the degree distribution has bounded support, two terms in the asymptotic expansion were identified under the quenched law, i.e., conditional on the realisation of the random tree and the random potential. The second term contains a variational formula indicating that the solution concentrates on a subtree with minimal degree according to a computable profile. The present paper extends the analysis to degree distributions with unbounded support. We identify the weakest condition on the tail of the degree distribution under which the arguments in [12] can be pushed through. To do so we need to control the occurrence of large degrees uniformly in large subtrees of the Galton-Watson tree.

§4.1 Introduction and main results

Section 4.1.1 provides a brief introduction to the parabolic Anderson model. Section 4.1.2 introduces basic notation and key assumptions. Section 4.1.3 states the main theorem and gives an outline of the remainder of the paper.

§4.1.1 The PAM and intermittency

The *parabolic Anderson model* (PAM) is the Cauchy problem

$$\partial_t u(x,t) = \Delta_{\mathscr{X}} u(x,t) + \xi(x)u(x,t), \qquad t > 0,\, x \in \mathscr{X},$$

where $\mathscr{X}$ is an ambient space, $\Delta_{\mathscr{X}}$ is a Laplace operator acting on functions on $\mathscr{X}$, and ξ is a random potential on $\mathscr{X}$. Most of the literature considers the setting where $\mathscr{X}$ is either $\mathbb{Z}^d$ or $\mathbb{R}^d$ with $d \geq 1$ (for mathematical surveys we refer the reader to [3], [29]). More recently, other choices for $\mathscr{X}$ have been considered as well: the complete graph [16], the hypercube [5], Galton-Watson trees [12], and random graphs with prescribed degrees [12].

The main target for the PAM is a description of *intermittency*: for large t the solution $u(\cdot,t)$ of (2.1) concentrates on well-separated regions in $\mathscr{X}$, called *intermittent islands*. Much of the literature has focussed on a detailed description of the size, shape and location of these islands, and the profiles of the potential $\xi(\cdot)$ and the solution $u(\cdot,t)$ on them. A special role is played by the case where ξ is an i.i.d. random potential with a *double-exponential* marginal distribution

$$\mathsf{P}(\xi(0) > u) = \mathrm{e}^{-\mathrm{e}^{u/\varrho}}, \qquad u \in \mathbb{R},$$

where $\varrho \in (0,\infty)$ is a parameter. This distribution turns out to be critical, in the sense that the intermittent islands neither grow nor shrink with time, and therefore represents a class of its own.

The analysis of intermittency typically starts with a computation of the large-time asymptotics of the total mass, encapsulated in what are called *Lyapunov exponents*. There is an important distinction between the *annealed* setting (i.e., averaged over the random potential) and the *quenched* setting (i.e., almost surely with respect to the random potential). Often both types of Lyapunov exponents admit explicit descriptions in terms of *characteristic variational formulas* that contain information about where and how the mass concentrates in $\mathscr{X}$. These variational formulas contain a *spatial part* (identifying where the concentration on islands takes place) and a *profile part* (identifying what the size and shape of both the potential and the solution are on the islands).

In the present paper we focus on the case where $\mathscr{X}$ is a Galton-Watson tree, in the quenched setting (i.e., almost surely with respect to the random tree and the random potential). In [12] the large-time asymptotics of the total mass was derived under the assumption that the degree distribution has bounded support. The goal of the present paper is to relax this assumption to unbounded degree distributions. In particular, we

identify the *weakest condition on the tail of the degree distribution* under which the arguments in [12] can be pushed through. To do so we need to control the occurrence of large degrees *uniformly in large subtrees* of the Galton-Watson tree.

§4.1.2 The PAM on a graph

We begin with some basic definitions and notations (and refer the reader to [3], [29] for more background).

Let $G = (V, E)$ be a *simple connected undirected* graph, either finite or countably infinite. Let Δ_G be the Laplacian on G, i.e.,

$$(\Delta_G f)(x) := \sum_{\substack{y \in V: \\ \{x,y\} \in E}} [f(y) - f(x)], \qquad x \in V,\ f\colon V \to \mathbb{R}. \tag{4.1}$$

Our object of interest is the non-negative solution of the Cauchy problem with localised initial condition,

$$\begin{array}{lcll} \partial_t u(x,t) & = & (\Delta_G u)(x,t) + \xi(x)u(x,t), & x \in V,\ t > 0, \\ u(x,0) & = & \delta_{\mathcal{O}}(x), & x \in V, \end{array} \tag{4.2}$$

where $\mathcal{O} \in V$ is referred to as the *root* of G. We say that G is *rooted at* $\mathcal{O}$ and call $G = (V, E, \mathcal{O})$ a *rooted graph*. The quantity $u(x,t)$ can be interpreted as the amount of mass present at time t at site x when initially there is unit mass at $\mathcal{O}$.

Criteria for existence and uniqueness of the non-negative solution to (4.2) are well known (see [24], [25] for the case $G = \mathbb{Z}^d$), and the solution is given by the *Feynman-Kac formula*

$$u(x,t) = \mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\, \mathbb{1}\{X_t = x\}\right], \tag{4.3}$$

where $X = (X_t)_{t \geq 0}$ is the continuous-time random walk on the vertices V with jump rate 1 along the edges E, and $\mathbb{P}_{\mathcal{O}}$ denotes the law of X given $X_0 = \mathcal{O}$. We are interested in the *total mass* of the solution,

$$U(t) := \sum_{x \in V} u(x,t) = \mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\right]. \tag{4.4}$$

Often we suppress the dependence on G, ξ from the notation. Note that, by time reversal and the linearity of (4.2), $U(t) = \hat{u}(0,t)$ with $\hat{u}$ the solution of (4.2) with a different initial condition, namely, $\hat{u}(x,0) = 1$ for all $x \in V$.

As in [12], throughout the paper we assume that the random potential $\xi = (\xi(x))_{x \in V}$ consists of i.i.d. random variables with marginal distribution satisfying:

Assumption 4.A. [Asymptotic double-exponential potential]
For some $\varrho \in (0, \infty)$,

$$\mathsf{P}\left(\xi(0) \geq 0\right) = 1, \qquad \mathsf{P}\left(\xi(0) > u\right) = \mathrm{e}^{-\mathrm{e}^{u/\varrho}} \ \text{ for } u \text{ large enough.} \tag{4.5}$$

■

The restrictions in (4.5) are helpful to avoid certain technicalities that require no new ideas. In particular, (4.5) is enough to guarantee existence and uniqueness of the non-negative solution to (4.2) on any graph whose largest degrees grow modestly with the size of the graph (as can be inferred from the proof in [25] for the case $G = \mathbb{Z}^d$; see Section 4.6 for more details). All our results remain valid under milder restrictions (e.g. [25, Assumption (F)] plus an integrability condition on the lower tail of $\xi(0)$).

The following *characteristic variational formula* is important for the description of the asymptotics of $U(t)$ when ξ has a double-exponential tail. Denote by $\mathcal{P}(V)$ the set of probability measures on V. For $p \in \mathcal{P}(V)$, define

$$I_E(p) := \sum_{\{x,y\}\in E} \Big(\sqrt{p(x)} - \sqrt{p(y)}\Big)^2, \qquad J_V(p) := -\sum_{x\in V} p(x)\log p(x),$$

and set

$$\chi_G(\varrho) := \inf_{p\in\mathcal{P}(V)} [I_E(p) + \varrho J_V(p)], \qquad \varrho \in (0,\infty). \tag{4.6}$$

The first term in (4.6) is the quadratic form associated with the Laplacian, describing the solution $u(\cdot,t)$ in the intermittent islands, while the second term in (4.6) is the Legendre transform of the rate function for the potential, describing the highest peaks of $\xi(\cdot)$ in the intermittent islands.

§4.1.3 The PAM on a Galton-Watson tree

Let D be a random variable taking values in $\mathbb{N}$. Start with a root vertex $\mathcal{O}$, and attach edges from $\mathcal{O}$ to D first-generation vertices. Proceed recursively: after having attached the n-th generation of vertices, attach to each one of them independently a number of vertices having the same distribution as D, and declare the union of these vertices to be the $(n+1)$-th generation of vertices. Denote by $\mathcal{GW} = (V, E)$ the graph thus obtained and by $\mathfrak{P}$ its probability law and $\mathfrak{E}$ the expectation. Write $\mathcal{P}$ and $\mathcal{E}$ to denote probability and expectation for D, and $\mathrm{supp}(D)$ to denote the support of $\mathcal{P}$. The law of D can be viewed as the offspring distribution of $\mathcal{GW}$, and the law of $D+1$ the degree distribution of $\mathcal{GW}$.

Throughout the paper, we assume that the degree distribution satisfies:

Assumption 4.B. [Exponential tails]
(1) $d_{\min} := \min \mathrm{supp}(D) \geq 2$ and $\mathcal{E}[D] \in (2,\infty)$.
(2) $\mathcal{E}\big[\mathrm{e}^{aD}\big] < \infty$ for all $a \in (0,\infty)$. ■

Under this assumption, $\mathcal{GW}$ is $\mathfrak{P}$-a.s. an infinite tree. Moreover,

$$\lim_{r\to\infty} \frac{\log|B_r(\mathcal{O})|}{r} = \log \mathcal{E}[D] =: \vartheta \in (0,\infty) \qquad \mathfrak{P}-a.s., \tag{4.7}$$

where $B_r(\mathcal{O}) \subset V$ is the ball of radius r around $\mathcal{O}$ in the graph distance (see e.g. [30, pp. 134–135]). Note that this ball depends on $\mathcal{GW}$ and therefore is random. Furthermore, under this assumption on the tail of D, the solution to the PAM exists and is unique.

Theorem 4.1.1. *Let $G = \mathcal{GW}$. Subject to Assumption 4.B(2),* (4.1) *has a unique non-negative solution* $(\mathsf{P} \times \mathfrak{P})$ *almost surely. This solution admits the Feynman-Kac representation* (4.3).

For our main result we need an assumption that is much stronger than Assumption 4.B(2).

Assumption 4.C. [Super-double-exponential tails] There exists a function $f\colon (0,\infty) \to (0,\infty)$ satisfying $\lim_{s\to\infty} f(s) = 0$, $\lim_{s\to\infty} f'(s) = 0$ and $\lim_{s\to\infty} f(s)\log s = \infty$ such that

$$\limsup_{s\to\infty} \mathrm{e}^{-s} \log \mathcal{P}(D > s^{f(s)}) < -2\vartheta. \tag{4.8}$$

■

To state our main result, we define the constant

$$\widetilde{\chi}(\varrho) := \inf\big\{\chi_T(\varrho)\colon\ T \text{ is an infinite tree with degrees in } \operatorname{supp}(D)\big\}, \tag{4.9}$$

with $\chi_G(\varrho)$ defined in (4.6), and abbreviate

$$\mathfrak{r}_t = \frac{\varrho t}{\log\log t}. \tag{4.10}$$

Theorem 4.1.2. *[Quenched Lyapunov exponent] Subject to Assumptions* 4.A–4.C*,*

$$\frac{1}{t}\log U(t) = \varrho\log(\vartheta\mathfrak{r}_t) - \varrho - \widetilde{\chi}(\varrho) + o(1), \quad t\to\infty, \qquad (\mathsf{P}\times\mathfrak{P})\text{-}a.s. \tag{4.11}$$

With Theorem 4.1.2 we have completed our task to relax the main result in [12] to degree distributions with unbounded support. The extension comes at the price of having to assume a tail that decays faster than double-exponential as shown in (4.8). This property is needed to control the occurrence of large degrees *uniformly in large subtrees* of $\mathcal{GW}$. No doubt Assumption 4.C is stronger than is needed, but to go beyond would require a major overhaul of the methods developed in [12], which remains a challenge.

In (4.2) the initial mass is located at the root. The asymptotics in (4.11) is robust against different choices. A heuristic explanation where the terms in (4.11) come from was given in [12, Section 1.5]. The asymptotics of $U(t)$ is controlled by random walk paths in the Feynman-Kac formula in (4.4) that run within time $\mathfrak{r}_t/\varrho\log\mathfrak{r}_t$ to an *intermittent island* at distance r_t from $\mathcal{O}$, and afterwards stay near that island for the rest of the time. The intermittent island turns out to consist of a subtree with degree $d_{\min}$ where the potential has a height $\varrho\log(\vartheta\mathfrak{r}_t)$ and a shape that is the solution of a variational formula restricted to that subtree. The first and third term in (4.11) are the contribution of the path after it has reached the island, the second term is the cost for reaching the island.

For $d \in \mathbb{N}\setminus\{1\}$, let $\mathcal{T}_d$ be the infinite homogeneous tree in which every node has downward degree d. It was shown in [12] that if $\varrho \geq 1/\log(d_{\min}+1)$, then

$$\widetilde{\chi}(\varrho) = \chi_{\mathcal{T}_{d_{\min}}}(\varrho). \tag{4.12}$$

Presumably $\mathcal{T}_{d_{\min}}$ is the *unique* minimizer of (4.9), but proving so would require more work.

Outline. The remainder of the chapter is organised as follows. Section 4.2 collects some structural properties of Galton-Watson trees. Section 4.3 contains several preparatory lemmas, which identify the maximum size of the islands where the potential is suitably high, estimate the contribution to the total mass in (4.4) by the random walk until it exits a subset of $\mathcal{GW}$, bound the principal eigenvalue associated with the islands, and estimate the number of locations where the potential is intermediate. Section 4.4 uses these preparatory lemmas to find the contribution to the Feynman-Kac formula in (4.4) coming from various sets of paths. Section 4.5 uses these contributions to prove Theorem 4.1.2. In Section 4.6 we prove Theorem 4.1.1.

Assumptions 4.A–4.B are needed throughout the paper. Only in Sections 4.4–4.5 do we need Assumption 4.C.

§4.2 Structural properties of the Galton-Watson tree

In the section we collect a few structural properties of $\mathcal{GW}$ that play an important role throughout the paper. None of these properties were needed in [12]. Section 4.2.1 looks at volumes, Section 4.2.2 at degrees, Section 4.2.3 at tree animals.

§4.2.1 Volumes

Let Z_k be the number of offspring in generation k, i.e.,

$$Z_k = |\{x \in V\colon\ d(x,\mathcal{O}) = k\}|, \tag{4.13}$$

where $d(x,\mathcal{O})$ is the distance from $\mathcal{O}$ to x. Let $\mu = \mathcal{E}[D]$. Then there exists a random variable $W \in (0,\infty)$ such that

$$W_k := \mathrm{e}^{-k\vartheta} Z_k = \mu^{-k} Z_k \to W \qquad \mathfrak{P}\text{-a.s. as } k \to \infty. \tag{4.14}$$

It is shown in [4, Theorem 5] that

$$\exists\, C < \infty, c > 0\colon \quad \mathfrak{P}(|W_k - W| \geq \varepsilon) \leq C\mathrm{e}^{-c\,\varepsilon^{2/3}\mu^{k/3}} \qquad \forall\, \varepsilon > 0,\ k \in \mathbb{N}. \tag{4.15}$$

In addition, it is shown in [6, Theorems 2–3] that if D is bounded, then

$$-\log \mathfrak{P}(W \geq x) = x^{\gamma^+/(\gamma^+-1)}\,[L^+(x) + o(1)], \qquad x \to \infty, \tag{4.16}$$

$$-\log \mathfrak{P}(W \leq x) = x^{-\gamma^-/(1-\gamma^-)}\,[L^-(x) + o(1)], \qquad x \downarrow 0, \tag{4.17}$$

where $\gamma^+ \in (1,\infty)$ and $\gamma^- \in (0,1)$ are the unique solutions of the equations

$$\mu^{\gamma^+} = d_{\max}, \qquad \mu^{\gamma^-} = d_{\min}, \tag{4.18}$$

with $L^+, L^-\colon (0,\infty) \to (0,\infty)$ real-analytic functions that are multiplicatively periodic with period μ^{γ^+-1}, respectively, $\mu^{1-\gamma^-}$. Note that Assumption 4.B(1) guarantees that $\gamma^- \neq 1$.

The tail behaviour in (4.16) requires that $d_{\max} < \infty$. In our setting we have $d_{\max} = \infty$, which corresponds to $\gamma^+ = \infty$, and so we expect exponential tail behaviour. The following lemma provides a rough bound.

Lemma 4.2.1. *[Exponential tail for generation sizes] If there exists an $a > 0$ such that $\mathcal{E}[\mathrm{e}^{aD}] < \infty$, then there exists an $a_* > 0$ such that $\mathfrak{E}[\mathrm{e}^{a_* W}] < \infty$.*

Proof. First note that if there exists an $a > 0$ such that $\mathcal{E}[\mathrm{e}^{aD}] < \infty$, then there exist $b > 0$ large and $c > 0$ small such that

$$\varphi(a) := \mathcal{E}[\mathrm{e}^{aD}] \leq \mathrm{e}^{\mu a + b a^2} \qquad \forall\, 0 < a < c. \tag{4.19}$$

Hence

$$\mathfrak{E}[\mathrm{e}^{aZ_{n+1}}] = \mathfrak{E}[\varphi(a)^{Z_n}] \leq \mathfrak{E}[\mathrm{e}^{(\mu a + b a^2) Z_n}] \tag{4.20}$$

and consequently, because $\mu > 1$,

$$\mathfrak{E}\left[\mathrm{e}^{aW_{n+1}}\right] \leq \mathfrak{E}\left[\mathrm{e}^{(a+ba^2\mu^{-(n+2)})W_n}\right] \leq \mathfrak{E}\left[\mathrm{e}^{a\exp(bc\mu^{-(n+2)})W_n}\right]. \tag{4.21}$$

Put $a_n := c\exp(-bc\sum_{k=0}^{n-1}\mu^{-(k+2)})$, which satisfies $0 < a_n \leq c$. From the last inequality in (4.21) it follows that

$$\mathfrak{E}\left[\mathrm{e}^{a_{n+1}W_{n+1}}\right] \leq \mathfrak{E}\left[\mathrm{e}^{a_n W_n}\right]. \tag{4.22}$$

Since $n \mapsto a_n$ is decreasing with $\lim_{n\to\infty} a_n = a_* > 0$, Fatou's lemma gives

$$\mathfrak{E}\left[\mathrm{e}^{a_* W}\right] \leq \mathfrak{E}\left[\mathrm{e}^{a_0 W_0}\right]. \tag{4.23}$$

Because $\mathcal{E}[\mathrm{e}^{a_0 W_0}] = \mathrm{e}^{a_0} < \infty$, we get the claim. □

The following lemma says that $\mathfrak{P}$-a.s. a ball of radius R_r centred anywhere in $B_r(\mathcal{O})$ has volume $\mathrm{e}^{\vartheta R_r + o(R_r)}$ as $r \to \infty$, provided R_r is large compared to $\log r$.

Lemma 4.2.2. *[Volumes of large balls] Subject to Assumption* 4.B(1)*, if there exists an $a > 0$ such that $\mathcal{E}[\mathrm{e}^{aD}] < \infty$, then for any R_r satisfying $\lim_{r\to\infty} R_r/\log r = \infty$,*

$$\liminf_{r\to\infty} \frac{1}{R_r} \log\Big(\inf_{x\in B_r(\mathcal{O})} |B_{R_r}(x)|\Big) = \limsup_{r\to\infty} \frac{1}{R_r} \log\Big(\sup_{x\in B_r(\mathcal{O})} |B_{R_r}(x)|\Big) = \vartheta \qquad \mathfrak{P}\text{–a.s.} \tag{4.24}$$

Proof. For $y \in \mathcal{GW}$ that lies k generations below $\mathcal{O}$, let $y[-i]$, $0 \leq i \leq k$ be the vertex that lies i generations above y. Define the *lower ball* of radius r around y as

$$B_r^{\downarrow}(y) := \{x \in V\colon\ \exists\, 0 \leq i \leq r \text{ with } x[-i] = y\}. \tag{4.25}$$

Note that $B_r^{\downarrow}(\mathcal{O}) = B_r(\mathcal{O})$.

We first prove the claim for lower balls. Afterwards we use a sandwich argument to get the claim for balls.

Let $\mathcal{Z}_k$ denote the vertices in the k-th generation. To get the upper bound, pick $\delta > 0$ and estimate

$$\begin{aligned}
&\mathfrak{P}\Big(\sup_{x\in B_r(\mathcal{O})} |B^{\downarrow}_{R_r}(x)| \geq \mathrm{e}^{(1+\delta)\vartheta R_r}\Big) \leq \sum_{k=0}^{r} \mathfrak{P}\Big(\sup_{x\in \mathcal{Z}_k} |B^{\downarrow}_{R_r}(x)| \geq \mathrm{e}^{(1+\delta)\vartheta R_r}\Big) \\
&\quad = \sum_{k=0}^{r}\sum_{l\in\mathbb{N}} \mathfrak{P}\Big(\sup_{x\in \mathcal{Z}_k} |B^{\downarrow}_{R_r}(x)| \geq \mathrm{e}^{(1+\delta)\vartheta R_r} \;\Big|\; Z_k = l\Big)\mathfrak{P}(Z_k = l) \\
&\quad \leq \sum_{k=0}^{r}\sum_{l\in\mathbb{N}} l\; \mathfrak{P}\Big(|B^{\downarrow}_{R_r}(\mathcal{O})| \geq \mathrm{e}^{(1+\delta)\vartheta R_r}\Big)\mathfrak{P}(Z_k = l) \\
&\quad = \mathfrak{P}\Big(|B^{\downarrow}_{R_r}(\mathcal{O})| \geq \mathrm{e}^{(1+\delta)\vartheta R_r}\Big) \sum_{k=0}^{r} \mathfrak{E}(Z_k).
\end{aligned} \tag{4.26}$$

By (4.7), $\sum_{k=0}^{r} \mathfrak{E}(Z_k) = \frac{\mathrm{e}^{\vartheta(r+1)}-1}{\mathrm{e}^{\vartheta}-1} = O(\mathrm{e}^{\vartheta r})$, and so in order to be able to apply the Borel-Cantelli lemma, it suffices to show that the probability in the last line decays faster than exponentially in r for any $\delta > 0$. To that end, estimate

$$\begin{aligned}
&\mathfrak{P}\Big(|B^{\downarrow}_{R_r}(\mathcal{O})| \geq \mathrm{e}^{(1+\delta)\vartheta R_r}\Big) = \mathfrak{P}\Big(\sum_{k=0}^{R_r} Z_k \geq \mathrm{e}^{(1+\delta)\vartheta R_r}\Big) \\
&\quad = \mathfrak{P}\Big(\sum_{k=0}^{R_r} W_k \geq \mathrm{e}^{\delta\vartheta R_r}\mathrm{e}^{\vartheta(R_r-k)}\Big) \leq \sum_{k=0}^{R_r} \mathfrak{P}\Big(W_k \geq \frac{1}{R_r+1}\mathrm{e}^{\delta\vartheta R_r}\mathrm{e}^{\vartheta(R_r-k)}\Big) \\
&\quad = \sum_{k=0}^{R_r} \mathfrak{P}\Big(W + (W_k - W) \geq \frac{1}{R_r+1}\mathrm{e}^{\delta\vartheta R_r}\mathrm{e}^{\vartheta(R_r-k)}\Big) \\
&\quad \leq \sum_{k=0}^{R_r} \mathfrak{P}\Big(W \geq \frac{1}{2(R_r+1)}\mathrm{e}^{\delta\vartheta R_r}\mathrm{e}^{\vartheta(R_r-k)}\Big) \\
&\qquad\qquad + \sum_{k=0}^{R_r} \mathfrak{P}\Big(|W_k - W| \geq \frac{1}{2(R_r+1)}\mathrm{e}^{\delta\vartheta R_r}\mathrm{e}^{\vartheta(R_r-k)}\Big) \\
&\quad \leq \mathfrak{E}[\mathrm{e}^{a_* W}] \sum_{k=0}^{R_r} \exp\Big(-a_* \frac{1}{2(R_r+1)}\mathrm{e}^{\delta\vartheta R_r}\mathrm{e}^{\vartheta(R_r-k)}\Big) \\
&\qquad\qquad + \sum_{k=0}^{R_r} C\exp\Big(-c\Big[\frac{1}{2(R_r+1)}\,\mathrm{e}^{\delta\vartheta R_r}\,\mathrm{e}^{\vartheta(R_r-k)}\Big]^{2/3}(\mathrm{e}^{\vartheta})^{k/3}\Big) \\
&\quad \leq \mathfrak{E}[\mathrm{e}^{a_* W}](R_r+1)\exp\Big(-a_* \frac{1}{2(R_r+1)}\mathrm{e}^{\delta\vartheta R_r}\Big) \\
&\qquad\qquad + C(R_r+1)\exp\Big(-c\Big[\frac{1}{2(R_r+1)}\,\mathrm{e}^{\delta\vartheta R_r}\Big]^{2/3}\Big),
\end{aligned} \tag{4.27}$$

where we use (4.15) with $\mu = \mathrm{e}^{\vartheta}$. This produces the desired estimate.

To get the lower bound, pick $0 < \delta < 1$ and estimate

$$\begin{aligned}
&\mathfrak{P}\Big(\inf_{x\in B_r(\mathcal{O})} |B^{\downarrow}_{R_r}(x)| \le \mathrm{e}^{(1-\delta)\vartheta R_r}\Big) \le \sum_{k=0}^{r} \mathfrak{P}\Big(\inf_{x\in\mathcal{Z}_k} |B^{\downarrow}_{R_r}(x)| \le \mathrm{e}^{(1-\delta)\vartheta R_r}\Big)\\
&\quad= \sum_{k=0}^{r}\sum_{l\in\mathbb{N}} \mathfrak{P}\Big(\inf_{x\in\mathcal{Z}_k} |B^{\downarrow}_{R_r}(x)| \le \mathrm{e}^{(1-\delta)\vartheta R_r} \;\Big|\; Z_k = l\Big)\mathfrak{P}(Z_k = l)\\
&\quad\le \sum_{k=0}^{r}\sum_{l\in\mathbb{N}} l\,\mathfrak{P}\Big(|B^{\downarrow}_{R_r}(\mathcal{O})| \le \mathrm{e}^{(1-\delta)\vartheta R_r}\Big)\mathfrak{P}(Z_k = l)\\
&\quad= \mathfrak{P}\Big(|B^{\downarrow}_{R_r}(\mathcal{O})| \le \mathrm{e}^{(1-\delta)\vartheta R_r}\Big)\sum_{k=0}^{r}\mathfrak{E}(Z_k).
\end{aligned} \tag{4.28}$$

It again suffices to show that the probability in the last line decays faster than exponentially in r for any $\delta > 0$. To that end, estimate

$$\begin{aligned}
&\mathfrak{P}\Big(|B^{\downarrow}_{R_r}(\mathcal{O})| \le \mathrm{e}^{(1-\delta)\vartheta R_r}\Big) = \mathfrak{P}\Big(\mathrm{e}^{-\vartheta R_r}\sum_{k=0}^{R_r} Z_k \le \mathrm{e}^{-\delta\vartheta R_r}\Big)\\
&\le \mathfrak{P}\Big(W_{R_r} \le \mathrm{e}^{-\delta\vartheta R_r}\Big) \le \mathfrak{P}(W \le 2\,\mathrm{e}^{-\delta\vartheta R_r}) + \mathfrak{P}(W - W_{R_r} \ge \mathrm{e}^{-\delta\vartheta R_r})\\
&\le \exp\Big(-c^{-}(2\mathrm{e}^{\delta\vartheta R_r})^{\frac{\gamma^-}{1-\gamma^-}}[1+o(1)]\Big) + C\exp\Big(-c\,[\mathrm{e}^{-\frac{2}{3}\delta\vartheta}(\mathrm{e}^{\vartheta})^{\frac{1}{3}}]^{R_r}\Big),
\end{aligned} \tag{4.29}$$

where we use (4.17), (4.15) with $\mu = \mathrm{e}^{\vartheta}$, and put $c^- := \inf L^- \in (0,\infty)$. For δ small enough this produces the desired estimate. This completes the proof of (4.24) for lower balls.

To get the claim for balls, we observe that

$$B^{\downarrow}_r(x) \subseteq B_r(x) \subseteq \bigcup_{k=0}^{r} B^{\downarrow}_r(x[-k]), \tag{4.30}$$

and therefore

$$|B^{\downarrow}_r(x)| \le |B_r(x)| \le \sum_{k=0}^{r} |B^{\downarrow}_r(x[-k])|. \tag{4.31}$$

It follows from (4.31) that

$$\inf_{x\in B_r(\mathcal{O})} |B^{\downarrow}_r(x)| \le \inf_{x\in B_r(\mathcal{O})} |B_r(x)| \le \sup_{x\in B_r(\mathcal{O})} |B_r(x)| \le (r+1)\sup_{x\in B_r(\mathcal{O})} |B^{\downarrow}_r(x)|. \tag{4.32}$$

Hence we get (4.24). □

§4.2.2 Degrees

Write D_x to denote the degree of vertex x. The following lemma implies that, $\mathfrak{P}$-a.s. and for $r \to \infty$, D_x is bounded by a vanishing power of $\log r$ for all $x \in B_{2r}(\mathcal{O})$.

Lemma 4.2.3. *[Maximal degree in a ball around the root]*
(a) Subject to Assumption 4.B(2)*, for every* $\delta > 0$*,*

$$\sum_{r\in\mathbb{N}} \mathfrak{P}\big(\exists\, x \in B_{2r}(\mathcal{O})\colon\ D_x > \delta r\big) < \infty. \tag{4.33}$$

(b) Subject to Assumption 4.C*, there exists a function* $\delta_r\colon\ (0,\infty) \to (0,\infty)$ *satisfying* $\lim_{r\to\infty} \delta_r = 0$ *and* $\lim_{r\to\infty} r\frac{\mathrm{d}}{\mathrm{d}r}\delta_r = 0$ *such that*

$$\sum_{r\in\mathbb{N}} \mathfrak{P}\big(\exists\, x \in B_{2r}(\mathcal{O})\colon\ D_x > (\log r)^{\delta_r}\big) < \infty. \tag{4.34}$$

Proof. (a) Estimate

$$\begin{aligned}
&\mathfrak{P}\big(\exists\, x \in B^{\downarrow}_{2r}(\mathcal{O})\colon\ D_x > \delta r\big) \le \sum_{k=0}^{2r} \mathfrak{P}\big(\exists\, x \in \mathcal{Z}_k\colon\ D_x > \delta r\big)\\
&= \sum_{k=0}^{2r}\sum_{l\in\mathbb{N}} \mathfrak{P}\big(\exists\, x \in \mathcal{Z}_k\colon\ D_x > \delta r \mid Z_k = l\big)\,\mathfrak{P}(Z_k = l) \qquad (4.35)\\
&\le \mathcal{P}(D > \delta r)\sum_{k=0}^{2r}\sum_{l\in\mathbb{N}} l\,\mathfrak{P}\big(Z_k = l\big) = \mathcal{P}(D > \delta r)\sum_{k=0}^{2r} \mathfrak{E}(Z_k).
\end{aligned}$$

Since $\sum_{k=0}^{2r} \mathfrak{E}(Z_k) = \frac{\mathrm{e}^{(2r+1)\vartheta}-1}{\mathrm{e}^{\vartheta}-1} = O(\mathrm{e}^{2r\vartheta})$, it suffices to show that $\mathcal{P}(D > \delta r) = O(\mathrm{e}^{-cr})$ for some $c > 2\vartheta$. Since $\mathcal{P}(D > \delta r) \le \mathrm{e}^{-a\delta r}\mathcal{E}(\mathrm{e}^{aD})$, the latter is immediate from Assumption 4.B(2) when we choose $a > 2\vartheta/\delta$.
(b) The only change is that in the last line $\mathcal{P}(D > \delta r)$ must be replaced by $\mathcal{P}(D > (\log r)^{\delta_r})$. To see that the latter is $O(\mathrm{e}^{-cr})$ for some $c > 2\vartheta$, we use the tail condition in (4.8) with $\delta_r = f(s)$ and $s = \log r$. □

§4.2.3 Tree animals

For $n \in \mathbb{N}_0$ and $x \in B_r(\mathcal{O})$, let

$$\mathcal{A}_n(x) = \{\Lambda \subset B_n(x)\colon\ \Lambda \text{ is connected}, \Lambda \ni x, |\Lambda| = n+1\} \tag{4.36}$$

be the set of *tree animals* of size $n+1$ that contain x. Put $a_n(x) = |\mathcal{A}_n(x)|$.

Lemma 4.2.4. *[Number of tree animals] Subject to Assumption* 4.B(2)*,* $\mathfrak{P}$*-a.s. there exists an* $r_0 \in \mathbb{N}$ *such that* $a_n(x) \le r^n$ *for all* $r \ge r_0$*,* $x \in B_r(\mathcal{O})$ *and* $0 \le n \le r$*.*

Proof. For $n \in \mathbb{N}_0$ and $x \in B^{\downarrow}_r(\mathcal{O})$, let

$$\mathcal{A}^{\downarrow}_n(x) = \{\Lambda \subset B^{\downarrow}_n(x)\colon\ \Lambda \text{ is connected}, \Lambda \ni x, |\Lambda| = n+1\} \tag{4.37}$$

be the set of *lower tree animals* of size $n+1$ that contain x. Put $a^{\downarrow}_n(x) = |\mathcal{A}^{\downarrow}_n(x)|$.

We first prove the claim for lower tree animals. Afterwards we use a sandwich argument to get the claim for tree animals.

Fix $\delta > 0$. By Lemma 4.2.3(a) and the Borel-Cantelli lemma, $\mathfrak{P}$-a.s. there exists an $r_0 = r_0(\delta) \in \mathbb{N}$ such that $D_x \leq \delta r$ for all $x \in B^{\downarrow}_{2r}(\mathcal{O})$. Any lower tree animal of size $n+1$ containing a vertex in $B^{\downarrow}_r(\mathcal{O})$ is contained in $B^{\downarrow}_{r+n}(\mathcal{O})$. Any lower tree animal of size $n+1$ can be created by adding a vertex to the outer boundary of a lower tree animal of size n. This leads to the recursive inequality

$$a^{\downarrow}_n(x) \leq (\delta r) a^{\downarrow}_{n-1}(x) \qquad \forall\, x \in B^{\downarrow}_r(\mathcal{O}), \quad 1 \leq n \leq r. \tag{4.38}$$

Since $a^{\downarrow}_0(x) = 1$, it follows that

$$a^{\downarrow}_n(x) \leq (\delta r)^n \qquad \forall\, x \in B^{\downarrow}_r(\mathcal{O}),\ 0 \leq n \leq r. \tag{4.39}$$

Pick $\delta \leq 1$ to get the claim for lower tree animals.

To get the claim for tree animals, pick $\delta \leq \frac{1}{(n+1)}$ and note that $a_n(x) \leq \sum_{k=0}^{n} a^{\downarrow}_n(x[-k])$ (compare with (4.31)), and so $a_n(x) \leq r^n$ for all $x \in B_r(\mathcal{O})$ and all $0 \leq n \leq r$. □

§4.3 Preliminaries

In this section we extend the lemmas in [12, Section 2]. Section 4.3.1 identifies the maximum size of the islands where the potential is suitably high. Section 4.3.2 estimates the contribution to the total mass in (4.4) by the random walk until it exits a subset of $\mathcal{GW}$. Section 4.3.3 gives a bound on the principal eigenvalue associated with the islands. Section 4.3.5 estimates the number of locations where the potential is intermediate.

Abbreviate $L_r = L_r(\mathcal{GW}) = |B_r(\mathcal{O})|$ and put

$$S_r := (\log r)^{\alpha}, \qquad \alpha \in (0,1). \tag{4.40}$$

§4.3.1 Maximum size of the islands

For every $r \in \mathbb{N}$ there is a unique a_r such that

$$\mathsf{P}(\xi(0) > a_r) = \frac{1}{r}. \tag{4.41}$$

By Assumption 4.A, for r large enough

$$a_r = \varrho \log\log r. \tag{4.42}$$

For $r \in \mathbb{N}$ and $A > 0$, let

$$\Pi_{r,A} = \Pi_{r,A}(\xi) := \{z \in B_r(\mathcal{O})\colon\ \xi(z) > a_{L_r} - 2A\} \tag{4.43}$$

be the set of vertices in $B_r(\mathcal{O})$ where the potential is close to maximal,

$$D_{r,A} = D_{r,A}(\xi) := \{z \in B_r(\mathcal{O})\colon\ \mathrm{dist}(z, \Pi_{r,A}) \leq S_r\} \tag{4.44}$$

be the S_r-neighbourhood of $\Pi_{r,A}$, and $\mathfrak{C}_{r,A}$ be the set of connected components of $D_{r,A}$ in $\mathcal{GW}$, which we think of as *islands*. For $M_A \in \mathbb{N}$, define the event

$$\mathcal{B}_{r,A} := \big\{\exists\, \mathcal{C} \in \mathfrak{C}_{r,A} \colon\ |\mathcal{C} \cap \Pi_{r,A}| > M_A\big\}. \tag{4.45}$$

Note that $\Pi_{r,A}, D_{r,A}, \mathcal{B}_{r,A}$ depend on $\mathcal{GW}$ and therefore are random.

Lemma 4.3.1. *[Maximum size of the islands] Subject to Assumptions* 4.A–4.B*, for every $A > 0$ there exists an $M_A \in \mathbb{N}$ such that*

$$\sum_{r\in\mathbb{N}} \mathsf{P}(\mathcal{B}_{r,A}) < \infty \qquad \mathfrak{P} - a.s. \tag{4.46}$$

Proof. We follow [8, Lemma 6.6]. By Assumption 4.A, for every $x \in V$ and r large enough,

$$\mathsf{P}(x \in \Pi_{r,A}) = \mathsf{P}(\xi(x) > a_{L_r} - 2A) = L_r^{-c_A} \tag{4.47}$$

with $c_A = e^{-2A/\varrho}$. By Lemma 4.2.2, $\mathfrak{P}$-a.s. for every $y \in B_r(\mathcal{O})$ and r large enough,

$$|B_{S_r}(y)| \le |B_{o(r)}(\mathcal{O})| = L_{o(r)} = L_r^{o(1)}, \tag{4.48}$$

where we use that $S_r = o(\log r) = o(r)$, and hence for every $m \in \mathbb{N}$,

$$\mathsf{P}(|B_{S_r}(y) \cap \Pi_{r,A}| \ge m) \le \binom{|B_{S_r}(y)|}{m} L_r^{-c_A m} \le (|B_{S_r}(y)| L_r^{-c_A})^m \le L_r^{-c_A m[1+o(1)]}. \tag{4.49}$$

Consequently, $\mathfrak{P}$-a.s.

$$\begin{aligned}\mathsf{P}(\exists\, \mathcal{C} \in \mathfrak{C}_{r,A} \colon\ |\mathcal{C} \cap \Pi_{r,A}| \ge m) &\le \mathsf{P}(\exists\, y \in B_r(\mathcal{O}) \colon\ |B_{S_r}(y) \cap \Pi_{r,A}| \ge m) \\ &\le |B_r(\mathcal{O})| L_r = L_r^{(1-c_A m)[1+o(1)]}.\end{aligned} \tag{4.50}$$

By choosing $m > 1/c_A$, we see that the above probability becomes summable in r, and so we have proved the claim with $M_A = \lceil 1/c_A \rceil$. □

Lemma 4.3.1 implies that $(\mathsf{P} \times \mathfrak{P})$-a.s. $\mathcal{B}_{r,A}$ does not occur eventually as $r \to \infty$. Note that $\mathfrak{P}$-a.s. on the event $[\mathcal{B}_{r,A}]^c$,

$$\forall\, \mathcal{C} \in \mathfrak{C}_{r,A} \colon\ |\mathcal{C} \cap \Pi_{r,A}| \le M_A,\ \mathrm{diam}_{\mathcal{GW}}(\mathcal{C}) \le 2M_A S_r,\ |\mathcal{C}| \le \mathrm{e}^{2\vartheta M_A S_r}, \tag{4.51}$$

where the last inequality follows from Lemma 4.2.2.

§4.3.2 Mass up to an exit time

Lemma 4.3.2. *[Mass up to an exit time] Subject to Assumption* 4.B(2)*, $\mathfrak{P}$-a.s. for any $\delta > 0$, $r \ge r_0$, $y \in \Lambda \subset B_r(\mathcal{O})$, $\xi \in [0,\infty)^V$ and $\gamma > \lambda_\Lambda = \lambda_\Lambda(\xi, \mathcal{GW})$,*

$$\mathbb{E}_y\left[\mathrm{e}^{\int_0^{\tau_{\Lambda^c}} (\xi(X_s)-\gamma)\,\mathrm{d}s}\right] \le 1 + \frac{(\delta r)\,|\Lambda|}{\gamma - \lambda_\Lambda}. \tag{4.52}$$

Proof. We follow the proof of [25, Lemma 2.18] and [19, Lemma 4.2]. Define

$$u(x) := \mathbb{E}_x \left[\mathrm{e}^{\int_0^{\tau_{\Lambda^c}} (\xi(X_s)-\gamma)\,\mathrm{d}s} \right]. \tag{4.53}$$

This is the solution to the boundary value problem

$$\begin{aligned} (\Delta + \xi - \gamma)u &= 0 \quad \text{on } \Lambda \\ u &= 1 \quad \text{on } \Lambda^c. \end{aligned} \tag{4.54}$$

Via the substitution $u =: 1 + v$, this turns into

$$\begin{aligned} (\Delta + \xi - \gamma)v &= \gamma - \xi \quad \text{on } \Lambda \\ v &= 0 \qquad \text{on } \Lambda^c. \end{aligned} \tag{4.55}$$

It is readily checked that for $\gamma > \lambda_\Lambda$ the solution exists and is given by

$$v = \mathcal{R}_\gamma(\xi - \gamma), \tag{4.56}$$

where $\mathcal{R}_\gamma$ denotes the resolvent of $\Delta + \xi$ in $\ell^2(\Lambda)$ with Dirichlet boundary condition. Hence

$$v(x) \le (\delta r)\,(\mathcal{R}_\gamma \mathbb{1})(x) \le (\delta r)\,\langle \mathcal{R}_\gamma \mathbb{1}, \mathbb{1}\rangle_\Lambda \le \frac{(\delta r)\,|\Lambda|}{\gamma - \lambda_\Lambda}, \quad x \in \Lambda, \tag{4.57}$$

where $\mathbb{1}$ denotes the constant function equal to 1, and $\langle \cdot, \cdot \rangle_\Lambda$ denotes the inner product in $\ell^2(\Lambda)$. To get the first inequality, we combine Lemma 4.2.3(a) with the lower bound in (B.2) from Lemma B.1.1, to get $\xi - \gamma \le \lambda_\Lambda + \delta r - \gamma \le \delta r$ on Λ. The positivity of the resolvent gives

$$0 \le [\mathcal{R}_\gamma(\delta r - (\xi - \gamma))](x) = (\delta r)\,[\mathcal{R}_\gamma \mathbb{1}](x) - [\mathcal{R}_\gamma(\xi - \gamma)](x). \tag{4.58}$$

To get the second inequality, we write

$$(\delta r)\,(\mathcal{R}_\gamma \mathbb{1})(x) \le (\delta r) \sum_{x \in \Lambda} (\mathcal{R}_\gamma \mathbb{1})(x) = (\delta r) \sum_{x \in \Lambda} (\mathcal{R}_\gamma \mathbb{1})(x) \mathbb{1}(x) = (\delta r)\,\langle \mathcal{R}_\gamma \mathbb{1}, \mathbb{1}\rangle_\Lambda. \tag{4.59}$$

To get the third inequality, we use the Fourier expansion of the resolvent with respect to the orthonormal basis of eigenfunctions of $\Delta + \xi$ in $\ell^2(\Lambda)$. □

§4.3.3 Principal eigenvalue of the islands

The following lemma provides a spectral bound.

Lemma 4.3.3. *[Principal eigenvalues of the islands] Subject to Assumptions* 4.A *and* 4.B(2)*, for any* $\varepsilon > 0$*,* $(\mathsf{P} \times \mathfrak{P})$*-a.s. eventually as* $r \to \infty$*,*

$$\textit{all } \mathcal{C} \in \mathfrak{C}_{r,A} \textit{ satisfy: } \lambda_{\mathcal{C}}(\xi; \mathcal{GW}) \le a_{L_r} - \widehat{\chi}_{\mathcal{C}}(\mathcal{GW}) + \varepsilon. \tag{4.60}$$

Proof. We follow the proof of [12, Lemma 2.6]. For $\varepsilon > 0$ and $A > 0$, define the event

$$\bar{\mathcal{B}}_{r,A} := \left\{ \begin{smallmatrix} \text{there exists a connected subset } \Lambda \subset V \text{ with } \Lambda \cap B_r(\mathcal{O}) \neq \emptyset, \\ |\Lambda| \leq \mathrm{e}^{2\vartheta M_A S_r},\, \lambda_\Lambda(\xi; \mathcal{GW}) > a_{L_r} - \widehat{\chi}_\Lambda(\mathcal{GW}) + \varepsilon \end{smallmatrix} \right\} \tag{4.61}$$

with M_A as in Lemma 4.3.1. Note that, by (4.A), $\mathrm{e}^{\xi(x)/\varrho}$ is stochastically dominated by $Z \vee N$, where Z is an Exp(1) random variable and $N > 0$ is a constant. Thus, for any $\Lambda \subset V$, using [12, Eq. (2.17)], putting $\gamma = \sqrt{\mathrm{e}^{\varepsilon/\varrho}} > 1$ and applying Markov's inequality, we may estimate

$$\begin{aligned} &\mathsf{P}\left(\lambda_\Lambda(\xi; \mathcal{GW}) > a_{L_r} - \widehat{\chi}_\Lambda(\mathcal{GW}) + \varepsilon\right) \leq \mathsf{P}\left(\mathcal{L}_\Lambda(\xi - a_{L_r} - \varepsilon) > 1\right) \\ &= \mathsf{P}\left(\gamma^{-1}\mathcal{L}_\Lambda(\xi) > \gamma \log L_r\right) \leq \mathrm{e}^{-\gamma \log L_r} \mathrm{E}[\mathrm{e}^{\gamma^{-1}\mathcal{L}_\Lambda(\xi)}] \leq \mathrm{e}^{-\gamma \log L_r} K_\gamma^{|\Lambda|} \end{aligned} \tag{4.62}$$

with $K_\gamma = \mathrm{E}[\mathrm{e}^{\gamma^{-1}(Z \vee N)}] \in (1, \infty)$. Next, by Lemma 4.2.4, for any $x \in B_r(\mathcal{O})$ and $1 \leq n \leq r$, the number of connected subsets $\Lambda \subset V$ with $x \in \Lambda$ and $|\Lambda| = n + 1$ is $\mathfrak{P}$-a.s. at most $(n+1)r^n \leq \mathrm{e}^{2n \log r}$ for $r \geq r_0$. Noting that $\mathrm{e}^{S_r} \leq r$, we use a union bound and that by Lemma 4.2.2 $\log L_r = \vartheta r + o(r)$ as $r \to \infty$ $\mathfrak{P}$-a.s., to estimate for r large enough,

$$\begin{aligned} \mathsf{P}(\bar{\mathcal{B}}_{r,A}) &\leq \mathrm{e}^{-(\gamma-1)\log L_r} \sum_{n=1}^{\lfloor \mathrm{e}^{2\vartheta M_A S_r} \rfloor} \mathrm{e}^{2n \log r} K_\gamma^n \\ &\leq \mathrm{e}^{2\vartheta M_A S_r} \exp\left\{-\vartheta(\gamma-1)r + o(r) + (2\log r + \log K_\gamma)\, \mathrm{e}^{2\vartheta M_A S_r}\right\} \\ &= r^{o(1)} \exp\left\{-\vartheta(\gamma-1)r + o(r) + (\log r)\, r^{o(1)}\right\} \leq \mathrm{e}^{-\frac{1}{2}\vartheta(\gamma-1)r}. \end{aligned} \tag{4.63}$$

Via the Borel-Cantelli lemma this implies that $(\mathsf{P} \times \mathfrak{P})$-a.s. $\bar{\mathcal{B}}_{r,A}$ does not occur eventually as $r \to \infty$. The proof is completed by invoking Lemma 4.3.1. □

Corollary 4.3.4. *[Uniform bound on principal eigenvalue of the islands] Subject to Assumptions* 4.A–4.B*, for ϑ as in* (4.7)*, and any $\varepsilon > 0$, $(\mathsf{P} \times \mathfrak{P})$-a.s. eventually as $r \to \infty$,*

$$\max_{\mathcal{C} \in \mathfrak{C}_{r,A}} \lambda^1_{\mathcal{C}}(\xi; G) \leq a_{L_r} - \widetilde{\chi}(\varrho) + \varepsilon. \tag{4.64}$$

Proof. See [12, Corollary 2.8]. The proof carries over verbatim because the degrees play no role. □

§4.3.4 Maximum of the potential

The next lemma shows that a_{L_r} is the leading order of the maximum of ξ in $B_r(\mathcal{O})$.

Lemma 4.3.5. *[Maximum of the potential] Subject to Assumptions* 4.A–4.B*, for any $\vartheta > 0$, $(\mathsf{P} \times \mathfrak{P})$-a.s. eventually as $r \to \infty$,*

$$\left| \max_{x \in B_r(\mathcal{O})} \xi(x) - a_{L_r} \right| \leq \frac{2\varrho \log r}{\vartheta r}. \tag{4.65}$$

Proof. See [12, Lemma 2.4]. The proof carries over verbatim and uses Lemma 4.2.2. □

§4.3.5 Number of intermediate peaks of the potential

We recall the following Chernoff bound for a binomial random variable with parameters n and p (see e.g. [9, Lemma 5.9]):

$$P\left(\mathrm{Bin}(n,p) \geq u\right) \leq \mathrm{e}^{-u[\log(\frac{u}{np})-1]}, \qquad u > 0. \tag{4.66}$$

Lemma 4.3.6. *[Number of intermediate peaks of the potential] Subject to Assumptions* 4.A *and* 4.B(2)*, for any $\beta \in (0,1)$ and $\varepsilon \in (0, \frac{1}{2}\beta)$ the following holds. For a self-avoiding path π in $\mathcal{GW}$, set*

$$N_\pi = N_\pi(\xi) := |\{z \in \mathrm{supp}(\pi)\colon\ \xi(z) > (1-\varepsilon)a_{L_r}\}|. \tag{4.67}$$

Define the event

$$\mathcal{B}_r := \left\{ \begin{smallmatrix} \text{there exists a self-avoiding path } \pi \text{ in } \mathcal{GW} \text{ with} \\ \mathrm{supp}(\pi)\cap B_r \neq \emptyset,\, |\,\mathrm{supp}(\pi)| \geq (\log L_r)^\beta \text{ and } N_\pi > \frac{|\,\mathrm{supp}(\pi)|}{(\log L_r)^\varepsilon} \end{smallmatrix} \right\}. \tag{4.68}$$

Then

$$\sum_{r\in\mathbb{N}_0} \mathsf{P}(\mathcal{B}_r) < \infty \qquad \mathfrak{P}-a.s. \tag{4.69}$$

Proof. We follow the proof of [12, Lemma 2.9]. Fix $\beta \in (0,1)$ and $\varepsilon \in (0, \frac{1}{2}\beta)$. (4.5) implies

$$p_r := \mathsf{P}(\xi(0) > (1-\varepsilon)a_{L_r}) = \exp\left\{-(\log L_r)^{1-\varepsilon}\right\}. \tag{4.70}$$

Fix $x \in B_r(\mathcal{O})$ and $k \in \mathbb{N}$. The number of self-avoiding paths π in $B_r(\mathcal{O})$ with $|\,\mathrm{supp}(\pi)| = k$ and $\pi_0 = x$ is at most $\mathrm{e}^{k\log r}$ by Lemma 4.2.4 for r sufficiently large. For such a π, the random variable N_π has a $\mathrm{Bin}(k, p_r)$-distribution. Using (4.66), we obtain

$$\begin{aligned} &\mathsf{P}\Big(\exists \text{ self-avoiding } \pi \text{ with } |\,\mathrm{supp}(\pi)| = k, \pi_0 = x \text{ and } N_\pi > k/(\log L_r)^\varepsilon\Big) \\ &\qquad \leq \exp\Big\{ -k\Big((\log L_r)^{1-2\varepsilon} - \log r - \frac{1+\varepsilon\log\log L_r}{(\log L_r)^\varepsilon}\Big)\Big\}. \end{aligned} \tag{4.71}$$

By the definition of ε, together with the fact that $L_r > r$ and $x \mapsto (\log\log x)/(\log x)^\varepsilon$ is eventually decreasing, the expression in parentheses above is at least $\frac{1}{2}(\log L_r)^{1-2\varepsilon}$. Summing over $k \geq (\log L_r)^\beta$ and $x \in B_r(\mathcal{O})$, we get $\mathfrak{P}-a.s.$

$$\mathsf{P}\left(\mathcal{B}_r\right) \leq 2L_r \exp\Big\{ -\tfrac{1}{2}(\log L_r)^{1+\beta-2\varepsilon}\Big\} \leq c_1 \exp\Big\{ -c_2(\log L_r)^{1+\delta}\Big\} \tag{4.72}$$

for some $c_1, c_2, \delta > 0$. Since $L_r > r$, (4.72) is summable in r. □

Lemma 4.3.6 implies that $(\mathsf{P}\times\mathfrak{P})$-a.s. for r large enough, all self-avoiding paths π in $\mathcal{GW}$ with $\mathrm{supp}(\pi)\cap B_r \neq \emptyset$ and $|\,\mathrm{supp}(\pi)| \geq (\log L_r)^\beta$ satisfy $N_\pi \leq \frac{|\,\mathrm{supp}(\pi)|}{(\log L_r)^\varepsilon}$.

Lemma 4.3.7. *[Number of high exceedances of the potential] Subject to Assumptions* 4.A *and* 4.B(2)*, for any $A > 0$ there is a $C \geq 1$ such that, for all $\delta \in (0,1)$, the following holds. For a self-avoiding path π in $\mathcal{GW}$, let*

$$N_\pi := |\{x \in \mathrm{supp}(\pi)\colon\ \xi(x) > a_{L_r} - 2A\}|. \tag{4.73}$$

Define the event

$$\mathcal{B}_r := \left\{ \begin{smallmatrix} \text{there exists a self-avoiding path } \pi \text{ in } G \text{ with} \\ \mathrm{supp}(\pi)\cap B_r \neq \emptyset,\, |\mathrm{supp}(\pi)| \geq C(\log L_r)^\delta \text{ and } N_\pi > \frac{|\mathrm{supp}(\pi)|}{(\log L_r)^\delta} \end{smallmatrix} \right\}. \tag{4.74}$$

Then $\sum_{r\in\mathbb{N}_0} \sup_{G\in\mathfrak{G}_r} \mathsf{P}(\mathcal{B}_r) < \infty$. In particular, $(\mathsf{P}\times\mathfrak{P})$-a.s. for r large enough, all self-avoiding paths π in $\mathcal{GW}$ with $\mathrm{supp}(\pi)\cap B_r \neq \emptyset$ and $|\mathrm{supp}(\pi)| \geq C(\log L_r)^\delta$ satisfy

$$N_\pi = |\{x \in \mathrm{supp}(\pi)\colon\ \xi(x) > a_{L_r} - 2A\}| \leq \frac{|\mathrm{supp}(\pi)|}{(\log L_r)^\delta}. \tag{4.75}$$

Proof. Proceed as for Lemma 4.3.6, noting that this time

$$p_r := \mathsf{P}\left(\xi(0) > a_{L_r} - 2A\right) = L_r^{-\epsilon} \tag{4.76}$$

where $\epsilon = \mathrm{e}^{-2A/\varrho}$, and taking $C > 2/\epsilon$. □

§4.4 Path expansions

In this section we extend [12, Section 3]. Section 4.4.1 proves three lemmas that concern the contribution to the total mass in (4.4) coming from various sets of paths. Section 4.4.2 proves a key proposition that controls the entropy associated with a key set of paths. The proof is based on the three lemmas in Section 4.4.1.

We need various sets of nearest-neighbour paths in $\mathcal{GW} = (V, E, \mathcal{O})$, defined in [12]. For $\ell \in \mathbb{N}_0$ and subsets $\Lambda, \Lambda' \subset V$, put

$$\begin{aligned} \mathscr{P}_\ell(\Lambda,\Lambda') &:= \left\{(\pi_0,\dots,\pi_\ell) \in V^{\ell+1}\colon \quad \begin{smallmatrix} \pi_0 \in \Lambda, \pi_\ell \in \Lambda', \\ \{\pi_i, \pi_{i-1}\} \in E\ \forall\, 1 \leq i \leq \ell \end{smallmatrix} \right\}, \\ \mathscr{P}(\Lambda,\Lambda') &:= \bigcup_{\ell\in\mathbb{N}_0} \mathscr{P}_\ell(\Lambda,\Lambda'), \end{aligned} \tag{4.77}$$

and set

$$\mathscr{P}_\ell := \mathscr{P}_\ell(V,V), \qquad \mathscr{P} := \mathscr{P}(V,V). \tag{4.78}$$

When Λ or Λ' consists of a single point, write x instead of $\{x\}$. For $\pi \in \mathscr{P}_\ell$, set $|\pi| := \ell$. Write $\mathrm{supp}(\pi) := \{\pi_0,\dots,\pi_{|\pi|}\}$ to denote the set of points visited by π.

Let $X = (X_t)_{t\geq 0}$ be the continuous-time random walk on G that jumps from $x \in V$ to any neighbour $y \sim x$ at rate 1. Denote by $(T_k)_{k\in\mathbb{N}_0}$ the sequence of jump times (with $T_0 := 0$). For $\ell \in \mathbb{N}_0$, let

$$\pi^\ell(X) := (X_0,\dots,X_{T_\ell}) \tag{4.79}$$

be the path in $\mathscr{P}_\ell$ consisting of the first ℓ steps of X. For $t \geq 0$, let

$$\pi(X_{[0,t]}) = \pi^{\ell_t}(X), \qquad \text{with } \ell_t \in \mathbb{N}_0 \text{ satisfying } T_{\ell_t} \leq t < T_{\ell_t+1}, \tag{4.80}$$

denote the path in $\mathscr{P}$ consisting of all the steps taken by X between times 0 and t.

Recall the definitions from Section 4.3.1. For $\pi \in \mathscr{P}$ and $A > 0$, define

$$\lambda_{r,A}(\pi) := \sup\big\{\lambda^1_{\mathcal{C}}(\xi; G)\colon\ \mathcal{C} \in \mathfrak{C}_{r,A},\, \mathrm{supp}(\pi) \cap \mathcal{C} \cap \Pi_{r,A} \neq \emptyset\big\}, \tag{4.81}$$

with the convention $\sup \emptyset = -\infty$. This is the largest principal eigenvalue among the components of $\mathfrak{C}_{r,A}$ in $\mathcal{GW}$ that have a point of high exceedance visited by the path π.

Lemma 4.4.1. *[Mass up to an exit time] Subject to Assumption* 4.C, $\mathfrak{P}$*-a.s. for any* $r \geq r_0$, $y \in \Lambda \subset B_r(\mathcal{O})$, $\xi \in [0,\infty)^V$ *and* $\gamma > \lambda_\Lambda = \lambda_\Lambda(\xi, \mathcal{GW})$,

$$\mathbb{E}_y\left[\mathrm{e}^{\int_0^{\tau_{\Lambda^c}} (\xi(X_s)-\gamma)\,\mathrm{d}s}\right] \leq 1 + \frac{(\log r)^{\delta r}\,|\Lambda|}{\gamma - \lambda_\Lambda}. \tag{4.82}$$

Proof. The proof is identical to that of Lemma 4.3.2, with δr replaced by $(\log r)^{\delta r}$ (recall Lemma 4.2.3). □

§4.4.1 Mass of the solution along excursions

Lemma 4.4.2. *[Path evaluation] For* $\ell \in \mathbb{N}_0$, $\pi \in \mathscr{P}_\ell$ *and* $\gamma > \max_{0\leq i<|\pi|}\{\xi(\pi_i) - D_{\pi_i}\}$,

$$\mathbb{E}_{\pi_0}\left[\mathrm{e}^{\int_0^{T_\ell} (\xi(X_s)-\gamma)\,\mathrm{d}s} \;\middle|\; \pi^\ell(X) = \pi\right] = \prod_{i=0}^{\ell-1} \frac{D_{\pi_i}}{\gamma - [\xi(\pi_i) - D_{\pi_i}]}. \tag{4.83}$$

Proof. The proof is identical to that of [12, Lemma 3.2]. The left-hand side of (4.83) can be evaluated by using the fact that T_ℓ is the sum of ℓ independent $\mathrm{Exp}(\deg(\pi_i))$ random variables that are independent of $\pi^\ell(X)$. The condition on γ ensures that all ℓ integrals are finite. □

For a path $\pi \in \mathscr{P}$ and $\varepsilon \in (0,1)$, we write

$$M^{r,\varepsilon}_\pi := \big|\big\{0 \leq i < |\pi|\colon\ \xi(\pi_i) \leq (1-\varepsilon)a_{L_r}\big\}\big|, \tag{4.84}$$

with the interpretation that $M^{r,\varepsilon}_\pi = 0$ if $|\pi| = 0$.

Lemma 4.4.3. *[Mass of excursions] Subject to Assumptions* 4.A–4.C, *for every* $A, \varepsilon > 0$, $(\mathsf{P}\times\mathfrak{P})$*-a.s. there exists an* $r_0 \in \mathbb{N}$ *such that, for all* $r \geq r_0$, *all* $\gamma > a_{L_r} - A$ *and all* $\pi \in \mathscr{P}(B_r(\mathcal{O}), B_r(\mathcal{O}))$ *satisfying* $\pi_i \notin \Pi_{r,A}$ *for all* $0 \leq i < \ell := |\pi|$,

$$\mathbb{E}_{\pi_0}\left[\mathrm{e}^{\int_0^{T_\ell} (\xi(X_s)-\gamma)\,\mathrm{d}s} \;\middle|\; \pi^\ell(X) = \pi\right] \leq q^\ell_{r,A}\mathrm{e}^{M^{r,\varepsilon}_\pi \log[(\log r)^{\delta r}/a_{L_r,A,\varepsilon}q_{r,A}]}, \tag{4.85}$$

where

$$a_{L_r,A,\varepsilon} := \varepsilon a_{L_r} - A, \qquad q_{r,A} := \left(1 + \frac{A}{(\log r)^{\delta_r}}\right)^{-1}. \tag{4.86}$$

Note that $\pi_\ell \in \Pi_{r,A}$ is allowed.

Proof. The proof is identical to that of [12, Lemma 3.3], with $d_{\max}$ replaced by $(\log r)^{\delta_r}$ (recall Lemma 4.2.3). □

We follow [12, Definition 3.4] and [9, Section 6.2]. Note that the distance between $\Pi_{r,A}$ and $D^{\mathrm{c}}_{r,A}$ in $\mathcal{GW}$ is at least $S_r = (\log L_r)^\alpha$ (recall (4.43)–(4.44)).

Definition 4.4.4. [Concatenation of paths] (a) When π and π' are two paths in $\mathscr{P}$ with $\pi_{|\pi|} = \pi'_0$, we define their *concatenation* as

$$\pi \circ \pi' := (\pi_0, \dots, \pi_{|\pi|}, \pi'_1, \dots, \pi'_{|\pi'|}) \in \mathscr{P}. \tag{4.87}$$

Note that $|\pi \circ \pi'| = |\pi| + |\pi'|$.

(b) When $\pi_{|\pi|} \neq \pi'_0$, we can still define the *shifted concatenation* of π and π' as $\pi \circ \hat{\pi}'$, where $\hat{\pi}' := (\pi_{|\pi|}, \pi_{|\pi|} + \pi'_1 - \pi'_0, \dots, \pi_{|\pi|} + \pi'_{|\pi'|} - \pi'_0)$. The shifted concatenation of multiple paths is defined inductively via associativity. ■

Now, if a path $\pi \in \mathscr{P}$ intersects $\Pi_{r,A}$, then it can be decomposed into an initial path, a sequence of excursions between $\Pi_{r,A}$ and $D^{\mathrm{c}}_{r,A}$, and a terminal path. More precisely, there exists $m_\pi \in \mathbb{N}$ such that

$$\pi = \check{\pi}^1 \circ \hat{\pi}^1 \circ \cdots \circ \check{\pi}^{m_\pi} \circ \hat{\pi}^{m_\pi} \circ \bar{\pi}, \tag{4.88}$$

where the paths in (4.88) satisfy

$$\begin{aligned}
\check{\pi}^1 &\in \mathscr{P}(V, \Pi_{r,A}) && \text{with} && \check{\pi}^1_i \notin \Pi_{r,A}, && 0 \le i < |\check{\pi}^1|,\\
\hat{\pi}^k &\in \mathscr{P}(\Pi_{r,A}, D^{\mathrm{c}}_{r,A}) && \text{with} && \hat{\pi}^k_i \in D_{r,A}, && 0 \le i < |\hat{\pi}^k|,\ 1 \le k \le m_\pi - 1,\\
\check{\pi}^k &\in \mathscr{P}(D^{\mathrm{c}}_{r,A}, \Pi_{r,A}) && \text{with} && \check{\pi}^k_i \notin \Pi_{r,A}, && 0 \le i < |\check{\pi}^k|,\ 2 \le k \le m_\pi,\\
\hat{\pi}^{m_\pi} &\in \mathscr{P}(\Pi_{r,A}, V) && \text{with} && \hat{\pi}^{m_\pi}_i \in D_{r,A}, && 0 \le i < |\hat{\pi}^{m_\pi}|,
\end{aligned} \tag{4.89}$$

while

$$\begin{aligned}
&\bar{\pi} \in \mathscr{P}(D^{\mathrm{c}}_{r,A}, V) \text{ and } \bar{\pi}_i \notin \Pi_{r,A}\ \forall\, i \ge 0 && \text{if } \hat{\pi}^{m_\pi} \in \mathscr{P}(\Pi_{r,A}, D^{\mathrm{c}}_{r,A}),\\
&\bar{\pi}_0 \in D_{r,A}, |\bar{\pi}| = 0 && \text{otherwise.}
\end{aligned} \tag{4.90}$$

Note that the decomposition in (4.88)–(4.90) is unique, and that the paths $\check{\pi}^1$, $\hat{\pi}^{m_\pi}$ and $\bar{\pi}$ can have zero length. If π is contained in $B_r(\mathcal{O})$, then so are all the paths in the decomposition.

Whenever $\mathrm{supp}(\pi) \cap \Pi_{r,A} \neq \emptyset$ and $\varepsilon > 0$, we define

$$s_\pi := \sum_{i=1}^{m_\pi} |\check{\pi}^i| + |\bar{\pi}|, \qquad k^{r,\varepsilon}_\pi := \sum_{i=1}^{m_\pi} M^{r,\varepsilon}_{\check{\pi}^i} + M^{r,\varepsilon}_{\bar{\pi}} \tag{4.91}$$

to be the total time spent in exterior excursions, respectively, on moderately low points of the potential visited by exterior excursions (without their last point).

In case $\mathrm{supp}(\pi) \cap \Pi_{r,A} = \emptyset$, we set $m_\pi := 0$, $s_\pi := |\pi|$ and $k_\pi^{r,\varepsilon} := M_\pi^{r,\varepsilon}$. Recall from (4.81) that, in this case, $\lambda_{r,A}(\pi) = -\infty$.

We say that $\pi, \pi' \in \mathscr{P}$ are *equivalent*, written $\pi' \sim \pi$, if $m_\pi = m_{\pi'}$, $\breve{\pi}'^i = \breve{\pi}^i$ for all $i = 1, \dots, m_\pi$, and $\bar{\pi}' = \bar{\pi}$. If $\pi' \sim \pi$, then $s_{\pi'}$, $k_{\pi'}^{r,\varepsilon}$ and $\lambda_{r,A}(\pi')$ are all equal to the counterparts for π.

To state our key lemma, we define, for $m, s \in \mathbb{N}_0$,

$$\mathscr{P}^{(m,s)} = \{\pi \in \mathscr{P} \colon\ m_\pi = m, s_\pi = s\}\,, \tag{4.92}$$

and denote by

$$C_{r,A} := \max\{|\mathcal{C}| \colon\ \mathcal{C} \in \mathfrak{C}_{r,A}\} \tag{4.93}$$

the maximal size of the islands in $\mathfrak{C}_{r,A}$.

Lemma 4.4.5. *[Mass of an equivalence class] Subject to Assumptions* 4.A *and* 4.C*, for every $A, \varepsilon > 0$, $(\mathsf{P} \times \mathfrak{P})$-a.s. there exists an $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$, all $m, s \in \mathbb{N}_0$, all $\pi \in \mathscr{P}^{(m,s)}$ with $\mathrm{supp}(\pi) \subset B_r(\mathcal{O})$, all $\gamma > \lambda_{r,A}(\pi) \vee (a_{L_r} - A)$ and all $t \geq 0$,*

$$\begin{aligned} &\mathbb{E}_{\pi_0}\left[\mathrm{e}^{\int_0^t (\xi(X_u) - \gamma)\,\mathrm{d}u}\, \mathbb{1}_{\{\pi(X_{[0,t]}) \sim \pi\}}\right] \\ &\leq \left(C_{r,A}^{1/2}\right)^{\mathbb{1}_{\{m>0\}}} \left(1 + \frac{(\log r)^{\delta_r}\, C_{r,A}}{\gamma - \lambda_{r,A}(\pi)}\right)^m \left(\frac{q_{r,A}}{d_{\min}}\right)^s \mathrm{e}^{k_\pi^{r,\varepsilon} \log[(\log r)^{\delta_r}/a_{L_r,A,\varepsilon} q_{r,A}]}. \end{aligned} \tag{4.94}$$

Proof. The proof is identical to that of [12, Lemma 3.5], with $d_{\max}$ is replaced by $(\log r)^{\delta_r}$ (recall Lemma 4.2.3). □

§4.4.2 Key proposition

The main result of this section is the following proposition.

Proposition 4.4.6. *[Entropy reduction] Let $\alpha \in (0,1)$ and $\kappa \in (\alpha, 1)$. Subject to Assumption* 4.C*, there exists an $A_0(r)$ such that, for all $A \geq A_0(r)$, with $\mathfrak{P}$-probability tending to one as $r \to \infty$, the following statement is true. For each $x \in B_r(\mathcal{O})$, each $\mathcal{N} \subset \mathscr{P}(x, B_r(\mathcal{O}))$ satisfying $\mathrm{supp}(\pi) \subset B_r(\mathcal{O})$ and $\max_{1 \leq \ell \leq |\pi|} \mathrm{dist}_G(\pi_\ell, x) \geq (\log L_r)^\kappa$ for all $\pi \in \mathcal{N}$, and each assignment $\pi \mapsto (\gamma_\pi, z_\pi) \in \mathbb{R} \times V$ satisfying*

$$\gamma_\pi \geq \left(\lambda_{r,A}(\pi) + \mathrm{e}^{-S_r}\right) \vee (a_{L_r} - A) \qquad \forall\, \pi \in \mathcal{N} \tag{4.95}$$

and

$$z_\pi \in \mathrm{supp}(\pi) \cup \bigcup_{\substack{\mathcal{C} \in \mathfrak{C}_{r,A}\,:\\ \mathrm{supp}(\pi) \cap \mathcal{C} \cap \Pi_{r,A} \neq \emptyset}} \mathcal{C} \qquad \forall\, \pi \in \mathcal{N}, \tag{4.96}$$

the following inequality holds for all $t \geq 0$:

$$\log \mathbb{E}_x\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbb{1}_{\{\pi(X_{[0,t]}) \in \mathcal{N}\}}\right] \leq \sup_{\pi \in \mathcal{N}} \left\{t\gamma_\pi + \mathrm{dist}_G(x, z_\pi) \log[(\log r)^{\delta_r}/a_{L_r,A,\varepsilon} q_{r,A}]\right\}. \tag{4.97}$$

Proof. The proof is based on [12, Section 3.4]. First fix $c_0 > 2$ and define

$$A_0(r) = (\log r)^{\delta_r} \left(\mathrm{e}^{4c_0 (\log r)^{1-\alpha}} - 1 \right). \tag{4.98}$$

Fix $A \geq A_0(r)$, $\beta \in (0, \alpha)$ and $\varepsilon \in (0, \frac{1}{2}\beta)$ as in Lemma 4.3.6. Let $r_0 \in \mathbb{N}$ be as given in Lemma 4.4.5, and take $r \geq r_0$ so large that the conclusions of Lemmas 4.2.3, 4.3.1, 4.3.3 and 4.3.6 hold, i.e., assume that the events $\mathcal{B}_r$ and $\mathcal{B}_{r,A}$ in these lemmas do not occur. Fix $x \in B_r(\mathcal{O})$. Recall the definitions of $C_{r,A}$ and $\mathscr{P}^{(m,s)}$. Note that the relation $\sim$ is an equivalence relation in $\mathscr{P}^{(m,s)}$, and define

$$\widetilde{\mathscr{P}}_x^{(m,s)} := \big\{ \text{equivalence classes of the paths in } \mathscr{P}(x, V) \cap \mathscr{P}^{(m,s)} \big\}. \tag{4.99}$$

The following bounded on the cardinality of this set is needed.

Lemma 4.4.7. *[Bound equivalence classes] Subject to Assumption* 4.C, $\mathfrak{P}$*-a.s.,* $|\widetilde{\mathscr{P}}_x^{(m,s)}| \leq (2C_{r,A})^m (\log r)^{\delta_r (m+s)}$ *for all* $m, s \in \mathbb{N}_0$.

Proof. We can copy the proof of [12, Lemma 3.6], replacing $d_{\max}$ by $(\log r)^{\delta_r}$.

The estimate is clear when $m = 0$. To prove that it holds for $m \geq 1$, write $\partial\Lambda := \{z \notin \Lambda \colon \operatorname{dist}_G(z, \Lambda) = 1\}$ for $\Lambda \subset V$. Then $|\partial\mathcal{C} \cup \mathcal{C}| \leq ((\log r)^{\delta_r} + 1)|\mathcal{C}| \leq 2(\log r)^{\delta_r} C_{r,A}$ by Lemma 4.2.3. Define the map $\Phi \colon \widetilde{\mathscr{P}}_x^{(m,s)} \to \mathscr{P}_s(x, V) \times \{1, \dots, 2(\log r)^{\delta_r} C_{r,A}\}^m$ as follows. For each $\Lambda \subset V$ with $1 \leq |\Lambda| \leq 2(\log r)^{\delta_r} C_{r,A}$, fix an injection $f_\Lambda \colon \Lambda \to \{1, \dots, 2(\log r)^{\delta_r} C_{r,A}\}$. Given a path $\pi \in \mathscr{P}^{(m,s)} \cap \mathscr{P}(x, V)$, decompose π, and denote by $\widetilde{\pi} \in \mathscr{P}_s(x, V)$ the shifted concatenation of $\check{\pi}^1, \dots, \check{\pi}^m, \bar{\pi}$. Note that, for $2 \leq k \leq m$, the point $\check{\pi}_0^k$ lies in $\partial\mathcal{C}_k$ for some $\mathcal{C}_k \in \mathfrak{C}_{r,A}$, while $\bar{\pi}_0 \in \partial\overline{\mathcal{C}} \cup \overline{\mathcal{C}}$ for some $\overline{\mathcal{C}} \in \mathfrak{C}_{r,A}$. Thus, it is possible to set

$$\Phi(\pi) := \big(\widetilde{\pi}, f_{\partial\mathcal{C}_2}(\check{\pi}_0^2), \dots, f_{\partial\mathcal{C}_m}(\check{\pi}_0^m), f_{\partial\bar{\mathcal{C}} \cup \bar{\mathcal{C}}}(\bar{\pi}_0) \big). \tag{4.100}$$

It is readily checked that $\Phi(\pi)$ depends only on the equivalence class of π and, when restricted to equivalence classes, Φ is injective. Hence the claim follows. □

Now take $\mathcal{N} \subset \mathscr{P}(x, V)$ as in the statement, and set

$$\widetilde{\mathcal{N}}^{(m,s)} := \big\{ \text{equivalence classes of paths in } \mathcal{N} \cap \mathscr{P}^{(m,s)} \big\} \subset \widetilde{\mathscr{P}}_x^{(m,s)}. \tag{4.101}$$

For each $\mathcal{M} \in \widetilde{\mathcal{N}}^{(m,s)}$, choose a representative $\pi_{\mathcal{M}} \in \mathcal{M}$, and use Lemma 4.4.7 to write

$$\begin{aligned} \mathbb{E}_x \left[\mathrm{e}^{\int_0^t \xi(X_u) \mathrm{d}u} \mathbb{1}_{\{\pi(X_{[0,t]}) \in \mathcal{N}\}} \right] &= \sum_{m,s \in \mathbb{N}_0} \sum_{\mathcal{M} \in \widetilde{\mathcal{N}}^{(m,s)}} \mathbb{E}_x \left[\mathrm{e}^{\int_0^t \xi(X_u) \mathrm{d}u} \mathbb{1}_{\{\pi(X_{[0,t]}) \sim \pi_{\mathcal{M}}\}} \right] \\ &\leq \sum_{m,s \in \mathbb{N}_0} (2(\log r)^{\delta_r} C_{r,A})^m ((\log r)^{\delta_r})^s \sup_{\pi \in \mathcal{N}^{(m,s)}} \mathbb{E}_x \left[\mathrm{e}^{\int_0^t \xi(X_u) \mathrm{d}u} \mathbb{1}_{\{\pi(X_{[0,t]}) \sim \pi\}} \right] \end{aligned} \tag{4.102}$$

with the convention $\sup\emptyset = 0$. For fixed $\pi \in \mathcal{N}^{(m,s)}$, by (4.95), apply (4.94) and Lemma 4.3.1 to obtain, for all r large enough and with $c_0 > 2$,

$$\begin{aligned}(2(c\log r)^{\delta_r})^m(\log r)^{\delta_r s}\,\mathbb{E}_x\left[\mathrm{e}^{\int_0^t \xi(X_u)\mathrm{d}u}\mathbb{1}_{\{\pi(X_{[0,t]})\sim\pi\}}\right]\\ \leq \mathrm{e}^{t\gamma_\pi}\mathrm{e}^{c_0 m\log r}[q_{r,A}(\log r)^{\delta_r}]^s\,\mathrm{e}^{k_\pi^{r,\varepsilon}\log[(\log r)^{\delta_r}/a_{L_r,A,\varepsilon}q_{r,A}]}.\end{aligned} \tag{4.103}$$

We next claim that, for r large enough and $\pi \in \mathcal{N}^{(m,s)}$,

$$s \geq [(m-1)\vee 1]\,S_r. \tag{4.104}$$

Indeed, when $m \geq 2$, $|\operatorname{supp}(\check\pi^i)| \geq S_r$ for all $2 \leq i \leq m$. When $m = 0$, $|\operatorname{supp}(\pi)| \geq \max_{1\leq\ell\leq|\pi|}|\pi_\ell - x| \geq (\log L_r)^\kappa \gg S_r$ by assumption. When $m = 1$, the latter assumption and Lemma 4.3.1 together imply that $\operatorname{supp}(\pi)\cap D_{r,A}^{\mathrm{c}} \neq \emptyset$, and so either $|\operatorname{supp}(\check\pi^1)| \geq S_r$ or $|\operatorname{supp}(\bar\pi)| \geq S_r$. Thus, (4.104) holds by the definition of S_r and s.

Note that $q_{r,A}^{S_r} < \mathrm{e}^{-4c_0\log r}$, so

$$\begin{aligned}&\sum_{m\geq 0}\sum_{s\geq[(m-1)\vee 1]S_r}\mathrm{e}^{c_0 m\log r}[q_{r,A}(\log r)^{\delta_r}]^s\\ &= \frac{[q_{r,A}(\log r)^{\delta_r}]^{S_r} + \mathrm{e}^{c_0\log r}[q_{r,A}(\log r)^{\delta_r}]^{S_r} + \sum_{m\geq 2}\mathrm{e}^{mc_0\log r}[q_{r,A}(\log r)^{\delta_r}]^{(m-1)S_r}}{1 - q_{r,A}(\log r)^{\delta_r}}\\ &\leq \frac{3\mathrm{e}^{-c_0\log r}}{1 - q_{r,A}(\log r)^{\delta_r}} < 1\end{aligned} \tag{4.105}$$

for r large enough. Inserting this back into (4.102), we obtain

$$\log\mathbb{E}_x\left[\mathrm{e}^{\int_0^t\xi(X_s)\mathrm{d}s}\mathbb{1}_{\{\pi(X_{0,t})\in\mathcal{N}\}}\right] \leq \sup_{\pi\in\mathcal{N}}\left\{t\gamma_\pi + k_\pi^{r,\varepsilon}\log[(\log r)^{\delta_r}/a_{L_r,A,\varepsilon}q_{r,A}]\right\}. \tag{4.106}$$

Thus the proof will be finished once we show that, for some $\varepsilon' > 0$ and whp, respectively, a.s. eventually as $r \to \infty$,

$$k_\pi^{r,\varepsilon} \geq \mathrm{dist}_G(x, z_\pi)(1 - 2(\log L_r)^{-\varepsilon'}) \qquad \forall\,\pi \in \mathcal{N}. \tag{4.107}$$

We can copy the argument at the end of [12, Section 3.4]. For each $\pi \in \mathcal{N}$ define an auxiliary path $\pi_\star$ as follows. First note that by using our assumptions we can find points $z', z'' \in \operatorname{supp}(\pi)$ (not necessarily distinct) such that

$$\mathrm{dist}_G(x, z') \geq (\log L_r)^\kappa, \qquad \mathrm{dist}_G(z'', z_\pi) \leq 2M_A S_r, \tag{4.108}$$

where the latter holds by (4.51). Write $\{z_1, z_2\} = \{z', z''\}$ with z_1, z_2 ordered according to their hitting times by π, i.e., $\inf\{\ell\colon \pi_\ell = z_1\} \leq \inf\{\ell\colon \pi_\ell = z_2\}$. Define π_e as the concatenation of the loop erasure of π between x and z_1 and the loop erasure of π between z_1 and z_2. Since π_e is the concatenation of two self-avoiding paths, it visits each point at most twice. Finally, define $\pi_\star \sim \pi_e$ by replacing the excursions of π_e from $\Pi_{r,A}$ to $D_{r,A}^{\mathrm{c}}$ by direct paths between the corresponding endpoints, i.e., replace

each $\hat{\pi}_e^i$ by $|\hat{\pi}_e^i| = \ell_i$, $(\hat{\pi}_e^i)_0 = x_i \in \Pi_{r,A}$, and $(\hat{\pi}_e^i)_{\ell_i} = y_i \in D_{r,A}^{\mathrm{c}}$ by a shortest-distance path $\widetilde{\pi}_\star^i$ with the same endpoints and $|\widetilde{\pi}_\star^i| = \mathrm{dist}_G(x_i, y_i)$. Since $\pi_\star$ visits each $x \in \Pi_{r,A}$ at most 2 times,

$$k_\pi^{r,\varepsilon} \geq k_{\pi_\star}^{r,\varepsilon} \geq M_{\pi_\star}^{r,\varepsilon} - 2|\operatorname{supp}(\pi_\star) \cap \Pi_{r,A}|(S_r+1) \geq M_{\pi_\star}^{r,\varepsilon} - 4|\operatorname{supp}(\pi_\star) \cap \Pi_{r,A}|S_r. \tag{4.109}$$

Note that $M_{\pi_\star}^{r,\varepsilon} \geq |\{x \in \operatorname{supp}(\pi_\star)\colon\ \xi(x) \leq (1-\varepsilon)a_{L_r}\}| - 1$ and, by (4.108), $|\operatorname{supp}(\pi_\star)| \geq \mathrm{dist}_G(x, z') \geq (\log L_r)^\kappa \gg (\log L_r)^{\alpha+2\varepsilon'}$ for some $0 < \varepsilon' < \varepsilon$. Applying Lemmas 4.3.6–4.3.7 and using (4.40) and $L_r > r$, we obtain, for r large enough,

$$k_\pi^{r,\varepsilon} \geq |\operatorname{supp}(\pi_\star)| \left(1 - \frac{2}{(\log L_r)^\varepsilon} - \frac{4S_r}{(\log L_r)^{\alpha+2\varepsilon'}}\right) \geq |\operatorname{supp}(\pi_\star)| \left(1 - \frac{1}{(\log L_r)^{\varepsilon'}}\right). \tag{4.110}$$

On the other hand, since $|\operatorname{supp}(\pi_\star)| \geq (\log L_r)^\kappa$, by (4.108) we have

$$\begin{aligned} |\operatorname{supp}(\pi_\star)| &= \big(|\operatorname{supp}(\pi_\star)| + 2M_A S_r\big) - 2M_A S_r \\ &= \big(|\operatorname{supp}(\pi_\star)| + 2M_A S_r\big)\left(1 - \frac{2M_A S_r}{|\operatorname{supp}(\pi_\star)| + 2M_A S_r}\right) \\ &\geq (\mathrm{dist}_G(x, z'') + 2M_A S_r)\left(1 - \frac{2M_A S_r}{(\log L_r)^\kappa}\right) \\ &\geq \mathrm{dist}_G(x, z_\pi)\left(1 - \frac{1}{(\log L_r)^{\varepsilon'}}\right), \end{aligned} \tag{4.111}$$

where the first inequality uses that the distance between two points on $\pi_\star$ is less than the total length of $\pi_\star$. Now (4.107) follows from (4.110)–(4.111). □

§4.5 Proof of the main theorem

Define

$$U^*(t) := \mathrm{e}^{t[\varrho \log(\vartheta \mathfrak{r}_t) - \varrho - \widetilde{\chi}(\varrho)]}, \tag{4.112}$$

where we recall (4.10). To prove Theorem 4.1.2 we show that

$$\frac{1}{t}\log U(t) - \frac{1}{t}\log U^*(t) = o(1), \quad t \to \infty, \qquad (\mathsf{P} \times \mathfrak{P})\text{-a.s.} \tag{4.113}$$

The proof proceeds via upper and lower bound, proved in Sections 4.5.1 and 4.5.2, respectively. Throughout this section, Assumptions 4.A, 4.B(1) and 4.C are in force.

§4.5.1 Upper bound

We follow [12, Section 4.2]. The proof of the upper bound in (4.113) relies on two lemmas showing that paths staying inside a ball of radius $\lceil t^\gamma \rceil$ for some $\gamma \in (0,1)$ or leaving a ball of radius $t \log t$ have a negligible contribution to (4.4), the total mass of the solution.

Lemma 4.5.1. *[No long paths] For any $\ell_t \geq t \log t$,*

$$\lim_{t\to\infty} \frac{1}{U^*(t)} \mathbb{E}_{\mathcal{O}} \left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbb{1}_{\{\tau_{[B_{\ell_t}]^{\mathrm{c}}} < t\}} \right] = 0 \quad (\mathsf{P} \times \mathfrak{P}) - a.s. \tag{4.114}$$

Proof. We follow [12, Lemma 4.2]. For $r \geq \ell_t$, let

$$\mathcal{B}_r := \left\{ \max_{x \in B_r(\mathcal{O})} \xi(x) \geq a_{L_r} + 2\varrho \right\}. \tag{4.115}$$

Since $\lim_{t\to\infty} \ell_t = \infty$, Lemma 4.3.5 gives that P-a.s.

$$\bigcup_{r \geq \ell_t} \mathcal{B}_r \text{ does not occur eventually as } t \to \infty. \tag{4.116}$$

Therefore we can work on the event $\bigcap_{r\geq\ell_t} [\mathcal{B}_r]^{\mathrm{c}}$. On this event, we write

$$\begin{aligned} \mathbb{E}_{\mathcal{O}} \left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbb{1}_{\{\tau_{[B_{\ell_t}]^{\mathrm{c}}} < t\}} \right] &= \sum_{r \geq \ell_t} \mathbb{E}_{\mathcal{O}} \left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbb{1}_{\{\sup_{s\in[0,t]} |X_s| = r\}} \right] \\ &\leq \mathrm{e}^{2\varrho t} \sum_{r \geq \ell_t} \mathrm{e}^{\varrho t \log r + \log(\delta_r \log\log r)} \, \mathbb{P}_{\mathcal{O}} \left(J_t \geq r \right), \end{aligned} \tag{4.117}$$

where J_t is the number of jumps of X up to time t, and we use that $|B_r(\mathcal{O})| \leq (\log r)^{\delta_r r}$. Next, J_t is stochastically dominated by a Poisson random variable with parameter $t(\log r)^{\delta_r}$. Hence

$$\mathbb{P}_{\mathcal{O}} \left(J_t \geq r \right) \leq \frac{[\mathrm{e}t \, (\log r)^{\delta_r}]^r}{r^r} \leq \exp\left\{ -r \log\left(\frac{r}{\mathrm{e}t \, (\log r)^{\delta_r}} \right) \right\} \tag{4.118}$$

for large r. Using that $\ell_t \geq t \log t$, we can easily check that, for $r \geq \ell_t$ and t large enough,

$$\varrho t \log r - r \log\left(\frac{r}{\mathrm{e}t \, (\log r)^{\delta_r}} \right) < -3r, \qquad r \geq \ell_t. \tag{4.119}$$

Thus (4.117) is at most

$$\mathrm{e}^{2\varrho t} \sum_{r \geq \ell_t} \mathrm{e}^{-3r + \log(\delta_r \log\log r)} \leq \mathrm{e}^{2\varrho t} \sum_{r \geq \ell_t} \mathrm{e}^{-2r} \leq 2 \, \mathrm{e}^{2\varrho t} \, \mathrm{e}^{-2\ell_t} \leq \mathrm{e}^{-\ell_t}. \tag{4.120}$$

Since $\lim_{t\to\infty} \ell_t = \infty$ and $\lim_{t\to\infty} U^*(t) = \infty$, this settles the claim. □

Lemma 4.5.2. *[No short paths] For any $\gamma \in (0,1)$,*

$$\lim_{t\to\infty} \frac{1}{U^*(t)} \mathbb{E}_{\mathcal{O}} \left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbb{1}_{\{\tau_{[B_{\lceil t^\gamma \rceil}]^{\mathrm{c}}} > t\}} \right] = 0 \quad (\mathsf{P} \times \mathfrak{P}) - a.s. \tag{4.121}$$

Proof. We follow [12, Lemma 4.3]. By Lemma 4.3.5 with $r = \lceil t^\gamma \rceil$, we may assume that

$$\max_{x \in B_{\lceil t^\gamma \rceil}} \xi(x) \leq \varrho \log\log L_{\lceil t^\gamma \rceil} + \frac{2\varrho \log \lceil t^\gamma \rceil}{\vartheta \lceil t^\gamma \rceil} \leq \gamma \varrho \log t + O(1), \quad t \to \infty, \tag{4.122}$$

where the second inequality uses that $\log L_{\lceil t^\gamma \rceil} \sim \log |B_{\lceil t^\gamma \rceil}(\mathcal{O})| \sim \vartheta \lceil t^\gamma \rceil$. Hence

$$\frac{1}{U^*(t)}\,\mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\mathbb{1}_{\{\tau_{[B_{\lceil t^\gamma\rceil}]^{\mathrm{c}}}>t\}}\right] \le \frac{1}{U^*(t)}\,\mathrm{e}^{\gamma\varrho t\log t+O(1)} \le \mathrm{e}^{(1-\gamma)\varrho t\log t + C\log\log\log t}, \quad t\to\infty, \tag{4.123}$$

for any constant $C > 1$. □

The proof of the upper bound in (4.113) also relies on a third lemma estimating the contribution of paths leaving a ball of radius $\lceil t^\gamma \rceil$ for some $\gamma \in (0,1)$ but staying inside a ball of radius $t \log t$. We slice the annulus between these two balls into layers, and derive an estimate for paths that reach a given layer but do not reach the next layer. To that end, fix $\gamma \in (\alpha, 1)$ with α as in (4.40), and let

$$K_t := \lceil t^{1-\gamma}\log t\rceil, \qquad r_t^{(k)} := k\lceil t^\gamma\rceil, \quad 1\le k\le K_t, \qquad \ell_t := K_t\lceil t^\gamma\rceil \ge t\log t. \tag{4.124}$$

For $1 \le k \le K_t$, define (recall (4.77))

$$\mathcal{N}_t^k := \Big\{\pi\in\mathscr{P}(\mathcal{O},V)\colon\ \mathrm{supp}(\pi)\subset B_{r_t^{k+1}}(\mathcal{O}),\, \mathrm{supp}(\pi)\cap B^{\mathrm{c}}_{r_t^k}(\mathcal{O})\neq\emptyset\Big\} \tag{4.125}$$

and set

$$U^k(t) := \mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\mathbb{1}_{\{\pi_{[0,t]}(X)\in\mathcal{N}_t^k\}}\right]. \tag{4.126}$$

Lemma 4.5.3. *[Upper bound on $U^k(t)$] For any $\varepsilon > 0$, $(\mathsf{P}\times\mathfrak{P})$-a.s. eventually as $t\to\infty$,*

$$\sup_{1\le k\le K_t}\frac{1}{t}\log U_t^k \le \frac{1}{t}\log U^*(t) + \varepsilon. \tag{4.127}$$

Proof. We follow [12, Lemma 4.4] Fix $k \in \{1,\dots,K_t\}$. For $\pi\in\mathcal{N}_t^k$, let

$$\gamma_\pi := \lambda_{r_t^{k+1},A}(\pi) + \mathrm{e}^{-S_{\lceil t^\gamma\rceil}}, \qquad z_\pi\in\mathrm{supp}(\pi), |z_\pi| > r_t^k, \tag{4.128}$$

chosen such that (4.95)–(4.96) are satisfied. By Proposition 4.4.6 and (4.86), $(\mathsf{P}\times\mathfrak{P})$-a.s. eventually as $t\to\infty$,

$$\frac{1}{t}\log U_t^k \le \gamma_\pi - \frac{|z_\pi|}{t}\Big(\log[\varepsilon\varrho\log(\vartheta r_t^{(k+1)})] - \delta_r\log[\log(r_t^{(k+1)})] + o(1)\Big). \tag{4.129}$$

Using Corollary 4.3.4 and $\log L_r \sim \vartheta r$, we bound

$$\gamma_\pi \le \varrho\log(\vartheta r_t^{(k+1)}) - \widetilde{\chi}(\varrho) + \tfrac{1}{2}\varepsilon + o(1). \tag{4.130}$$

Moreover, $|z_\pi| > r_t^{k+1} - \lceil t^\gamma\rceil$ and

$$\begin{aligned}\frac{\lceil t^\gamma\rceil}{t}&\Big(\log[\varepsilon\varrho\log(\vartheta r_t^{(k+1)})] - \delta_r\log[\log(r_t^{(k+1)})]\Big)\\ &\le \frac{1}{t^{1-\gamma}}\log\log(2t\log t) = o(1).\end{aligned} \tag{4.131}$$

Hence

$$\gamma_\pi \le F_t(r_t^{(k+1)}) - \widetilde{\chi}(\varrho) + \tfrac{1}{2}\varepsilon + o(1) \tag{4.132}$$

with

$$F_t(r) := \varrho \log(\vartheta r) - \frac{r}{t}\big[\log(\varepsilon\varrho\log(\vartheta r)) - \delta_r \log(\log r)\big], \qquad r > 0. \tag{4.133}$$

The function F_t is maximized at any point r_t satisfying

$$\varrho t = r_t \left[\log(\varepsilon\varrho\log(\vartheta r_t)) - (\delta_r + r_t \tfrac{\mathrm{d}}{\mathrm{d}r}\delta_r)\log\log r_t + \frac{1}{\log(\vartheta r_t)} - \frac{\delta_r}{\log r_t}\right]. \tag{4.134}$$

In particular, $r_t = \mathfrak{r}_t[1 + o(1)]$, which implies that

$$\sup_{r>0} F_t(r) \le \varrho \log(\vartheta \mathfrak{r}_t) - \varrho + o(1), \qquad t \to \infty. \tag{4.135}$$

Inserting (4.135) into (4.132), we obtain $\frac{1}{t}\log U_t^k < \varrho\log(\vartheta\mathfrak{r}_t) - \varrho - \widetilde{\chi}(\varrho) + \varepsilon$, which is the desired upper bound because $\varepsilon > 0$ is arbitrary. □

Proof of the upper bound in (4.113). To avoid repetition, all statements hold $(\mathfrak{P}\times\mathsf{P})$-a.s. eventually as $t \to \infty$. Set

$$U^0(t) := \mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\mathbb{1}_{\{\tau_{[B_{\lceil t^\gamma\rceil}]^{\mathrm{c}}} > t\}}\right], \quad U^\infty(t) := \mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\mathbb{1}_{\{\tau_{[B_{\lceil t\log t\rceil}]^{\mathrm{c}}} \le t\}}\right]. \tag{4.136}$$

Then

$$U(t) \le U^0(t) + U^\infty(t) + K_t \max_{1\le k\le K_t} U^k(t). \tag{4.137}$$

From Lemmas 4.5.1–4.5.3 and the fact that $K_t = o(t)$, we get

$$\limsup_{t\to\infty}\left\{\frac{1}{t}\log U(t) - \frac{1}{t}\log U^*(t)\right\} \le \varepsilon. \tag{4.138}$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of the upper bound in (4.11). □

§4.5.2 Lower bound

We first introduce an alternative representation for χ in (2.11) in terms of a 'dual' variational formula. Fix $\varrho \in (0,\infty)$ and a graph $G = (V, E)$. The functional

$$\mathcal{L}(q; G) := \sum_{x\in V} \mathrm{e}^{q(x)/\varrho} \in [0,\infty], \qquad q\colon V \to [-\infty,\infty), \tag{4.139}$$

plays the role of a large deviation rate function for the potential ξ in V (compare with (2.7)). For $\Lambda \subset V$, define

$$\widehat{\chi}_\Lambda(G) := -\sup_{\substack{q\colon V\to[-\infty,\infty),\\ \mathcal{L}(q;G)\le 1}} \lambda_\Lambda(q; G) \in [0,\infty). \tag{4.140}$$

The condition $\mathcal{L}(q; G) \le 1$ under the supremum ensures that the potentials q have a fair probability under the i.i.d. double-exponential distribution. Write $\widehat{\chi}(G) = \widehat{\chi}_V(G)$.

Proposition 4.5.4. *[Alternative representations for χ] For any graph $G = (V, E)$ and any $\Lambda \subset V$,*

$$\widehat{\chi}_\Lambda(\varrho; G) \geq \widehat{\chi}_V(\varrho; G) = \widehat{\chi}_G(\varrho) = \chi_G(\varrho). \tag{4.141}$$

Proof. See [12, Section A.1] □

For the lower bound we follow [12, Section 4.1]. Fix $\varepsilon > 0$. By the definition of $\widetilde{\chi}$, there exists an infinite rooted tree $T = (V', E', \mathcal{Y})$ with degrees in $\mathrm{supp}(D_g)$ such that $\chi_T(\varrho) < \widetilde{\chi}(\varrho) + \frac{1}{4}\varepsilon$. Let $Q_r = B_r^T(\mathcal{Y})$ be the ball of radius r around $\mathcal{Y}$ in T. By Proposition 4.5.4 and (4.140), there exist a radius $R \in \mathbb{N}$ and a potential profile $q\colon B_R^T \to \mathbb{R}$ with $\mathcal{L}_{Q_R}(q; \varrho) < 1$ (in particular, $q \leq 0$) such that

$$\lambda_{Q_R}(q; T) \geq -\widehat{\chi}_{Q_R}(\varrho; T) - \tfrac{1}{2}\varepsilon > -\widetilde{\chi}(\varrho) - \varepsilon. \tag{4.142}$$

For $\ell \in \mathbb{N}$, let $B_\ell = B_\ell(\mathcal{O})$ denote the ball of radius ℓ around $\mathcal{O}$ in $\mathcal{GW}$. We will show next that, $(\mathfrak{P} \times \mathsf{P})$-a.s. eventually as $\ell \to \infty$, B_ℓ contains a copy of the ball Q_R where the potentail ξ is bounded from below by $\varrho \log\log |B_\ell| + q$.

Proposition 4.5.5. *[Balls with high exceedances] $(\mathfrak{P} \times \mathsf{P})$-almost surely eventually as $\ell \to \infty$, there exists a vertex $z \in B_\ell$ with $B_{R+1}(z) \subset B_\ell$ and an isomorphism $\varphi : B_{R+1}(z) \to Q_{R+1}$ such that $\xi \geq \varrho \log\log |B_\ell| + q \circ \varphi$ in $B_R(z)$. In particular,*

$$\lambda_{B_R(z)}(\xi; \mathcal{GW}) > \varrho \log\log |B_\ell| - \widetilde{\chi}(\varrho) - \varepsilon. \tag{4.143}$$

Any such z necessarily satisfies $|z| \geq c\ell$ $(\mathfrak{P} \times \mathsf{P})$-a.s. eventually as $\ell \to \infty$ for some constant $c = c(\varrho, \vartheta, \widetilde{\chi}(\varrho), \varepsilon) > 0$.

Proof. See [12, Proposition 4.1]. The proof carries over verbatim because the degrees play no role. □

Proof of the lower bound in (4.11). Let z be as in Proposition 4.5.5. Write τ_z for the hitting time of z by the random walk X. For $s \in (0, t)$, we estimate

$$\begin{aligned} U(t) &\geq \mathbb{E}_{\mathcal{O}}\Big[\mathrm{e}^{\int_0^t \xi(X_u)\,\mathrm{d}u}\, \mathbb{1}_{\{\tau_z \leq s\}}\, \mathbb{1}_{\{X_u \in B_R(z)\ \forall u \in [\tau_z, t]\}}\Big] \\ &= \mathbb{E}_{\mathcal{O}}\Big[\mathrm{e}^{\int_0^{\tau_z} \xi(X_u)\,\mathrm{d}u}\, \mathbb{1}_{\{\tau_z \leq s\}}\, \mathbb{E}_z\Big[\mathrm{e}^{\int_0^v \xi(X_u)\,\mathrm{d}u}\, \mathbb{1}_{\{X_u \in B_R(z)\ \forall u \in [0,v]\}}\Big]\Big|_{v = t - \tau_z}\Big], \end{aligned} \tag{4.144}$$

where we use the strong Markov property at time τ_z. We first bound the last term in the integrand in (4.144). Since $\xi \geq \varrho \log\log |B_\ell| + q$ in $B_R(z)$,

$$\begin{aligned} \mathbb{E}_z\Big[\mathrm{e}^{\int_0^v \xi(X_u)\,\mathrm{d}u} \mathbb{1}_{\{X_u \in B_R(z)\ \forall u \in [0,v]\}}\Big] &\geq \mathrm{e}^{v\varrho \log\log |B_\ell|} \mathbb{E}_{\mathcal{Y}}\Big[\mathrm{e}^{\int_0^v q(X_u)\,\mathrm{d}u} \mathbb{1}_{\{X_u \in Q_R\ \forall u \in [0,v]\}}\Big] \\ &\geq \mathrm{e}^{v\varrho \log\log |B_\ell|} \mathrm{e}^{v\lambda_{Q_R}(q;T)} \phi^1_{Q_R}(\mathcal{Y})^2 \\ &> \exp\big\{v\,(\varrho \log\log |B_\ell| - \widetilde{\chi}(\varrho) - \varepsilon)\big\} \end{aligned} \tag{4.145}$$

for large v, where we used that $B_{R+1}(z)$ is isomorphic to Q_{R+1} for the indicators in the first inequality, and applied Lemma B.1.2 and (4.142) to obtain the second and third inequalities, respectively. On the other hand, since $\xi \geq 0$,

$$\mathbb{E}_{\mathcal{O}}\Big[\mathrm{e}^{\int_0^{\tau_z} \xi(X_u)\,\mathrm{d}u} \mathbb{1}\{\tau_z \leq s\}\Big] \geq \mathbb{P}_{\mathcal{O}}(\tau_z \leq s), \tag{4.146}$$

and we can bound the latter probability from below by the probability that the random walk runs along a shortest path from the root $\mathcal{O}$ to z within a time at most s. Such a path $(y_i)_{i=0}^{|z|}$ has $y_0 = \mathcal{O}$, $y_{|z|} = z$, $y_i \sim y_{i-1}$ for $i = 1, \dots, |z|$, has at each step from y_i precisely $\deg(y_i)$ choices for the next step with equal probability, and the step is carried out after an exponential time E_i with parameter $\deg(y_i)$. This gives

$$\mathsf{P}_{\mathcal{O}}(\tau_z \leq s) \geq \Big(\prod_{i=1}^{|z|} \frac{1}{\deg(y_i)}\Big) P\Big(\sum_{i=1}^{|z|} E_i \leq s\Big) \geq ((\log|z|)^{\delta_\ell})^{-|z|} \mathrm{Poi}_{d_{\min} s}([|z|, \infty)), \tag{4.147}$$

where Poi_γ is the Poisson distribution with parameter γ, and P is the generic symbol for probability. Summarising, we obtain

$$\begin{aligned} U(t) &\geq ((\log|z|)^{\delta_l})^{-|z|} \mathrm{e}^{-d_{\min} s} \frac{(d_{\min} s)^{|z|}}{|z|!} \mathrm{e}^{(t-s)\left[\varrho \log\log|B_\ell| - \widetilde{\chi}(\varrho) - \varepsilon\right]} \\ &\geq \exp\left\{-d_{\min} s + (t-s)\left[\varrho \log\log|B_\ell| - \widetilde{\chi}(\varrho) - \varepsilon\right] - |z| \log\left(\frac{(\log|z|)^{\delta_\ell}}{d_{\min}} \frac{|z|}{s}\right)\right\} \\ &\geq \exp\left\{-d_{\min} s + (t-s)\left[\varrho \log\log|B_\ell| - \widetilde{\chi}(\varrho) - \varepsilon\right] - \ell \log\left(\frac{(\log \ell)^{\delta_\ell}}{d_{\min}} \frac{\ell}{s}\right)\right\}, \end{aligned} \tag{4.148}$$

where in the last inequality we use that $s \leq |z|$ and $\ell \geq |z|$. Further assuming that $\ell = o(t)$, we see that the optimum over s is obtained at

$$s = \frac{\ell}{d_{\min} + \varrho \log\log|B_\ell| - \widetilde{\chi}(\varrho) - \varepsilon} = o(t). \tag{4.149}$$

Note that, by Proposition 4.5.5, this s indeed satisfies $s \leq |z|$. Applying (4.7) we get, after a straightforward computation, $(\mathfrak{P} \times \mathsf{P})$-a.s. eventually as $t \to \infty$,

$$\frac{1}{t} \log U(t) \geq \varrho \log\log|B_\ell| - \frac{\ell}{t} \log\log \ell - \frac{\ell}{t} \delta_\ell \log\log \ell - \widetilde{\chi}(\varrho) - \varepsilon + O\left(\frac{\ell}{t}\right). \tag{4.150}$$

Inserting $\log|B_\ell| \sim \vartheta \ell$, we get

$$\frac{1}{t} \log U(t) \geq F_\ell - \widetilde{\chi}(\varrho) - \varepsilon + o(1) + O\left(\frac{\ell}{t}\right) \tag{4.151}$$

with

$$F_\ell = \varrho \log(\vartheta \ell) - \frac{\ell}{t} \log\log \ell - \frac{\ell}{t} \delta_\ell \log\log \ell. \tag{4.152}$$

The optimal ℓ for F_ℓ satisfies

$$\varrho t = \ell\big[1 + (\delta_\ell + \ell \tfrac{\mathrm{d}}{\mathrm{d}\ell} \delta_\ell)\big] \log\log \ell + \frac{\ell \delta_\ell}{\log \ell} + \frac{\ell}{\log \ell}, \tag{4.153}$$

i.e., $\ell = \mathfrak{r}_t[1 + o(1)]$. For this choice we obtain

$$\frac{1}{t}\log U(t) \geq \varrho \log(\vartheta \mathfrak{r}_t) - \varrho - \widetilde{\chi}(\varrho) - \varepsilon + o(1). \tag{4.154}$$

Hence $(\mathfrak{P} \times \mathsf{P})$-a.s.

$$\liminf_{t\to\infty} \left\{ \frac{1}{t}\log U(t) - \frac{1}{t}\log U^*(t) \right\} \geq -\varepsilon. \tag{4.155}$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of the lower bound in (4.11). □

Remark: It is clear from (4.134) and (4.153) that, in order to get the correct asymptotics, it is crucial that both δ_r and $r\frac{\mathrm{d}}{\mathrm{d}r}\delta_r$ tend to zero as $r \to \infty$. This is why Assumption 4.C is the weakest condition on the tail of the degree distribution under which the arguments in [12] can be pushed through.

§4.6 Existence and uniqueness of the Feynman-Kac formula

We follow the argument in [24, Section 2], where existence and uniqueness of the Feynman-Kac formula in (4.3) was shown for $G = \mathbb{Z}^d$.

Theorem 4.6.1. *Subject to Assumptions 4.A and 4.B,* (4.2) *has a unique nonnegative solution* $(P \times \mathfrak{P})$*-almost surely. This solution admits the Feynman-Kac representation in* (4.3)*.*

We note that, due to the exponential growth of the Galton-Watson tree, the condition on the potential needed here is stronger than the one required in [24] on $\mathbb{Z}^d$.

The proof of Theorem 4.6.1 requires several preparatory results. Lemmas 4.6.2 and 4.6.3 below show the existence and uniqueness, respectively, of the Feynman-Kac solution for a deterministic potential. Lemma 4.6.4 extends this to a random potential.

Consider the problem

$$\begin{array}{lcll} \partial_t u(x,t) & = & (\Delta_G u)(x,t) + q(x)u(x,t), & x \in V,\, t > 0, \\ u(x,0) & = & \delta_{\mathcal{O}}(x), & x \in V, \end{array} \tag{4.156}$$

where q is a deterministic potential that is bounded from below. Without loss of generality, we may assume that q is nonnegative.

Define

$$v(x,t) = \mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t q(X_s)\mathrm{d}s}\, \mathbb{1}\{X_t = x\} \right]. \tag{4.157}$$

Lemma 4.6.2. *[Existence]* (4.156) *admits at least one nonnegative solution if and only if*

$$v(x,t) < \infty \qquad \forall\ (t,x) \in \mathbb{R}_+ \times V. \tag{4.158}$$

If (4.158) *is fulfilled, then* v *is the minimal nonnegative solution of* (4.156).

Proof. See [12, Lemma 2.2]. The proof relies on restricting the Feynman-Kac functional in (4.158) to cubes of length $2N$ around the origin and letting $N \to \infty$. On the tree we restrict to balls of radius R around the root and let $R \to \infty$. The arguments carry over with this change. □

Lemma 4.6.3. *[Uniqueness] If* q *is bounded from below, then* (4.156) *admits at most one nonnegative solution* $\mathfrak{P}$*-almost surely.*

Proof. It suffices to show that (4.156) with initial condition $u(x,0) = 0$, $x \in V$, only has the 0 solution. We follow the proof of [24, Lemma 2.3]. For $R \in \mathbb{N}$, define Γ_R to be the set of paths

$$\gamma : \mathcal{O} = x_0 \to x_1 \to \cdots \to x_n \tag{4.159}$$

consisting of neighbouring vertices in V such that $x_0, \cdots, x_{n-1} \in B_R(\mathcal{O})$ and $x_n \in Z_{R+1}$. Furthermore, define

$$\gamma_+ = \{x \in \gamma : q(x) > 0\}. \tag{4.160}$$

Let τ_R be the first time when the random walk hits Z_{R+1}, and let v be a solution of (4.156). The Feynman-Kac representation of v reads

$$v(t,0) = \mathbb{E}_{\mathcal{O}} \left[\mathrm{e}^{\int_0^{\tau_R} q(X_s)\mathrm{d}s} v\Big(t - \tau_R, X(\tau_R)\Big) \mathbb{1}\{\tau_R \le t\} \right]. \tag{4.161}$$

We are done once we show that

$$v(T,0) \ge \left(\frac{T}{t}\right)^{R-1} \mathrm{e}^{-(\log R)(T-t)} v(t,0) \tag{4.162}$$

for all $0 < t \le T$ and all $R \in \mathbb{N}$. Indeed, in that case the right-hand side tends to infinity as $R \to \infty$, and therefore so does the left-hand side, which leaves $v(t,0) = 0$ for all $t \in (0,T]$ as the only possible solution.

To prove (4.162), fix an arbitrary path $\gamma \in \Gamma_R$. The contribution of the random walk moving along the path γ equals $\chi_\gamma(t)$ with

$$\chi_\gamma(t) = \left(\prod_{i=0}^{n-1} \frac{1}{D_i}\right) \mathbb{E}_{\mathcal{O}} \left(\exp\left\{\sum_{i=0}^{n-1} q_i \sigma_i\right\} v\left(t - \sum_{i=0}^{n-1} \sigma_i, x_n\right) \mathbb{1}\left\{\sum_{i=0}^{n-1} \sigma_i \le t\right\}\right), \tag{4.163}$$

where $q_i = q(x_i)$ and the σ_i are the successive waiting times of the random walk, which are independent and exponentially distributed with parameter D_i. Letting m be such

that $q(x_m) = \min_{x\in\gamma} q(x)$, we can rewrite (4.163) as

$$\begin{aligned}\chi_\gamma(T) &= \int\limits_{\sum_{i=0}^{n-1} s_i \le T} \cdots \int \sum_{i=0}^{n-1} \mathrm{d}s_i \exp\left\{\sum_{i=0}^{n-1} s_i(q_m - D_i)\right\} v\left(T - \sum_{i=0}^{n-1} s_i, x_n\right) \exp\left\{\sum_{i=0}^{n-1} s_i(q_i - q_m)\right\} \\ &\ge \int\limits_{\sum_{i=0}^{n-1} s_i \le T} \cdots \int \sum_{i=0}^{n-1} \mathrm{d}s_i \exp\left\{\sum_{i=0}^{n-1} s_i(q_m - \log R)\right\} v\left(T - \sum_{i=0}^{n-1} s_i, x_n\right) \exp\left\{\sum_{i=0}^{n-1} s_i(q_i - q_m)\right\},\end{aligned} \tag{4.164}$$

where the inequality uses Lemma 4.2.3(a), which shows that the maximal degree is $o(R)$ as $R \to \infty$. After some straightforward manipulations and making a change of integration variables (for full details see [24, (2.14)–(2.15)]), we arrive at (4.162). □

Lemma 4.6.4. *For each $t > 0$,*

$$\mathbb{P}_{\mathcal{O}}\left(\max_{s\in[0,t]} |X_s| = R\right) \le \mathrm{e}^{-[1+o(1)]R\log R}, \qquad R \to \infty, \quad \mathfrak{P}-a.s., \tag{4.165}$$

where, for $x \in V$, $|x| = \mathrm{dist}(\mathcal{O}, x)$ denotes the distance between x and the root $\mathcal{O}$.

Proof. For fixed R, let $(\tilde{X}_t)_{t\ge0}$ be the random walk on the regular tree with offspring D_R such that $D_x \le D_R$ for all $x \in B_R(\mathcal{O})$. From Lemma 4.2.3, $D_R = o(R)$. Define $(N(t))_{t\ge0}$ to be the Poisson process with rate D_R, associated with the jumps of $\tilde{X}$. We estimate

$$\mathbb{P}_{\mathcal{O}}\left(\max_{s\in[0,t]} |X_s| = R\right) \le \mathbb{P}_{\mathcal{O}}\left(\max_{s\in[0,t]} |\tilde{X}_s| = R\right) \le \mathbb{P}_{\mathcal{O}}(N(t) = R).$$

Since

$$\mathbb{P}_{\mathcal{O}}(N(t) = R) = \frac{(D_R)^R}{R!}\mathrm{e}^{-tD_R},$$

(4.165) follows from Stirling's formula. □

Proof of Theorem 4.6.1. We follow the proof of [24, Theorem 2.1 a)]. We need to check that the expression in (4.3) is finite for arbitrary $(t, x) \in \mathbb{R}_+ \times V$. To that end we estimate

$$\mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t q(X_s)\mathrm{d}s}\, \mathbb{1}\{X_t = x\}\right] \le \sum_{R\in\mathbb{N}} \mathbb{P}_{\mathcal{O}}\left(\max_{s\in[0,t]} |X_s| = R\right) \exp\left\{t \max_{y\in B_R(\mathcal{O})} \xi(y)\right\}.$$

We know from Lemma 4.3.5 that $\max_{y\in B_R(\mathcal{O})} \xi(y) \sim \varrho\log(\theta R)$ as $R \to \infty$, $(P \times \mathfrak{P})$-a.s. (recall (4.7) and (4.42)). Applying Lemma 4.6.4, we see that the sum on the right-hand side is indeed finite. □