The parabolic Anderson model on Galton-Watson trees

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CHAPTER 3

The parabolic Anderson model on a periodic Galton-Watson tree

Abstract

In [14], the annealed total mass of the solution to the parabolic Anderson model on a regular tree with an i.i.d. double-exponential random potential was studied. The first two terms in the asymptotic expansion for large time of the total mass was identified. This chapter extends the analysis to a periodic Galton-Watson tree with large periodicity and is therefore a crucial step towards understanding the annealed total mass on the regular Galton-Watson tree. To do this we need to carefully deal with the non-homogeneity of the periodic Galton-Watson tree.

§3.1 Introduction and main results

In [12], the quenched asymptotic growth rate of the total mass of the PAM on a Galton-Watson tree was indentified. The ultimate goal in this chapter is to obtain the corresponding result in the annealed setting, i.e. the growth rate when averaged over the potential. Chapter 2 made the first step in this direction by considering the regular tree. This chapter considers the Galton-Watson tree with large periodicity, which approximates the full Galton-Watson tree and is therefore another step towards our goal. The periodicity allows us to build on the previous chapter, where the key techniques developed heavily rely on an underlying periodic structure. The main challenge here is to navigate the non-homogeneity of the periodic Galton-Watson tree, which has to be dealt with carefully.

In Section 3.1, the periodic Galton-Watson tree is defined, and the PAM and key quantities are introduced. In Section 3.2 the main theorem is stated. Sections 3.3 and 3.4 concern the proof of the main theorem through a lower, and respectively an upper bound.

§3.1.1 Definition of the periodic Galton-Watson tree

We analyse the PAM on the graph generated by a 'periodic' Galton-Watson tree. The graph is generated by first taking $\mathbb{Z}$, and looking at $[N] = \{0, 1, \cdots, N\} \subset \mathbb{Z}$. From each $x \in [N]$, independently generate a Galton-Watson tree with offspring distribution D, except at the roots $x \in [N]$ where the offspring distribution is $D - 1$. Let the tree rooted at x be denoted by $\mathcal{T}_x$. For $x \in \mathbb{Z} \setminus [N]$, repeat the same Galton-Watson tree that was generated at $x \mod (N+1)$. In other words, the trees $\mathcal{T}_x$ and $\mathcal{T}_y$ are equal if $x = y \mod (N+1)$. This can be equivalently viewed as the infinite concatenation of $[N]$ with all the generated Galton-Watson trees hanging off. Denote the resulting tree by $\mathcal{GW} = (V, E, 0)$ and the probability with respect to $\mathcal{GW}$ by $\mathfrak{P}$. Note that the standard Galton-Watson tree corresponds to $N = \infty$, so that there is no periodicity, or alternatively, each vertex in $\mathbb{Z}$ has an independent Galton-Watson hanging off it, and not just the vertices $[N]$.

§3.1.2 The PAM on a periodic Galton-Watson tree

See (1.14) for the definition of the PAM on a general (locally-finite) graph and other relevant notation. Recall that the annealed total mass is given by the Feynman-Kac representation

$$\langle U(t) \rangle = \left\langle \mathbb{E}_0 \left(\mathrm{e}^{\int_0^t \xi(X_s) \mathrm{d}s} \right) \right\rangle. \tag{3.1}$$

where $\langle \cdot \rangle$ denotes expectation with respect to the potential $\xi = (\xi(x))_{x \in \mathcal{GW}}$, $X = (X_t)_{t \geq 0}$ a continuous-time random walk on the vertices V with jump rate 1 along the edges E, and $\mathbb{P}_0$ denotes the law of X given $X_0 = 0$.

§3.1.3 Assumptions on the potential

We make the same assumptions as in the previous chapter. Throughout the paper we assume that the random potential ξ consists of i.i.d. random variables with a marginal distribution whose cumulant generating function

$$H(u) = \log \left\langle \mathrm{e}^{u\xi(\mathcal{O})} \right\rangle \tag{3.2}$$

satisfies the following:

Assumption 3.A. [Asymptotic double-exponential potential]
There exists a $\varrho \in (0, \infty)$ such that

$$\lim_{u\to\infty} uH''(u) = \varrho. \tag{3.3}$$

■

It will be useful later on to observe that, since ξ is assumed to be i.i.d., we have from (3.1)-(3.2) that

$$\langle U(t)\rangle = \mathbb{E}_{\mathcal{O}}\left(\exp\left[\sum_{x\in V} H(\ell_t(x))\right]\right), \tag{3.4}$$

where

$$\ell_t(x) = \int_0^t \mathbb{1}\{X_s = x\}\,\mathrm{d}s, \qquad x \in V,\, t \geq 0,$$

is the local time of X at vertex x up to time t. See Chapter 2.1.2 for further details.

§3.2 Main theorem: annealed total mass for large times

To state the main theorem, we introduce the following *characteristic variational formula*. Denote by $\mathcal{P}(V)$ the set of probability measures on V. For $p \in \mathcal{P}(V)$, define

$$I_E(p) = \sum_{\{x,y\}\in E} \left(\sqrt{p(x)} - \sqrt{p(y)}\right)^2, \qquad J_V(p) = -\sum_{x\in V} p(x)\log p(x), \tag{3.5}$$

and set

$$\chi_{\mathcal{GW}}(\varrho) = \inf_{p\in\mathcal{P}(V)} [I_E(p) + \varrho J_V(p)], \qquad \varrho \in (0,\infty). \tag{3.6}$$

The first term in (3.5) is the quadratic form associated with the Laplacian, which is the large deviation rate function for the *empirical distribution*

$$L_t = \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s = \frac{1}{t}\sum_{x\in V} \ell_t(x)\delta_x \in \mathcal{P}(V). \tag{3.7}$$

The following lemma links the Feynman-Kac formula (3.4) to the main theorem, as well as introduces several quantities that are used later on in the proof. The lemma pulls the leading order term out of the expansion and shows that the second order term is controlled by the large deviation principle for the empirical distribution of the normalised local times.

Lemma 3.2.1. *[Key object for the expansion] If $G = (V, E)$ is finite, then*

$$\langle U(t) \rangle = \mathrm{e}^{H(t)+o(t)} \, \mathbb{E}_{\mathcal{O}} \left(\mathrm{e}^{-\varrho t J_V(L_t)} \right), \qquad t \to \infty, \tag{3.8}$$

where J_V is the functional in (3.6) *and L_t is the empirical distribution in* (3.7).

Proof. See Lemma 2.1.2 in Chapter 2 □

For the offspring distribution D, denote its support by $\mathrm{supp}(D)$. For technical reasons we require the following assumption.

Assumption 3.B. [Offspring distribution]
There exist $4 \leq d^- \leq d^+ < \infty$ such that $\mathrm{supp}(D) \subseteq [d^-, d^+]$. ■

Theorem 3.2.2. *Subject to Assumptions 3.A and 3.B, as $t \to \infty$,*

$$\frac{1}{t} \log \langle U(t) \rangle = \varrho \log(\varrho t) - \varrho - \chi_{\mathcal{GW}}(\varrho) + o(1), \qquad \mathfrak{P}\text{-a.s.} \tag{3.9}$$

The variational formula depends on the realisation of $\mathcal{GW}$ and hence is a random object. However, as will become clear from later analysis, it is actually deterministic and equal to the one of the embedded regular tree with degrees $d^- + 1$. The latter has been studied in Chapter 2.5.

§3.3 Proof of main theorem: lower bound

A lower bound for (3.9) in Theorem 3.2.2 is obtained though a standard and straightforward argument. Let $B_R(0) \subset \mathcal{GW}$ be the ball of radius R around 0 in the graph distance. We consider a random walk that is killed when it leaves $B_R(0)$. This gives

$$\frac{1}{t} \log \langle U(t) \rangle \geq \varrho \log(\varrho t) - \varrho - \chi_R^-(\varrho) + o(1), \qquad t \to \infty,$$

with $\chi_R^-(\varrho)$ the variational formula on $B_R(0)$ with zero boundary condition. It is easily shown that $\chi_R^-(\varrho) \to \chi_{\mathcal{GW}}(\varrho)$ as $R \to \infty$. Hence, letting $R \to \infty$ we get the desired lower bound. See Chapter 2.2 for technical details. The inhomogeneity of the periodic Galton-Watson tree plays no role in the argument, which carries over exactly.

§3.4 Proof of main theorem: upper bound

In this section we prove the upper bound for (3.9) in Theorem 3.2.2. We try to follow the argument used on the regular tree in the previous chapter. Again, the argument is comprised of four steps:

(I) *Condition on the backbone* of X (Section 3.5.1).

(II) *Project* X onto a concatenation of finite subtrees attached to this backbone that have depth R and special *tadpoles* at the bottom. (Section 3.5.2).

(III) *Periodise* the projected X to obtain a *Markov renewal process* on a finite graph (Section 3.5.3).

(IV) Use the *large deviation principle* for the empirical distribution of *Markov renewal processes* derived in [31] to obtain a variational formula on a single subtree (Section 3.5.4).

Finally, in Section 3.6 we derive the upper bound of the expansion by letting $R \to \infty$ in the variational formula.

§3.5 Backbone, projection, periodisation and upper variational formula

§3.5.1 Backbone

Although the intuition is the same as on the regular tree, the backbone has to be defined more carefully due to the inhomogeneity. Since X is transient, it escapes to infinity along a path in $\mathcal{T}_n$ for some $n \in \mathbb{Z}$. We can assume $n \in \mathbb{N}_0$, for if not, we can reflect the labelling in Section 3.1.1 so that this is the case. For $k \in \mathbb{Z}_+$, let Z_k denote all the vertices at distance k from 0 *and* belonging to $\cup_{l\in\mathbb{Z}_+}\mathcal{T}_l$. Similarly for $k \in \mathbb{Z}_-$, Z_k denotes all the vertices at distance k from 0, *and* belonging to $\cup_{l\in\mathbb{Z}_-}\mathcal{T}_l$. For fixed $k \in \mathbb{Z}$, define T_k to be the last time X visits Z_k, i.e.

$$T_k = \sup\{t > 0 : X_t \in Z_k\}.$$

The backbone is the sequence of vertices $(X_{T_k})_{k\geq 0}$ with the convention that $X_{T_0} = 0$, and can be interpreted as the path along which the random walk escapes to infinity. The following lemma shows that we may assume that the backbone can be taken equal to $\mathbb{Z}$ without loss of generality.

Lemma 3.5.1. *For every $t \geq 0$,*

$$\mathbb{E}_{\mathcal{O}}\left(\exp\left[\sum_{x\in V(\mathcal{GW})} H(\ell_t(x))\right]\right) = \mathbb{E}_{\mathcal{O}}\left(\exp\left[\sum_{x\in V(\mathcal{GW})} H(\ell_t(x))\right] \,\middle|\, (X_{T_k})_{k\geq 0} = \mathbb{Z}\right)$$

in distribution.

Proof. We apply the following permutations to $\mathcal{GW}$ that turns the backbone into $\mathbb{Z}$. We do the permutations inductively on k as follows:

- Since X_{T_0} is already in $\mathbb{Z}$ by convention, we do not permute it.

- Suppose that $\mathcal{GW}$ has been permuted such that $X_{T_i} = i \in \mathbb{Z}$ for all $i = 0, \cdots, j-1$. Then X_{T_j} is in the subtree rooted at $X_{T_{j-1}}$. If $X_{T_j} \neq j \in \mathbb{Z}$, then we can swap it and the tree hanging below it with j and the tree hanging below it. In other words, we swap $\mathcal{T}_{X_{T_j}}$ with $\mathcal{T}_j$.

This permutation procedure preserves the edges and the vertices and so the resulting tree is isomorphic to $\mathcal{GW}$. Therefore, conditioning on the backbone being $\mathbb{Z}$ does not affect the distribution of the total mass given in (3.4). □

§3.5.2 Projection

For every vertex that is distance R from the backbone, replace the tree hanging below it by a special *tadpole* vertex. R is chosen such that it is a multiple of N, i.e. $R = nN$ for some $n \in \mathbb{N}$, with N from Section 3.1.1. Denote this truncated version of $\mathcal{GW}$ by $\mathcal{GW}_R$. We apply the following map to X. Whenever X travels farther than distance R from the backbone, its excursion is cut off and replaced by a sojourn time at the corresponding tadpole. The resulting path, which we call $X^R = (X_t^R)_{t\geq 0}$, is a Markov renewal process on $\mathcal{GW}_R$ with the following properties:

- The sojourn times in all the vertices that are not tadpoles have distribution $\mathrm{EXP}(D_x + 1)$.
- The sojourn times in all the tadpole attached to vertex x have distribution ψ_x, defined as the conditional distribution of the *return time* τ_x of the random walk on the Galton-Watson tree rooted at x *given* that $\tau_x < \infty$ (again see [30] for a proper definition).
- The transitions into the tadpoles have probability $\frac{D_x}{D_x+1}$, the transitions out of the tadpoles have probability 1 (because of the conditioning on the backbone). Here D_x denotes the number of offspring of x, which has the same distribution as D (and $D_x + 1$ is the degree of the vertex).

Write $(\ell_t^{\mathcal{GW}_R}(x))_{x\in V_{\mathcal{GW}_R}}$ to denote the local times of X^R at time t.

Lemma 3.5.2. *[Projection onto tadpoles] For every $R \in (N\mathbb{Z}) \cap \mathbb{N}$ and $t \geq 0$,*

$$\mathbb{E}_0\left(\exp\left[\sum_{x\in V(\mathcal{GW})} H(\ell_t(x))\right]\right) \leq \mathbb{E}_0\left(\exp\left[\sum_{x\in V(\mathcal{GW}_R)} H(\ell_t^{\mathcal{GW}_R}(x))\right]\right).$$

Proof. See Lemma 2.3.3. The map stacks local times on top of each other and the inequality follows from the convexity of H defined in (3.2). □

§3.5.3 Periodisation

We cut $\mathcal{GW}_R$ into periodic units of length $R+1$ with the same R as in Section 3.5.2. We do this in the natural way following the inherent periodic structure.By the contruction of the perodic Galton-Waton tree, we can fold all paths of X^R that are in $\mathcal{T}_m$ into paths in $\mathcal{T}_x$ where $m = x \mod (R+1)$ for all $m \in \mathbb{Z}$ since the trees are identical. Finally, paths that go from $R \mod (R+1)$ to $0 \mod (R+1)$ are folded by adding an additional edge between 0 and R. Denote the periodised graph by $\mathcal{GW}_{\pi,R}$ and the random walk on $\mathcal{GW}_{\pi,R}$ by $X^{\pi,R}$, which is obtained by folding X^R. Write $(\ell_t^{\pi,R}(x))_{x\in V_{\mathcal{GW}_R}}$ to denote the local times of $X^{\pi,R}$ at time t.

Lemma 3.5.3. *[Periodisation to a finite graph] For every $R \in (N\mathbb{Z}) \cap \mathbb{N}$ and $t \geq 0$,*

$$\mathbb{E}_0\left(\exp\left[\sum_{x\in V(\mathcal{GW}_R)} H(\ell_t^{GW_R}(x))\right]\right) \leq \mathbb{E}_0\left(\exp\left[\sum_{x\in V(\mathcal{GW}_{\pi,R})} H(\ell_t^{\pi,R}(x))\right]\right).$$

Proof. The periodisation again stacks local times on top of each other. □

Chapter 3

Crucial observation. Let ∂V_R be the set of vertices to which a tadpole is attached. Due to shift invariance, we may assume that the total local time spent at $\partial V_R \cup 0 \cup \star$, is at most t/R, without loss of generality. See Lemma 2.3.6 for more details.

§3.5.4 Large deviation rate function

We use the following large deviation principle for Markov renewal processes derived in [31].

To simplify notation, we define

$\star$	=	vertex R on the backbone,
$\square$	=	set of *tadpoles*,
∂V_R	=	set of vertices neighbouring $\square$,
$\mathrm{int}(V_R)$	=	$V_R \setminus (\square \cup \partial V_R)$,
$\square_x$	=	tadpole attached to $x \in \partial V_R \setminus \star$.

1. For $x \notin \square$, $\psi_x = \mathrm{EXP}(D_x + 1)$, and so

$$\lambda_x(\theta) = \begin{cases} \log\left(\frac{D_x+1}{D_x+1-\theta}\right), & \theta < D_x + 1, \\ \infty, & \theta \geq D_x + 1. \end{cases}$$

To find $\lambda_x^*(\alpha)$ we compute

$$\frac{\partial}{\partial\theta}[\alpha\theta - \log(\tfrac{D_x+1}{D_x+1-\theta})] = \alpha - \frac{1}{D_x+1-\theta}, \qquad \frac{\partial^2}{\partial\theta^2}[\alpha\theta - \log(\tfrac{D_x}{D_x+1-\theta})] = -\frac{1}{(D_x+1-\theta)^2} < 0.$$

This gives that the supremum in (A.4) is uniquely taken at

$$\theta^* = D_x + 1 - \tfrac{1}{\alpha}, \qquad \alpha > 0,$$

so that

$$\lambda_x^*(\alpha) = \alpha(D_x+1) - \log[\alpha(D_x+1)] - 1, \qquad \alpha > 0. \tag{3.10}$$

2. Inserting (3.10) into (A.1)–(A.3), we get

$$I^{\dagger}_{E_R}(p) = \sum_{x\in V_R}\sum_{y\sim x} p(x) + \inf_{\beta\in(0,\infty)} \inf_{q\in\mathcal{P}(V_R)} \sup_{\hat{q}\in\mathcal{P}(V_R)} L(\beta, q, \hat{q} \mid p),$$

where we recall that $y \sim x$ means that x and y are connected by an edge in $\mathcal{GW}_{\pi,R}$ (denoted by E_R), and

$$L(\beta, q, \widehat{q} \mid p) = -A - B - C - D,$$

where

$$\begin{aligned}
A &= \sum_{x\in\mathrm{int}(V_R)} \beta q(x) \left\{1 + \log\left(\frac{\sum_{y\sim x}\widehat{q}(y)}{\widehat{q}(x)} \frac{p(x)}{\beta q(x)}\right)\right\},\\
B &= \sum_{x\in\partial V_R\setminus\star} \beta q(x) \left\{1 + \log\left(\frac{\widehat{q}(x^{\uparrow}) + D_x\widehat{q}(\Box_x)}{\widehat{q}(x)} \frac{p(x)}{\beta q(x)}\right)\right\},\\
C &= \beta q(\star) \left\{1 + \log\left(\frac{\widehat{q}(\star^{\uparrow}) + D_\star\widehat{q}(\mathcal{O})}{\widehat{q}(\star)} \frac{p(\star)}{\beta q(\star)}\right)\right\},\\
D &= \sum_{x\in\Box} \beta q(x) \left\{\log\left(\frac{\widehat{q}(x^{\uparrow})}{\widehat{q}(x)}\right) - (\mathcal{L}\lambda_x)\left(\frac{p(x)}{\beta q(x)}\right)\right\},
\end{aligned}$$

with $\mathcal{L}\lambda_x$ the Legendre transform of the cumulant generating function of ψ_x and $x^{\uparrow}$ the unique vertex to which x is attached upwards. Note that A, B, C each combine two terms, and that A, B, C, D depend on p. We suppress this dependence because p is fixed.

3. Inserting the parametrisation $\widehat{q} = u/\|u\|_1$ and $q = v/\|v\|_1$ with $u, v\colon V_R \to (0, \infty)$ and putting $\beta q = v$, we may write

$$I^{\dagger}_{E_R}(p) = \sum_{x\in V_R} (D_x + 1)p(x) + \inf_{v\colon V_R\to(0,\infty)} \sup_{u\colon V_R\to(0,\infty)} L(u, v) \tag{3.11}$$

with

$$L(u, v) = -A - B - C - D,$$

where

$$\begin{aligned}
A &= \sum_{x\in\mathrm{int}(V_R)} v(x) \left\{1 + \log\left(\frac{\sum_{y\sim x} u(y)}{u(x)} \frac{p(x)}{v(x)}\right)\right\},\\
B &= \sum_{x\in\partial V_R\setminus\star} v(x) \left\{1 + \log\left(\frac{u(x^{\uparrow}) + D_x u(\Box_x)}{u(x)} \frac{p(x)}{v(x)}\right)\right\},\\
C &= v(\star) \left\{1 + \log\left(\frac{u(\star^{\uparrow}) + D_x u(\mathcal{O})}{u(\star)} \frac{p(\star)}{v(\star)}\right)\right\},\\
D &= \sum_{x\in\Box} v(x) \left\{\log\left(\frac{u(x^{\uparrow})}{u(x)}\right) - (\mathcal{L}\lambda_x)\left(\frac{p(x)}{v(x)}\right)\right\}.
\end{aligned} \tag{3.12}$$

Our task is to carry out the supremum over u and the infimum over v in (3.11).

4. First, we compute the infimum over v for fixed u. (Later we will make a judicious choice for u to obtain a lower bound.) Abbreviate

$$\begin{aligned} A_u(x) &= \frac{\sum_{y\sim x} u(y)}{u(x)}\, p(x), && x \in \mathrm{int}(V_R), \\ B_u(x) &= \frac{u(x^{\uparrow}) + D_x u(\square_x)}{u(x)}\, p(x), && x \in \partial V_R \backslash \star, \\ C_u(\star) &= \frac{u(\star^{\uparrow}) + D_\star u(\mathcal{O})}{u(\star)}\, p(\star). \end{aligned} \tag{3.13}$$

• For $z \in V_R$, the first derivatives of L are

$$\begin{aligned} z \in \mathrm{int}(V_R)&: \quad \frac{\partial L(u,v)}{\partial v(z)} = -\log\left(\frac{A_u(z)}{v(z)}\right), \\ z \in \partial V_R \backslash \star&: \quad \frac{\partial L(u,v)}{\partial v(z)} = -\log\left(\frac{B_u(z)}{v(z)}\right), \\ z = \star&: \quad \frac{\partial L(u,v)}{\partial v(z)} = -\log\left(\frac{C_u(z)}{v(z)}\right), \end{aligned}$$

while the second derivatives of L equal $1/v(z) > 0$. Hence the infimum is uniquely taken at

$$\begin{aligned} x \in \mathrm{int}(V_R)&: \quad \bar{v}(x) = A_u(x), \\ x \in V_R \backslash \star&: \quad \bar{v}(x) = B_u(x), \\ x = \star&: \quad \bar{v}(x) = C_u(x). \end{aligned}$$

• For $z \in \square$, the computation is more delicate. Define

$$\mu_x(\alpha) = \alpha(\mathcal{L}\lambda_x)'(\alpha) - (\mathcal{L}\lambda_x)(\alpha).$$

The function μ_x has range $(-\infty, \log M_x]$ for some $M_x < \infty$. The maximal value is uniquely taken at $\alpha = \infty$. Therefore there are two cases.

► $u(z)/u(z^{\uparrow}) \le M_z$: Abbreviate $\alpha_u(z) = p(z)/v(z)$. For $z \in \square$,

$$\begin{aligned} \frac{\partial L(u,v)}{\partial v(z)} &= \log\left(\frac{u(z)}{u(z^{\uparrow})}\right) + (\mathcal{L}\lambda_z)\left(\frac{p(z)}{v(z)}\right) - \frac{p(z)}{v(z)}(\mathcal{L}\lambda_z)'\left(\frac{p(z)}{v(z)}\right) \\ &= \log\left(\frac{u(z)}{u(z^{\uparrow})}\right) - \mu_z(\alpha_u(z)), \\ \frac{\partial^2 L(u,v)}{v(z)^2} &= \frac{p^2(z)}{v^3(z)}(\mathcal{L}\lambda_z)''\left(\frac{p(z)}{v(z)}\right) > 0, \end{aligned}$$

where we use that $\mathcal{L}\lambda_x$, being a Legendre transform, is strictly convex. Hence the infimum is uniquely taken at

$$\bar{v}(x) = \frac{p(x)}{\alpha_u(x)}, \qquad x \in \square,$$

with $\alpha_u(x)$ solving the equation

$$\log\left(\frac{u(x)}{u(x^{\uparrow})}\right) = \mu_x(\alpha_u(x)), \qquad x \in \square.$$

Since $\mu_x'(\alpha) = \alpha(\mathcal{L}\lambda_x)''(\alpha)$ and $\mathcal{L}\lambda_x$ is strictly convex (see Fig. A.3 in Appendix A.2), μ_x is strictly increasing and therefore invertible. Consequently,

$$\alpha_u(x) = \mu_x^{-1}\left(\log\left(\frac{u(x)}{u(x^{\uparrow})}\right)\right), \qquad x \in \square.$$

Putting the above formulas together, we arrive at (recall (3.13))

$$\begin{aligned} L(u) &= \inf_{v\colon V_R \to (0,\infty)} L(u,v) \\ &= -\sum_{x \in \mathrm{int}(V_R)} A_u(x) \quad - \sum_{x \in \partial V_R \setminus \star} B_u(x) \quad - C_u(\star) \quad + \sum_{x \in \square} D_u(x) \end{aligned} \tag{3.14}$$

with (recall (3.12))

$$\begin{aligned} D_u(x) &= -\frac{p(x)}{\alpha_u(x)}\left[\log\left(\frac{u(x^{\uparrow})}{u(x)}\right) - (\mathcal{L}\lambda_x)(\alpha_u(x))\right] \\ &= \frac{p(x)}{\alpha_u(x)}\left[(\mathcal{L}\lambda_x)(\alpha_u(x)) - \mu_x(\alpha_u(x))\right] \\ &= p(x)\,(\mathcal{L}\lambda_x)'(\alpha_u(x)) = p(x)\left((\mathcal{L}\lambda_x)' \circ \mu_x^{-1}\right)\left(\log\left(\frac{u(x)}{u(x^{\uparrow})}\right)\right). \end{aligned}$$

▶ $u(x)/u(x^{\uparrow}) > M_z$: In this case $\frac{\partial L(u,v)}{\partial v(z)} > 0$, the infimum is uniquely taken at $\bar{v}(x) = 0$, and

$$D_u(x) = p(x)\,\theta_{c,x}, \qquad x \in \square,$$

where we use (A.13). Note that the right-hand side does not depend on u.

5. Recall that $(\mathcal{L}\lambda_x)'(\alpha) = \theta_x^*(\alpha)$ and let $\alpha_x = \mu_x^{-1}(\log(\frac{u(x)}{u(x^{\uparrow})}))$. For $u(x)/u(x^{\uparrow}) \in [1, M_x]$,

$$\theta_x^*(\alpha_x) \geq \theta_{\min}^*(\alpha_x),$$

while for $u(x)/u(x^{\uparrow}) \in (-\infty, 1)$,

$$\theta_x^*(\alpha_x) < \theta^*(\alpha_x).$$

This follows from the fact that the sojourn time at x is stochastically smaller than that on the minimal tree. It also follows that

$$\theta_{c,x} \geq \theta_{c,\min}.$$

We may therefore estimate

$$L(u) \geq \bar{L}(u) := -\sum_{x \in \mathrm{int}(V_R)} A_u(x) \quad - \sum_{x \in \partial V_R \setminus \star} B_u(x) \quad - C_u(\star) \quad + \sum_{x \in \square} \bar{D}_u(x),$$

where

$$\bar{D}_u(x) = \begin{cases} p(x)\left((\mathcal{L}\lambda_{\max})' \circ \mu_{\max}^{-1}\right)\left(\log\left(\frac{u(x)}{u(x^\uparrow)}\right)\right), & u(x)/u(x^\uparrow) \in (-\infty, 1), \\ p(x)\left((\mathcal{L}\lambda_{\min})' \circ \mu_{\min}^{-1}\right)\left(\log\left(\frac{u(x)}{u(x^\uparrow)}\right)\right), & u(x)/u(x^\uparrow) \in [1, \sqrt{d^-}], \\ p(x)\theta_{c,\min}, & u(x)/u(x^\uparrow) > \sqrt{d^-}. \end{cases}$$

To simplify notation, we suppress min from the notation. For $u(x)/u(x^\uparrow) \in [1, d^-]$, it can be shown that $(\mathcal{L}\lambda)' \circ \mu^{-1} = \lambda^{-1}$. Moreover, $(\lambda^{-1} \circ \log) = S$ with

$$S(\beta) = d^- + 1 - \beta - \frac{d^-}{\beta}, \qquad \beta \in (0, \sqrt{d^-}\,]. \tag{3.15}$$

Since S has domain $(0, \sqrt{d^-}\,]$, $D_u(x)$ is only defined when $u(x)/u(x^\uparrow) \leq \sqrt{d^-}$, in which case

$$\bar{D}_u(x) = p(x)\, S\left(\frac{u(x)}{u(x^\uparrow)}\right), \qquad x \in \square. \tag{3.16}$$

On the other hand for $u(x)/u(x^\uparrow) > d^-$,

$$\bar{D}_u(x) = p(x)\,(\sqrt{d^-} - 1)^2 = p(x)\, S(\sqrt{d^-}), \qquad x \in \square.$$

For $u(x)/u(x^\uparrow) \in (-\infty, 1)$, (3.15) and (3.16) hold with d^- replaced by d^+.

6. Next, we compute the supremum over u for $\bar{L}$. The first derivatives of $\bar{L}$ are

$$\begin{aligned}
z \in \mathrm{int}(V_R)\backslash\mathcal{O}: \quad & \frac{\partial \bar{L}(u)}{\partial u(z)} = \frac{\sum_{y\sim z} u(y)}{u^2(z)}\, p(z) - \sum_{y\sim z} \frac{1}{u(y)}\, p(y), \\
z = \mathcal{O}: \quad & \frac{\partial \bar{L}(u)}{\partial u(\mathcal{O})} = \frac{\sum_{y\sim\mathcal{O}} u(y)}{u(\mathcal{O})^2}\, p(\mathcal{O}) - \sum_{y:y^\uparrow=\mathcal{O}} \frac{1}{u(y)} p(y) - \frac{D_\star}{u(\star)}\, p(\star), \\
z = \star: \quad & \frac{\partial \bar{L}(u)}{\partial u(\star)} = -\frac{1}{u(\mathcal{O})}\, p(\mathcal{O}) + \frac{u(\star^\uparrow) + D_\star u(\mathcal{O})}{u(\star)^2}\, p(\star), \\
z \in \partial V_R\backslash\star: \quad & \frac{\partial \bar{L}(u)}{\partial u(z)} = -\frac{1}{u(z^\uparrow)}\, p(z^\uparrow) + \frac{u(z^\uparrow) + D_z u(\square_z)}{u(z)^2}\, p(z) \\
& \qquad + \left[\frac{u(\square_z)}{u(z)^2} - \frac{d^-}{u(\square_z)}\right] p(\square_z) 1_{\left\{\frac{u(z)}{u(z^\uparrow)} \in [1,\sqrt{d}]\right\}} \\
& \qquad + \left[\frac{u(\square_z)}{u(z)^2} - \frac{d^+}{u(\square_z)}\right] p(\square_z) 1_{\left\{\frac{u(z)}{u(z^\uparrow)} < 1\right\}}, \\
z \in \square: \quad & \frac{\partial \bar{L}(u)}{\partial u(z)} = -\frac{D_{z^\uparrow}}{u(z^\uparrow)}\, p(z^\uparrow) + \left[-\frac{1}{u(z^\uparrow)} + \frac{d^- u(z^\uparrow)}{u(z)^2}\right] p(z)\, 1_{\left\{\frac{u(z)}{u(z^\uparrow)} \in [1,\sqrt{d}]\right\}} \\
& \qquad + \left[-\frac{1}{u(z^\uparrow)} + \frac{d^+ u(z^\uparrow)}{u(z)^2}\right] p(z)\, 1_{\left\{\frac{u(z)}{u(z^\uparrow)} < 1\right\}}.
\end{aligned} \tag{3.17}$$

Chapter 3

The second derivatives of L are all < 0. The first line in (3.17) can be rewritten as

$$\sum_{y\sim z} u(y)\left[\frac{p(z)}{u^2(z)}-\frac{p(y)}{u^2(y)}\right],$$

which is zero when

$$\bar{u}(x)=\sqrt{p(x)}, \qquad x\in \mathrm{int}(V_R)\setminus \mathcal{O}. \tag{3.18}$$

Given the choice in (3.18), the fourth line in (3.17) is zero when

$$\bar{u}(x)=\sqrt{\frac{d^-p(x^\uparrow)p(x)}{D_{x^\uparrow}p(x^\uparrow)+p(x)}}1_{\left\{\frac{u(x)}{u(x^\uparrow)}\in[1,\sqrt{d^-}]\right\}}+\sqrt{\frac{d^+p(x^\uparrow)p(x)}{D_{x^\uparrow}p(x^\uparrow)+p(x)}}1_{\left\{\frac{u(x)}{u(x^\uparrow)}<1\right\}}, \qquad x\in\Box. \tag{3.19}$$

Furthermore, the derivative in the fifth line is strictly negative when both indicators are 0 and therefore at least one indicator must be 1. This is guaranteed since the quotient

$$\frac{\bar{u}(x)}{\bar{u}(x^\uparrow)}=\sqrt{\frac{d^-p(x)}{D_{x^\uparrow}p(x^\uparrow)+p(x)}}1_{\left\{\frac{u(x)}{u(x^\uparrow)}\in[1,\sqrt{d^-}]\right\}}+\sqrt{\frac{d^+p(x^\uparrow)p(x)}{D_{x^\uparrow}+p(x)}}1_{\left\{\frac{u(x)}{u(x^\uparrow)}<1\right\}}, \qquad x\in\Box,$$

is bounded from above by $\sqrt{d}$ for all $D_{x^\uparrow}$. In addition, we can rewrite the indicators in (3.19) to get

$$\bar{u}(x)=\sqrt{\frac{d^-p(x^\uparrow)p(x)}{D_{x^\uparrow}p(x^\uparrow)+p(x)}}1_{\left\{\frac{p(x)}{p(x^\uparrow)}\geq\frac{D_{x^\uparrow}}{d^--1}\right\}}+\sqrt{\frac{d^+p(x^\uparrow)p(x)}{D_{x^\uparrow}p(x^\uparrow)+p(x)}}1_{\left\{\frac{p(x)}{p(x^\uparrow)}<\frac{D_{x^\uparrow}}{d^+-1}\right\}}, \qquad x\in\Box. \tag{3.20}$$

Given the choice in (3.18)–(3.19), also the fourth line in (3.17) is zero. Thus, only the second and third line in (3.17) are non-zero, but this is harmless because $\mathcal{O},\star$ carry a negligible weight in the limit as $R\to\infty$,because of the constraint $p(\partial V_R\cup\mathcal{O})\leq 1/R$ (recall Section 3.5.3). To simplify notation, define

$$\Omega_\Box^-(x)=\sqrt{\frac{d^-p(x^\uparrow)p(x)}{D_{x^\uparrow}p(x^\uparrow)+p(x)}}, \qquad \Omega_\Box^+(x)=\sqrt{\frac{d^+p(x^\uparrow)p(x)}{D_{x^\uparrow}p(x^\uparrow)+p(x)}},$$

and

$$\Omega^-(x)=\sqrt{\frac{d^-p(x)p(\Box_x)}{D_xp(x)+p(\Box_x)}}, \qquad \Omega^+(x)=\sqrt{\frac{d^+p(x)p(\Box_x)}{D_xp(x)+p(\Box_x)}}.$$

Inserting (3.18)–(3.19) into (3.14) and using (3.13) and (3.16), we get the following lower bound:

$$
\begin{aligned}
&\sup_{u\colon V_R\to(0,\infty)} \bar{L}(u)\\
&\geq -\sum_{x\in\mathrm{int}(V_R)} A_{\bar{u}}(x) \quad - \sum_{x\in\partial V_R\setminus\star} B_{\bar{u}}(x) \quad - C_{\bar{u}}(\star) + \sum_{x\in\Box} \bar{D}_{\bar{u}}(x)\\
&= -\sum_{x\in\mathrm{int}(V_R)} \sum_{y\sim x} \sqrt{p(y)p(x)}\\
&\quad -\sum_{x\in\partial V_R\setminus\star} \sqrt{p(x)}\left(\sqrt{p(x^\uparrow)} + D_x\Omega^-(x)1_{\left\{\frac{p(\Box_x)}{p(x)}\geq\frac{D_x}{d^--1}\right\}} + D_x\Omega^+(x)1_{\left\{\frac{p(\Box_x)}{p(x)}<\frac{D_x}{d^--1}\right\}}\right)\\
&\quad -\sqrt{p(\star)}\left(\sqrt{p(\star^\uparrow)} + D_\star\sqrt{p(\mathcal{O})}\right)\\
&\quad +\sum_{x\in\Box} p(x)\left[S^-\left(\frac{\Omega^-_\Box(x)}{\sqrt{p(x^\uparrow)}}\right)1_{\left\{\frac{p(x)}{p(x^\uparrow)}\geq\frac{D_{x^\uparrow}}{d^--1}\right\}} + S^+\left(\frac{\Omega^+_\Box(x)}{\sqrt{p(x^\uparrow)}}\right)1_{\left\{\frac{p(x)}{p(x^\uparrow)}<\frac{D_{x^\uparrow}}{d^+-1}\right\}}\right]
\end{aligned}
$$

7. Using the relation $(D_x+1)p(x) = \sum_{y\sim x} p(x)$, $x\in\mathrm{int}(V_R)$, we get from (3.11) that

$$I^\dagger_{E_R}(p) \geq K^1_R(p) + K^2_R(p)$$

with

$$
\begin{aligned}
K^1_R(p) &= \sum_{x\in\mathrm{int}(V_R)} \sum_{y\sim x}\left[p(x) - \sqrt{p(x)p(y)}\right]\\
&= \sum_{\{x,y\}\in E_R\setminus\{\mathcal{O},\star\}} \left(\sqrt{p(x)} - \sqrt{p(y)}\right)^2 + \left[p(\mathcal{O}) - \sqrt{p(\mathcal{O})p(\star)}\right] - \sum_{x\in\partial V_R}\left[p(x) - \sqrt{p(x)p(x^\uparrow)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
K^2_R(p) = \sum_{x\in\partial V_R\setminus\star} \Bigg[&(D_x+1)p(x)\\
&-\sqrt{p(x)}\left(\sqrt{p(x^\uparrow)} + D_x\Omega^-(x)1_{\left\{\frac{p(\Box_x)}{p(x)}\geq\frac{D_x}{d^--1}\right\}} + D_x\Omega^+(x)1_{\left\{\frac{p(\Box_x)}{p(x)}<\frac{D_x}{d^--1}\right\}}\right)\Bigg]\\
&+(D_\star+1)p(\star) - \sqrt{p(\star)}\left(\sqrt{p(\star^\uparrow)} + D_\star\sqrt{p(\mathcal{O})}\right)\\
+\sum_{x\in\Box} p(x)&\left[S^-\left(\frac{\Omega^-_\Box(x)}{\sqrt{p(x^\uparrow)}}\right)1_{\left\{\frac{p(x)}{p(x^\uparrow)}\geq\frac{D_{x^\uparrow}}{d^--1}\right\}} + S^+\left(\frac{\Omega^+_\Box(x)}{\sqrt{p(x^\uparrow)}}\right)1_{\left\{\frac{p(x)}{p(x^\uparrow)}<\frac{D_{x^\uparrow}}{d^+-1}\right\}}\right].
\end{aligned}
$$

The first sum in the right-hand side of $K^1_R(p)$ equals the *standard rate function* $I_{\widehat{E}_R}(p)$ without $\{\mathcal{O},\star\}$ and the tadpoles. Rearranging and simplifying terms, we arrive at

$$I^\dagger_{E_R}(p) \geq I_{\widehat{E}_R}(p) + K^3_R(p) \tag{3.21}$$

with

$$K^3_R(p) = S_{\partial V_R\setminus\star}(p) + S_{\mathcal{O},\star}(p) + S_{(\partial V_R\setminus\star)\cup\Box}(p),$$

where

$$S_{\partial V_R\setminus\star}(p) = \sum_{x\in\partial V_R\setminus\star} D_x p(x),$$

$$S_{\mathcal{O},\star}(p) = \left(\sqrt{p(\mathcal{O})} - \sqrt{p(\star)}\right)^2 + (D_\star - 1)\left[p(\star) - \sqrt{p(\mathcal{O})p(\star)}\right],$$

$$S_{(\partial V_R\setminus\star)\cup\square}(p) = -\sum_{x\in\partial V_R\setminus\star} p(x)\, D_x \left(\frac{\Omega^-(x)}{\sqrt{p(x)}} 1_{\left\{\frac{p(\square_x)}{p(x)} \geq \frac{D_x}{d^- -1}\right\}} + \frac{\Omega^+(x)}{\sqrt{p(x)}} 1_{\left\{\frac{p(\square_x)}{p(x)} < \frac{D_x}{d^- -1}\right\}}\right)$$
$$+ \sum_{x\in\square} p(x)\left[S^-\left(\frac{\Omega^-_\square(x)}{\sqrt{p(x^\uparrow)}}\right) 1_{\left\{\frac{p(x)}{p(x^\uparrow)} \geq \frac{D_{x^\uparrow}}{d^- -1}\right\}} + S^+\left(\frac{\Omega^+_\square(x)}{\sqrt{p(x^\uparrow)}}\right) 1_{\left\{\frac{p(x)}{p(x^\uparrow)} < \frac{D_{x^\uparrow}}{d^+ -1}\right\}}\right]. \tag{3.22}$$

8. Since $\sqrt{p(\mathcal{O})p(\star)} \leq \frac{1}{2}[p(\mathcal{O})+p(\star)]$, the boundary constraint $\sum_{x\in\partial V_R\cup\mathcal{O}} p(x) \leq 1/R$ (recall Section 3.5.3) implies that $S_{\partial V_R\setminus\star}(p)+S_{\mathcal{O},\star}(p) = O(1/R)$. The same constraint implies that the first sum in $S_{(\partial V_R\setminus\star)\cup\square}(p)$ is $O(1/R)$. Hence

$$K_R^3(p) \geq O(1/R) + \sum_{x\in\partial V_R\setminus\star} p(x)\, F\left(\tfrac{p(\square_x)}{p(x)}\right)$$

with

$$F(w) = w\left(d^- + 1 - \sqrt{d}\left[\sqrt{\frac{w}{d+w}} + \sqrt{\frac{d^+ + w}{w}}\right]\right) 1_{\left\{\frac{p(\square_x)}{p(x)} \geq \frac{D_x}{d^- -1}\right\}}$$
$$+ w\left(d^+ + 1 - \sqrt{d^+}\left[\sqrt{\frac{w}{d+w}} + \sqrt{\frac{d^+ + w}{w}}\right]\right) 1_{\left\{\frac{p(\square_x)}{p(x)} < \frac{D_x}{d^- -1}\right\}}.$$

The map $w \mapsto F(w)$ is continuous on $(0,\infty)$ with

$$F(w) = \begin{cases} -\frac{\sqrt{d^-}}{\sqrt{d^+}}\sqrt{w} + (d^- + 1)w + O(w^{3/2}), & w \downarrow 0,\\ [(d^- + 1) - 2\sqrt{d^-}\,]\,w + \sqrt{d^-}(d^- - d^+)/2 + O(w^{-1}), & w\to\infty, \end{cases}$$

on the first indicator, while

$$F(w) = \begin{cases} -\sqrt{w} + (d^+ + 1)w + O(w^{3/2}), & w \downarrow 0,\\ [(d^+ + 1) - 2\sqrt{d^+}\,]\,w + \sqrt{d^-}(d^- - d^+)/2 + O(w^{-1}), & w\to\infty, \end{cases}$$

on the second indicator. From this we see that if $d^+ \geq d^- \geq 4$, then there exists a $C \in (1,\infty)$ such that

$$F(w) + C \geq \left(1 - \sqrt{w}\right)^2, \qquad w\in[0,\infty).$$

Hence we have the lower bound

$$K_R^3(p) \geq O(1/R) + \sum_{x\in\partial V_R\setminus\star} p(x)\left[-C + \left(1 - \sqrt{\tfrac{p(\square_x)}{p(x)}}\right)^2\right]$$
$$= O(1/R) + \sum_{x\in\partial V_R\setminus\star}\left(\sqrt{p(x)} - \sqrt{p(\square_x)}\right)^2.$$

Via (3.21)–(3.22), it follows that

$$I^{\dagger}_{E_R} \geq O(1/R) + I_{\widetilde{E}_R}(p), \qquad R \in \mathbb{N},$$

with $I_{\widetilde{E}_R}(p)$ the *standard rate function* and

$$\widetilde{E}_R = \widehat{E}_R \cup \big[\cup_{x\in\partial V_R\setminus\star} \{x, \Box_x\}\big]$$

the set of edges obtained by *removing the edge* $\{\mathcal{O}, \star\}$.

§3.6 Limit of the upper variational formula

With the above, we have shown that

$$\frac{1}{t}\log\langle U(t)\rangle \leq \varrho\log(\varrho t) - \varrho - \chi^{+}_{R}(\varrho) + o(1), \qquad t\to\infty,$$

with

$$\chi^{+}_{R}(\varrho) = \inf_{p\in\mathcal{P}(\mathcal{GW}_{\pi,R})} \Big\{ I^{\dagger}_{E}(p) + \varrho J_{V(\mathcal{GW}_{\pi,R})}(p)\Big\}. \tag{3.23}$$

It only remains to show that $\liminf_{R\to\infty}\chi^{+}_{R}(\varrho) \geq \chi_{\mathcal{GW}}(\varrho)$. Due to shift invariance (recall Section 3.5.3), we may assume that the minimiser in (3.23) satisfies $\sum_{x\in\partial V_R\cup\mathcal{O}} p(x) \leq 1/R$. Next

$$\mathcal{GW}_{\pi,R} \subseteq \mathcal{GW}.$$

Consequently,

$$I_{\widetilde{E}_R}(p) = I_E(p) - \sum_{x\in\partial V_R\setminus\star} (D_x - 1)p(x), \qquad \forall\, p\in\mathcal{P}(\mathcal{GW})\colon\ \mathrm{supp}(p)\subseteq\mathcal{GW}_{\pi,R},$$

where the sum compensates for the contribution coming from the edges in $\mathcal{GW}$ that link the vertices in $\partial V_R\setminus\star$ to the vertices one layer deeper in $\mathcal{GW}$ that are not tadpoles. Since this sum is $O(1/R)$, we obtain

$$\begin{aligned}\chi^{+}_{R}(\varrho) &= \inf_{p\in\mathcal{P}(V(\mathcal{GW}_{\pi,R}))} \Big\{ I^{\dagger}_{E}(p) + \varrho J_{V(\mathcal{GW}_{R})}(p)\Big\}\\ &\geq O(1/R) + \inf_{p\in\mathcal{P}(V(\mathcal{GW}))\colon\ \mathrm{supp}(p)\subseteq V(\mathcal{GW}_{\pi,R})} \big\{ I_{E(\mathcal{GW})}(p) + \varrho J_{V(\mathcal{GW})}(p)\big\}\\ &\geq O(1/R) + \chi_{\mathcal{GW}}(\varrho),\end{aligned}$$

where the last inequality follows after dropping the constraint under the infimum.

Chapter 3