The parabolic Anderson model on Galton-Watson trees
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CHAPTER 1
Introduction

§1.1 Introduction

§1.1.1 Definitions, intermittency and problems

The parabolic Anderson model (abbreviated PAM) is the Cauchy problem for the heat equation with random potential. It is given by the parabolic differential equation

$$\begin{array}{lcll} \frac{\partial}{\partial t}u(x,t) & = & \kappa\Delta u(x,t)+\xi(x)u(x,t), & t>0,\, x\in\mathscr{X}, \\ u(x,0) & = & f(x), & x\in\mathscr{X}, \end{array} \tag{1.1}$$

where $\kappa > 0$ is the diffusion constant, $\mathscr{X}$ an ambient space, Δ the Laplace operator acting on functions on $\mathscr{X}$, ξ the random potential, and f the initial condition. The ambient space $\mathscr{X}$ can be taken to be continuous, and/or the potential taken to be time dependent. However, we shall restrict to discrete spaces (graphs) and a static potential in this thesis. See [34], [18], [20] for more background on $\mathbb{R}^d$, and [10], [17] for works on time-dependent potentials.

Under mild conditions on the initial condition and potential, the unique non-negative solution to (1.1) admits the the well-known *Feynman-Kac* representation

$$u(x,t) = \mathbb{E}_x\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s} f(X_t)\right], \tag{1.2}$$

where $X = (X_t)_{t\geq 0}$ is the Markov process with generator $\kappa\Delta$, i.e. a continuous-time random walk with jump rate κ to each neighbour. In addition, $\mathbb{P}_x$ denotes probability with respect to X given that $X_0 = x$. Since the equations

$$\frac{\partial}{\partial t}v(x,t) = \kappa\Delta v(x,t), \qquad \frac{\partial}{\partial t}w(x,t) = \xi(x)w(x,t),$$

have solutions

$$v(x,t) = \mathbb{E}_x(f(X_t)), \qquad w(x,t) = \mathrm{e}^{\int_0^t \xi(x)\mathrm{d}s} f(x),$$

the solution in (1.2) is heuristically plausible.

One interpretation of the PAM concerns the following non-interacting particle system. At time t and site x, particles are killed with rate $\xi^-(x)$ or are split into two with rate $\xi^+(x)$, where $\xi(x) = \xi^+(x) - \xi^-(x)$. At the same time, each particle jumps independently with $\kappa\Delta$ as generator. The solution $u(x,t)$ is the average number of particles (or *mass*) at site x and time t, with initial condition $f(x)$ that is integer valued. The parabolic problem is a classical model of a system evolving in an inhomogeneous random medium and has wide ranging applications. By considering the particles above as organisms, the PAM has an application in population dynamics. The PAM is also related to other physical problems, including the Burger's equation, and the advection-convection equation for a temperature field. See [10] for more details and more applications.

Another contributing factor to the popularity of PAM is that the model exhibits *intermittency*, rather than being spatially homogeneous. This is related to Anderson localisation observed in the associated Anderson Hamiltonian $\kappa\Delta + \xi$ studied in [1], where the eigenvectors of the operator are concentrated around a point and decay exponentially. Heuristically, intermittency is the related phenomenon where the majority of the total mass

$$U(t) = \sum_{x \in \mathscr{X}} u(x,t)$$

is concentrated in specific regions of $\mathscr{X}$ as $t \to \infty$ known as *intermittent islands*. See [29][Section 2.2.4] for further details on how Anderson localisation relates to intermittency. The natural questions regarding intermittency are therefore:

(a) What is the asymptotics of $U(t)$ as $t \to \infty$?

(b) Where are the intermittent island(s) situated in $\mathscr{X}$? How does ξ affect the intermittent islands?

(c) What do ξ and $u(t,x)$ look like on the intermittent islands?

The above intuition is too imprecise to provide a rigorous definition of intermittency. In the literature, intermittency is defined using the moments of the total mass

$$m_p(t) = \frac{1}{t} \log\langle U(t)^p \rangle, \quad p \in \mathbb{N}, \tag{1.3}$$

where $\langle \cdot \rangle$ denotes expectation with respect to the potential ξ. Here $m_p(t)$ is known as the p^{th} *Lyapunov exponent*, and the model is said to be intermittent if

$$m_1 < \frac{m_2}{2} < \cdots < \frac{m_p}{p} < \cdots \tag{1.4}$$

as $t \to \infty$. We refer to [29][Section 1.4] for a more complete picture as to why this definition captures the phenomenon described before. Loosely speaking, (1.4) says that moments grow faster than previous moments by an exponential factor. This can only happen if large amounts of mass concentrate on small regions of $\mathscr{X}$, so that the main contribution to the moments comes from regions where the solution takes large values. In other words, the overwhelming part of the mass has to be concentrated in these regions for large t. This means that the definition in terms of moments is consistent with the heuristics given before.

Understanding the asymptotics of $m_p(t)$ is often the first step in the analysis of the PAM, as it quantifies the intermittency phenomenon and resolves the first question above. There is a distinction between the *quenched* total mass and the *annealed* total mass, i.e. the total mass taken almost surely with respect to the potential or averaged over the potential. In (1.3) and (1.4) the total mass was taken annealed, although the quenched asymptotics of $U(t)$ is also relevant. For the other questions regarding the intermittent islands, further analysis is required. The analysis in this case is done through a *characteristic variational formula* that optimises the large deviation probability of realisations of the potential and the principal eigenvalue of the Anderson

Hamiltonian on the intermittent islands. The variational formula also serves as the second order correction term in the asymptotics of both the quenched and the annealed total mass. As will become clear later, this is because potential profiles that are 'close' to the minimiser of the variational formula contribute most to the total mass. There is again a distinction between the annealed and the quenched variational formulae despite their similarities. In essence they approach the problem from different starting points and hence differ in their interpretation and are also formulated slightly differently.

§1.1.2 The parabolic Anderson model on a lattice

The study of the PAM on discrete spaces originated on the lattice $\mathbb{Z}^d$, $d \geq 1$, through the seminal works [24] and [25] by Gärtner and Molchanov. In particular, they showed that the (non-negative) solution to (1.1) exists and is unique under mild conditions, namely non-negative bounded initial conditions and that the potential does not percolate from below. In addition, the solution has the simple representation given by (1.2).

Henceforward, we shall assume that $f(x) = \delta_0(x)$, the potential ξ to be independent and identically distributed, and $\kappa = 1$. In this case, the PAM in (1.1) reads

$$\begin{array}{lcll} \frac{\partial}{\partial t} u(x,t) & = & \Delta u(x,t) + \xi(x) u(x,t), & t > 0,\, x \in \mathbb{Z}^d, \\ u(x,0) & = & \delta_0(x), & x \in \mathbb{Z}^d, \end{array} \tag{1.5}$$

where the Laplacian is given by

$$(\Delta f)(x) = \sum_{\substack{y \in \mathbb{Z}^d \\ |y-x|=1}} [f(y) - f(x)]. \tag{1.6}$$

The solution is then given by

$$u(x,t) = \mathbb{E}_0 \left[\mathrm{e}^{\int_0^t \xi(X_s) \mathrm{d}s} \mathbb{1}\{X_t = x\} \right], \tag{1.7}$$

where we have used time reversal. Taking a point mass at 0 is the standard choice of initial condition, as it allows for the total mass $U(t)$ to be expressed as

$$U(t) = \sum_{x \in \mathbb{Z}^d} u(x,t) = \mathbb{E}_0 \left[\mathrm{e}^{\int_0^t \xi(X_s) \mathrm{d}s} \right]. \tag{1.8}$$

The result for arbitrary κ can be obtained from those for $\kappa = 1$, by scaling time.

Since [24] and [25], the PAM on $\mathbb{Z}^d$ has been extensively studied and is now well understood. We give a brief summary of the main results under the above assumptions. We borrow from [29].

Suppose that the cumulant generating function of the potential

$$H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$$

belongs to the *de Haan* class of regular functions, i.e. that

$$\lim_{t\to\infty} \frac{H(ct) - cH(t)}{g(t)} = \hat{H}(t) \not\equiv 0 \tag{1.9}$$

for all $c \neq 1$, and that $\lim_{t\to\infty} g(t)/t$ exists. It is known from [35] that there are 4 qualitatively different regimes (known as *universality classes*) for the behaviour of the total mass, as well as the size and number of intermittent islands. Precisely which regime the asymptotics belong to depends on the the limit $\hat{H}$, which is determined by the upper tail of the distribution of $\xi(0)$. The 4 classes are:

(a) *Double exponential.* This is the regime that was originally studied in [25]. Potentials in this class have upper tails approximately equal to

$$\mathsf{P}(\xi(0) > u) = \mathrm{e}^{-\mathrm{e}^{u/\varrho}}, \qquad u \in \mathbb{R}, \tag{1.10}$$

for a parameter $\varrho \in (0,\infty)$. The unique feature of this class is that the intermittent islands do not shrink nor grow with time, while also being non-trivial. In this class, the moments of the total mass satisfy, as $t \to \infty$,

$$\langle U(t)^p\rangle = \exp[H(pt) - pt\chi(\varrho) + o(t)], \tag{1.11}$$

where $\chi(\varrho)$ is the annealed characteristic variational formula given by

$$\chi(\varrho) := \inf_{p\in\mathcal{P}(\mathbb{Z}^d)} [I(p) + \varrho J(p)], \qquad \varrho \in (0,\infty), \tag{1.12}$$

with

$$I(p) = \sum_{x,y\in\mathbb{Z}^d : x\sim y} \left(\sqrt{p(x)} - \sqrt{p(y)}\right)^2, \qquad J(p) = -\sum_{x\in\mathbb{Z}^d} p(x)\log p(x).$$

In this regime, $H(t) = \varrho t\log(\varrho t) - \varrho t + o(t)$ as $t \to \infty$. Furthermore, the variational formula was studied in [23] and it is known that the minimiser decomposes into the d-fold tensor product of the minimiser for the case $d = 1$. The minimiser of the one-dimensional problem is unimodal (the maximum can be taken at 0 due to translation invariance), and decreasing with the distance to the maximum. Furthermore, it is known that the minimiser is unique up to translations for ϱ sufficiently large.

For the quenched case, [25] showed that as, $t \to \infty$,

$$\frac{1}{t}\log U(t) = \frac{H(d\log t)}{d\log t} - \chi_\star(\varrho) + o(1), \tag{1.13}$$

where $\chi_\star(\varrho)$ is the quenched variational formula. In this case, the annealed and the quenched variational formulae are equal and given by (1.12). However, we emphasise that the two have different interpretations despite them being equal in this case.

(b) *Single point.* This class comprises potentials with heavier tails and corresponds to the double exponential class with $\varrho = \infty$. This class was also analysed in [25], as a limiting case for the double exponential class. Examples of distributions in this class include the normal distribution, the Weibull distribution with

$$\mathsf{P}(\xi(0) > u) = \mathrm{e}^{-Cu^{\alpha}}, \qquad \alpha > 1,$$

where the condition $\alpha > 1$ ensures all exponential moments are finite. In this class, the results are consistent with the double exponential class with $\varrho = \infty$. The moments of the total mass satisfy, as $t \to \infty$,

$$\langle U(t)^p \rangle = \exp[H(pt) - pt\chi(\infty) + o(t)].$$

The variational formula is also the same as (1.12) at $\varrho = \infty$, in which case the minimiser is taken at a single point and the value is equal to $2d$. Therefore the intermittent island is a single site.

(c) *Bounded Potential.* The classical example is

$$\mathsf{P}(\xi(0) = 0) = p, \qquad \mathsf{P}(\xi(0) = -\infty) = 1 - p,$$

for $p \in [0, 1]$ which corresponds to Bernoulli traps. In this case the total mass can be interpreted as the survival probability of the random walk. In this class, the diameter of the intermittent islands diverges to ∞ at least as fast as some power of t as $t \to \infty$. Furthermore, it is known from [32] that the minimiser of the variational formula exists, is unique and is compactly supported on a ball. The quenched and annealed asymptotics for the total mass are complicated and involve the diameter of the intermittent islands. See [7] for details.

(d) *Almost bounded.* This class lies between the bounded and the double exponential classes in terms of the thickness of the tails. An example is obtained after ϱ is replaced by a regular function $\varrho(u)$ that tends to 0 as $u \to \infty$. Here, the diameter of the intermittent islands diverges to ∞, but slower than any power of t as $t \to \infty$. See [21] for further details.

§1.1.3 The parabolic Anderson model on random graphs

On a general graph $G = (V, E)$, the PAM is defined by

$$\begin{aligned} \tfrac{\partial}{\partial t} u(x,t) &= \Delta u(x,t) + \xi(x) u(x,t), && t > 0,\, x \in V, \\ u(x,0) &= \delta_{\mathcal{O}}(x), && x \in V. \end{aligned} \tag{1.14}$$

The Laplacian is given by

$$(\Delta f)(x) = \sum_{y \sim x} [f(y) - f(x)], \tag{1.15}$$

where the sum runs over all neighbours of x, i.e. $y \sim x$ means that $\{x, y\} \in E$. Note that this is precisely equal to the graph Laplacian matrix on G given by $D - A$, where D is the degree matrix and A is the adjacency matrix. The solution is given by

$$u(x,t) = \mathbb{E}_{\mathcal{O}}\left[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\,\mathbb{1}\{X_t = x\}\right], \tag{1.16}$$

where the random walk has jump rate 1 along the edges E, and $\mathcal{O} \in V$. In the case of a rooted graph, $\mathcal{O}$ is chosen to be the root.

There has been little work in the area of deterministic graphs. The most impactful works in this direction are [16] and [5]. The former concerned the PAM on a complete graph of size N with an i.i.d. exponentially distributed potential. All moments are eventually infinite as $t \to \infty$ under this potential, hence intermittency was quantified by showing that

$$\frac{u(t,y)}{U(t)} \to 1 \quad a.s. \tag{1.17}$$

as $t \to \infty$ and $N \to \infty$ simultaneously (in any relation), where y is the site with the largest value of ξ. This is to be expected, as the exponential distribution has even thicker tails than the single point class above. Furthermore, it was shown that there is a phase transition depending on whether $t/\log N \to 0$ or not, i.e. whether time grows faster than the logarithm of the size of the graph or not. In the case of the hypercube $\{0,1\}^n$, the potential was taken to be i.i.d. normally distributed with mean 0 and variance n. In this case, the PAM exhibits a similar behaviour as on the complete graph: as $t, n \to \infty$, most of the mass is concentrated on one site in the same sense as (1.17). This site is again where the potential is maximal. Moreover, there is again a phase transition, except that the critical time scale is $n \log n$. The PAM on the hypercube has biological applications, as it can model the occurrence of mutants in a large population. The hypercube can be interpreted as the set of all possible gene sequences of 2 alleles (the gene pool), and the potential can be seen as a random fitness landscape.

For random graphs, even less is known. The study of the PAM on random graphs was initiated in [12] on the Galton-Watson tree $(G, V, \mathcal{O})$ with offspring distribution bounded below by a constant $d \geq 2$, and with expectation ϑ. Consequently, the Galton-Watson trees considered are infinite with probability 1. For the double exponential class of potentials, the almost sure asymptotics of the total mass was shown to be

$$\frac{1}{t}\log U(t) = \varrho \log\left(\frac{\varrho t \vartheta}{\log\log t}\right) - \varrho - \chi(\varrho) + o(1), \tag{1.18}$$

almost surely with respect to the potential and the random tree. The argument roughly follows the ideas from the lattice. The total mass is estimated by considering random walk paths that very quickly reach an intermittent island with high values of the potential and subsequently remain there. The probabilistic cost to travel to the island and the maximum of the potential on the island were calculated and optimised over. The first term in (1.18) corresponds to the maximal value of the potential in the island, which is obtained through extreme-value analysis. Since neighbourhoods grow exponentially on the Galton-Watson tree, the first term is larger than (1.13) on $\mathbb{Z}^d$, where neighbourhoods grow polynomially, because there are more sites for the potential to take larger values. It was also shown that the intermittent islands are equal to an infinite regular tree with minimal degree when ϱ is sufficiently large. This was done by analysing the variational formula

$$\chi(\varrho) := \inf_{p\in\mathcal{P}(V)} [I_E(p) + \varrho J_V(p)], \qquad \varrho \in (0,\infty), \tag{1.19}$$

which is the same as (1.12) but now on V, and showing that it is non-increasing under successive ‘trimming’ of excess branches.

More recently, in [2], the PAM with i.i.d. Pareto distributed potentials was analysed on a critical Galton-Watson tree conditioned on survival. The authors followed the arguments presented in [28] for the lattice and obtained that the asymptotics of the total mass satisfies

$$\lim_{\lambda\to\infty} \liminf_{t\to\infty} \mathsf{P}(\log U(t) \in [\lambda^{-1} t a(t), \lambda t a(t)]) = 1,$$

where λ is a parameter coming from the offspring distribution, and $a(t)$ is the order of the maximal value of the potential in a growing ball that was explicitly identified. Furthermore, it was shown that the solution localises on 2 points almost surely, and on a single point with high probability. Formally, there exist processes V_t and W_t on the vertices such that as $t \to \infty$,

$$\frac{u(t,V_t) + u(t,W_t)}{U(t)} \to 1$$

almost surely with respect to the tree and the potential, and

$$\frac{u(t,V_t)}{U(t)} \to 1$$

in probability with respect to the tree and the potential.

The main difficulty of moving to random graphs is that they complicate the analysis by adding another layer of randomness. Furthermore, many properties of the lattice, including degree regularity and translation invariance that previous works relied on, are no longer present. Another difficulty is the exponentially growing nature of the Galton-Watson tree, which means that any subtree has a boundary comparable to its volume in size. Consequently, bounds that rely on reflection techniques that fold $\mathbb{Z}^d$ into a box of an appropriate size at the cost of a negligible boundary term are no longer applicable. Even though the Galton-Watson tree is not regular, the degrees are i.i.d. so that there is still some regularity. Hence, the Galton-Watson tree is the natural first step towards extending the results beyond the lattice. Furthermore, random graphs that are locally tree-like may share the the same behaviour as the Galton-Watson tree. An example is the *configuration model*, as shown in [12].

§1.1.4 Overview of the results

This thesis aims to contribute to the PAM on random trees. In this section we give an overview of the main results. In all cases, we work under the i.i.d. double exponential class of potentials. We also assume that the minimal offspring is at least 2, so that we always have an infinite tree.

Chapter 2

In Chapter 2 we derive the annealed asymptotics of the total mass on a regular tree with degree $d+1$.

Theorem 1.1.1. *[Growth rate of the total mass] For any $d \geq 4$,*

$$\frac{1}{t}\log\langle U(t)\rangle = \varrho\log(\varrho t) - \varrho - \chi(\varrho) + o(1), \qquad t \to \infty.$$

This shows that the asymptotics on the regular tree exactly matches with the lattice given by (1.11). We try to mimic the argument from [25] for the lattice, which involves finding matching upper lower and upper bounds. For the lattice, the lower bound is obtained by approximating the random walk on the infinite lattice by the random walk on box of length R with zero boundary conditions. The upper bound is obtained by periodising the random walk onto a torus of length R. The bounds coincide in the limit $R \to \infty$, which gives the result. The upper bound heavily relies on translation invariance in the periodisation procedure, which is not present on the regular tree. To overcome this, we devise a *novel procedure* to periodise the random walk onto a subtree analogous to the torus for the lattice. This procedure relies on new ideas as well as delicate computations.

Subsequently, we analysed the variational formula on the regular tree. In [12] it was already shown that the minimiser is attained by the regular tree with minimal degree. The theorem below collects its key properties.

Theorem 1.1.2. *[Properties of the variational formula] For any $d \geq 2$ the following hold:*
(a) The infimum in (1.12) *may be restricted to the set*

$$\mathcal{P}^{\downarrow}_{\mathcal{O}}(V) = \{p \in \mathcal{P}(V)\colon\ \mathrm{argmax}\, p = \mathcal{O},\ p \text{ is non-increasing along backbones.} \tag{1.20}$$

(b) For every $\varrho \in (0,\infty)$, the infimum in (1.19) *restricted to $\mathcal{P}^{\downarrow}_{\mathcal{O}}(V)$ is attained, every minimiser $\bar{p}$ is such that $\bar{p} > 0$ on V, and $\partial S_R = \sum_{\partial B_R(\mathcal{O})} \bar{p}(x)$, $R \in \mathbb{N}_0$, satisfies*

$$\sum_{R\in\mathbb{N}_0} \partial S_R \log(R+1) \leq \frac{d+1}{\varrho},$$

where $B_R(\mathcal{O})$ is the ball of radius R centred at $\mathcal{O}$.
(c) The function $\varrho \mapsto \chi(\varrho)$ is strictly increasing and globally Lipschitz continuous on $(0,\infty)$, with

$$\lim_{\varrho\downarrow 0} \chi(\varrho) = d-1, \qquad \lim_{\varrho\to\infty} \chi(\varrho) = d+1.$$

Chapter 3

We would like to have a result analogous to Theorem 1.1.1 on the Galton-Watson tree. Chapter 3 makes another step in this direction by extending the results of Chapter 2 to a *periodic* Galton-Watson tree, i.e. a Galton-Watson tree up to distance R from the root, which is repeated every R generations. This is a close approximation of the Galton-Watson tree for R large. On this tree, we again derive the asymptotics of the annealed total mass. We require the following assumption on the offspring distribution D.

Assumption 1.A. [Offspring distribution]
There exist $4 \le d^- \le d^+ < \infty$ such that $\mathrm{supp}(D) \subseteq [d^-, d^+]$. ■

Theorem 1.1.3. *Subject to Assumption 1.A, as $t \to \infty$,*

$$\frac{1}{t} \log\langle U(t)\rangle = \varrho \log(\varrho t) - \varrho - \chi_{\mathcal{GW}}(\varrho) + o(1)$$

almost surely with respect to the tree.

The proof relies on the novel techniques developed in the previous chapter, but all the technicalities and fine details needed to be carefully adapted. Obtaining the result requires careful navigation with the degrees, which are now random.

Chapter 4

Chapter 4 concerns the aspymptotics of the total mass, but in the quenched setting. This chapter follows the framework that was set up in [12], which required the offspring distribution to be bounded. This goal of this chapter is to identify the weakest condition on the degree distribution under which the arguments can be pushed through. We require the following assumption on the offspring distribution.

Assumption 1.B. [Super-double-exponential tails] There exists a function $f\colon (0,\infty) \to (0,\infty)$ satisfying $\lim_{s\to\infty} f(s) = 0$, $\lim_{s\to\infty} f'(s) = 0$ and $\lim_{s\to\infty} f(s)\log s = \infty$ such that

$$\limsup_{s\to\infty} \mathrm{e}^{-s} \log \mathcal{P}(D > s^{f(s)}) < -2\vartheta.$$

■

The assumption is slightly weaker than $\mathcal{E}\big(\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{aD}}}\big) < \infty$ for some $a < \infty$.

Theorem 1.1.4. *Subject to Assumption 1.B, as $t \to \infty$,*

$$\frac{1}{t} \log U(t) = \varrho \log\left(\frac{\varrho t \vartheta}{\log\log t}\right) - \varrho - \chi(\varrho) + o(1).$$

The asymptotics are the same as in [12], but we have relaxed the assumption to offspring distributions with unbounded support. Obtaining the result requires a careful analysis of the structural properties of the Galton-Watson tree and control on the occurrence of large degrees uniformly in large subtrees. Vertices with large degrees are problematic and Assumption 1.B is the weakest condition needed to control them.

As mentioned above, the existence and uniqueness of the Feynman-Kac solution to PAM on the lattice was proven in [24]. The existence and uniqueness on trees should hold as well, although was never explicitly proven. In this chapter we follow the arguments of [24] and rigorously prove the existence and uniqueness on Galton-Watson trees provided, the offspring distribution has all exponential moments.

Assumption 1.C. [Exponential tails]
$\mathcal{E}\left[e^{aD}\right] < \infty$ for all $a \in (0,\infty)$. ■

Theorem 1.1.5. *Subject to Assumption 1.C,* (1.14) *has a unique non-negative solution for all most all realisations of the potential and the tree. This solution admits the Feynman-Kac representation in* (1.16)*.*

Chapter 5

Chapter 5 again considers the PAM on a Galton-Watson tree, but with the normalised Laplacian

$$(\Delta f)(x) = \frac{1}{\deg(x)} \sum_{y\sim x} [f(y) - f(x)],$$

instead of (1.15). The aim of this chapter is to investigate the effect of the normalisation of the Laplacian. In the case of the lattice or regular graphs, the choice does not matter as the normalisation is uniform across all the vertices and is therefore simply a constant. This constant can be absorbed into the diffusion constant κ and all results can be easily inferred. In [5] and [16], the degrees approach infinity with the size of the graph and hence the normalised Laplacian was the only viable choice. In the case of inhomogoenous graphs, which random graphs are, the choice of Laplacian plays a role. We again consider the quenched asymptotics of the total mass.

Assumption 1.D. [Exponential tails]
$\mathcal{E}\left[e^{aD}\right] < \infty$ for all $a \in (0,\infty)$. ■

Theorem 1.1.6. *[Total mass asymptotics] Subject to Assumption* 1.D*,*

$$\frac{1}{t}\log U(t) = \varrho \log\left(\frac{\varrho t \vartheta}{\log\log t}\right) - \varrho - \widetilde{\chi}(\varrho) + o(1), \quad t \to \infty, \tag{1.21}$$

almost surely with respect to the graph and the potential.

In this case, the variational formula is given by

$$\widetilde{\chi}(\varrho) = \inf_{p\in\mathcal{P}(V)} [I_E(p) + \varrho J_V(p)], \qquad \varrho \in (0,\infty), \tag{1.22}$$

with

$$I_E(p) = \sum_{\{x,y\}\in E} \left(\sqrt{\frac{p(x)}{\deg(x)}} - \sqrt{\frac{p(y)}{\deg(y)}}\right)^2, \qquad J_V(p) = -\sum_{x\in V} p(x)\log p(x).$$

Note that I_E is different compared with (1.19). Hence, the leading-order terms in (1.21) and (1.1.4) are the same, however the normalised Laplacian results in a different the second-order term. Furthermore, comparing Assumption 1.A with Assumption 1.D we see that the asymptotics now holds under vastly milder conditions on the offspring distribution. We again follow the framework of [12]. The main challenge is to investigate the spectral properties of the Anderson Hamiltonian $\Delta+\xi$, which are now different.

It was found in [12] that the minimiser of the variational formula on the Galton-Watson tree is equal to the variational formula on the regular tree minimal degrees $\chi_{\mathcal{T}_d}$. We show that, with the normalised Laplacian and the different I_E-function this is again true, and under weaker conditions on ϱ.

Theorem 1.1.7. *[Identification of the minimiser] If* $\varrho \geq \frac{1}{d_{\min} \log(d_{\min}+1)}$, *then* $\widetilde{\chi}(\varrho) = \chi_{\mathcal{T}_d}(\varrho)$.

§1.1.5 Open problems

We end this introduction by stating a few open problems. The ones that are within reach include:

- Extend the techniques from Chapters 2 and 3 to the Galton-Watson tree.
- For the variational formula on the regular tree analysed in Chapter 2, determine whether the minimiser $\bar{p}$ is unique modulo translation, $\bar{p}(x)$ satisfies $\lim_{|x|\to\infty} |x|^{-1} \log \bar{p}(x) = -\infty$, $\bar{p}$ is radially symmetric, and whether $\varrho \mapsto \chi_{\mathcal{T}}(\varrho)$ is analytic on $(0,\infty)$.
- Identify the minimiser of the variational formula for all ϱ, and show that it is unique.
- Calculate the asymptotics of $\langle U(t)^p \rangle$ on the Galton-Watson tree and see whether this equal to
$$\exp\{H(pt) - pt\chi(\varrho) + o(t)\},$$
which is the same form as (1.11) on the lattice.

As stated before, very little is known about the PAM on general graphs. More challenging open problems include:

- Can we prove all our results under a different class of potential?
- What can we say about the PAM on other random graph models?