AMERICAN PSYCHOLOGICAL ASSOCIATION

Psychological Methods

ISSN: 1082-989X

https://doi.org/10.1037/met0000642

Latent Growth Factors as Predictors of Distal Outcomes

Ethan M. McCormick[1, 2], Patrick J. Curran[3], and Gregory R. Hancock[4]
[1] Methodology and Statistics Department, Institute of Psychology, Leiden University
[2] Cognitive Neuroscience Department, Donders Institute for Brain, Cognition and Behavior, Radboud University Medical Center
[3] Department of Psychology and Neuroscience, University of North Carolina
[4] Department of Human Development and Quantitative Methodology, University of Maryland

OPEN DATA OPEN MATERIALS

Abstract

A currently overlooked application of the latent curve model (LCM) is its use in assessing the consequences of development patterns of change—that is as a predictor of distal outcomes. However, there are additional complications for appropriately specifying and interpreting the distal outcome LCM. Here, we develop a general framework for understanding the sensitivity of the distal outcome LCM to the choice of time coding, focusing on the regressions of the distal outcome on the latent growth factors. Using artificial and real-data examples, we highlight the unexpected changes in the regression of the slope factor which stand in contrast to prior work on time coding effects, and develop a framework for estimating the distal outcome LCM at a point in the trajectory—known as the aperture—which maximizes the interpretability of the effects. We also outline a prioritization approach developed for assessing incremental validity to obtain consistently interpretable estimates of the effect of the slope. Throughout, we emphasize practical steps for understanding these changing predictive effects, including graphical approaches for assessing regions of significance similar to those used to probe interaction effects. We conclude by providing recommendations for applied research using these models and outline an agenda for future work in this area.

Translational Abstract

Growth models which estimate the developmental trajectory of phenomenon of interest (i.e., unconditional models), and then potentially use covariates to predict individual variability in growth (i.e., conditional models), are common in both applied and methodological work. However, models which in turn use individual variability in growth to predict a distal outcome remain relatively rare. In both unconditional and conditional models, one well-described decision point is how to code time in order to set an intercept location, with known effects on the parameters associated with the intercept based on that decision. However, in distal outcome growth models, changes in time coding instead impact the effect of the slope on the distal outcome—not the effect of the intercept—due to changes in the growth factor correlation. We propose two solutions to address this issue and generate maximally interpretable effects: (a) a framework for estimating the distal outcome LCM at a point known as the aperture, and (b) a prioritization approach for assessing incremental validity to obtain invariant and unique effects of both intercept and slope. Throughout the article, we emphasize practical steps for understanding these changing predictive effects, including graphical approaches for assessing regions of significance, similar to those used to probe interaction effects. Code for estimating these model is provided to assist readers in implementing these models with their own data.

Keywords: latent curve model, distal outcome, time coding, aperture, incremental validity

Supplemental materials: https://doi.org/10.1037/met0000642.supp

Ethan M. McCormick https://orcid.org/0000-0002-7919-4340

Ethan M. McCormick was supported with funds from the NWO (Nederlandse Organisatie voor Wetenschappelijk Onderzoek) Domain Social Sciences and Humanities (SSH) and Jacobs Foundation Research Fellowship Program (2023-1510-00).

The authors declare no competing interests. All data sets used in the current article are publicly available or may be regenerated using the code provided at https://osf.io/fzubt/. A preprint version of the manuscript is available at https://psyarxiv.com/fevra/.

Ethan M. McCormick served as lead for conceptualization, formal analysis, methodology, software, visualization, writing–original draft, and writing–review and editing. Patrick J. Curran contributed equally to conceptualization, and served in a supporting role for methodology, and writing–review and editing. Gregory R. Hancock served in a supporting role for writing–review and editing.

The data are available at https://doi.org/10.17605/OSF.IO/FZUBT.

The experimental materials are available at https://doi.org/10.17605/OSF.IO/FZUBT.

Correspondence concerning this article should be addressed to Ethan M. McCormick, Methodology and Statistics Department, Institute of Psychology, Leiden University, Wassenaarseweg 52, 2333 AK Leiden, The Netherlands. Email: e.m.mccormick@fsw.leidenuniv.nl

Motivating goals common to nearly all longitudinal research involve the evaluation of the course, cause, and consequence of change over time (Curran et al., 2010; Singer & Willett, 2003). This is particularly true in the examination of interindividual differences in intraindividual change, primarily in the form of the broad class of growth curve models. For example, we might be interested in modeling trajectories of reading ability in a sample of children followed throughout the primary school years. Each individual trajectory represents intraindividual (or within-person) change, while the means and variances of the set of sample trajectories represents interindividual (or between-person) differences in change over time. The first goal of establishing the course of reading ability includes characterizing the shape of the trajectory (e.g., linear vs. quadratic vs. exponential); the fixed and random effects associated with the trajectory (e.g., mean and variance in starting point and rate of change over time); and the potential covariation of reading ability with some other developmental process (e.g., how reading and math ability "travel together" through time, Curran & Hancock, 2021). These considerations all involve characterizing the pattern of change over time for a construct, and we have well-developed and widely used methods for examining these issues in practice (Bollen & Curran, 2006; Grimm et al., 2016; Meredith & Tisak, 1990; Raudenbush & Bryk, 2002; see McCormick, Byrne et al., 2023, for an overview).

The second goal of growth modeling is the systematic evaluation of the causes of developmental change.[1] In practice, one often first establishes the course of development (i.e., identifying the optimal functional form that captures the structure of the repeated assessments) and then extends the model to include one or more exogenous covariates that predict different components of change over time. Covariates can be time-invariant and represent person-level characteristics that are constant over time (e.g., biological sex, birth order, and treatment condition) or time-varying and represent person- and time-specific characteristics that can take on different numerical values and predictive impact of the repeated outcome over time (e.g., stress, negative affect, and substance use). Time-invariant covariates shift the conditional mean of the trajectories themselves (e.g., starting point and/or rate of change), while time-varying covariates directly influence the time-specific assessments above-and-beyond the underlying trajectory. As with modeling the course of trajectories, there are readily available methods for including both time-invariant and time-varying covariates into our models of growth (Curran et al., 2004; McNeish & Matta, 2020; Stoel & Hox, 2004).

However, despite decades of remarkable methodological progress in nearly all applications of the growth curve modeling, the third goal remains vexingly elusive[2]: namely, the principled modeling of the consequences of individual variability in developmental change. Whereas the course reflects the pattern of change unfolding over time, and the causes reflect potential determinants of the developmental process, we can think of the consequences such that the growth trajectories themselves serve as predictors of one or more distal outcomes. To continue our example, we might be interested in predicting the consequence of whether children performed at grade level in mathematics upon transition to middle school using their trajectories of math and reading ability from kindergarten to the fifth grade (e.g., initial abilities in kindergarten and rates of change over primary school) in order to understand how ability development might influence later success.

Models of growth with distal outcomes are not new and have been hypothesized within the methodological literature for quite some time (e.g., Muthén & Curran, 1997, Figure 3; Seltzer et al., 1997; von Soest & Hagtvet, 2011), and published substantive applications of these trajectories-as-predictors models exist in numerous fields. For example, studies in education are often concerned with predicting end-of-schooling achievement or employment status (e.g., Hammer et al., 2007; Rowe et al., 2012), while clinical applications have studied the effect of early psychopathology on later mental health (Koukounari et al., 2017) and substance use trajectories on rates of risky sexual behavior (Spoth et al., 2014). In political science, this approach has been used to model the effect of adolescent trajectories of prosocial behavior on later civic engagement (Taylor et al., 2018). These studies concern the prediction of observed variables at some time either concurrent with the final repeated measure or truly distal from the estimated trajectories.[3] A related approach known as the parallel process latent growth curve mediation model (Cheong et al., 2003; O'Laughlin et al., 2018) uses factors from the latent growth curve models fit for two constructs as the mediator and outcome in a single model. While this approach has been used to investigate the mediating role of trauma on externalizing behavior (Barboza et al., 2017), as well as pathways to depression (Koukounari et al., 2017) and nonsuicidal self-injury (Gandhi et al., 2019), there is a lack of temporal precedence between intercepts and slopes across constructs when the repeated measures of the two constructs co-occur, potentially hampering valid causal conclusions.

Despite the obvious importance of this entire class of potential research hypothesis, virtually nothing is known about the optimal estimation and subsequent interpretation of the consequences of intraindividual change over time. Our motivating goal is to systematically examine the complexities that underlie this critically important pillar of the course, cause, and consequence triad. We begin with a review of the latent curve model (LCM, Meredith & Tisak, 1990). While growth curve models can also be estimated within the multilevel modeling framework (Bauer, 2003; Curran, 2003), the standard multilevel growth model does not allow for simultaneous estimation of both the growth model and the distal outcome prediction (Liu et al., 2021; McNeish & Matta, 2018). Given the ubiquity of this method in practice (for a full treatment, see Bollen & Curran, 2006; Grimm et al., 2016; Newsom, 2015), we will focus on the elements most relevant to the use of trajectories in the prediction of distal outcomes. We also briefly review what is known about the rescaling of time in growth models and its well-understood impact on unconditional and conditional growth trajectories. We then expand our notational scheme to define the distal outcome growth model and demonstrate the unexpected and concerning impact that the choice of time coding can exert on the use of growth trajectories in the prediction of distal

[1] Here we use the term cause broadly, fully realizing that the validity of causal inferences lies predominantly in the features of the experimental design and not in the statistical model (e.g., Shadish et al., 2002). A less loaded term might be determinants, or—even more anemic still—predictors of change over time.

[2] As an anecdotal example of the long-standing elusiveness of this topic, Patrick J. Curran received a grant to study these models in 1999 that ultimately did not achieve its stated goals due to limitations in computational methods at the time. Ethan M. McCormicks anecdotal examples from this time are slightly less relevant as they were in the second grade.

[3] While truly distal outcomes are clearly preferable for causal inference, it is likely that outcomes concurrent with the final time point will continue to be prevalent due to the constraints of funding and effort.

outcomes. Using both artificial and real empirical data, we highlight how this issue can be identified, outline a set of potential solutions and recommendations for applied researchers, and suggest future directions for work within this area.

The Latent Curve Model

Many approaches exist for modeling interindividual differences in intraindividual change, including key exemplars within the multilevel model (MLM) and the structural equation model (SEM). Whereas the MLM approaches the repeated measures data as a source of nonindependence among observations (multiple assessments nested within individuals, Bryk & Raudenbush, 1987), the SEM approaches the same repeated measures data from a latent variable perspective (the repeated measures serve as manifest indicators to define the underlying latent growth factors, Meredith & Tisak, 1990). Decades of research has shown that there is significant overlap—and often isomorphism—between these two approaches (Bauer, 2003; Curran, 2003), but key points of difference do exist in their ability to model certain effects (McCormick, Byrne et al., 2023; McNeish & Matta, 2018). While methods have been proposed for using MLM to estimate individual trajectories that are then used as predictors in subsequent models, this requires a two-step procedure that is characterized by a number of unavoidable limitations (Liu et al., 2021). The SEM-based growth model has been referred to by many names, but the most common is the LCM, first developed and described by Meredith and Tisak (1990; but also see McArdle & Epstein, 1987). The LCM represents a powerful analytic framework for assessing a broad class of questions relating to interindividual differences in developmental trajectories over time. LCMs are strikingly flexible and have been a mainstay of longitudinal modeling for several decades. Given the ubiquity of these methods in practice, we will leave a comprehensive review of the LCM to prior work (Biesanz et al., 2004; Bollen & Curran, 2006; Curran et al., 2010; Ghisletta & McArdle, 2012; Hancock et al., 2013; MacCallum et al., 1997; McCormick, Byrne et al., 2023). Instead, here we focus on the notational scheme that will allow us to develop the LCM with distal outcomes in subsequent sections.

The Unconditional LCM

We begin by defining $\mathbf{y}_i$ to represent a vector of a repeatedly assessed outcome y that is unique to individual i ($i = 1, 2, ..., N$) of length T ($t = 1, 2, ..., T$). We assume the outcome to be continuously scaled, but several options exist for estimating LCMs with discrete outcome that we do not discuss here (Masyn et al., 2014; Mehta et al., 2004). We can define one or more latent growth factors underlying the observed repeated measures as

$$\mathbf{y}_i = \mathbf{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i, \tag{1}$$

where $\mathbf{\Lambda}$ is a $T \times K$ factor loading matrix containing the numerical values of time for $k = 1, 2, ..., K$ growth factors, $\boldsymbol{\eta}_i$ is a K-length vector of latent growth factor scores, and the time-specific residuals are distributed as $\boldsymbol{\varepsilon}_i \sim \text{MVN}(0, \mathbf{\Theta}_{\boldsymbol{\varepsilon}})$. In SEM parlance, we can think of this as the measurement equation that maps the observed items onto the latent factor(s) (Bollen, 1989). We can also express the structural equation that defines and related the latent factors themselves. The unconditional model (in which there are no exogenous predictors of the latent factors) is defined as

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \zeta_i, \tag{2}$$

where $\boldsymbol{\alpha}$ is a vector of length K containing the means of the latent scores, and $\boldsymbol{\zeta}_i$ is a vector of length K and captures the individual deviations around the factor means that are assumed distributed $\zeta_i \sim \text{MVN}(0, \mathbf{\Psi})$.

Following standard rules of covariance algebra, we can gather all of the model parameters into vector $\boldsymbol{\theta}$ and derive the model-implied covariance and mean structure (e.g., Bollen, 1989, pp. 85–88):

$$\mathbf{\Sigma}(\boldsymbol{\theta}) = \mathbf{\Lambda}\mathbf{\Psi}\mathbf{\Lambda}' + \mathbf{\Theta}_{\boldsymbol{\varepsilon}}, \tag{3}$$

$$\boldsymbol{\mu}(\boldsymbol{\theta}) = \mathbf{\Lambda}\boldsymbol{\alpha}, \tag{4}$$

where $\mathbf{\Sigma}(\boldsymbol{\theta})$ and $\boldsymbol{\mu}(\boldsymbol{\theta})$ represent the implied covariance and mean structures, respectively. Estimates of the parameters in $\boldsymbol{\theta}$ can be obtained in a variety of ways, the most ubiquitous of which is the maximum likelihood (ML). Given the plethora of existing resources on model estimation and evaluation, we do not dwell on this topic here; see Bollen (1989) and Enders (2001) for additional details.

The model defined thus far is sometimes called an unconditional growth model in that we have yet to incorporate any exogenous predictors of growth. This structure represents the first component of our triad: the unconditional model captures the course of change over time. The latent factor means (or the "fixed effects") of the growth trajectories reside in $\boldsymbol{\alpha}$, the individual variability around these means (or the "random effects") reside in $\mathbf{\Psi}$, and the residuals of the time-specific repeated measures are captured in $\mathbf{\Theta}_{\boldsymbol{\varepsilon}}$. The unconditional LCM might consist of just an intercept factor (reflecting the lack of systematic change over time), or it may be characterized by some polynomial linear or quadratic change, or even a more-complex exponential, piecewise, or latent basis change model (e.g., see Bollen & Curran, 2006, Chapter 4). Regardless of specific structure, this unconditional model provides an estimate of where individuals begin and where they go over time.

The Conditional LCM

The unconditional model can easily be extended to include one or more exogenous predictors, either of the growth factors themselves (i.e., time-invariant covariates [TICs]) or of the time-specific repeated measures (i.e., time-varying covariates [TVCs]). Given our focus is on the latent factors we only consider TICs here, but all of our developments readily extend to the TVC model as well. We can expand Equation 2 to include one or more TICs such that

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \mathbf{\Gamma}\mathbf{x}_i + \zeta_i, \tag{5}$$

where $\mathbf{\Gamma}$ is a $K \times Q$ matrix of regression coefficients and $\mathbf{x}_i$ is a vector of length $q = 1, 2, ..., Q$ of exogenous predictors. The optimal linear combination of these predictors serves to shift the conditional means of the latent factors while assuming the residuals' distributions are constant over $\mathbf{x}_i$. Whether a set of exogenous covariates predicts higher versus lower initial levels or steeper versus less steep changes over time, the conditional model attempts to characterize the optimal linear combination of determinants of change over time. Much is known about the estimation, evaluation, and interpretation of conditional LCMs (for further details, we refer the reader to

Biesanz et al., 2004; Curran et al., 2004). Importantly, these conditional models represent the second component of our triad: the conditional LCM captures the causes of change over time. However, before we proceed on to the third component—consequences—we must first think a bit more carefully about how time is scaled within the LCM.

The Scaling of Time

Within the LCM the numerical values assessing the passage of time are typically set as fixed quantities in the factor loading matrix $\mathbf{\Lambda}$, although in some cases, a subset of values can be estimated from the sample data (Meredith & Tisak, 1990). The first column of $\mathbf{\Lambda}$ is set to unity to represent the intercept of the trajectory, and additional columns can be used to define a wide variety of functional forms of interest. For example, for a linear growth model with time originally coded as $t = 1, 2, ..., T$, $\mathbf{\Lambda}$ is defined as

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 1+a \\ 1 & 2+a \\ \vdots & \vdots \\ 1 & T+a \end{bmatrix}, \tag{6}$$

where the first column defines the intercept factor and the second the linear slope factor. Here we introduce a scaling factor denoted a that allows us to systematically scale time in a variety of principled ways by changing the zero point of the second column in $\mathbf{\Lambda}$ (we expand on this further below). As with any regression-like equation, the latent intercept factor captures the mean and variance of the trajectory when time is equal to zero. Thus, if time is originally coded as $t = 1, 2, ..., T$, setting $a = -1$ will define the intercept as the expectation of the outcome at the initial time point; setting $a = -T$ will define the intercept as the expectation of the outcome at the final time point; and setting $a = -\bar{t}$ will define the intercept as the expectation of the outcome at the middle time point. The impact of changing the scale of time was a vexing issue in early applications of the LCM but is now well understood (Biesanz et al., 2004; for a detailed summary, see Bollen & Curran, 2006, pp. 113–120).

Biesanz et al. (2004) thoroughly explored this issue and allowed several general conclusions to be drawn. For both the unconditional and the conditional LCMs, any choice of a results in an equivalently fitting model; that is, any choice of a results in the same log-likelihood value and no scaling fits any better or worse than any other scaling. However, seemingly paradoxically, despite identical model fit, changes in certain model parameter estimates and associated standard errors often result. For example, in an unconditional linear LCM, the mean and variance of the slope factor are unchanged across different values of a. However, the mean and variance of the intercept factor, as well as the covariance between the intercept and slope factor differ over different choices of a, sometimes substantially so. For instance, the covariance between the intercept and slope (ψ_{21}) changes across different time codings ($\psi_{21} \rightarrow \psi^*_{21}$) by the magnitude of the slope factor variance (ψ_{22}) per unit change in a (Δa; we will delve into where these transformations originate in greater detail in later sections):

$$\psi^*_{21} = \psi_{21} + \psi_{22}\Delta a. \tag{7}$$

These parameter changes make logical sense because if there is random variability in the slope then the relative ordering of individuals at any given time point can, and almost always does, change. As such, the mean and variance of the intercept factor, as well as the covariance of the intercept with other growth factors (of key importance in later developments) change as a function of where the zero-point of time is defined.

These relations extend to the conditional LCM in which any predictors of the slope factor are invariant to the choice of a and predictors of the intercept factor vary as a function of a. As with the unconditional LCM, the conditional LCM remains likelihood equivalent over choice of a and the changes in the effects of the predictors (both the parameter estimates and the standard errors) are deterministically scaled as a function of the numerical values of time, as discussed comprehensively by Biesanz et al. (2004). Indeed, everything we have described thus far is fully developed and well understood. We review this here because we next extend the model to include a distal outcome where we will find that the choice of a exerts a substantial and somewhat unexpected impact on both parameter estimates and on substantive conclusions.

The Distal Outcome LCM

We return to the unconditional LCM and expand the notation to include a single distal outcome measure z_i. For simplicity, we assume the distal outcome is continuously scaled, although discrete outcomes could easily be incorporated. Similarly, we assume a single outcome, although this too could easily be extended to a vector of observed outcomes, or even one or more multiple-indicator latent factors. The measurement model defined in Equation 1 remains unchanged, but we expand the structural equation (Equation 2) such that

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta}_i + \zeta_i. \tag{8}$$

For a linear model with a single distal outcome, the elements of Equation 8 are

$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ z_i \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta_{z1} & \beta_{z2} & 0 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ z_i \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \\ \zeta_{zi} \end{bmatrix}, \tag{9}$$

where the distal outcome is included as a third element of vector $\boldsymbol{\eta}_i$.[4] This model is presented in Figure 1.

As before, we assume $\zeta_i \sim \text{MVN}(0, \mathbf{\Psi})$ is partitioned such that part relates to the growth factors and part to the distal outcome. For our example, this partitioned matrix takes the following form:

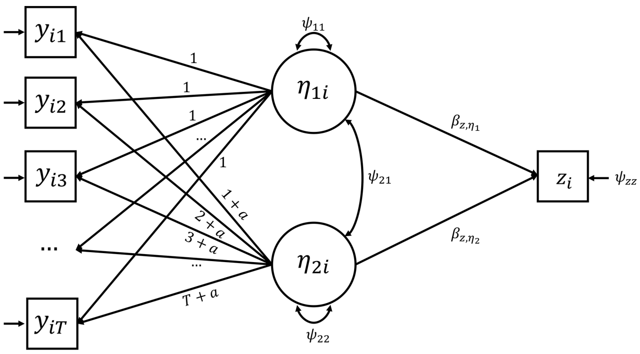
$$\mathbf{\Psi} = \begin{bmatrix} \psi_{11} & \psi_{12} & 0 \\ \psi_{21} & \psi_{22} & 0 \\ 0 & 0 & \psi_{zz} \end{bmatrix}, \tag{10}$$

where the upper two-by-two quadrant represents the variances and covariance among the growth factors, and the lower ψ_{zz} represents the residual variance of the distal outcome z_i. This example is for a simple linear LCM with a single manifest distal outcome, but $\mathbf{\Psi}$ would expand accordingly to accommodate additional growth factors or additional outcomes.

[4] We could have defined z_i as a constrained single-indicator latent factor so that the third element was denoted as η_{3i}, but we see this as an unnecessary complication for now, and will return to this idea in when we consider an alternative incremental validity specification.

Figure 1

The Latent Curve Model With a Distal Outcome



Note. In this extension of the unconditional model, we include a distal outcome z_i which is predicted by both the intercept (η_{1i}) and slope (η_{2i}) factors. Of key interest here are the regressions of the distal outcome on the growth factors (β_{z,η_1} and β_{z,η_2}, respectively). Note that the factor loadings follow the form in Equation 6 where we can arbitrarily rescale the location of the intercept.

Of key interest here are the two parameters capturing the regression of the outcome on the intercept factor (β_{z,η_1}) and on the slope factor (β_{z,η_2}). As with any regression coefficient, these capture the prediction of outcome z_i per unit-change in each growth factor net the effects of the other growth factor. We italicize this because here is where things begin to depart from the unconditional and conditional models we have examined thus far. As we noted earlier, each component of the growth trajectory (e.g., the intercept and linear slope) jointly define the trajectory in its entirety; the trajectory can only be reconstituted by the combination of the set of components. Imagine two trajectories with linear slope scores both equal to 5.0, yet one has an intercept of 1.0 and the other of 10; it makes no sense to treat the two slopes as equivalent without also considering the associated intercepts. What then does it mean to examine the prediction of a distal outcome from one trajectory component net the effects of the other component(s)? Of critical importance when thinking about this issue is the relation between the intercept and slope factor, a relation that we earlier showed is dependent on the scaling of time (Equation 7).

To see this more clearly, let us briefly consider a simple ordinary least squares (OLS) regression with two correlated predictors:

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i. \tag{11}$$

Note the similarity in form to the two correlated growth factors predicting the distal outcome in Equation 8. The expression of the standardized regression coefficients based on the correlations among the two predictors and the outcome are:

$$\beta_{\text{std},1} = \frac{\rho_{y_1} - \rho_{y_2}\rho_{12}}{1-\rho_{12}^2}, \quad \beta_{\text{std},2} = \frac{\rho_{y_2} - \rho_{y_1}\rho_{12}}{1-\rho_{12}^2}. \tag{12}$$

It is clear that the correlation between the two predictors (ρ_{12}) plays a critical role in the calculation of the partial regression coefficients. Indeed, no partialing occurs only when $\rho_{12} = 0$. However, whereas in the OLS regression model, it is well known that the correlation between the predictors is invariant to linear transformations of the predictors (e.g., centering or standardizing x_{1i} or x_{2i} does not affect ρ_{12}), the correlation between the growth factors can—and will—change as a function of linear transformations of time (e.g., placing zero at the beginning, middle, or end of the span of time). If the correlation between growth factors is in part determined by choice of zero-point, and this same correlation plays a critical role in the calculation of the partial regression coefficients, this begs the logical question: At what time point should the effects be partialed? Or, even more vexingly, should the effects be partialed at all? To better appreciate these challenges, we briefly turn to an illustrative, simulated data set.

Impact of Time Coding on Distal Outcome Regressions

To demonstrate the impact of the changing correlation on the calculation of partial regression coefficients, we began by creating a single simulated data set. We simulated data from known population moments corresponding to a linear LCM and a single continuous distal outcome reflected in Figure 1.[5] We defined the intercept factor to have a mean (α_1) of 3.0 and a variance (ψ_{11}) of 1.0, the slope factor to have a mean (α_2) of 0.20 and a variance (ψ_{22}) of 0.25, and the intercept and slope factor to have a covariance (ψ_{21}) of 0.075 (representing a small positive correlation of 0.15). We then generated five continuous repeated measures with a normally distributed, heteroscedastic residual variance that resulted in a constant communality of 0.50 for each repeated measure in a sample of 5,000 cases. Finally, defining the intercept to represent the beginning of the trajectory, we generated a single continuous and normal dependent distal outcome variable (z_i) that was in part determined by the intercept and slope factors. More specifically,

$$z_i = 5 + 0.1\eta_{1i} + 0.2\eta_{2i} + \zeta_{z_i}, \tag{13}$$

with a residual variance ζ_{zi} scaled to result in a multiple R^2 value of 0.50. We fit two standard linear LCMs in which the growth factors predicted the distal outcome: one in which the intercept was set to the initial time point (t_1) and one in which the intercept was set to the final time point (t_5). As expected, results confirm that the recoding of time has no effect on model fit (see Table 1). However, there are several striking differences in parameter estimates that warrant closer consideration.

To begin, some parameters behave in predictable ways similar to the unconditional and conditional LCMs across changing time coding. For example, the intercept mean (α_{η_1}), intercept variance (ψ_{11}), and intercept-slope covariance (ψ_{21}) all differ as a function of choice of time scale (given that they reflect estimates of level at different points in the growth trajectory), whereas the slope mean (α_{η_2}) and slope variance (ψ_{22}) remain invariant. This is as expected. However, examining the effects of the growth factors on the distal outcome reveals a wholly different finding. Whereas the conditional linear LCM in which the prediction of the intercept factor by an exogenous predictor varies as a function of the scaling of time and the prediction of the slope does not (Biesanz et al., 2004), here the opposite holds: the effect of the intercept on the distal outcome is invariant to scale of time whereas the effect of the slope is not.[6] Indeed, not only does the effect of the slope factor in the prediction of the outcome change across different time coding schemes, in our simulated example it even reverses in sign. If we were to define the intercept to represent the initial time point,

[5] Code to replicate all analyses is available online: https://osf.io/fzubt/.

[6] Note that this invariance applies to the unstandardized effects—the standardized effect is non-invariant due to the change in the variance of the intercept factor.

Table 1
Parameter Recovery: Initial and Final Status Models

		Initial status				Final status			
		T	*df*	*p*		*T*	*df*	*p*	
		9.908	13	.701		9.908	13	.701	
Param.	Pop. θ	Est.	*SE*	Std. Est.	*p*	Est.	*SE*	Std. Est.	*p*
β_{z,η_1}	0.100	0.098	0.003	0.465	<.001	0.098	0.003	1.096	<.001
β_{z,η_2}	0.200	0.196	0.007	0.468	<.001	−0.195	0.018	−0.465	<.001
α_{η_1}	3.000	2.991	0.019	2.966	<.001	3.817	0.040	1.606	<.001
α_{η_2}	0.200	0.207	0.010	0.408	<.001	0.207	0.010	0.408	<.001
ψ_{11}	1.000	1.017	0.041	1.000	<.001	5.652	0.163	1.000	<.001
ψ_{21}	0.075	0.068	0.016		<.001	1.091	0.039		<.001
r_{21}	0.150	0.133	0.035		<.001	0.907	0.005		<.001
ψ_{22}	0.250	0.256	0.011	1.000	<.001	0.256	0.011	1.000	<.001
R_z^2	0.500	0.493				0.493			

Note. *T* is the χ^2 test statistic and *df* is the model degrees of freedom. Pop. θ is the population parameter used to generate the data. Est. is the sample-recovered parameter. *SE* is the standard error of the estimate. Std. Est. is the standardized estimate. Beta (β) denotes the regression coefficients, alpha (α) denotes factor means, psi (ψ) denotes the factor variances and covariances, and r_{21} denotes the factor correlation. R_z^2 is the proportion of variance explained in the distal outcome.

we would interpret a positive effect of the slope on the outcome (indicating steeper slopes predict higher levels of z_i) whereas if we were to define the intercept to represent the last time point we would obtain the opposite interpretation (indicating that steeper slopes predict lower levels of z_i), even though the parameters of the slope factor itself are invariant to the different codings of time. To compound the issue, both estimates would be deemed "highly significant" with large, standardized effect sizes. To understand these unexpected results, we need to start by expanding the derivations outlined by Biesanz et al. (2004) to apply to the distal-outcome model.

Deriving Parameter Estimate Transformations

Although these initial results may be somewhat surprising, the parameter transformations are far from mysterious and can be straightforwardly derived in a similar way to the unconditional and conditional LCM transformations (Biesanz et al., 2004). For any arbitrary coding of time in a linear growth model, we can linearly transform either the zero-point or the spacing interval such that $t^* = a + bt$. The matrix expression for these coefficients is

$$\mathbf{T} = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}, \tag{14}$$

where a rescales the intercept and b rescales the slope (or interval of time). We can generalize this to the LCM in matrix form, such that for a given growth model specification with factor loading matrix $\mathbf{\Lambda}$, we can define a general transformation matrix denoted $\mathbf{T}$ that allows us to reexpress the parameters as a function of the choice of time scale via a from Equation 6.[7] As a simple example say that for the linear function defined in Equation 6 that defines the initial time assessment as zero, we can rescale this to end in zero by defining the transformation matrix as[8]:

$$\mathbf{T} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \tag{15}$$

such that

$$\mathbf{\Lambda}^* = \mathbf{\Lambda T} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{16}$$

Biesanz et al. (2004, p. 35) then showed that if the transformed factor-loading matrix can be expressed as a function of the original loading matrix and transformation matrix

$$\mathbf{\Lambda}^* = \mathbf{\Lambda T}, \tag{17}$$

then the transformed covariance matrix ($\mathbf{\Psi}^*$) can likewise be expressed as

$$\mathbf{\Psi}^* = \mathbf{T}^{-1}\mathbf{\Psi}\mathbf{T}^{-1\prime}, \tag{18}$$

and the transformed mean vector ($\boldsymbol{\alpha}^*$) can be computed as

$$\boldsymbol{\alpha}^* = \mathbf{T}^{-1}\boldsymbol{\alpha}, \tag{19}$$

where the inverse transformation matrix ($\mathbf{T}^{-1}$) has the form

$$\mathbf{T}^{-1} = \left(\mathbf{\Lambda}^{*\prime}\mathbf{\Lambda}^*\right)^{-1}\mathbf{\Lambda}^{*\prime}\mathbf{\Lambda}. \tag{20}$$

These expressions demonstrate the deterministic effect of rescaling time on the resulting latent covariance matrix and mean vector. The above expressions are for the unconditional LCM, and these can be expanded further to apply to the conditional LCM as well as resulting standard errors. Important to our discussion here, however, these can also be expanded to include the effect of the transformation of time on the regression of the distal outcome z_i on the latent growth factors.

[7] For those familiar with exploratory factor analysis, this is similar to the idea of factor rotation.

[8] For our purposes here, we will retain the metric of time by setting $b = 1$, although it is possible to compute these transformations across different units of time (e.g., monthly vs. annual vs. biannual change).

Here we present the final results, with associated details for the derivations of both parameter estimates and standard errors delineated in the Appendix. As above, define $\mathbf{\Lambda}$ as the original coding of time (whatever that might be), $\mathbf{T}$ as the transformation matrix, and $\mathbf{\Lambda}^*$ as the rescaled factor loading matrix—just as in Equation 17. Define $\mathbf{B}$ to represent the regression of the distal outcome on the set of growth factors under the original coding of time (as was defined in Equation 8). We can show that

$$\mathbf{B}^* = \mathbf{T}'\mathbf{B}. \tag{21}$$

If we then use $\mathbf{T}$ from Equation 15, we can show that with an original

$$\mathbf{B} = \begin{bmatrix} \beta_{z,\eta_1} \\ \beta_{z,\eta_2} \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \tag{22}$$

then $\mathbf{B}^*$ is

$$\mathbf{B}^* = \mathbf{T}'\mathbf{B} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}' \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix}. \tag{23}$$

This expression thus follows from the process outlined by Biesanz et al. (2004), but with results that are opposite of what Biesanz et al. (2004) described. Specifically, whereas parameters for predictors of the slope factor are invariant as a function of the rescaling of time, parameters for the regression of distal outcomes on the slope factor vary as a function of the rescaling of time.

We can further show (see the Appendix for full details) that the effect of the slope on the outcome is

$$\beta^*_{z,\eta_2} = \beta_{z,\eta_2} + \beta_{z,\eta_1}\Delta a, \tag{24}$$

where Δa represents the shift in the intercept of the growth trajectory between alternative time codings.[9] In comparison, we can show that the effect of the intercept on the outcome is invariant across time codings, or:

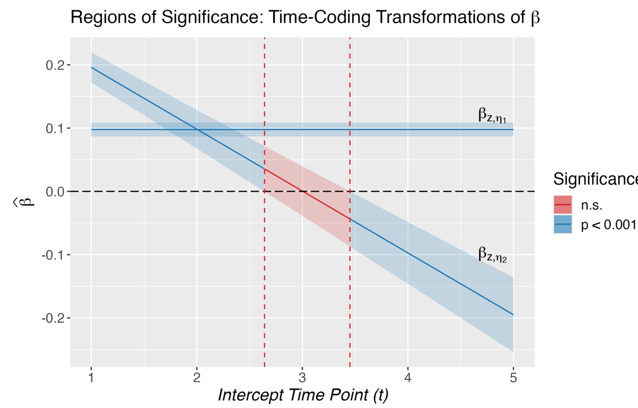
$$\beta^*_{z,\eta_1} = \beta_{z,\eta_1}. \tag{25}$$

These equations show that we can estimate the regression of the distal outcome on the slope and intercept factors across the continuum of zero points of time, and for each of these infinite points we can compute a confidence interval. Using existing methods to compute and plot regions of significance (e.g., Bauer & Curran, 2005; Curran et al., 2004, see online supplemental material for practical implementation), we can graphically display these effects over the continuum of potential zero-points (Figure 2).

When considering the difference in substantive interpretation that adopting different time coding schemes would engender, these results are stunning. For our simulated data—which were generated to be typical of what might be encountered in practice—if we define the intercept to represent the initial value, the slope has a significant and positive prediction of the distal outcome; as we move the zero-point to the middle assessment, this effect becomes nonsignificant; and as we move toward defining the intercept as the final time point, the effect becomes significant and negative. In contrast, the effect of the intercept on the distal outcome is invariant across all

Figure 2

Regions of Significance for the Effect of the Slope on the Distal Outcome



Note. The effect of the slope on the distal outcome β_{z,η_2} changes as a function of where in time (t) the intercept is coded, while the effect of the intercept β_{z,η_1} is invariant. The shaded blue regions represent the regions of significance for the 99.9% confidence bands of each effect. The region in which these confidence bands include zero are shaded red and the value of t at the boundaries of this region are noted by the vertical dashed red lines. See the online article for the color version of this figure.

ONLINE COLOR

choices of a ($\hat{\beta}_{z,\eta_1} = 0.098\ \forall{:}a$). Although this pattern of numerical results reflects the characteristics of our simulated data, changes in β_{z,η_2} relative to β_{z,η_1} generalize, and highlight the deterministic relation between the centering of time and the role of the slope as a predictor of some distal outcome. Under conditions where the effect of the intercept is substantial, changes across time codings this large—or even larger—can be expected (Equation 7). Additionally, across many repeated measures, even small effect sizes of β_{z,η_1} can compound, as the change in slope effect is linear and unbounded. As such, we recommend that this regions-of-significance approach should be adopted as standard practice in these models to better understand the range of potential effects across time coding alternatives.

The challenge remains then, that if all time coding schemes in real data are equally valid from a model-fit perspective, how do we best adjudicate among alternative codings without bringing additional information or priorities to bear? One such priority might be generating the most interpretable parameters, begging the question of whether there is a coding scheme that would theoretically optimize this criterion of the parameter interpretability in the model. We next propose a set of standard approaches that aim to minimize the potential confusion about parameter interpretation and maximize the information that each parameter conveys about the relation between the growth factors and the distal outcome.

[9] One possibility this expression raises is that β_{z,η_2} is invariant if we omit the predictive effect of the intercept (β_{z,η_1}). Unfortunately, unless the effect of β_{z,η_1} is zero, or the factors are orthogonalized, omitting this pathway will result in bias in β_{z,η_2} as the model attempts to reproduce that relationship through the slope factor (see Section 3 of the online supplemental materials for complete details).

A Principled Time Coding Approach

It is useful here to strip away some of the apparent complexity of the LCM in order to address the issues raised in the prior section. As mentioned earlier, at the between-person factor level, we have a two-predictor regression of the distal outcome. It is well-known that correlated predictors can be problematic in multiple regression, resulting in parameter estimate instability, lowering power, and introducing challenges for interpretation of unique effects (Cohen et al., 2003; see McCormick, 2021, e.g., in the longitudinal case). As such, the goal in multiple regression is to include (relatively) weakly correlated predictors to yield stable parameter estimates across samples, small standard errors, and interpretable regression coefficients. Unlike observed variable regression, however, where the correlation between predictors is invariant to linear transformations, in the LCM, the correlation between intercept and slope varies across the different time coding schemes (Biesanz et al., 2004). If we are unaware of the impacts that time-coding choice has on the effect of the slope factor on the distal outcome, this can result in highly correlated predictors and regression effects that are challenging to interpret meaningfully. Once properly understood, however, we can use this knowledge to our advantage to specify the LCM in such a way that we maximize the stability, power, and interpretability of the regression effects for the distal outcome. For instance, in our artificial data example (Table 1), the initial status intercept model has a relatively low intercept-slope correlation ($\hat{r}_{21} = 0.133$), while the final status model has a very high correlation ($\hat{r}_{21} = 0.907$). Based on our criteria for a desirable set of predictors, we would prefer the former model because the unique effects of the intercept and slope will be more stable and meaningful However, these time codings are only two of an infinite set of potential schemes that we could adopt, and we need a principled, model-based method for deriving the optimal time coding scheme that avoids searching this infinite space with needing to fit a model multiple times.

Fortunately, such a principled approach is readily available. For any linear growth model with a random slope ($\psi_{22} > 0$), there is a time point where the factor covariance is minimized. This point is conveniently also where the variance of the intercept factor is minimized, a location in the trajectory known as the aperture point (Hancock & Choi, 2006). Following derivations from Hancock and Choi (2006), we can compute the value of a (Equation 26) needed to shift the intercept from its current coded location to the aperture. In a linear LCM, this value can be expressed as the ratio of the factor covariance over the variance of the slope factor, or:

$$a = \frac{\psi_{\eta_1,\eta_2}}{\psi_{\eta_2}} = \frac{\psi_{21}}{\psi_{22}}, \tag{26}$$

which can be computed as a part of model estimation. For our simulated data, both the initial ($\hat{a} = 0.265$, $SE = 0.069$) and final ($\hat{a} = 4.265$, $SE = 0.069$) status models' estimates of a suggests that the aperture is 0.265 time-units prior to the first observation (note that positive a values shift the intercept backwards in time, and vice versa). Although this aperture resides outside the range of observed time points, we will briefly ignore this issue and first outline how this a shift is implemented generally. Recoding time to estimate the intercept at the aperture point results in the following factor loading matrix:

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & [\lambda_{\text{initial},1} + \hat{a}] \\ 1 & [\lambda_{\text{initial},2} + \hat{a}] \\ 1 & [\lambda_{\text{initial},3} + \hat{a}] \\ 1 & [\lambda_{\text{initial},4} + \hat{a}] \\ 1 & [\lambda_{\text{initial},5} + \hat{a}] \end{bmatrix} = \begin{bmatrix} 1 & [0 + 0.265] \\ 1 & [1 + 0.265] \\ 1 & [2 + 0.265] \\ 1 & [3 + 0.265] \\ 1 & [4 + 0.265] \end{bmatrix} = \begin{bmatrix} 1 & 0.265 \\ 1 & 1.265 \\ 1 & 2.265 \\ 1 & 3.265 \\ 1 & 4.265 \end{bmatrix}. \tag{27}$$

When we fit this aperture time coding to the data, we can generate a solution that minimizes the factor covariances—indeed it orthogonalizes the intercept and slope covariance completely ($\hat{\psi}_{21}$= 0.000, SE = 0.018, p = 1.000; Table 2).[10] Naturally, this leads to an estimate of $\hat{a}$ = 0, indicating that we have estimated the model at the aperture point. This solution allows us to interpret the two distal outcome regression coefficients independently as there is no overlapping variance between the intercept and slope factors in this model.

Despite everything outlined thus far, we do need to exercise caution and not indiscriminately follow this aperture estimate without considering the structure of our sample data. As we noted, the model-implied aperture in this example is outside the range of the observed data, and we should not expect that our model adequately characterizes a process outside this range of time. As such, we might alternatively estimate the intercept at the point nearest the aperture that remains bounded within the time window covered by our measurements. In this example, that would suggest that we should retain the initial status model. Indeed for many applications where trajectories diverge over time (i.e., the right side of a bowtie, Hancock & Choi, 2006), the common practice of estimating the intercept at the initial time point may have inadvertently been maximizing the interpretability of these regression slopes by estimating nearest the aperture time coding. However, this should not be taken as a given in these models, especially when such a readily available estimation procedure can confirm the model-implied location of the aperture in any given data set.

Prioritizing the Slope Using Incremental Validity

We have seen that the unique effect of the slope—captured by the regression coefficient—on the distal outcome changes across recodings of time because of the change in covariance between intercept and slope factors, while the unique effect of the intercept is invariant. However, we might prefer to estimate the model in such a way that the effect of the slope remains invariant. While the intercept is an integral part of the growth model, the slope effect is often of greater theoretical interest. This is in part due to the allure of predicting based on change, however, there are other reasons to prefer prioritizing the slope effect over that of the intercept. First, while often referred to as the "starting point" (indeed we have used this language

[10] If we do not restrict the range of a, this procedure will always orthogonalize the factors; however, there may be good reasons to restrict this range in practice so that a does not fall outside the observed time frame.

Table 2
Parameter Recovery: Aperture Model

	Aperture point			
Param.	Est.	*SE*	Std. Est.	*p*
β_{z,η_1}	0.098	0.003	0.461	<.001
β_{z,η_2}	0.222	0.007	0.530	<.001
α_{η_1}	2.936	0.020	2.938	<.001
α_{η_2}	0.207	0.010	0.408	<.001
ψ_{11}	0.999	0.046	1.000	<.001
ψ_{21}	0.000	0.018		1.000
r_{21}	0.000	0.035		1.000
ψ_{22}	0.256	0.011	1.000	<.001
R_z^2	0.493			
a	0.000	0.069		1.000

Note. Est. is the sample-recovered parameter. *SE* is the standard error of the estimate. Std. Est. is the standardized estimate. Beta (β) denotes the regression coefficients, alpha (α) denotes the factor means, psi (ψ) denotes the factor variances and covariances, and r_{21} denotes the factor correlation. R_z^2 is the proportion of variance explained in the distal outcome.

ourselves), the intercept is best characterized as the point at which we began measuring a process already in progress—that is, we may begin measuring trajectories of mathematics skills at age 10, but the growth and development of numerical cognition pre-dates our study. Exceptions to this general rule are instructive. For instance, Huttenlocher et al. (1991) investigated trajectories of vocabulary, but importantly began observing children before they began speaking. This means that at the initial time point, no children were speaking—that is there was zero variance in initial level—giving a natural, substantively meaningful, "true" aperture at which to estimate the growth model where the effect of the slope is given theoretical priority (a zero-variance intercept cannot predict a distal outcome). However, for most applications, we need a model-based method for prioritizing the slope if we cannot estimate at the model-implied aperture.

Fortunately, such a way exists, and involves the specification of additional latent factors that measure the unique variance of each predictor in a prespecified order (Feng & Hancock, 2021). We can see a visual representation of this alternative approach below:

The key to setting predictive priority is to regress lower-priority predictors on higher-priority ones (Feng & Hancock, 2021) and thereby isolate the shared variance to the higher-priority predictor; here we will regress the intercept factor on the slope factor (β_{η_1,η_2}) to prioritize the slope. Note that our goal in doing this regression differs fundamentally from regressing one factor on another in a standard LCM. Here, we do not wish to interpret this regression per se (although its standardized coefficient is equal to the factor correlation), we only use it to remove any shared variance between the two factors from the lower-priority factor (here the intercept). This partialing of variance from the intercept then allows us to construct uniqueness factors (ξ) which are orthogonal versions of the original growth factors. To build these uniqueness factors, we constrain the variances of the original latent growth factors (η) to zero, estimate the variance of the uniqueness factors (ξ), and then regress the growth factors on their respective uniqueness factors ($\gamma_{\eta,\xi}$) with factor loadings of 1. We also specify a single-indicator latent factor (η_{3i}) for the distal outcome (z_i) with a factor loading of 1, estimate the factor variance, and set the variance of the observed distal outcome to zero (Figure 3). While this model specification is much more verbose than the standard distal outcome LCM, we can conceptualize the incremental validity model as accomplishing two primary things: (a) orthogonalizing the growth factors by removing all shared variance from the lower-priority intercept factor, and (b) regressing the distal outcome on the unique factors to preserve time coding-invariant predictive relationships.

This architecture allows us to obtain regression coefficients between the unique factors and the outcome factor ($\gamma_{\eta_3,\xi}$) that exactly correspond to those coefficients obtained in the aperture model ($\hat{\gamma}_{\eta_3,\xi_2} = 0.222$ and $\hat{\gamma}_{\eta_3,\xi_1} = 0.098$; see Table 2). Unlike the regressions in the usual model specification, however, these coefficients are invariant across time coding choices (see Section 4 of the online supplemental materials in the code output for model specification and full output). This approach is a particularly attractive model under a range of conditions. For instance, if the aperture falls substantially outside the observed range of the data, then the transformation of the covariance between factors into a regression preserves the interpretability of the slope effect (given that it is the highest-priority in the model) without sacrificing intepretability of other parameters in the model (e.g., the factor means). This model specification also has potential for generalized use in more complex models (e.g., higher-order polynomials or piecewise functions) where we might wish to prioritize different sequences of factors depending on our purposes, or indeed to the inclusion of additional covariates predicting the distal outcome in a specific prioritized order. Overall, applying this incremental validity approach to the problem of correlated growth factors allows us maximal flexibility in specifying the growth model while maintaining interpretable regression coefficients between the components of growth and the distal outcome.

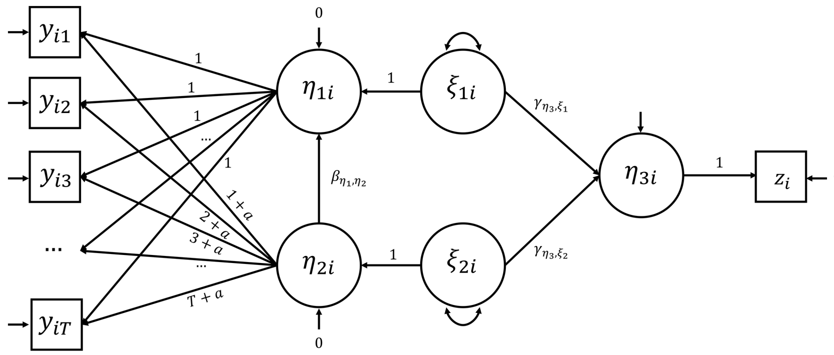
Real-Data Example

Thus far we have demonstrated our findings using artificial data; however, we also wish to highlight how these issues might be encountered in real-data settings. To this end, we drew publicly available longitudinal data from the Midlife in the United States (MIDUS; https://www.midus.wisc.edu/) study ($n = 3{,}294$) including three repeated measures (spaced nine ears apart) of individuals' subjective negative affect, and a measure of the number of chronic health conditions experienced taken concurrent with the third repeated measure—hence not a truly distal outcome but likely representative of many potential applications of these models (see Willroth et al., 2020, for a description of these data). We can first fit an unconditional LCM to these data with time centered at the first time point, which converges without issue and describes negative affect trends which on average decrease slowly over time ($\hat{\alpha}_{\eta_2} = -0.002, p = .002$) but with significant person-to-person variation ($\hat{\psi}_{22} = 3.15 \times 10^{-4}$; $p < .001$; see Section 6.1 and Figure S6 of the online supplemental materials for complete output). If we estimate the aperture as part of this unconditional model, we see that the model-implied aperture point is 6.479-time units after our initial zero point ($SE = 1.228$), between the first and second observation (recall that observations are spaced 9 years apart).

We then fit the LCM with distal outcome model with the same time coding scheme. While the unconditional model converged without issue, this model fails to converge. Inspecting the output reveals that we are encountering an offending estimate ($\hat{r}_{21} = 1.048$) and other extreme parameter estimates (e.g., the aperture is estimated to be >193-time units before the first observation). Given what we

Figure 3

Incremental Validity Specification



Note. By regressing the lower-priority intercept factor (η_1) on the higher-priority slope factor (η_2) and estimating a new set of latent variables (ξ), we can obtain regression estimates ($\gamma_{\eta_{3,\xi}}$) which are not affected by the choice of time coding scheme. Note that the variances of the growth factors and observed distal outcome are set to zero, while the variances of the ξ factors and η_3 are freely estimated.

know from the methods outlined above, we can alter our time coding approach to instead have the intercept at the final time point, which should reduce this problematic interfactor correlation. As expected, this final status model converges without issue, and the parameter estimates return to plausible values (Table 3), consistent with those seen in the unconditional model. We will primarily focus here on the regressions of chronic health conditions on the growth factor which indicate that greater negative affect at the final time point relates to higher levels chronic health conditions ($\hat{\beta}_{z,\eta_1} = 3.089$, $SE = 0.200$, $p < .001$), but the rate of change over time does not significantly predict levels of chronic health conditions ($\hat{\beta}_{z,\eta_2} = 8.142$, $SE = 10.570$, $p = .441$). The model-implied aperture falls 11.472-time units before the final time point, again somewhere between the first and second repeated measure, and we can explain 23.4% of the variance in the chronic health condition outcome.

Given the magnitude of the unstandardized effect of the intercept, we can expect meaningful change in the effect of the slope across different time coding schemes. To visualize this, we can take the same regions-of-significance approach that we did in Figure 2. Doing so shows that the magnitude of the slope effect would increase—and become significant—as we shift the intercept coding from the final towards the initial time point (Figure 4). Here, we do not have the dramatic swing in sign that we did in the example data, but nevertheless this example highlights many of the challenges associated with the time coding-induced changes in parameters. Estimating the intercept at the first time point—the most widely used approach in practice—results in a model that will not even converge, while estimating it at the final time point results in a nonsignificant effect.

To resolve these challenges, we can reestimate our model at the aperture point estimate in our final status model ($\hat{a} = 11.472$). Unlike in our original simulation example, this aperture point is within the range of observed data, and so we can proceed with this model directly. Here we see yet another pattern of effects, where the intercept ($\hat{\beta}_{z,\eta_1} = 3.089$, $SE = 0.200$, $p < .001$) *and* the slope ($\hat{\beta}_{z,\eta_2} = 43.579$, $SE = 10.552$, $p < .001$), significantly predict the distal outcome. In this aperture model, we are able to orthogonalize the growth factors ($\hat{\psi}_{21} = 0.000$, $SE = 0.001$, $p = 1.000$), which give

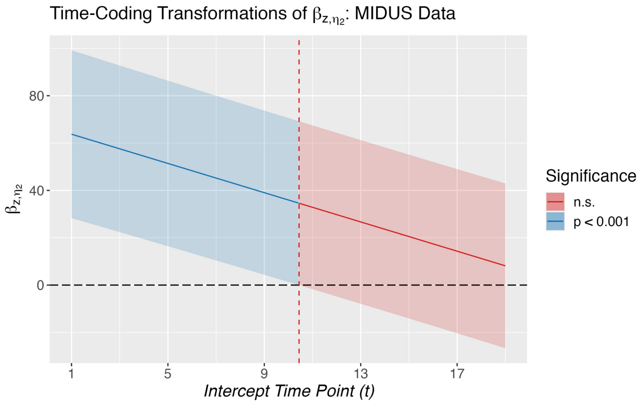
Table 3

Parameter Recovery: MIDUS Models

	Final status				Aperture status			
Param.	Est.	*SE*	Std. Est.	*p*	Est.	*SE*	Std. Est.	*p*
β_{z,η_1}	3.089	0.200	0.459	<.001	3.089	0.200	0.406	<.001
β_{z,η_2}	8.142	10.570	0.049	.441	43.579	10.552	0.262	<.001
α_{η_1}	1.471	0.010	3.143	<.001	1.494	0.008	3.605	<.001
α_{η_2}	−0.002	0.001	−0.106	.002	−0.002	0.001	−0.106	.002
ψ_{11}	0.219	0.017	1.000	<.001	0.172	0.010	1.000	<.001
ψ_{21}	0.004	0.001		<.001	0.000	0.001		1.000
r_{21}	0.465	0.058		<.001	0.000	0.078		1.000
ψ_{22}	0.000	0.000	1.000	<.001	0.000	0.000	1.000	<.001
R^2_z	0.234				0.234			
a	−11.472	1.708		<.001	−0.000	1.708		1.000

Note. Est. is the sample-recovered parameter. *SE* is the standard error of the estimate. Std. Est. is the standardized estimate. Beta (β) denotes the regression coefficients, alpha (α) denotes the factor means, psi (ψ) denotes the factor variances and covariances, and r_{21} denotes the factor correlation. R^2_z is the proportion of variance explained in the distal outcome. MIDUS = Midlife in the United States.

Figure 4
Regions of Significance for the MIDUS Data



Note. When we estimate our model at the final time point, we get a non-significant effect of the slope on the distal outcome (red shaded region), while if we estimate the model at the aperture ($t = 7.528$), we get a significant positive effect of the slope (blue shaded). Even though we get model-implied values of the effect of the slope for an intercept at the first time point, in practice this model results in an improper solution ($\hat{r}_{21} > 1$). MIDUS = Midlife in the United States. See the online article for the color version of this figure.

us the independent contributions of intercept and slope—that is the unique and zero-order relationships are the same (see Table 3 for model results).

While the aperture being within the observed time range might remove some of the motivation for adopting the incremental validity approach, we present it here for completeness. When we use the final time point as the intercept and adopt the incremental validity model (Figure 3; the full model specification and results can be seen in Section 6.2.4 of the online supplemental material), we obtain a significant and positive effect of both slope and intercept on the distal outcome ($\hat{\gamma}_{\eta_3,\xi_2} = 3.089$ and $\hat{\gamma}_{\eta_3,\xi_1} = 43.579$), which match exactly the regression coefficients obtained in the aperture model (Table 3) as expected.

Recommendations for Applied Research

We have seen unexpected complexity in modeling the effect of interindividual differences in growth model parameters on a distal outcome. While these challenges are surmountable, they do require greater care and a clear understanding of how time coding choices will impact the results of a given model. Here, we provide several practical steps that can be applied in substantive work to avoid pitfalls of interpretation. The over-arching conclusion that can be drawn from the current work is that we cannot expect to simply fit a single distal outcome model and interpret its results without considering the impact of time coding on the parameter estimates. Even for researchers who have a strong theoretical goal for placing the intercept at a given point in the growth process, the realization that the sign and significance of their distal outcome prediction could change if they were to make an alternative decision should give pause. In work without this strong theoretical imperative—which is likely a greater percentage than would be optimal—the default placement of the intercept should be treated with even greater skepticism. Graphical output like a regions-of-significance plot provide valuable information about the range of effects we can expect to see across different codings of time, to ensure that our interpretations of the slope effect are not contingent on arbitrary decisions. Generating these plots will also prevent post hoc placement of the intercept to obtain notionally significant effects. While here we only plot the time coding-dependency of the regression of the distal outcome on the slope, these plots would also be useful as diagnostic information for other parameters in the model (e.g., the mean and variance of the intercept). The transformations of these other parameters have already been described in other work (Biesanz et al., 2004), but it is unclear how deeply these conclusions have permeated into the broader community of applied researchers. As such, making these plots a standard feature of output in widely available software packages would help to raise awareness of these effects and improve general understanding on how modeling decisions impact the substantive interpretation of results.

When selecting a model for interpretation, our results strongly preference one of two approaches: (a) a model estimated at the aperture point, or (b) a model estimated within the incremental validity framework. From a theoretical standpoint, both of these approaches ensure the most stable, powered, and interpretable effects out of possible time coding alternatives. The aperture approach is attractive due to the practical ease with which we can compute the shift from any arbitrary starting point model needed to reach the aperture and the ability to easily obtain standardized and unstandardized effect estimates. Additionally, by maintaining the relatively simple standard structure of the growth model, it may be a more tractable model for more complex applications (e.g., with the inclusion of both predictors and distal outcomes, or in multivariate models). However, in situations where the aperture falls outside the observed time range, we would strongly recommend against coding the intercept at the aperture. In such instances, placing the intercept at the closest point to the aperture possible might be a logical compromise. Alternatively, the incremental validity model (Figure 3) is a welcomed alternative which can return optimal results with any time coding approach. Based on the results we present, alternatives to these two time coding approaches would require compelling justification.

Finally, while our focus has been on determining optimal strategies for specifying and interpreting the effects of the growth factors on the distal outcome, the general distal outcome LCM framework can be expanded to include additional covariates, including direct effects on the distal outcome and indirect effects through prediction of the growth factors (Biesanz et al., 2004). The direct effects of additional covariates for the distal outcome are not impacted by time coding decisions, as the joint variance explained by the factors is always invariant across time codings. In contrast, combining the distal outcome model with the conditional growth model with indirect covariate effects on the distal outcome through the growth factors requires additional care because time coding choices influence the effect of the covariate on the intercept factor, but then the effect of the slope on the distal outcome.

Directions for Future Research

The effects outlined here are suggestive of several productive directions for future research. First, the impacts of time coding decisions on distal outcome regressions were only explored in linear models. Future work should seek to generalize these principles, not only to higher-order polynomial models (e.g., quadratic and cubic), but also to more complex developmental models of change (e.g., piecewise and latent basis). Secondly, we thresholded our regions-of-significance plots at *p*-values that seemed reasonable given the relatively large sample sizes; however, there is room for much more rigorous research into how to threshold these plots in a meaningful way. If the aperture time coding is used by default, however, this may alleviate some of these concerns, and regression coefficients can then be interpreted as usual. Another potential area for future consideration is whether the focus here on unique effects of the growth factors is optimal for understanding the consequences of development. Growth factors jointly determine the developmental trajectory—for instance, individuals with the same initial status but different slopes progressively diverge over time—and considering these factors as independent predictors might miss important interaction effects (e.g., Kelava et al., 2011), similar to the ideas motivating the use of latent class growth models (Muthén & Muthé, 2000; Nylund-Gibson et al., 2019). Finally, it remains to be seen how these effects might interact with the effects that Biesanz et al. (2004) detailed when combining a conditional and distal outcome LCM—as would be the case in a parallel process mediation model (Cheong et al., 2003; O'Laughlin et al., 2018)—as conditioning the factors on a exogenous variable would change the model-implied aperture point.

Summary and Conclusions

Our goal here was to develop a framework for modeling the consequences of intraindividual differences in intraindividual trajectories of change, complementing prior work on methods to assess the course and causes of developmental patterns. We demonstrated that—in contrast to prior work on time coding effects in unconditional and conditional growth models (Biesanz et al., 2004)—the unstandardized effect of the intercept on the distal outcome remains invariant, while the effect of the slope changes systematically across different time codings. We analytically derived the parameter transformations that govern these time coding effects and show that the effect of the slope factor on the distal outcome changes linearly at a magnitude equal to the effect of the intercept factor. We provided a graphical approach for assessing these changes across plausible alternative codings of time. To adjudicate between the possible time coding approaches, we propose estimating the intercept at the aperture (Hancock & Choi, 2006)—or the point where the variance of the intercept, and consequently the covariance between intercept and slope are minimized—in order to facilitate the clearest interpretation of the unique effects of each factor, or adopting an incremental validity approach which produces orthogonal predictive effects regardless of time coding. We then demonstrated how these issues would be confronted in real data and provided recommendations for best practice in substantive research using the distal outcome LCM.

While we have focused primarily on technical challenges related to obtaining valid and interpretable parameter estimates from the latent curve model with distal outcomes, we believe these models—and measuring distal outcomes generally—to be strong tools for testing substantive causal theories. What indeed are the consequences for faster versus slower math skill development in primary school, or of the course of depression in adolescence? Distal outcome models allow for important contextualization of individual variability in growth trajectories by assessing the downstream consequences of developmental change. Valid causal estimates depend, as always, on design consideration as much as statistical modeling, and so true distal outcomes that are temporally removed from the growth model will aid in these efforts. Overall, we believe that these models have enormous potential for addressing novel developmental questions of downstream consequences and are a fertile area for subsequent methodological development.

References

Barboza, G. E., Dominguez, S., & Pinder, J. (2017). Trajectories of post-traumatic stress and externalizing psychopathology among maltreated foster care youth: A parallel process latent growth curve model. *Child Abuse & Neglect*, *72*, 370–382. https://doi.org/10.1016/j.chiabu.2017.09.007

Bauer, D. J. (2003). Estimating multilevel linear models as structural equation models. *Journal of Educational and Behavioral Statistics*, *28*(2), 135–167. https://doi.org/10.3102/10769986028002135

Bauer, D. J., & Curran, P. J. (2005). Probing interactions in fixed and multilevel regression: Inferential and graphical techniques. *Multivariate Behavioral Research*, *40*(3), 373–400. https://doi.org/10.1207/s15327906mbr4003_5

Biesanz, J. C., Deeb-Sossa, N., Papadakis, A. A., Bollen, K. A., & Curran, P. J. (2004). The role of coding time in estimating and interpreting growth curve models. *Psychological Methods*, *9*(1), 30–52. https://doi.org/10.1037/1082-989X.9.1.30

Bollen, K. A. (1989). *Structural equations with latent variables*. WileyJohn Wiley & Sons.

Bollen, K. A., & Curran, P. J. (2006). *Latent curve models: A structural equation perspective [Google-Books-ID: CDnG15sPvigC]*. John Wiley & Sons.

Bryk, A. S., & Raudenbush, S. W. (1987). Application of hierarchical linear models to assessing change. *Psychological Bulletin*, *101*(1), 147–158. https://doi.org/10.1037/0033-2909.101.1.147

Cheong, J., MacKinnon, D. P., & Khoo, S. T. (2003). Investigation of mediational processes using parallel process latent growth curve modeling. *Structural Equation Modeling: A Multidisciplinary Journal*, *10*(2), 238–262. https://doi.org/10.1207/S15328007SEM1002_5

Cohen, J., Cohen, P., West, S. G., & Aiken, L. S. (2003). *Applied multiple regression/correlation analysis for the behavioral sciences* (3rd ed.). Lawrence Erlbaum Associates.

Curran, P. J. (2003). Have multilevel models been structural equation models all along? *Multivariate Behavioral Research*, *38*(4), 529–569. https://doi.org/10.1207/s15327906mbr3804_5

Curran, P. J., Bauer, D. J., & Willoughby, M. T. (2004). Testing main effects and interactions in latent curve analysis. *Psychological Methods*, *9*(2), 220–237. https://doi.org/10.1037/1082-989X.9.2.220

Curran, P. J., & Hancock, G. R. (2021). The challenge of modeling co-developmental processes over time. *Child Development Perspectives*, *15*(2), 67–75. https://doi.org/10.1111/cdep.12401

Curran, P. J., Obeidat, K., & Losardo, D. (2010). Twelve frequently asked questions about growth curve modeling. *Journal of Cognition and Development*, *11*(2), 121–136. https://doi.org/10.1080/15248371003699969

Enders, C. K. (2001). A primer on maximum likelihood algorithms available for use with missing data. *Structural Equation Modeling: A Multidisciplinary Journal*, *8*(1), 128–141. https://doi.org/10.1207/S15328007SEM0801_7

Feng, Y., & Hancock, G. R. (2021). Model-based incremental validity. *Psychological Methods*, *27*(6), 1039–1060. https://doi.org/10.1037/met0000342

Gandhi, A., Luyckx, K., Molenberghs, G., Baetens, I., Goossens, L., Maitra, S., & Claes, L. (2019). Maternal and peer attachment, identity formation, and non-suicidal self-injury: A longitudinal mediation study. *Child and Adolescent Psychiatry and Mental Health*, *13*(1), Article 7. https://doi.org/10.1186/s13034-019-0267-2

Ghisletta, P., & McArdle, J. J. (2012). Latent curve models and latent change score models estimated in R. *Structural Equation Modeling: A Multidisciplinary Journal*, *19*(4), 651–682. https://doi.org/10.1080/10705511.2012.713275

Grimm, K. J., Ram, N., & Estabrook, R. (2016). *Growth modeling: Structural equation and multilevel modeling approaches*. Guilford Publications.

Hammer, C. S., Lawrence, F. R., & Miccio, A. W. (2007). Bilingual children's language abilities and early reading outcomes in head start and kindergarten. *Language, Speech, and Hearing Services in Schools*, *38*(3), 237–248. https://doi.org/10.1044/0161-1461(2007/025)

Hancock, G. R., & Choi, J. (2006). A vernacular for linear latent growth models. *Structural Equation Modeling: A Multidisciplinary Journal*, *13*(3), 352–377. https://doi.org/10.1207/s15328007sem1303_2

Hancock, G. R., Harring, J. R., & Lawrence, F. R. (2013). *Using latent growth modeling to evaluate longitudinal change*. IAP Information Age Publishing.

Huttenlocher, J., Haight, W., Bryk, A., Seltzer, M., & Lyons, T. (1991). Early vocabulary growth: Relation to language input and gender. *Developmental Psychology*, *27*(2), 236–248. https://doi.org/10.1037/0012-1649.27.2.236

Kelava, A., Werner, C. S., Schermelleh-Engel, K., Moosbrugger, H., Zapf, D., Ma, Y., Cham, H., Aiken, L. S., & West, S. G. (2011). Advanced nonlinear latent variable modeling: Distribution analytic LMS and QML estimators of interaction and quadratic effects. *Structural Equation Modeling: A Multidisciplinary Journal*, *18*(3), 465–491. https://doi.org/10.1080/10705511.2011.582408

Koukounari, A., Stringaris, A., & Maughan, B. (2017). Pathways from maternal depression to young adult offspring depression: An exploratory longitudinal mediation analysis. *International Journal of Methods in Psychiatric Research*, *26*(2), Article e1520. https://doi.org/10.1002/mpr.1520

Liu, S., Kuppens, P., & Bringmann, L. (2021). On the use of empirical bayes estimates as measures of individual traits. *Assessment*, *28*(3), 845–857. https://doi.org/10.1177/1073191119885019

MacCallum, R. C., Kim, C., Malarkey, W. B., & Kiecolt-Glaser, J. K. (1997). Studying multivariate change using multilevel models and latent curve models. *Multivariate Behavioral Research*, *32*(3), 215–253. https://doi.org/10.1207/s15327906mbr3203_1

Masyn, K. E., Petras, H., & Liu, W. (2014). Growth curve models with categorical outcomes. In G. Bruinsma & D. Weisburd (Eds.), *Encyclopedia of criminology and criminal justice* (pp. 2013–2025). Springer. https://doi.org/10.1007/978-1-4614-5690-2_404

McArdle, J. J., & Epstein, D. (1987). Latent growth curves within developmental structural equation models. *Child Development*, *58*(1), 110–133. https://doi.org/10.2307/1130295

McCormick, E. M. (2021). Multi-level multi-growth models: New opportunities for addressing developmental theory using advanced longitudinal designs with planned missingness. *Developmental Cognitive Neuroscience*, *51*, Article 101001. https://doi.org/10.1016/j.dcn.2021.101001

McCormick, E. M. (2023). *Latent growth factors as predictors of distal outcomes: Completing the triad*. https://osf.io/fzubt/

McCormick, E. M., Byrne, M. L., Flournoy, J. C., Mills, K. L., & Pfeifer, J. H. (2023). The Hitchhiker's guide to longitudinal models: A primer on model selection for repeated-measures methods. *Developmental Cognitive Neuroscience*, *63*, Article 101281. https://doi.org/10.1016/j.dcn.2023.101281

McCormick, E. M., Curran, P. J., & Hancock, G. R. (2023). *Latent growth factors as predictors of distal outcomes: Completing the triad*. PsyArXiv. https://doi.org/10.31234/osf.io/fevra

McNeish, D., & Matta, T. (2018). Differentiating between mixed-effects and latent-curve approaches to growth modeling. *Behavior Research Methods*, *50*(4), 1398–1414. https://doi.org/10.3758/s13428-017-0976-5

McNeish, D., & Matta, T. H. (2020). Flexible treatment of time-varying covariates with time unstructured data. *Structural Equation Modeling: A Multidisciplinary Journal*, *27*(2), 298–317. https://doi.org/10.1080/10705511.2019.1627213

Mehta, P. D., Neale, M. C., & Flay, B. R. (2004). Squeezing interval change from ordinal panel data: Latent growth curves with ordinal outcomes. *Psychological Methods*, *9*(3), 301–333. https://doi.org/10.1037/1082-989X.9.3.301

Meredith, W., & Tisak, J. (1990). Latent curve analysis. *Psychometrika*, *55*(1), 107–122. https://doi.org/10.1007/BF02294746

Muthén, B. O., & Curran, P. J. (1997). General longitudinal modeling of individual differences in experimental designs: A latent variable framework for analysis and power estimation. *Psychological Methods*, *2*(4), 371–402. https://doi.org/10.1037/1082-989X.2.4.371

Muthén, B. O., & Muthé, L. K. (2000). Integrating person-centered and variable-centered analyses: Growth mixture modeling with latent trajectory classes. *Alcoholism: Clinical and Experimental Research*, *24*(6), 882–891. https://doi.org/10.1111/j.1530-0277.2000.tb02070.x

Newsom, J. T. (2015). *Longitudinal structural equation modeling: A comprehensive introduction* (pp. xxiii, 411). Routledge.

Nylund-Gibson, K., Grimm, R. P., & Masyn, K. E. (2019). Prediction from latent classes: A demonstration of different approaches to include distal outcomes in mixture models. *Structural Equation Modeling: A Multidisciplinary Journal*, *26*(6), 967–985. https://doi.org/10.1080/10705511.2019.1590146

O'Laughlin, K. D., Martin, M. J., & Ferrer, E. (2018). Cross-sectional analysis of longitudinal mediation processes. *Multivariate Behavioral Research*, *53*(3), 375–402. https://doi.org/10.1080/00273171.2018.1454822

Raudenbush, S. W., & Bryk, A. S. (2002). *Hierarchical linear models: Applications and data analysis methods*. Sage.

Rowe, M. L., Raudenbush, S. W., & Goldin-Meadow, S. (2012). The pace of vocabulary growth helps predict later vocabulary skill. *Child Development*, *83*(2), 508–525. https://doi.org/10.1111/j.1467-8624.2011.01710.x

Seltzer, M. M., Greenberg, J. S., Krauss, M. W., & Hong, J. (1997). Predictors and outcomes of the end of co-resident caregiving in aging families of adults with mental retardation or mental illness. *Family Relations*, *46*(1), 13–22. https://doi.org/10.2307/585602

Shadish, W. R., Cook, T. D., & Campbell, D. T. (2002). *Experimental and quasi-experimental designs for generalized causal inference*. Houghton, Mifflin.

Singer, J. D., & Willett, J. B. (2003). *Applied longitudinal data analysis: Modeling change and event occurrence*. Oxford University Press.

Spoth, R., Clair, S., & Trudeau, L. (2014). Universal family-focused intervention with young adolescents: Effects on health-risking sexual behaviors and STDs among young adults. *Prevention Science*, *15*(S1), 47–58. https://doi.org/10.1007/s11121-012-0321-2

Stoel, R. D., & Hox, J. (2004). Including time-invariant covariates in the latent growth curve model. *Structural Equation Modeling: A Multidisciplinary Journal*, *11*(2), 155–167. https://doi.org/10.1207/s15328007sem1102_1

Taylor, L. K., Merrilees, C. E., Baird, R., Goeke-Morey, M. C., Shirlow, P., & Cummings, E. M. (2018). Impact of political conflict on trajectories of adolescent prosocial behavior: Implications for civic engagement. *Developmental Psychology*, *54*(9), 1785–1794. https://doi.org/10.1037/dev0000552

von Soest, T., & Hagtvet, K. A. (2011). Mediation analysis in a latent growth curve modeling framework. *Structural Equation Modeling: A Multidisciplinary Journal*, *18*(2), 289–314. https://doi.org/10.1080/10705511.2011.557344

Willroth, E. C., Ong, A. D., Graham, E. K., & Mroczek, D. K. (2020). Being happy and becoming happier as independent predictors of physical health and mortality. *Psychosomatic Medicine*, *82*(7), 650–657. https://doi.org/10.1097/PSY.0000000000000832

(*Appendix follows*)

Appendix

Parameter Estimate Transformations

The model-implied covariance matrix for all observed measures (including the repeated measures and the distal outcome) with a given time coding is:

$$\mathbf{\Sigma}_{\text{observed}} = \mathbf{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Psi}(\mathbf{I} - \mathbf{B})^{-1\prime}\mathbf{\Lambda}'. \tag{A1}$$

Biesanz et al. (2004) showed that we can compute the parameters associated with an alternative time coding using the inverse transformation matrix

$$\mathbf{T}^{-1} = (\mathbf{\Lambda}^{*\prime}\mathbf{\Lambda}^{*})^{-1}\mathbf{\Lambda}^{*\prime}\mathbf{\Lambda}. \tag{A2}$$

Because the observed repeated measures (y_{it}) and the distal outcome are linked only through the latent variables in this model structure, we can simplify matters by defining the considering the covariance matrix among y variables and the covariance matrix between the y and z variables separately. The covariance matrix among the y variables then follows exactly the derivations from Biesanz et al. (2004), where

$$\mathbf{\Lambda}^{*}\mathbf{\Psi}^{*}\mathbf{\Lambda}^{*\prime} = \mathbf{\Lambda}\mathbf{\Psi}\mathbf{\Lambda}' \quad \& \quad \mathbf{\Lambda}^{*}\boldsymbol{\alpha}^{*} = \mathbf{\Lambda}\boldsymbol{\alpha}, \tag{A3}$$

such that:

$$\mathbf{\Psi}^{*} = \mathbf{T}^{-1}\mathbf{\Psi}\mathbf{T}^{-1\prime} \quad \& \quad \boldsymbol{\alpha}^{*} = \mathbf{T}^{-1}\boldsymbol{\alpha}. \tag{A4}$$

While these matrix expressions generalize to higher-order models as well, in the linear LCM case we can solve these expressions for simple scalar equations in terms of Δa (i.e., the shift in where the intercept is estimated between alternate time codings) for both the means:

$$\alpha^{*}_{\eta_1} = \alpha_{\eta_1} + \alpha_{\eta_2}\Delta a \quad \& \quad \alpha^{*}_{\eta_2} = \alpha_{\eta_2}, \tag{A5}$$

and (co)variances of the latent growth factors:

$$\begin{aligned} \psi^{*}_{\eta_1} &= \psi_{\eta_1} + 2\psi_{\eta_1,\eta_2}\Delta a + \psi_{\eta_2}(\Delta a)^2 \quad \& \\ \psi^{*}_{\eta_1,\eta_2} &= \psi_{\eta_1,\eta_2} + \psi_{\eta_2}\Delta a \quad \& \quad \psi^{*}_{\eta_2} = \psi_{\eta_2}. \end{aligned} \tag{A6}$$

When we consider the covariance of the y variables and the distal outcome (z_i), the matrix expression is:

$$\mathbf{\Sigma}_{yz} = \mathbf{\Lambda}\mathbf{\Psi}\mathbf{B}. \tag{A7}$$

When considering alternative time codings, the following holds:

$$\mathbf{\Lambda}^{*}\mathbf{\Psi}^{*}\mathbf{B}^{*} = \mathbf{\Lambda}\mathbf{\Psi}\mathbf{B}, \tag{A8}$$

meaning that we can solve for $\mathbf{B}^{*}$ using Equations 16 and A4, which we detail below:

$$\begin{aligned} (\mathbf{\Lambda}\mathbf{T})(\mathbf{T}^{-1}\mathbf{\Psi}\mathbf{T}^{-1\prime})\mathbf{B}^{*} &= \mathbf{\Lambda}\mathbf{\Psi}\mathbf{B}, \\ (\mathbf{\Lambda}\mathbf{\Psi}\mathbf{T}^{-1\prime})\mathbf{B}^{*} &= \mathbf{\Lambda}\mathbf{\Psi}\mathbf{B}, \\ \mathbf{T}^{-1\prime}\mathbf{B}^{*} &= \mathbf{B}, \\ \mathbf{B}^{*} &= \mathbf{T}'\mathbf{B}, \end{aligned} \tag{A9}$$

resulting in:

$$\mathbf{B}^{*} = \mathbf{T}'\mathbf{B}. \tag{A10}$$

We reiterate Equation 23, and lay out the matrix expression for change in the regressions of the distal outcome on the growth factors to illustrate this transformation in practice:

$$\begin{aligned} \mathbf{B}^{*} = \mathbf{T}'\mathbf{B} &= \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}' \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix}. \end{aligned} \tag{A11}$$

Working through these matrices gives us the scalar expressions in Equations 24 and 25:

$$\beta^{*}_{z,\eta_2} = \beta_{z,\eta_2} + \beta_{z,\eta_1}\Delta a \quad \& \quad \beta^{*}_{z,\eta_1} = \beta_{z,\eta_1}. \tag{A2}$$

Standard Error Transformations

We can also derive the standard errors for these transformed parameter estimates with a straightforward application of the principles in prior work (Biesanz et al., 2004; Curran et al., 2004). Namely that we can use the Jacobian matrix of partial derivatives to estimate the change in the variances of parameter estimates. Biesanz et al. (2004) showed that the Jacobian matrix for the factor covariance matrix is as follows:

$$\begin{aligned} \mathbf{J}_{\text{vec}(\mathbf{\Psi})\rightarrow\text{vec}(\mathbf{\Psi}^{*})} &= \left[\frac{\delta\text{vec}(\mathbf{\Psi}^{*})}{\delta\text{vec}(\mathbf{\Psi})}\right]' \\ &= \left[\frac{\delta(\mathbf{T}^{-1}\otimes\mathbf{T}^{-1})\text{vec}(\mathbf{\Psi})}{\delta\text{vec}(\mathbf{\Psi})}\right]' \\ &= (\mathbf{T}^{-1}\otimes\mathbf{T}^{-1})', \end{aligned} \tag{A13}$$

where $\otimes$ is the outer product. Applied to the factor covariance matrix $\mathbf{\Psi}$, this allows us to calculate the asymptotic covariance matrix for the new time coding as:

$$\text{ACOV}(\mathbf{\Psi}^{*}) = \mathbf{J}'_{\text{vec}(\mathbf{\Psi})\rightarrow\text{vec}(\mathbf{\Psi}^{*})}\text{ACOV}(\mathbf{\Psi})\mathbf{J}_{\text{vec}(\mathbf{\Psi})\rightarrow\text{vec}(\mathbf{\Psi}^{*})}. \tag{A14}$$

Working out this matrix expression in the linear LCM gives us a set of scalar equations for the new asymptotic parameter variances, of which we can take the square root to obtain the standard errors. Some asymptotic variances do not change across time coding transformations, including for the variance of the linear slope or distal outcome.

$$\text{VAR}(\psi^{*}_{22}) = \text{VAR}(\psi_{22}) \quad \& \quad \text{VAR}(\psi^{*}_{zz}) = \text{VAR}(\psi_{zz}). \tag{A15}$$

The asymptotic variance for the covariance between intercept and slope changes via the following quadratic expression

$$\begin{aligned} \text{VAR}(\psi^{*}_{21}) = \text{VAR}(\psi_{21}) &+ 2\text{COV}(\psi_{21}, \psi_{22})\Delta a \\ &+ \text{VAR}(\psi_{22})(\Delta a)^2, \end{aligned} \tag{A16}$$

and the asymptotic variance for the variance of the intercept factor

(*Appendix continues*)

changes as a function of a complex quartic expression.

$$\begin{aligned}\mathrm{VAR}(\psi_{11}^*) = {} & \mathrm{VAR}(\psi_{11}) \\ & + 4\mathrm{COV}(\psi_{11}, \psi_{21})\Delta a \\ & + 4\mathrm{VAR}(\psi_{21})(\Delta a)^2 \\ & + 2\mathrm{COV}(\psi_{11}, \psi_{22})(\Delta a)^2 \\ & + 4\mathrm{COV}(\psi_{21}, \psi_{22})(\Delta a)^3 \\ & + \mathrm{VAR}(\psi_{22})(\Delta a)^4.\end{aligned} \tag{A17}$$

We can apply the same method to the matrix of regression coefficients between the distal outcome and latent growth factors. The result is remarkable simple

$$\begin{aligned}\mathbf{J}_{\mathrm{vec}(\mathbf{B})\rightarrow\mathrm{vec}(\mathbf{B}^*)} &= \left[\frac{\delta\mathrm{vec}(\mathbf{B}^*)}{\delta\mathrm{vec}(\mathbf{B})}\right]' \\ &= \left[\frac{\delta\mathrm{vec}(\mathbf{T}'\mathbf{B})}{\delta\mathrm{vec}(\mathbf{B})}\right]' \\ &= [\mathbf{T}']' \\ &= \mathbf{T},\end{aligned} \tag{A18}$$

being just the original transformation matrix (**T**). We can then pre and postmultiply this Jacobian with the asymptotic covariance matrix of the regression coefficients to give the transformed variances.

$$\begin{aligned}\mathrm{ACOV}(\mathbf{B}^*) &= \mathbf{J}'_{\mathrm{vec}(\mathbf{B})\rightarrow\mathrm{vec}(\mathbf{B}^*)}\mathrm{ACOV}(\mathbf{B})\mathbf{J}_{\mathrm{vec}(\mathbf{B})\rightarrow\mathrm{vec}(\mathbf{B}^*)} \\ &= \mathbf{T}'\mathrm{ACOV}(\mathbf{B})\mathbf{T}.\end{aligned} \tag{A19}$$

Finally, working out this matrix expression in the linear LCM gives us a set of scalar equations for the new regression coefficients

$$\begin{aligned}&\mathrm{VAR}(\beta^*_{z,\eta_1}) = \mathrm{VAR}(\beta_{z,\eta_1}), \\ &\mathrm{VAR}(\beta^*_{z,\eta_2}) = \mathrm{VAR}(\beta_{z,\eta_2}) + 2\mathrm{COV}(\beta_{z,\eta_1}, \beta_{z,\eta_2})\Delta a + \beta_{z,\eta_1}(\Delta a)^2.\end{aligned} \tag{A20}$$

Taking the square root of these transformed asymptotic variances yields the relevant standard errors.

Received April 14, 2023
Revision received September 18, 2023
Accepted November 14, 2023 ■