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Semiconductor laser with filtered optical feedback: from optical injection to conventional feedback

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We study a model for a semiconductor laser subject to filtered optical feedback, i.e. a system of delay differential equations (DDEs). In this model, the filter is characterized by a mean frequency Ω_m and a filter width λ. In the limit of a narrow filter ($\lambda \to 0$), the laser equations reduce to the equations for a laser with optical injection, whereas they become the Lang–Kobayashi equations in the limit of an unbounded filter width ($\lambda \to \infty$). We perform a bifurcation analysis of steady-state solutions with the parameter λ as main bifurcation parameter. In this way, we get an insight in the relation between the filter width and the number of possible steady states. In particular, we obtain the result that there exist parameter regions in which the number of possible steady states decreases when λ is increased. From a mathematical point of view, our approach to combine information about limiting systems with precise bifurcation results may be of interest for a wider class of systems of DDEs.

Keywords: semiconductor laser; delay differential equations; bifurcations.

1. Introduction

Semiconductor lasers are being used in many technological applications, ranging from CD players and laser printers to optical telecommunications. In many of these systems, reflections from an external mirror, back into the laser, are often unavoidable. Upon re-entry into the laser, the light has acquired a certain delay time due to the travel outside the laser. This delay time is in general large relative to the fast internal time-scales of the laser and must therefore be taken into account.

Lasers with conventional optical feedback (COF), where the spectral content is not changed before re-entry, have been studied quite extensively. More recently, filtered external optical feedback (FOF) has become a topic of study, with the first experiments dating from the late 1980s (Zorabedian *et al.*, 1987; Li & Abraham, 1989; Etrich *et al.*, 1991; Zorabedian, 1992; Yan *et al.*, 1996). In a FOF laser, the light is filtered before it is led back into the laser. Light with certain frequencies can pass the filter, while other frequencies are filtered out. There are several advantages of FOF over COF. Filtering can be used to control the output of the laser because of the external tunable parameters and it also leads to more stable dynamics. In particular, filtering can make a laser operate in a single longitudinal mode.

The laser with FOF is in its simplest version modelled mathematically by three equations. The first two, for the (complex) electric field E and the (real) inversion n, are given by a rescaled version of the

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so-called Lang–Kobayashi (LK) equations (Lang & Kobayashi, 1980; Krauskopf & Lenstra, 2000) that were derived for the COF laser:

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{1}{2}(1+\mathrm{i}\alpha)\xi n(t)E(t) + \kappa E(t-\tau)\mathrm{e}^{-\mathrm{i}\omega_0\tau},\\ \frac{\mathrm{d}n}{\mathrm{d}t} &= J - J_{\mathrm{thr}} - \frac{n(t)}{T_1} - (\varGamma_0 + \xi n(t))|E(t)|^2.\end{aligned} \tag{1.1}$$

Here, time t is scaled with ω_{r}^{-1}, where ω_{r} is the frequency of the relaxation oscillation, an intrinsic resonance of the laser. The parameter κ measures the injected field strength. The parameter α is the linewidth enhancement factor, ξ is the differential gain, ω_0 is the solitary laser angular frequency, $\varGamma_0$ is the photon decay rate, T_1 is the carrier lifetime and J and J_{thr} are the pump current and its value at solitary laser threshold. In practical experiments, the parameter ω_0 often depends on J, see for instance Fischer *et al.* (2000a).

In these LK equations, the delay term $\kappa E(t-\tau)\mathrm{e}^{-\mathrm{i}\omega_0\tau}$ appears, which models feedback from an external flat mirror. To incorporate the filter, centred at frequency ω_{m} with width $\varLambda$, this term is replaced by a term γF. The filtered electric field F satisfies

$$\frac{\mathrm{d}F}{\mathrm{d}t} - (\mathrm{i}\omega_{\mathrm{m}} - \varLambda)F = \varLambda E(t-\tau)\mathrm{e}^{-\mathrm{i}\omega_0\tau}. \tag{1.2}$$

Hence, the full model we analyse is

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{1}{2}(1+\mathrm{i}\alpha)\xi nE + \gamma F,\\ \frac{\mathrm{d}n}{\mathrm{d}t} &= J - J_{\mathrm{thr}} - \frac{n(t)}{T_1} - (\varGamma_0 + \xi n(t))|E(t)|^2,\\ \frac{\mathrm{d}F}{\mathrm{d}t} &- (\mathrm{i}\omega_{\mathrm{m}} - \varLambda)F = \varLambda E(t-\tau)\mathrm{e}^{-\mathrm{i}\omega_0\tau}.\end{aligned} \tag{1.3}$$

See Yousefi & Lenstra (1999) and references therein for an explanation and for a derivation and numerical studies of the resulting system.

The first studies were followed by a series of numerical and experimental studies (Fischer *et al.*, 2000a, 2004; Yousefi *et al.*, 2000, 2002, 2003) in which various types of behaviour were observed. However, analytically, even the simplest behaviour of the laser with filtered feedback was hardly understood so that additional analysis was needed. One attempt for an analytical approach is to study the limit cases of the problem. This is the strategy chosen in this paper. Other recent approaches are, for instance, a simplification of the rate equations via asymptotic analysis (Nizette & Erneux, 2006) or detailed bifurcation analysis using continuation methods (Erzgräber *et al.*, 2006, 2007).

The first limit case we consider is the limit $\varLambda \to \infty$, in which the problem reduces to a laser with conventional feedback, i.e. without filtering. In this limit, the equation for F simply becomes the algebraic relation $F = E(t-\tau)\mathrm{e}^{-\mathrm{i}\omega_0\tau}$, which means that the equations reduce to the LK equations (1.1). The second limit case is the limit $\varLambda \to 0$, the limit of the small filter width. We show that in this limit, the laser is in fact a single optically injected semiconductor laser. However, since the injection is still obtained through feedback, the feedback delay appears to play a role in the selection of frequency and phase shift of the electric field E of the laser.

The LK equations and the extended version presented here are models of delay differential equations (DDEs) type that are really of interest from an applicational viewpoint. Therefore, the questions that arise for these equations differ from the main research questions and known results for DDEs. Often, the studied topics are of a more fundamental, mathematical nature, such as existence and uniqueness questions or long-time behaviour of solutions, and appear to be very hard to analyse even for single DDEs. See, for instance, Fiedler & Mallet-Paret (1989) and Polner (2002) for examples of results about long-time behaviour. In the case of a very large delay, that may even be state dependent, more analytical results can be obtained. We refer to the series Mallet-Paret & Nussbaum (1992, 1996, 2003) and references therein for examples. Many of the results are still of a rather theoretical nature, and deal, for instance, only with (very) slowly oscillating solutions with a period larger than the delay time.

It is in general very difficult to extend these results to higher-order systems of DDEs (but see, for instance, Mallet-Paret & Sell (1996a,b) for some fundamental results or Hale & Verduyn-Lunel (2003) for an example of a more practical, but linear, application). The only way to really analyse more complicated systems seems to perform numerical analysis. We, however, choose a different approach.

Since the current system has, as compared to the LK equations, two new parameters that are adjustable in an experiment, it is very natural to perform bifurcation analysis in these parameters for certain types of (basic) solutions. This will be the focus of this paper. Although it is possible to perform such analysis numerically (see, for instance, Engelborghs *et al.*, 2001, 2002; Green & Krauskopf, 2006, and references therein), we feel that precise analytical results help to get a better understanding. We choose to combine analysis of two singularly perturbed limits mentioned above with bifurcation analysis in order to obtain answers to certain physical questions. Our approach differs from the average one in that we do not attempt to determine how the dynamics of the limiting systems mirror the dynamics for more moderate choices of the bifurcation parameter, but instead combine knowledge about the limiting behaviour with bifurcation information about the route between these limits. We think that this can also be an interesting approach for other, rather involved, systems of DDEs.

From a physical point of view, analysis of the parameter regions $0 < \Lambda \ll 1$ and $0 < \frac{1}{\Lambda} \ll 1$ *a priori* seems not to be of any interest since filters in experiments will satisfy neither of the limits. However, analysis of these parameter regions may help to understand the dynamics that are observed in other regions. The aim of this paper is to try and understand how the filter equations form a bridge between these two limits and thus between the standard single injection laser equations and LK equations. To do so, we rescale System (1.3) to a set of dimensionless equations:

$$\begin{aligned} E' &= (1 + \mathrm{i}\alpha)NE + \Gamma F, \\ TN' &= P - N - (1 + 2N)|E|^2, \\ F' &= \lambda E(s - \theta)\mathrm{e}^{-\mathrm{i}\Omega_0\theta} + (\mathrm{i}\Omega_{\mathrm{m}} - \lambda)F. \end{aligned} \tag{1.4}$$

See Appendix A for the details of this rescaling. From now on, N is called the inversion, s is the rescaled time variable and E and F are the (complex) electric field and filtered electric field. The prime means differentiation with respect to s. The parameters intrinsic to the laser are the ratio of decay times T and the linewidth enhancement factor α. The parameter P is the pump current related to the amount of electrical energy supplied to the laser. Related to the filter are Ω_{m} its mean frequency and the filter width λ. Other parameters are the delay θ, the feedback strength Γ and the solitary laser frequency Ω_0.

Equations (1.4) form a 5D system of DDEs with a delay θ. Their phase space is the infinitely dimensional space $C[-\theta, 0]$ of continuous functions with values in the 5D (E, N, F)-space. See Diekmann *et al.* (1995) for a general treatment of such equations.

We restrict ourselves to the study of 'fixed points' or the so-called external filtered modes (EFMs) and their bifurcations as the filter parameter λ (Λ) is varied. Although the equations for the fixed points, that form the backbone of the laser dynamics, have been written down many times before, they have not been analysed in a concise way until recently. In this article, we assemble various pieces of the EFM puzzle.

In Section 2, we start with the analysis of the basic state $(0, P, 0)$ and its bifurcations, and relate the Hopf bifurcation that it undergoes to the physical relevance of computed EFM solutions (see also Rottschäfer & Krauskopf, 2006b). We relate this analysis to the physical concept of 'threshold reduction', see e.g. Fischer *et al.* (2000b). We also describe the converse, a threshold increment (see Section 2.5).

Then, we carefully analyse the EFMs in the limit $\lambda \to 0$ in Section 3, and find that solutions have either a frequency Ω with $|\Omega - \Omega_{\mathrm{m}}| = \mathrm{O}(\lambda)$ or a frequency $\Omega = \mathrm{O}(\lambda)$. In the first case, the laser synchronizes with the filter as one wishes in this set-up. In the second case, only a small portion of the electric field E passes the filter, i.e. $|F| = \mathrm{O}(\lambda)$ while $|E| = \mathrm{O}(1)$, and the laser operates close to its solitary laser frequency.

We also find that for $0 < \lambda \ll 1$, the laser can only operate $\mathrm{O}(\lambda)$ close to mean filter frequency Ω_{m} if Ω_{m} is chosen such that it lies in one of a number of bounded intervals $I^{j}_{\Omega_{\mathrm{m}}}$. The number of such intervals varies depending on the parameters α, Γ, θ and Ω_0. One interval, $I^{0}_{\Omega_{\mathrm{m}}}$, is centred around $\Omega_{\mathrm{m}} = 0$ and exists for all parameter choices.

In Section 5, we compare the limits $\lambda \to 0$ and $\lambda \to \infty$. We find that there exist parameter regions in which the injection limit $\lambda \to 0$ has three EFM solutions (steady states, named locked modes if they are stable), whereas the LK limit $\lambda \to \infty$ has only one such solution (named external cavity mode (ECM) for a laser with COF). This is somewhat surprising since numerically, so far, only increments of the number of EFMs to (1.4) had been observed for increasing λ.

Section 6 consists of a bifurcation analysis, in which we analyse saddle-node (SN) bifurcations of EFMs that occur as λ is varied. We determine conditions under which the bifurcation is creating of nature (two EFMs appear as λ is increased) or annihilating of nature (two EFMs disappear as λ is increased). The main conclusions are stated in Section 6.1. We conclude that all bifurcations of EFMs with frequency Ω between 0 (solitary laser frequency) and Ω_{m} (mean frequency of the filter) are of creating nature, which corresponds to observations in numerical studies (Yousefi & Lenstra, 1999; Yousefi *et al.*, 2001): the two 'islands' of EFMs that have been observed for low values of λ get connected in a series of bifurcations, in which additional EFMs appear for increasing λ and no annihilations occur in between.

Our analysis is complementary to that of Green & Krauskopf (2006), where the EFMs are studied for a fixed value of λ (and fixed α, Γ and T) and varying detuning Δ between the laser and the filter (i.e. varying Ω_{m} in our case). This study focuses on the islands (Yousefi & Lenstra, 1999) or 'EFM components' (Green & Krauskopf, 2006) and the parameter sets for which these get connected (see Remark 6.3).

2. Existence of EFMs

2.1 *The basic solution*

The basic solution of (1.4) is the trivial solution $(E, N, F) = (0, P, 0)$. When this basic solution is stable, there is no lasing activity; the laser is off. Equations (1.4) are rescaled so that $P = 0$ is the solitary laser threshold. This means that, for a solitary laser, $(0, P, 0)$ destabilizes when the pump current increases above $P = 0$. It is well-known that delayed optical feedback can cause a reduction of this threshold, see Fischer *et al.* (2000b). Therefore, one may expect that in (1.4), with non-zero Γ and λ, the trivial solution already undergoes a destabilizing bifurcation for some $P^* < 0$. We study the stability properties of $(0, P, 0)$ below and in Section 2.4.

With $E = E_1 + \mathrm{i}E_2$ and $F = F_1 + \mathrm{i}F_2$, we write (1.4) as

$$\frac{\mathrm{d}}{\mathrm{d}s}(E_1(s), E_2(s), N(s), F_1(s), F_2(s))^\top$$
$$= G(E_1(s), E_2(s), N(s), F_1(s), F_2(s), E_1(s-\theta), E_2(s-\theta)).$$

The linear variational equation around the fixed point $(E, N, F) = (0, P, 0)$ then becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}(E_1(s), E_2(s), N(s), F_1(s), F_2(s))^\top$$
$$= M(E_1(s), E_2(s), N(s), F_1(s), F_2(s), E_1(s-\theta), E_2(s-\theta))^\top, \tag{2.1}$$

where M denotes the Jacobian DG of G evaluated at the fixed point $(0, 0, P, 0, 0, 0, 0)$. With $\xi(s) = (E_1(s), E_2(s), N(s), F_1(s), F_2(s))^\top$, (2.1) is written in the standard way as

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = A\xi(s) + B\xi(s-\theta), \tag{2.2}$$

where A and B are given by

$$A = \begin{pmatrix} P & -\alpha P & 0 & \Gamma & 0 \\ \alpha P & P & 0 & 0 & \Gamma \\ 0 & 0 & -\frac{1}{T} & 0 & 0 \\ 0 & 0 & 0 & -\lambda & -\Omega_\mathrm{m} \\ 0 & 0 & 0 & \Omega_\mathrm{m} & -\lambda \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \lambda\cos\Omega_0\theta & \lambda\sin\Omega_0\theta & 0 & 0 & 0 \\ -\lambda\sin\Omega_0\theta & \lambda\cos\Omega_0\theta & 0 & 0 & 0 \end{pmatrix}.$$

Now, for each root μ_i of the characteristic equation

$$\det\Delta(\mu) = 0, \quad \text{where } \Delta(\mu) = \mu I - A - \mathrm{e}^{-\mu\theta}B,$$

the exponential $\xi(s) = \bar{\xi}_i\,\mathrm{e}^{\mu_i s}$, with $\bar{\xi}_i$ a constant vector, is a solution of (2.2). We compute

$$\det\Delta(\mu) = \left(\mu + \frac{1}{T}\right)[(\Gamma\lambda\,\mathrm{e}^{-\mu\theta}\sin\Omega_0\theta - \Omega_\mathrm{m}(\mu - P) - \alpha P(\mu+\lambda))^2$$
$$+ (\Gamma\lambda\,\mathrm{e}^{-\mu\theta}\cos\Omega_0\theta - (\mu+\lambda)(\mu-P) + \Omega_\mathrm{m}\alpha P)^2]. \tag{2.3}$$

For delay equations, the characteristic equation has, in general, infinitely (but countably) many solutions, of which finitely many satisfy $\mathrm{Re}(\mu_i) > 0$ (see Diekmann *et al.*, 1995). Here, there is only a finite number of eigenvalues if $\lambda = 0$ or $\Gamma = 0$, the eigenvalues being $\mu = -\frac{1}{T}$, $\mu = \pm\mathrm{i}\Omega_\mathrm{m}$ and $\mu = P \pm \mathrm{i}\alpha P$ when $\lambda = 0$ and $\mu = -\frac{1}{T}$, $\mu = -\lambda \pm \mathrm{i}\Omega_\mathrm{m}$ and $\mu = P \pm \mathrm{i}\alpha P$ when $\Gamma = 0$. Since $\lambda \geqslant 0$, this means that the state $(0, P, 0)$ is stable in these two cases if $P < 0$ and unstable if $P > 0$.

In case $\lambda \neq 0$ and $\Gamma \neq 0$, all eigenvalues, apart from the eigenvalue $\mu = -\frac{1}{T}$, must satisfy

$$\Gamma\lambda\,\mathrm{e}^{-\mu\theta}\sin\Omega_0\theta - \Omega_\mathrm{m}(\mu - P) - \alpha P(\mu+\lambda)$$
$$= \pm\mathrm{i}(\Gamma\lambda\,\mathrm{e}^{-\mu\theta}\cos\Omega_0\theta - (\mu+\lambda)(\mu-P) + \Omega_\mathrm{m}\alpha P).$$

To compute for which P the state $(0, P, 0)$ destabilizes, we write $\mu = \mathrm{i}\nu$, $\nu \in \mathbf{R}$, and find

$$\begin{aligned} -\Gamma\lambda \sin(\nu\theta - \Omega_0\theta) &= (\lambda - P)\nu - \Omega_{\mathrm{m}} P + \alpha P\lambda, \\ -\Gamma\lambda \cos(\nu\theta - \Omega_0\theta) &= \nu^2 + (\Omega_{\mathrm{m}} + \alpha P)\nu + P\lambda + \alpha P \Omega_{\mathrm{m}} \end{aligned} \tag{2.4}$$

as condition for a Hopf bifurcation of $(0, P, 0)$. We already refer to Fig. 3 for plots of the Hopf curves for various choices of P and λ. The basic solution $(0, P, 0)$ is stable inside the loop that appears for small $P > 0$. For $P < 0$, the solution is stable below the Hopf curve.

If $\nu = 0$, the Hopf bifurcation coincides with a SN bifurcation. This is actually an SN bifurcation of an EFM. We now address this type of solutions.

2.2 *External filtered modes*

If the basic solution $(0, P, 0)$ becomes unstable, the laser starts lasing. EFMs represent the simplest type of lasing behaviour. These modes (fixed points) have constant intensities and inversion and a phase that depends linearly on time:

$$E(s) = R\,\mathrm{e}^{\mathrm{i}(\Omega s + \phi_0)}, \quad F(s) = S\,\mathrm{e}^{\mathrm{i}\Omega s} \quad \text{and} \quad N(s) = N, \tag{2.5}$$

where R, S, N, Ω and ϕ_0 are constants. Since the fields E and F are optically related, they have the same frequency, possibly with a phase shift ϕ_0. These solutions are easiest studied in polar coordinates, so away from $|E| = |F| = 0$, we rewrite (1.4) as

$$\begin{aligned} E(s) &= R(s)\mathrm{e}^{\mathrm{i}\tilde{\phi}(s)} = R(s)\mathrm{e}^{\mathrm{i}(\Omega s + \phi(s))}, \\ F(s) &= S(s)\mathrm{e}^{\mathrm{i}\tilde{\psi}(s)} = S(s)\mathrm{e}^{\mathrm{i}(\Omega s + \psi(s))}, \end{aligned} \tag{2.6}$$

with Ωs defined as the linear part of the phase of F. The laser operates at $\Omega_{\mathrm{L}} = \Omega + \Omega_0\ \Omega$, i.e. Ω is the detuning of the frequencies of E and F with respect to the solitary laser frequency Ω_0.

REMARK 2.1 In standard modelling of lasers, injection Ω is defined as the detuning of the 'injected field' from the solitary laser frequency. Here, F is in fact the injected field, and it is thus natural to put Ωs as the full linear part of F rather than E. This choice means that $\psi(s)$ is purely non-linear and that $\phi(s)$ may contain constant terms. Note, however, that in, for instance, Yousefi & Lenstra (1999) and Fischer *et al.* (2000a), the opposite choice has been made in the definition of the fixed points: there the phase shift ϕ_0 in (2.5) is incorporated in F.

With the decomposition (2.6), for $R, S \neq 0$, we write (1.4) as

$$\begin{aligned} R' &= NR + \Gamma S \cos(\psi - \phi), \\ \phi' + \Omega &= \alpha N + \Gamma \frac{S}{R} \sin(\psi - \phi), \\ S' &= \lambda R(s-\theta)\cos(\phi(s-\theta) - \psi(s) - \Omega_0\theta - \Omega\theta) - \lambda S, \\ \psi' + \Omega &= \lambda \frac{R(s-\theta)}{S(s)} \sin(\phi(s-\theta) - \psi(s) - \Omega_0\theta - \Omega\theta) + \Omega_{\mathrm{m}}, \\ TN' &= P - N - (1 + 2N)R^2. \end{aligned} \tag{2.7}$$

EFMs (2.5) satisfy $R' = \phi' = S' = \psi' = N' = 0$. The resulting conditions can, for $\lambda \neq 0$, be written as

$$\Gamma^2 \frac{S^2}{R^2} = N^2 + (\Omega - \alpha N)^2, \tag{2.8}$$

$$\tan(\phi_0) = \frac{\Omega - \alpha N}{N}, \tag{2.9}$$

$$\frac{S^2}{R^2} = \frac{\lambda^2}{\lambda^2 + (\Omega - \Omega_{\mathrm{m}})^2}, \tag{2.10}$$

$$\tan(\phi_0 - \Omega_0\theta - \Omega\theta) = \frac{\Omega - \Omega_{\mathrm{m}}}{\lambda}, \tag{2.11}$$

$$R^2 = \frac{P - N}{1 + 2N}. \tag{2.12}$$

In the LK limit $\lambda \to \infty$, (2.10) reduces to $S^2 = R^2$ so that (2.8) is an ellipse in the (Ω, N)-plane: it resembles the fixed-point ellipse that defines the locus of fixed points of the LK equations. Combining (2.8) and (2.10), the locus of EFMs of (2.7) in (Ω, N)-space is found as is shown in Fig. 1.

The relation (2.12) between R and N is also found in the LK and injection laser equations. It implies that solutions with real amplitude R, the physically relevant solutions (Rottschäfer & Krauskopf, 2006a), should for $P > -\frac{1}{2}$ satisfy

$$-\frac{1}{2} < N < P. \tag{2.13}$$

With $N = -\Gamma \frac{S}{R} \cos(\phi_0) = -\Gamma \cos(\phi_0 - \Omega_0\theta - \Omega\theta)\cos(\phi_0)$, (2.9) and (2.11) form a set of two equations for Ω and ϕ_0:

$$\Omega = -\Gamma \cos(\phi_0 - \Omega_0\theta - \Omega\theta)\cos(\phi_0)(\tan(\phi_0) + \alpha), \tag{2.14}$$

$$\Omega - \Omega_{\mathrm{m}} = \lambda \tan(\phi_0 - \Omega_0\theta - \Omega\theta), \tag{2.15}$$

from which one derives

$$\phi_0 = \Omega_0\theta + \Omega\theta + \arctan\left(\frac{\Omega - \Omega_{\mathrm{m}}}{\lambda}\right)$$

and

$$\Omega = -\Gamma \sqrt{\frac{\lambda^2(1+\alpha^2)}{\lambda^2 + (\Omega - \Omega_{\mathrm{m}})^2}} \sin\left(\Omega_0\theta + \Omega\theta + \arctan\left(\frac{\Omega - \Omega_{\mathrm{m}}}{\lambda}\right) + \arctan(\alpha)\right). \tag{2.16}$$

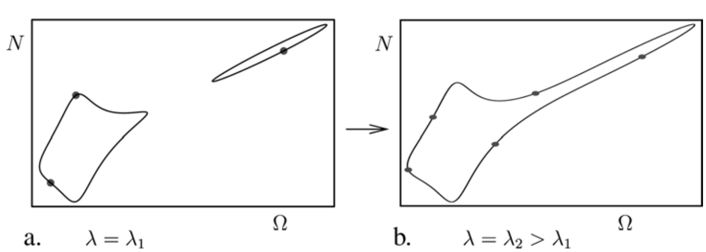

FIG. 1. Locus of EFMs in (Ω_{m}, N)-space as the parameter λ is increased. EFMs are plotted as dots. The locus always consists of one connected component as $\lambda \to \infty$ (b); depending on the parameters, it may consist of one or two components for smaller λ (a).

This corresponds to the calculations in Yousefi & Lenstra (1999) and Green & Krauskopf (2006). Note here that the equality $\frac{S}{R} = \cos(\phi_0 - \Omega_0\theta - \Omega\theta)$ that is used to derive this expression is not necessarily satisfied in the limit $\lambda = 0$, but it is as long as $\lambda > 0$. Note as well that $\phi_0^k := \phi_0 + k\pi$, $k \in \mathbf{Z}$, also solves (2.15) but leads to the same formula for Ω.

Every solution Ω of (2.16) corresponds to exactly one EFM. Once a value for Ω is determined, the unknowns R, S, N and ϕ_0 are obtained via the equations above. However, (2.16) is transcendental in Ω and it can therefore not be solved explicitly.

2.3 *Bifurcations of fixed points*

In order to study the EFMs, we define the functions

$$f(\Omega;\lambda) = -\Gamma\sqrt{\frac{\lambda^2(1+\alpha^2)}{\lambda^2+(\Omega-\Omega_{\rm m})^2}}\sin\left(\Omega_0\theta + \Omega\theta + \arctan\left(\frac{\Omega-\Omega_{\rm m}}{\lambda}\right) + \arctan(\alpha)\right), \quad (2.17)$$

$$g(\Omega;\lambda) = \Omega. \quad (2.18)$$

Solutions of (2.16) satisfy $f = g$, and SN bifurcations of fixed points occur when both $f = g$ and $f_\Omega = g_\Omega$. A combination of these two requirements will lead to our bifurcation results. In Fig. 2, the functions f and g are plotted as λ is increased.

The main question in this article is how the number of EFMs varies with λ, i.e. when the set-up is varied from an injection laser to a laser with external feedback. It is well-known that the model for the injection laser only exhibits parameter regions with one or three fixed points, while the LK equations can have n fixed points (which may not be all physically relevant) for any odd $n > 0$ (Rottschäfer & Krauskopf, 2006a; van Tartwijk & Lenstra, 1995; Wieczorek *et al.*, 1999), and it is interesting to see how these fixed points are related. Section 6 is devoted to this analysis.

2.4 *The physical relevance bifurcation*

As mentioned in (2.13), the amplitude of the EFMs becomes complex and the EFMs are no longer ‘physically relevant’ when $-\frac{1}{2} < N < P$. In order to study the related ‘physical relevance bifurcation’ at which N passes through the value P, we study a neighbourhood of this transition. It occurs when the amplitudes R and S of the EFM get small, and simultaneously N goes to P. Therefore, we set

$$N = P + c_1\delta^2, \quad R = c_3\delta, \quad S = c_2\delta, \quad z = \frac{c_2}{c_3} > 0, \quad (2.19)$$

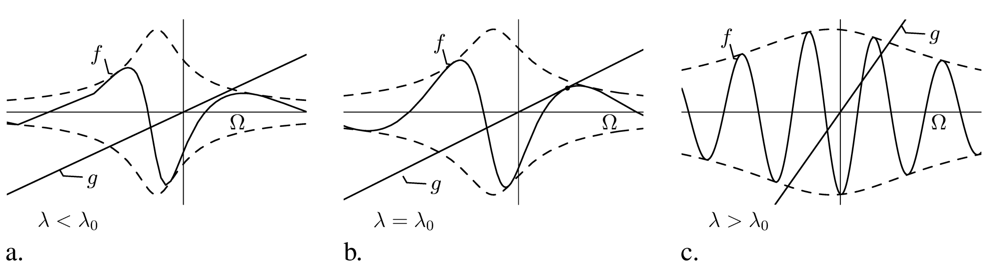

FIG. 2. Plots of f and g as functions of Ω as the parameter λ is increased. (a) Only one EFM exists. (b) Two EFMs are formed in a SN bifurcation at $\lambda = \lambda_0$. (c) There are three EFMs.

where the c_i are constants. An EFM solution is only physically relevant if the constants c_2 and c_3 are positive, so we take $c_2, c_3 > 0$. It follows from (2.12) that $c_1 = -(1 + 2P)c_3^2$. Substituting this assumption into the equilibrium equations for the EFMs, we obtain

$$N = -\Gamma z \cos(\phi_0), \tag{2.20}$$

$$\Omega = \alpha N + N \tan(\phi_0), \tag{2.21}$$

$$z = \cos(\Omega_0\theta + \Omega\theta - \phi_0), \tag{2.22}$$

$$\Omega - \Omega_{\mathrm{m}} = -\lambda \tan(\Omega_0\theta + \Omega\theta - \phi_0). \tag{2.23}$$

Filling in (2.19) in (2.20), one finds as leading-order equations in δ

$$\phi_0 = \pm \arccos\left(-\frac{P}{\Gamma z}\right) + 2k\pi, \quad k \in \mathbf{Z}, \quad \text{or} \quad \sin(\phi_0) = \pm\sqrt{1 - \frac{P^2}{\Gamma^2 z^2}}, \tag{2.24}$$

where the plus and minus signs correspond to each other. Substitution of (2.20) and (2.24) into (2.21) yields

$$\Omega = \alpha P \mp \sqrt{\Gamma^2 z^2 - P^2}. \tag{2.25}$$

Furthermore, (2.22) gives that

$$\Omega_0\theta + \Omega\theta - \phi_0 = \pm \arccos(z) + 2k\pi, \quad k \in \mathbf{Z}. \tag{2.26}$$

Here, the plus or minus sign is independent of those in (2.24). Equations (2.22), (2.26) and (2.23) together yield

$$\Omega - \Omega_{\mathrm{m}} = \mp\frac{\lambda}{z}\sqrt{1 - z^2}, \tag{2.27}$$

and combining (2.24) and (2.26) gives the four cases

$$\Omega_0\theta + \Omega\theta \mp \arccos\left(-\frac{P}{\Gamma z}\right) = \pm \arccos(z) + 2k\pi, \quad k \in \mathbf{Z}. \tag{2.28}$$

(Note that the signs change independently.) Combining (2.25), (2.27) and (2.28) leads to the four cases

$$\mp\sqrt{\Gamma^2 z^2 - P^2} \pm \frac{\lambda}{z}\sqrt{1 - z^2} = \Omega_{\mathrm{m}} - \alpha P \tag{2.29}$$

and to each of them one related equation

$$\Omega_0\theta = \pm \arccos\left(-\frac{P}{\Gamma z}\right) \pm \left[\arccos(z) + \theta\frac{\lambda}{z}\sqrt{1 - z^2}\right] - \theta\Omega_{\mathrm{m}} + 2k\pi, \quad k \in \mathbf{Z}. \tag{2.30}$$

(Note that if a sign changes in (2.29), the corresponding sign changes in (2.30) as well.)

Summarizing, once $z > 0$ is solved from (2.29), (2.30) gives the relation the parameters satisfy in the physical relevance bifurcation, i.e. when an EFM disappears getting a complex-valued amplitude.

It follows from (2.12) that the closer an EFM moves to the boundary where it ceases to be physically relevant, the smaller its ‘radius’ R becomes. Physically, this means that its intensity $|E|^2 = R^2$ goes to zero. When the boundary is reached, the radius has become zero, which suggests a Hopf bifurcation of a solution on the subspace $\{E = 0,\ F = 0\}$ in (E, N, F)-space. The solution involved can only be the basic solution discussed in Section 2.1. Indeed, we find numerically that the expressions as found for the physical relevance bifurcation and the Hopf bifurcation (2.4) coincide. We prove this coincidence algebraically in the two limits $\lambda \to \infty$ and $\lambda \to 0$.

In the limit $\lambda \to \infty$, the variable z, introduced in (2.19), must satisfy $z = 1$. Hence, (2.30) reduces to exactly the same equation as found for both the Hopf curve and the curve of physical relevant bifurcations in the LK equations in Rottschäfer & Krauskopf (2006b):

$$\Omega_0\theta = \pm\left[\arccos\left(-\frac{P}{\Gamma}\right) + \theta\sqrt{\Gamma^2 - P^2}\right] - \alpha\theta P + 2k\pi.$$

On the other hand, for $\lambda = 0$, (2.4) yields that the purely imaginary eigenvalue $\mu = \mathrm{i}\nu$ satisfies $\nu = -\Omega_\mathrm{m}$, which ensures by continuity that $|\nu + \Omega_\mathrm{m}| = \mathrm{O}(\lambda)$ for $0 < \lambda \ll 1$. Therefore, we put $\nu = -\Omega_\mathrm{m} + a\lambda$ in (2.4) and find as leading-order terms in λ

$$\Gamma \sin(\theta(\Omega_0 + \Omega_\mathrm{m})) = -aP - \Omega_\mathrm{m} + \alpha P, \tag{2.31}$$

$$-\Gamma \cos(\theta(\Omega_0 + \Omega_\mathrm{m})) = a(\alpha P - \Omega_\mathrm{m}) + P. \tag{2.32}$$

Squaring both equations, adding them and solving the resulting equation for a give

$$a^2 = \frac{\Gamma^2}{P^2 + (\alpha P - \Omega_\mathrm{m})^2} - 1.$$

Multiplying (2.31) by a and subtracting (2.32) gives, after some manipulations,

$$\theta(\Omega_0 + \Omega_\mathrm{m}) + \arctan\alpha = \pm\arccos\left[-\frac{P}{\Gamma}\sqrt{1 + a^2}\right] + 2k\pi. \tag{2.33}$$

Equation (2.33) coincides with the curve (2.30) obtained for the physical relevance bifurcation when $\lambda \ll 1$: for $\lambda \ll 1$, the leading-order terms in (2.29) give that

$$z^2 = \frac{P^2 + (\Omega_\mathrm{m} - \alpha P)^2}{\Gamma^2} \quad \text{and hence } a^2 + 1 = \frac{1}{z^2}.$$

This implies that $\arctan a = \arccos z$. Using the two relations between a and z, one observes that (2.30) and (2.33) indeed coincide for $0 < \lambda \ll 1$.

In the above analysis, we heavily rely on the fact that $\lambda \ll 1$. For more general λ, this result is difficult to obtain, but plots of both curves indeed overlap, as they should from a physical point of view (see Fig. 3).

From this analysis, we conclude that the curve of Hopf bifurcations of the solution $(0, P, 0)$ not only gives information about the stability of the basic solution itself but also tells us whether or not all the computed EFMs are physically relevant.

2.5 *The Hopf curve and change of the lasing threshold*

We proceed by analysing the various properties of the Hopf curve that will give information about the so-called threshold reduction. For any parameter combination, the curves satisfying (2.4) are symmetric under $(\Gamma, \Omega_0\theta) \to (-\Gamma, \Omega_0\theta + \pi)$.

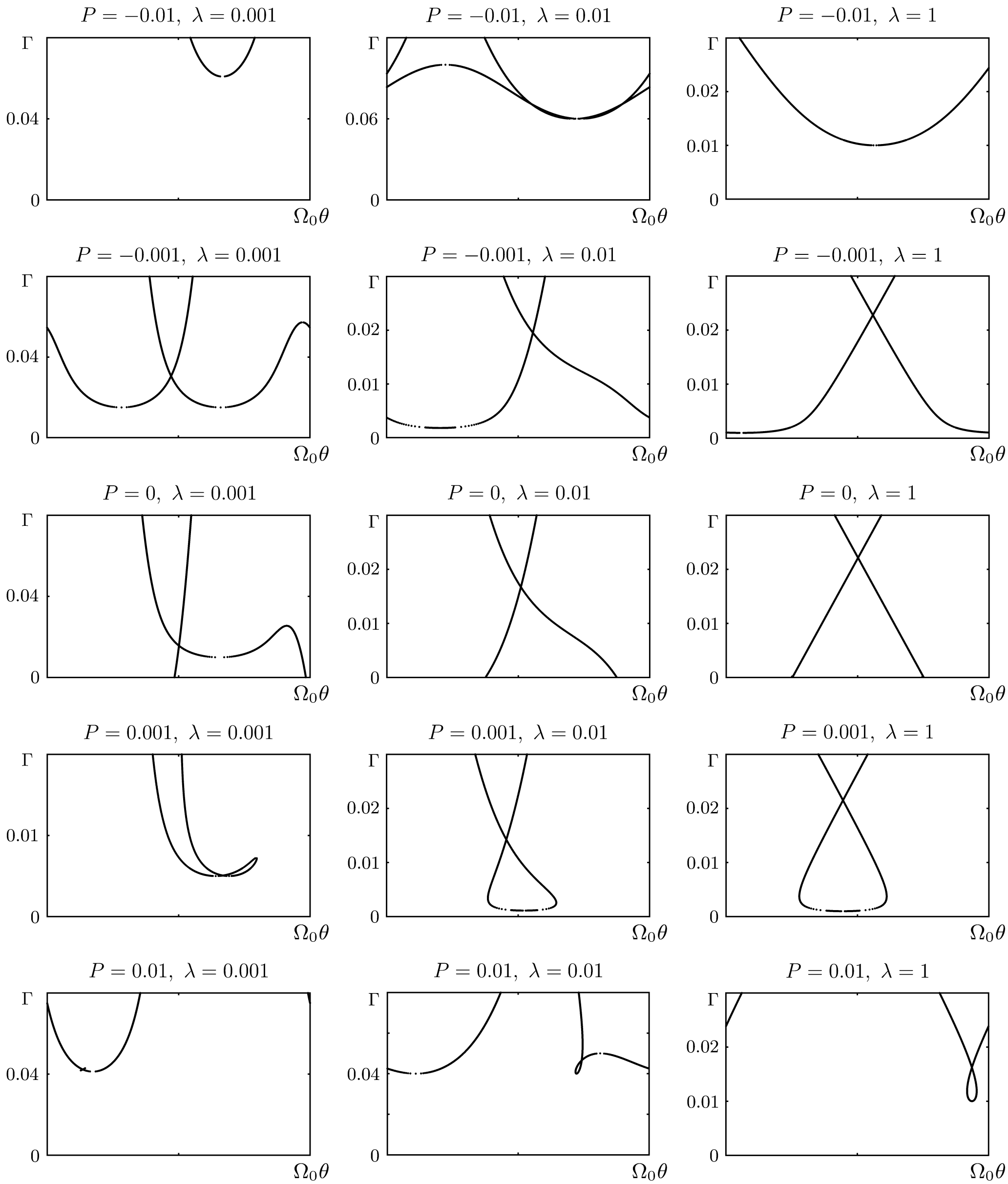

FIG. 3. Plots of the Hopf bifurcation curves (2.4) of (0, P, 0), which coincide with the curves of physical relevance bifurcations (2.30). In all panels, the value $\Omega_0\theta$ runs from 0 to 2π. From left to right, λ is varied from the injection to the LK limit and in the vertical direction, P is varied. The latter illustrates the threshold reduction or increment, see the text.

For $P = 0$, the Hopf bifurcations form curves in the $(\Omega_0\theta, \Gamma)$-plane that intersect the $\Gamma = 0$ axis; see the $P = 0$ panels in Fig. 3. We compute the intersection points in the limit $\Gamma \downarrow 0$ as follows: By (2.4), the eigenvalue $\mu = \mathrm{i}\nu$ satisfies

$$\nu^2 + \frac{\nu^2(\nu + \Omega_\mathrm{m})^2}{\lambda^2} = \Gamma^2, \tag{2.34}$$

so for $\lambda \neq 0$ fixed, ν behaves in the limit $\Gamma \downarrow 0$ as $\nu = \mathrm{O}(\Gamma)$. With $\nu = w\Gamma$ in (2.34), we find $w^2(1 + \frac{(w\Gamma+\Omega_\mathrm{m})^2}{\lambda^2}) = 1$, which results in the limit $\Gamma = 0$ in $w = \pm(1 + \frac{\Omega_\mathrm{m}^2}{\lambda^2})^{-1/2}$. Back in (2.4), this gives for $P = 0$ at leading order $\Omega_0\theta = a_\pm + 2k\pi$ or $\Omega_0\theta = \pi - a_\pm + 2k\pi$, $k \in \mathbf{Z}$, with

$$a_\pm = \arcsin\left(\pm\left(1 + \frac{\Omega_\mathrm{m}^2}{\lambda^2}\right)^{-\frac{1}{2}}\right) \tag{2.35}$$

as intersection points of (2.4) with the $\Gamma = 0$ axis.

In the limit $\lambda \to \infty$, (2.4) obeys the symmetries $(\Gamma, \Omega_0\theta) \to (-\Gamma, \Omega_0\theta + \pi)$ and $(\Gamma, \Omega_0\theta) \to (-\Gamma, \pi - \Omega_0\theta)$ and becomes degenerate curves $\Gamma = \frac{1}{\theta}\Omega_0\theta - \frac{\pi}{2\theta} + \frac{2k\pi}{\theta}$ and $\Gamma = -\frac{1}{\theta}\Omega_0\theta - \frac{\pi}{2\theta} + \frac{2k\pi}{\theta}$, which are, up to 2π-periodicity, two lines that intersect each other at $(\Omega_0\theta, \Gamma) = (\pi, \frac{2\pi}{\theta})$, and that intersect the $\Gamma = 0$ axis at $\Omega_0\theta = \frac{\pi}{2}$ and $\Omega_0\theta = \frac{3\pi}{2}$, respectively.

For $P \neq 0$, the singular situation, with two lines that intersect $\Gamma = 0$, breaks down. The breakdown is always in the same manner: for small $P > 0$, the lines connect in a loop, that disappears as P increases further, while for small $P < 0$, the lines connect the other way around (via $\Omega_0\theta = 0$—recall the 2π-periodicity). In the limit $\lambda \to \infty$, it is easily calculated that the minimum of (2.4) restricted to the positive half-plane is always $\Gamma = |P|$ and that the loop disappears at $P = \frac{1}{\tau}$. For general λ, the minimum also lies at $\Gamma = |P|$.

The configuration of this Hopf curve and its change as P varies across the solitary laser threshold $P = 0$ perfectly explain the concept of threshold reduction: the laser with optical feedback may be lasing below its solitary laser threshold because of possible positive interference between the laser field and the feedback field. By such interference, the light that is fed back may counter the losses that need to be overcome for lasing. The converse, a threshold increment, can theoretically also occur as a consequence of negative interference.

For $P < 0$, the $(0, P, 0)$-solution is always stable for values $(\Gamma, \Omega_0\theta)$ below the lowest connected part of the Hopf curves and unstable above this curve: a high enough feedback rate Γ can cause lasing below threshold. As P decreases further, the minimum of this curve increases (at $\Gamma = |P|$) and a higher feedback rate Γ is required for lasing.

For $P = 0$, the region in which $(0, P, 0)$ is stable is reduced to the (deformed) triangle, which in turn becomes the closed loop for $P > 0$. If there are no other stable solutions for the parameters $(\Omega_0\theta, \Gamma)$ within this 'triangle', then the laser has an 'increased threshold' for these parameters: the laser remains off until the loop has, for increasing P, shrinked or moved far enough so that $(\Omega_0\theta, \Gamma)$ lies outside the loop. In Rottschäfer & Krauskopf (2006b), it is shown that in the LK limit such situations indeed occur, assuming that the only alternative stable solutions are EFMs.

3. Analysis in the limit $\lambda \to 0$

In this section, we study System (1.4) in the limit $\lambda \to 0$ and especially focus on the number of EFMs. For $0 < \lambda \ll 1$, System (1.4) is a singularly perturbed system. Depending on the magnitude of Ω_m, there are different cases to study. We focus on the case $\Omega_{\mathrm{m}} = \mathrm{O}(1)$. If the solution moreover satisfies $|F| = \mathrm{O}(1)$ with respect to λ, the third equation in (1.4) resembles in first-order standard circular motion with frequency Ω_{m}; $F(s) = C\,\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{m}} s}$.

REMARK 3.1 If $\Omega_{\mathrm{m}} = \mathrm{O}(\lambda)$ and/or $|F| = \mathrm{O}(\lambda)$, the EFMs limit either at the trivial solution $(0, P, 0)$ or at one of the cases described in Lemma 3.4 and Remark 3.6.

First, we study the case $\lambda = 0$ for which (2.7) yields

$$S' = 0 \quad \text{and} \quad \psi' = \Omega_{\mathrm{m}} - \Omega,$$

which results, by the requirement that ψ contains only non-linear terms in s, in $\Omega = \Omega_{\mathrm{m}}$ and $S =$ constant. This implies that in this case $F(s) = S\,\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{m}} s}$. Substituting this into the remaining equations of (1.4) yields

$$\begin{aligned} E' &= (1 + \mathrm{i}\alpha)NE + \Gamma S\,\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{m}} s}, \\ TN' &= P - N - (1 + 2N)|E|^2, \end{aligned} \tag{3.1}$$

which are exactly the equations for a single injection laser (see Krauskopf & Lenstra, 2000, for various versions). Its EFMs or fixed points $E(s) = R\,\mathrm{e}^{\mathrm{i}(\Omega s + \phi_0)}$ and $N(s) = N$ are given by

$$\Omega = \Omega_{\mathrm{m}}, \tag{3.2}$$

$$0 = NR + \Gamma S \cos(-\phi_0), \tag{3.3}$$

$$\Omega_{\mathrm{m}} = \alpha N + \Gamma \frac{S}{R} \sin(-\phi_0), \tag{3.4}$$

$$0 = P - N - (1 + 2N)R^2. \tag{3.5}$$

Note that this corresponds to zero derivatives in (2.7) for $\lambda = 0$. Hence when $\lambda = 0$, the injected field strength $|F|$ is fixed and the detuning frequency Ω equals Ω_{m}. The 5D system (1.4) for the filtered laser reduces to the 3D system describing the single injection laser. The consequence is that the phase space of a single injection laser serves as a skeleton for the behaviour of the semiconductor laser with filtered feedback, as long as the filter width is sufficiently small ($0 < \lambda \ll 1$). In the limit $\lambda \to 0$, System (1.4) can be seen as a 2D stack of 3D systems that each describes an injection laser for a fixed injected field strength $|F|$ and detuning frequency Ω_{m}.

However, one should be careful here. Equations (2.10), (2.11) and (2.16) suggest a singular perturbation analysis and introduction of a new independent variable $y = (\Omega - \Omega_{\mathrm{m}})/\lambda$. Considering y as an O(1) independent variable corresponds well with the intuitive expectation that the laser will operate at a frequency close to the frequency of the injected field: $\Omega - \Omega_{\mathrm{m}} = \mathrm{O}(\lambda)$. In the sequel, we analyse the fixed points, mainly by studying (2.16) for $\Omega = \Omega_{\mathrm{m}} + \lambda y$, with $y = \mathrm{O}(1)$ and $\lambda \ll 1$. In the case that $\Omega - \Omega_{\mathrm{m}} = \mathrm{O}(1)$, however, we introduce $x = \Omega - \Omega_{\mathrm{m}}$ and study (2.16) for $\Omega = \Omega_{\mathrm{m}} + x$, with $x = \mathrm{O}(1)$.

3.1 *The case $\Omega - \Omega_{\mathrm{m}} = \mathrm{O}(\lambda)$ and $\Omega_{\mathrm{m}} = \mathrm{O}(1)$*

For $\Omega - \Omega_{\mathrm{m}} = \mathrm{O}(\lambda)$, we set $\Omega = \Omega_{\mathrm{m}} + \lambda y$ so that (2.16) for the Ω-coordinate of the EFMs becomes

$$\Omega_{\mathrm{m}} + \lambda y = -\Gamma \sqrt{\frac{1 + \alpha^2}{1 + y^2}} \sin[(\Omega_0 + \Omega_{\mathrm{m}} + \lambda y)\theta + \arctan(y) + \arctan(\alpha)], \tag{3.6}$$

which is in leading order

$$\Omega_{\rm m} = -\Gamma\sqrt{\frac{1+\alpha^2}{1+y^2}}\sin[(\Omega_0+\Omega_{\rm m})\theta + \arctan(y) + \arctan(\alpha)]. \tag{3.7}$$

We write f and g as functions of y:

$$f(y;\lambda) = -\Gamma\sqrt{\frac{1+\alpha^2}{1+y^2}}\sin[(\Omega_0+\Omega_{\rm m}+\lambda y)\theta + \arctan(y) + \arctan(\alpha)], \tag{3.8}$$

$$g(y;\lambda) = \lambda y + \Omega_{\rm m} \tag{3.9}$$

so that EFMs for zero or small λ can be studied as intersections of f and g. With

$$\begin{aligned} C &:= \Gamma\theta\sqrt{1+\alpha^2}, \\ D &:= (\Omega_0+\Omega_{\rm m})\theta + \arctan(\alpha) \end{aligned} \tag{3.10}$$

and $\lambda = 0$, we write

$$f(y;0) = -\frac{C}{\theta\sqrt{1+y^2}}\sin[\arctan(y)+D] = \frac{-C(y\cos D + \sin D)}{\theta(1+y^2)}, \quad g(y;0) = \Omega_{\rm m}.$$

Here, C is the classical feedback strength (van der Graaf, 1997). The function $f(y;0)$ has, unless $\cos D = 0$, a single zero at $y = -\tan D$ and two extreme values, see Fig. 4(b). Furthermore, f satisfies $\lim_{|y|\to\infty} f(y;0) = 0$. We first treat the case $\cos D \neq 0$. In this case, the extrema are

$$f_{\rm min} = f(y_+) = \frac{-C\cos^2 D}{\theta(2-2\sin D)} = -\frac{C}{2\theta}(1+\sin D) < 0 \quad \text{with } y_+ = \frac{-\sin D + 1}{\cos D},$$

$$f_{\rm max} = f(y_-) = \frac{C\cos^2 D}{\theta(2+2\sin D)} = \frac{C}{2\theta}(1-\sin D) > 0 \quad \text{with } y_- = \frac{-\sin D - 1}{\cos D},$$

see Fig. 4(b). Note that $|f(y_+)| \neq |f(y_-)|$ unless $\sin D = 0$, in which case also $f(0;0) = 0$ and f is an odd function, see Fig. 4(c) where $\sin D = 0$.

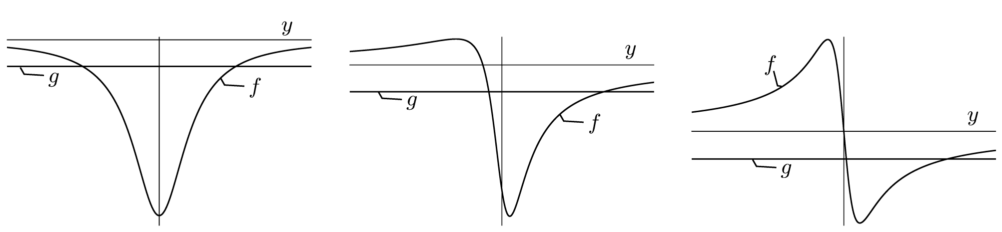

FIG. 4. Plots of $f(y;0)$ and $g(y;0)$ as functions of y for various values of Ω_0. The parameter $\Omega_{\rm m}$ is taken negative. When $\Omega_{\rm m}$ is varied, the function g moves up or down and f changes from, which leads to creation or annihilation of intersection points of f and g, the EFMs. (a) Ω_0 is such that $\cos D = 0$ and $\sin D > 0$. (b) Ω_0 is such that $\cos D \neq 0$, $\sin D \neq 0$ and $\cos D > 0$. (c) The case when $\sin D = 0$ and $\cos D > 0$.

Since $g(y;0) = \Omega_{\rm m}$ for $\lambda = 0$, it is clear that $f(y;0) = g(y;0)$ has two solutions when $\Omega_{\rm m}$ lies between the two extrema of f:

$$-\frac{C}{2\theta}(1+\sin D) < \Omega_{\rm m} < \frac{C}{2\theta}(1-\sin D) = -\frac{C}{2\theta}(1+\sin D) + \frac{C}{\theta}. \tag{3.11}$$

An equality on either side corresponds to a bifurcation value: SN bifurcations where two EFMs are formed or disappear take place at $\Omega_{\rm m} = \pm\frac{C}{2\theta}(1 \mp \sin D)$. Substituting the expressions for C and D, we see that the bifurcations occur for those $\Omega_{\rm m}$ satisfying

$$-\frac{1}{2}\Gamma\sqrt{1+\alpha^2}(\pm 1 + \sin((\Omega_0 + \Omega_{\rm m})\theta + \arctan\alpha)) = \Omega_{\rm m}. \tag{3.12}$$

So depending on the parameters, there are a number of bounded intervals in $\Omega_{\rm m}$-space in which $f = g$ has, for $\lambda = 0$, two solutions. The parameter $\Omega_{\rm m}$ can be varied to enter or leave these intervals, i.e. to gain or lose two EFMs via a SN bifurcation, see Fig. 4(b).

Furthermore, the laser operates at a frequency $\Omega = \Omega_{\rm m} + \lambda y$. Hence, the fact that there are only a number of bounded intervals in $\Omega_{\rm m}$-space in which $f(y;0) = g(y;0)$ has two solutions, while $f(y;0) = g(y;0)$ has no solutions for parameter values $\Omega_{\rm m}$ outside these intervals, says the following: given fixed parameters α, Γ, θ and Ω_0, the laser can in the small-λ regime only operate at frequencies Ω in a number of bounded intervals. If $\Omega_{\rm m}$ is chosen such that it lies in such interval, the laser will (probably) operate at a frequency $O(\lambda)$ close to this $\Omega_{\rm m}$. If $\Omega_{\rm m}$ is more than $O(\lambda)$ away from these intervals, there are no EFM solutions with $\Omega = \Omega_{\rm m} + \lambda y$.

We now study the $\Omega_{\rm m}$-intervals where two EFMs exist. Their number changes when both (3.12) holds and the derivatives with respect to $\Omega_{\rm m}$ at both sides of (3.12) are equal. Differentiating (3.12) with respect to $\Omega_{\rm m}$ yields

$$\cos(\Omega_{\rm m}\theta + D_1) = -\frac{2}{C},$$

with C as above and $D_1 := \Omega_0\theta + \arctan(\alpha)$. Therefore,

$$\Omega_{\rm m}\theta + D_1 = \pm\arccos\left(-\frac{2}{C}\right) + 2k\pi$$

and

$$\sin(\Omega_{\rm m}\theta + D_1) = \pm\sqrt{1-\frac{4}{C^2}}.$$

These two equations combined with (3.11) yield that an $\Omega_{\rm m}$-interval is created or annihilated precisely when

$$-\frac{1}{2}C\left[1 \pm \sqrt{1-\frac{4}{C^2}}\right] + D_1 = \Omega_{\rm m}\theta + D_1 = \pm\arccos\left(-\frac{2}{C}\right) + 2k\pi$$

or

$$-\frac{1}{2}C\left[1 \pm \sqrt{1-\frac{4}{C^2}}\right] + D_1 + C = \Omega_\mathrm{m}\theta + D_1 = \pm\arccos\left(-\frac{2}{C}\right) + 2k\pi,$$

where the plus and minus signs correspond to each other. Rewriting these equations and substituting D_1, we see that creation/annihilation of an interval is in $(\alpha, \Gamma, \theta, \Omega_0)$-parameter space determined by

$$\begin{aligned}\Omega_0\theta &= \pm\arccos\left(-\frac{2}{C}\right) + \frac{1}{2}C \pm \sqrt{\frac{C^2}{4}-1} - \arctan\alpha + 2k\pi,\\ \Omega_0\theta &= \pm\arccos\left(-\frac{2}{C}\right) - \frac{1}{2}C \pm \sqrt{\frac{C^2}{4}-1} - \arctan\alpha + 2k\pi.\end{aligned} \tag{3.13}$$

Note that these expressions do not depend on the parameter P, the pump, and that these equalities can only occur when $\frac{C^2}{4} - 1 \geqslant 0$, hence when $C > 2$. When $C < 2$, the number of Ω_m-intervals remains fixed to one that always exists: an interval around $\Omega_\mathrm{m} = 0$ for which the size decreases to zero as $\Gamma \to 0$. The curves (3.13) are plotted in Fig. 5 in $(\Omega_0\theta, \Gamma)$-parameter space; the number of Ω_m-intervals is given in the various regions.

From (3.12) and (3.13), we conclude that the number of EFMs with $\frac{\Omega-\Omega_\mathrm{m}}{\lambda} = O(1)$ changes between zero and two as Ω_m is varied.

LEMMA 3.2 Let $\lambda \downarrow 0$ and fix the parameters α, Ω_0, Γ and θ such that $\cos D := \cos(\Omega_0 + \Omega_\mathrm{m})\theta + \arctan(\alpha) \neq 0$. If $C = \Gamma\theta\sqrt{1+\alpha^2} < 2$, there is a single open interval $I^0_{\Omega_\mathrm{m}}$ around $\Omega_\mathrm{m} = 0$ such that there are two solutions Ω to (2.16), hence two EFMs, with $y = \frac{\Omega-\Omega_\mathrm{m}}{\lambda} = O(1)$ when $\Omega_\mathrm{m} \in I^0_{\Omega_\mathrm{m}}$ and none if $\Omega_\mathrm{m} \notin \overline{I^0_{\Omega_\mathrm{m}}}$.

For $C = \Gamma\theta\sqrt{1+\alpha^2} > 2$, there are a number of intervals $I^j_{\Omega_\mathrm{m}}$ such there are two solutions Ω to (2.16) with $y = \frac{\Omega-\Omega_\mathrm{m}}{\lambda} = O(1)$ if $\Omega_\mathrm{m} \in I^j_{\Omega_\mathrm{m}}$ for some j and none if $\Omega_\mathrm{m} \notin \overline{I^j_{\Omega_\mathrm{m}}}$. The locus in parameter space of the creations of these regions is given by (3.13).

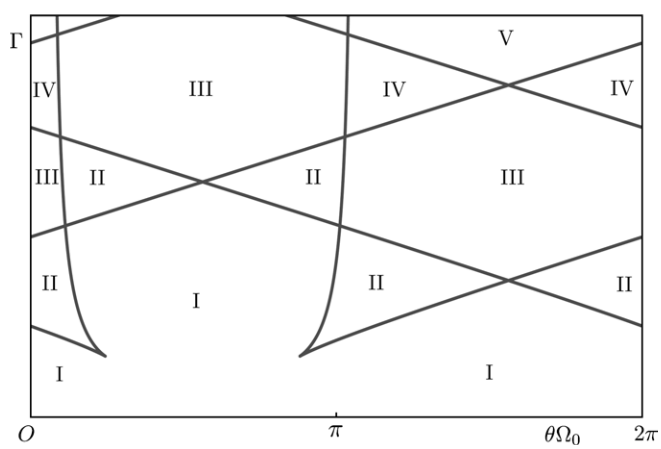

FIG. 5. The curves (3.13) and the number of intervals $I^j_{\Omega_\mathrm{m}}$ for Ω_m in $(\Omega_0\theta, \Gamma)$-parameter space for $\lambda \to 0$. When Ω_m in chosen in these intervals, there exist three EFMs.

Next, we consider the case $\cos D = 0$. In this case, $f(y;0)$ has a single extremum $f(0;0) = -\frac{C}{\theta}\sin D = \pm\frac{C}{\theta}$, and f is an even function. The equation $f(y;0) = g(y;0)$ has two solutions if $\Omega_\mathrm{m} \in (0, -\frac{C}{\theta}\sin D)$, and SN bifurcations of EFMs occur for $\Omega_\mathrm{m} = 0$ and for $\Omega_\mathrm{m} = -\frac{C}{\theta}\sin D$. From $\cos D = 0$, it follows that either $D = \frac{3}{2}\pi + 2k\pi$ or $D = \frac{1}{2}\pi + 2k\pi$. Hence, there exist two solutions when $\Omega_\mathrm{m} \in (0, \pm\frac{C}{\theta})$, where the positive sign corresponds with $D = \frac{3}{2}\pi + 2k\pi$ and the negative sign with $D = \frac{1}{2}\pi + 2k\pi$. The SN bifurcations of EFMs take place exactly when $\Omega_\mathrm{m} = \pm\frac{C}{\theta} = \pm\Gamma\sqrt{1+\alpha^2}$. Inserting this into the expression for D, one sees that this is only possible if

$$\Omega_\mathrm{m} = \frac{3\pi}{2\theta} + \frac{2k\pi}{\theta} - \Omega_0 - \frac{1}{\theta}\arctan\alpha = \Gamma\sqrt{1+\alpha^2} \quad \text{or} \tag{3.14}$$

$$\Omega_\mathrm{m} = \frac{\pi}{2\theta} + \frac{2k\pi}{\theta} - \Omega_0 - \frac{1}{\theta}\arctan\alpha = -\Gamma\sqrt{1+\alpha^2}. \tag{3.15}$$

Hence, this bifurcation takes place when Ω_m is varied and the remaining parameters satisfy

$$\frac{\pi}{2} - \theta\Omega_0 - \arctan\alpha = \begin{cases} \Gamma\theta\sqrt{1+\alpha^2} - \pi + 2k\pi & \text{or} \\ -\Gamma\theta\sqrt{1+\alpha^2} + 2k\pi. \end{cases} \tag{3.16}$$

These expressions correspond to those obtained after letting $C \to \infty$ in (3.13). The bifurcation that occurs when Ω_m crosses $\Omega_\mathrm{m} = 0$ is a special one: two EFMs either vanish or appear at $\Omega = \pm\infty$.

From (3.12) and (3.16), we conclude the following lemma.

LEMMA 3.3 Let $\lambda \downarrow 0$ and fix the parameters α, Ω_0, Γ and θ such that (3.16) holds. Then, there is a single open interval $J^0_{\Omega_\mathrm{m}}$ bounded on one side by $\Omega_\mathrm{m} = 0$ and on the other side by (3.14) or (3.15) such that there are two solutions Ω to (2.16), hence two EFMs, with $y = \frac{\Omega-\Omega_\mathrm{m}}{\lambda} = \mathrm{O}(1)$ when $\Omega_\mathrm{m} \in J^0_{\Omega_\mathrm{m}}$ and none if $\Omega_\mathrm{m} \notin \overline{J^0_{\Omega_\mathrm{m}}}$.

3.2 *The case* $\Omega - \Omega_\mathrm{m} = \mathrm{O}(1)$

To analyse the case $\Omega - \Omega_\mathrm{m} = \mathrm{O}(1)$, we introduce $\Omega = \Omega_\mathrm{m} + x$ with $\Omega_\mathrm{m} = \mathrm{O}(1)$ and $x = \mathrm{O}(1)$ into (2.16), and write the functions f and g as functions of x:

$$f(x;\lambda) = -\lambda\frac{C}{\theta\sqrt{\lambda^2+x^2}}\sin\left(x\theta + \arctan\left(\frac{x}{\lambda}\right) + D\right), \tag{3.17}$$

$$g(x;\lambda) = x + \Omega_\mathrm{m}, \tag{3.18}$$

where again $C = \Gamma\theta\sqrt{1+\alpha^2}$ and $D = (\Omega_0 + \Omega_\mathrm{m})\theta + \arctan\alpha$. It is necessary to use these scalings if $y \to \frac{-\Omega_\mathrm{m}}{\lambda}$, i.e. if $\Omega \to 0$. The limit of $f(x;\lambda)$ as $\lambda \to 0$ is well-defined for all $|x| > \mathrm{O}(\lambda)$ and satisfies $f(x;0) = 0$ for every $|x| > \mathrm{O}(\lambda)$. For $x = \mathrm{O}(\lambda)$, one should consider $x = \lambda y$ as was done in the previous section and (3.6) would be obtained. The function f in (3.17) has its zeroes where $\arctan\left(\frac{x}{\lambda}\right) + x\theta + D = k\pi$. As $\lambda \to 0$, $\arctan\left(\frac{x}{\lambda}\right) \to \frac{\pi}{2}$ for $x > 0$ and $\arctan\left(\frac{x}{\lambda}\right) \to -\frac{\pi}{2}$ for $x < 0$, hence the zeroes of $f(x;\lambda)$ at either side of $x = 0$ are π/θ apart as $\lambda \to 0$.

For $\lambda = 0$, the functions $f(x;0)$ and $g(x;0)$ intersect at $x = -\Omega_\mathrm{m}$ since f is identically zero. By applying the implicit function theorem, it follows that there exist λ_0 and x_0 such that $\forall\,\lambda \in [0,\lambda_0)$, $x \in (-\Omega_\mathrm{m} - x_0, -\Omega_\mathrm{m} + x_0) : |\frac{\partial f}{\partial x}(x;\lambda)| < 1$. Hence, since $\frac{\partial g}{\partial x}(x;\lambda) = 1$, there exists a unique intersection point of $f(x;\lambda)$ and $g(x;\lambda)$ with $x \in (-\Omega_\mathrm{m} - x_0, -\Omega_\mathrm{m} + x_0)$. The uniqueness proof can be extended to $(x;\lambda) \in (\mathbf{R}\backslash[-X, X]) \times [0, \lambda_0)$ for some $X = \mathrm{O}(\lambda)$ since $|\frac{\partial f}{\partial x}(x;\lambda)| < 1$ for all these x as long as λ is small enough.

This intersection point can be approximated using that its x-coordinate must lie close to $-\Omega_{\mathrm{m}}$, hence, we set $x = -\Omega_{\mathrm{m}} + \lambda z$ or $z = \frac{\Omega}{\lambda}$. Then, $f(z;\lambda) = g(z;\lambda)$ implies

$$-\frac{C}{\theta\sqrt{\lambda^2 + (-\Omega_{\mathrm{m}} + \lambda z)^2}} \sin\left((-\Omega_{\mathrm{m}} + \lambda z)\theta + \arctan\left(\frac{-\Omega_{\mathrm{m}} + \lambda z}{\lambda}\right) + D\right) = z,$$

which leads in the limit $\lambda \to 0$, with C and D substituted, to

$$\begin{aligned} z &= \lim_{\lambda \to 0} \frac{\Gamma\sqrt{1+\alpha^2}}{|\Omega_{\mathrm{m}}|} \sin\left(\Omega_0\theta + \arctan\alpha + \arctan\left(\frac{-\Omega_{\mathrm{m}} + \lambda z}{\lambda}\right)\right) \\ &= -\frac{\Gamma\sqrt{1+\alpha^2}}{\Omega_{\mathrm{m}}} \cos(\Omega_0\theta + \arctan\alpha). \end{aligned} \tag{3.19}$$

LEMMA 3.4 For $\lambda \to 0$ and α, Ω_0, $\Omega_{\mathrm{m}} = \mathrm{O}(1)$, Γ and θ fixed, there exists a solution Ω to (2.16) with $x = \Omega - \Omega_{\mathrm{m}} = \mathrm{O}(1)$. This solution is constant up to and including order $\mathrm{O}(\lambda)$, and is approximated by $\Omega = -\lambda\frac{\Gamma\sqrt{1+\alpha^2}}{\Omega_{\mathrm{m}}}\cos(\Omega_0\theta + \arctan\alpha)$.

REMARK 3.5 This solution, where $x = \Omega - \Omega_{\mathrm{m}} = \mathrm{O}(1)$, corresponds to a solution with $|\Omega_{\mathrm{L}} - \Omega_0| = \mathrm{O}(\lambda)$ in Erneux *et al.* (2004), whereas the solutions with $\Omega - \Omega_{\mathrm{m}} = \mathrm{O}(\lambda) = \lambda y$ correspond to the limit $|\Omega_{\mathrm{L}} - \Omega_{\mathrm{f}}| = \mathrm{O}(\lambda)$.

REMARK 3.6 The solution with $\Omega - \Omega_{\mathrm{m}} = \mathrm{O}(1)$ must by (2.10) necessarily satisfy $\frac{S}{R} = \mathrm{O}(\lambda)$. This corresponds to the physical intuition that if the injected field strength S is (too) small, the laser prefers to operate at its own solitary laser frequency Ω_0 instead of the frequency of the injected light (with centre frequency $\Omega_{\mathrm{f}} = \Omega_{\mathrm{m}} + \Omega_0$).

Lemmas 3.2 and 3.4 together give a complete overview of the number of EFMs (number of solutions of (2.16)) that exist for small λ in different parameter regions. From Lemma 3.4, it follows that for $\lambda \to 0$ and the other parameters fixed, there always exists one EFM. Combining this with Lemma 3.2, where depending on Ω_{m}, zero or two EFMs exist, we find that as Ω_{m} is varied, the total number of EFMs for (1.4) in the limit $\lambda \to 0$ changes between one and three.

In the case of one EFM, the locus of EFMs always consists of a single component that grows with λ. In the case of three EFMs, there are two components for small λ that get connected as λ increases, as plotted in Fig. 1 (see also Corollary 6.2).

4. Analysis in the limit $\lambda \to \infty$

In this section, we study System (1.4) as $\lambda \to \infty$. Again, there are two cases to study, related to different orders of magnitude of Ω_{m}. We, however, focus on the case $\Omega_{\mathrm{m}} = \mathrm{O}(1)$.

In the limit $\lambda \to \infty$, with $\Omega_{\mathrm{m}} = \mathrm{O}(1)$, the equation for F in (1.4) simply reduces to the algebraic relation $F(s) = E(s-\theta)\mathrm{e}^{-\mathrm{i}\Omega_0\theta}$ as long as $F = \mathrm{O}(1)$. This means that (1.3) reduces to the LK equations (1.1), and System (1.4) becomes the rescaled form of the LK equations as studied in Rottschäfer & Krauskopf (2006a).

With $\mu \equiv \frac{1}{\lambda}$, the rescaled equations (1.4) are rewritten as

$$\begin{aligned} E' &= (1+\mathrm{i}\alpha)NE + \varGamma F, \\ TN' &= P - N - (1+2N)|E|^2, \\ \mu F' &= E(s-\theta)\mathrm{e}^{-\mathrm{i}\varOmega_0\theta} + (\mathrm{i}\mu\varOmega_\mathrm{m} - 1)F. \end{aligned} \tag{4.1}$$

By introducing rescaled time $s = \mu\tilde{s}$ and $\theta = \mu\tilde{\theta}$, we obtain the equivalent system

$$\begin{aligned} \dot{E} &= \mu(1+\mathrm{i}\alpha)NE + \mu\varGamma F, \\ T\dot{N} &= \mu P - \mu N - \mu(1+2N)|E|^2, \\ \dot{F} &= E(\tilde{s}-\tilde{\theta})\mathrm{e}^{-\mathrm{i}\varOmega_0\theta} + (\mathrm{i}\mu\varOmega_\mathrm{m} - 1)F, \end{aligned} \tag{4.2}$$

where the dot denotes differentiation with respect to $\tilde{s}$.

Taking the limit $\lambda \to \infty$ corresponds to $\mu \to 0$. For $\mu = 0$, (4.2) reduces to

$$\dot{E} = \dot{N} = 0 \quad \text{and} \quad \dot{F} = E(\tilde{s}-\tilde{\theta})\mathrm{e}^{-\mathrm{i}\varOmega_0\theta} - F = E\,\mathrm{e}^{-\mathrm{i}\varOmega_0\theta} - F,$$

as long as $\varOmega_\mathrm{m} = \mathrm{O}(1)$ and $|F| = \mathrm{O}(1)$ with respect to μ. So, E and N are constant on the $\tilde{s}$-time-scale. For $\mu = 0$, the last equation of System (4.1) yields $F(s) = E(s-\theta)\mathrm{e}^{-\mathrm{i}\varOmega_0\theta}$, meaning that the dynamics of (4.1) is only defined when this holds. The flow is then prescribed by the first two equations of System (4.1), the (rescaled) LK equation:

$$\begin{aligned} E' &= (1+\mathrm{i}\alpha)NE + \varGamma E(s-\theta)\mathrm{e}^{-\mathrm{i}\varOmega_0\theta}, \\ TN' &= P - N - (1+2N)|E|^2. \end{aligned}$$

Hence, the fixed points, EFMs, of System (4.1) are the fixed points, ECMs, of the LK equation, with the additional restriction that they lie in $\{F = E\,\mathrm{e}^{-\mathrm{i}\varOmega_0\theta}\}$. In the coordinates (2.5), these fixed points should therefore satisfy $R = S$ and $\phi_0 = \varOmega_0\theta$.

Indeed, the relation for the $\varOmega$-values of the EFMs (2.16) reduces in the limit $\lambda \to \infty$ to the equation $f(\varOmega, \infty) = g(\varOmega, \infty)$ or

$$\varOmega = -\varGamma\sqrt{1+\alpha^2}\sin(\varOmega_0\theta + \varOmega\theta + \arctan(\alpha)), \tag{4.3}$$

which equals, after a rescaling of parameters, the relation for EFMs in the LK equation given in Rottschäfer & Krauskopf (2006a). Hence, the results from Rottschäfer & Krauskopf (2006a) concerning EFMs and their formation in SN bifurcations immediately apply. As concluded there, SN bifurcations of EFMs should also satisfy $f_\varOmega = g_\varOmega$, which together with (4.3) yields the locus of SN bifurcations in $(\varOmega_0\theta, \varGamma)$-parameter space:

$$\varOmega_0\theta^{\pm} = \pm\left[\sqrt{C^2-1} + \arccos\left(-\frac{1}{C}\right)\right] - \arctan\alpha + 2k\pi. \tag{4.4}$$

Note that this condition does not depend on the parameter P, the pump. This is irrespective of Condition (2.13).

5. Comparison between the limits $\lambda \to 0$ and $\lambda \to \infty$

In this section, we compare the results we obtained in the limits $\lambda \to 0$ (injection limit) and $\lambda \to \infty$ (LK limit). We first state a relation between the equations found for the Ω-values of the EFMs in both limits. Comparing (4.3) and (3.7), one concludes the following lemma.

LEMMA 5.1 The $\lambda \to 0$-limit equation (3.7) has a solution $y = 0$ if and only if Ω_{m} is a solution for the $\lambda \to \infty$-limit equation (4.3).

This means that although the FOF laser in general has a frequency $\mathrm{O}(\lambda)$ off the filter mean Ω_{m} if $0 < \lambda \ll 1$, its frequency Ω can satisfy $\Omega = \Omega_{\mathrm{m}}$ exactly (the laser with filtered feedback acts exactly as a laser with injected optical field $\Gamma S\,\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{m}}\theta}$). This happens if and only if the parameters α, Γ and θ are chosen such that one of the EFMs (ECMs) that exist in the limit $\lambda \to 0$ has frequency $\Omega = \Omega_{\mathrm{m}}$.

REMARK 5.2 Although this may seem striking, it is clear from the physics that there should be such relation between the injection limit and the LK limit. A semiconductor laser normally lases at its solitary laser frequency Ω_0. If light with a different frequency Ω_1 is injected, the laser may or may not 'lock' to this frequency, i.e. operate at $\Omega = \Omega_1$ instead of $\Omega = \Omega_0$. For a COF laser, the situation is similar if the laser operates at one of its ECMs. In an ECM with Ω_1, the laser (again with solitary laser frequency Ω_0) operates at $\Omega = \Omega_1$ and this light re-enters (so is in fact 'injected') after an external round trip.

Hence, if the parameters are such that in the injection limit $\lambda \to 0$ the laser locks to the injected frequency $\Omega = \Omega_{\mathrm{m}}$, the same semiconductor laser would in the LK limit $\lambda \to \infty$ have an EFM with $\Omega = \Omega_{\mathrm{m}}$. Vice versa, if the parameters are such that there exists, among others, an EFM with $\Omega = \Omega_{\mathrm{m}}$ in the LK limit, then a narrow ($\lambda \to 0$) filter centred at Ω_{m} only allows this EFM to persist and this EFM also exists in the injection limit.

For a laser operating at a fixed point determined by (3.6), we have by (2.7)

$$\lambda y = \Omega - \Omega_{\mathrm{m}} = \lambda \frac{R}{S} \sin(\phi_0 - (\Omega_0 + \Omega_{\mathrm{m}} + \lambda y)\theta),$$

from which the angle ϕ_0 can be solved. It follows that if it exists, so if the parameters are chosen as in Lemma 5.1, an EFM with $y = 0$ satisfies $\phi_0 = (\Omega_{\mathrm{m}} + \Omega_0)\theta + k\pi$, $k \in \mathbf{Z}$.

A second comparison between the LK limit and the injection limit can be made by looking at the curves of SN bifurcations of EFMs in both limits, i.e. by comparing (3.13) to (4.4).

Recall that the curves (3.13) for $\lambda \to 0$ separate regions in the $(\Omega_0\theta, \Gamma)$-plane where the number of Ω_{m}-intervals $I^j_{\Omega_{\mathrm{m}}}$ changes. When Ω_{m} is chosen in an interval $I^j_{\Omega_{\mathrm{m}}}$, there are three EFMs of which two satisfy $|\Omega - \Omega_{\mathrm{m}}| = \mathrm{O}(\lambda)$ and one satisfies $\Omega = \mathrm{O}(\lambda)$. When Ω_{m} is chosen outside any interval $I^j_{\Omega_{\mathrm{m}}}$, only the one EFM with $\Omega = \mathrm{O}(\lambda)$ exists. The curves (4.4) in the LK limit on the other hand separate regions in the $(\Omega_0\theta, \Gamma)$-plane with different number of EFMs.

In Fig. 6, we plotted (3.13) and (4.4) in the $(\Omega_0\theta, \Gamma)$-parameter space for an $\Omega_0\theta$-window of length 2π. When a black (thinner) line is crossed, a pair of EFMs is created or annihilated in the LK limit $\lambda \to \infty$. When a blue (thicker) line is crossed, a region $I^j_{\Omega_{\mathrm{m}}}$, where three EFMs exist, appears or disappears in the injection limit $\lambda \to 0$. In the figure, the number of EFMs for the LK limit is given in the different regions that are separated by the curves. For the $\lambda \to 0$-limit, the number of Ω_{m}-intervals where three EFMs exist are given (roman numerals).

We define the various regions in $(\Omega_0\theta, \Gamma)$-parameter space as follows:

$$\begin{aligned} \mathcal{V}_k &:= \{(\Omega_0\theta, \Gamma) | (4.3) \text{ has } k \text{ solutions}\}, \quad k = 1, 3, 5, \ldots, \\ \mathcal{W}_l &:= \{(\Omega_0\theta, \Gamma) | (3.11) \text{ is satisfied in } l \text{ regions}\}, \quad l = \mathrm{I}, \mathrm{II}, \mathrm{III}, \ldots. \end{aligned} \tag{5.1}$$

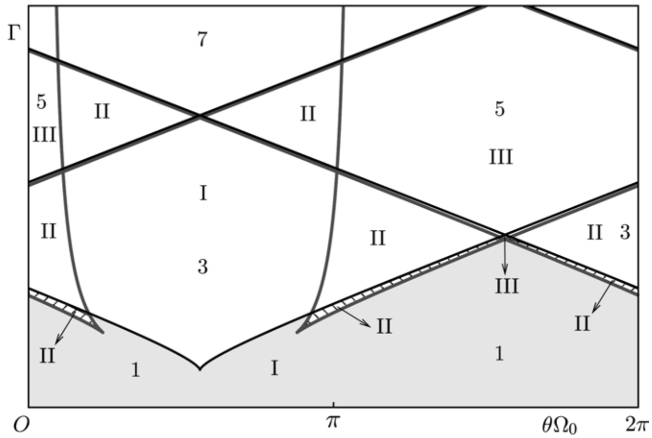

FIG. 6. The SN bifurcation curves (3.13) (blue thicker lines) and the number of intervals $I^j_{\Omega_m}$ (roman numerals) in $(\Omega_0\theta, \Gamma)$-parameter space for $\lambda \to 0$. For $\lambda \to \infty$, the SN bifurcation curves (4.4) (black thinner curves) and the number of EFMs are given. The tiny region $\mathcal{V}_1 \cap \mathcal{W}_{III}$ is denoted by the arrow pointing down to III. See the text for further explanations.

The bifurcation curves in Fig. 6 do not depend on the pump parameter P, so the figure is the same for any value of P. Moreover, the cusp point of the LK limit is found at $(\theta\Omega_0, \Gamma) = \left(\pi - \arctan\alpha, \frac{1}{\theta\sqrt{\alpha^2+1}}\right)$ and the cusp point of the $\lambda \to 0$-limit lies at $(\theta\Omega_0, \Gamma) = \left(\pm 1 + \pi - \arctan\alpha, \frac{2}{\theta\sqrt{\alpha^2+1}}\right)$; hence the arrangement of the curves is exactly as given in Fig. 6. Furthermore, it can be shown that the first LK curve lies 'above' the curve for $\lambda \to 0$ for any choice of the parameters. We thus conclude the following theorem.

THEOREM 5.3 Let $\alpha, P > 0$ be given. Then, the sets $\mathcal{V}_1 \cap \mathcal{W}_I$, $\mathcal{V}_1 \cap \mathcal{W}_{II}$ and $\mathcal{V}_1 \cap \mathcal{W}_{III}$ in $(\Omega_0\theta, \Gamma)$-parameter space are all non-empty.

Theorem 5.3 and Fig. 6 can be explained as follows.

Regardless of $\Omega_0\theta$, the LK limit, $\lambda \to \infty$, has one EFM in the limit $\Gamma \downarrow 0$ and a second and third solution appear when Γ is increased above the first bifurcation curve. Also regardless of $\Omega_0\theta$, the injection limit, $\lambda \to 0$, has one window $I^0_{\Omega_m}$ such that in the limit $\Gamma \downarrow 0$, there is a single EFM when $\Omega_m \notin I^0_{\Omega_m}$ and there are three EFMs if $\Omega_m \in I^0_{\Omega_m}$, see Lemma 3.4. The same is true in the total (large) region $\mathcal{V}_1 \cap \mathcal{W}_I$, the shaded part of Fig. 6. A second region $I^1_{\Omega_m}$ appears when Γ is increased above the first bifurcation curve so that $(\Omega_0\theta, \Gamma) \in \mathcal{V}_1 \cap \mathcal{W}_{II}$. Here, three EFMs exist for $\Omega_m \in I^0_{\Omega_m} \cup I^1_{\Omega_m}$ and one EFM otherwise. There is even a tiny region $\mathcal{V}_1 \cap \mathcal{W}_{III}$ where a third Ω-interval $I^2_{\Omega_m}$ exists.

The boundaries of the intervals $I^j_{\Omega_m}$ are given by the (transcendental) equation (3.12).

6. Bifurcation analysis for varying λ

In this section, we study bifurcations of EFMs as λ varies from $\lambda = 0$ to $\lambda \to \infty$. As these states are characterized by their Ω-coordinates determined by $f = g$ (2.17, 2.18), we study SN bifurcations of EFMs as solutions of $f(\Omega; \lambda) = g(\Omega; \lambda)$ and $f_\Omega(\Omega; \lambda) = g_\Omega(\Omega; \lambda)$ (or the same characterization in one of the alternative independent variables x or y). When both equations are satisfied, EFMs are created

or disappear at a SN bifurcation in $(\Omega;\lambda)$. Solutions of this pair of equations are not easily found, so we alternatively use either one of the following sufficient conditions to characterize SN bifurcations at which two EFMs are 'created' as λ is increased, here stated in the y variable.
At a SN bifurcation $(y=y_0;\lambda=\lambda_0)$, two EFMs are created (supercritical bifurcation) as λ is increased if one of the following three statements holds:

$$\begin{aligned}
&\text{(i)}\ \ f(y_0;\lambda_0)=g(y_0;\lambda_0),\ \ f_y(y_0;\lambda_0)=g_y(y_0;\lambda_0)\ \text{and}\\
&\qquad (f-g)_{yy}(y_0;\lambda_0)(f-g)_\lambda(y_0;\lambda_0)<0,\\
&\text{(ii)}\ \ \forall\,\delta>0\ \text{small enough},\ f(y_0;\lambda_0)=g(y_0;\lambda_0)>0,\ \ f_y(y_0;\lambda_0)=g_y(y_0;\lambda_0)\ \text{and}\\
&\qquad f(y_0;\lambda_0+\delta)>g(y_0;\lambda_0+\delta),\\
&\text{(iii)}\ \ \forall\,\delta>0\ \text{small enough},\ f(y_0;\lambda_0)=g(y_0;\lambda_0)<0,\ \ f_y(y_0;\lambda_0)=g_y(y_0;\lambda_0)\ \text{and}\\
&\qquad f(y_0;\lambda_0+\delta)<g(y_0;\lambda_0+\delta).
\end{aligned} \tag{6.1}$$

Two EFMs will 'vanish' (subcritical bifurcation) under these conditions with $(f-g)_{yy}(y_0;\lambda_0)(f-g)_\lambda(y_0;\lambda_0)>0$ in case (i), $f(y_0;\lambda_0+\delta)<g(y_0;\lambda_0+\delta)$ in case (ii) and $f(y_0;\lambda_0+\delta)>g(y_0;\lambda_0+\delta)$ in case (iii). Note here that a non-degeneracy condition $f_{yy}\neq g_{yy}=0$ should also hold to assure occurrence of a SN bifurcation.

Of these three characterizations, the first is the most standard one (cf. Guckenheimer & Holmes, 1983, Theorem 3.4.1, or Hale & Koçak, 1991, Theorem 10.9). It will, however, turn out that it is not always easy to determine the sign of $(f-g)_{yy}(y_0;\lambda_0)$ so that the two other characterizations will also prove useful.

Recall that the function g is given by $g(y;\lambda)=\lambda y+\Omega_m$. Hence, at the point $y_0=-\frac{\Omega_m}{\lambda}$, it changes sign providing a boundary to when $g(y;\lambda)$ is positive or negative as in cases (ii) or (iii) of (6.1), respectively. Therefore, we define the interval I_y by

$$I_y:=\left(-\frac{\Omega_m}{\lambda_0},0\right),\quad \text{for } \Omega_m>0,\qquad I_y:=\left(0,-\frac{\Omega_m}{\lambda_0}\right),\quad \text{for } \Omega_m<0. \tag{6.2}$$

6.1 *Main results*

We are now ready to state the main results of this section, concerning SN bifurcations of both creating and annihilating nature.

THEOREM 6.1 Let α, Γ, θ, Ω_m and Ω_0 be fixed and let $y_\pm=-\frac{\Omega_m}{2\lambda_0}\pm\frac{1}{2\lambda_0}\sqrt{\Omega_m^2+4\lambda_0/\theta}$. Consider a bifurcation in the point (y_0,λ_0) with $\lambda_0>0$. Two EFMs are created in (y_0,λ_0) when λ is increased through λ_0 if $y_0\in(-\infty,y_-)\cup I_y\cup(y_+,\infty)$, and two EFMs disappear at (y_0,λ_0) if y_0 is located outside these intervals.

The basic ingredients for the proof of this theorem are Lemmas 6.8 and 6.9 that will be proved in Section 6.2.

The SN bifurcations of EFMs with $y_0\in I_y$ form a bifurcation sequence that, as λ is increased from 0 to ∞, results in connection of the two islands of fixed points mentioned in Yousefi & Lenstra (1999), see Fig. 1. The $\lambda\downarrow 0$ limiting EFMs close to $\Omega=0$ and $\Omega=\Omega_m$ (if they exist) are the limiting island configuration. From Theorem 6.1, one immediately deduces the following corollary that corresponds to observations in Yousefi & Lenstra (1999).

COROLLARY 6.2 The SN bifurcations of EFMs with $y_0\in I_y$, that together form a sequence resulting in the connection of the two islands of fixed points, are all of 'creating' nature.

REMARK 6.3 The two islands or EFM components (Green & Krauskopf, 2006) become connected when $g(y)$ and the envelope of $f(y)$ are tangent in some $y_c \in I_y$, i.e. for $\lambda = \lambda_c$ such that

$$\pm \frac{C}{\theta\sqrt{1+y_c^2}} = \lambda_c y_c + \Omega_m \tag{6.3}$$

and

$$\mp \frac{C y_c}{\theta}(1+y_c^2)^{-3/2} = \lambda_c. \tag{6.4}$$

Although Theorem 6.1 gives a characterization of SN bifurcations for which the number of EFMs increases, respectively, decreases, it does not state whether such bifurcations indeed do take place or not. In Corollary 6.7, we however conclude that for certain choices of the parameters at least one annihilating bifurcation occurs.

In Figs 7 and 8, examples of annihilating bifurcations are shown.

Further results on the occurrence of bifurcations depend strongly on the value of the effective feedback strength $C = \Gamma\theta\sqrt{1+\alpha^2}$ in the LK equations, introduced in van der Graaf (1997). The first result is as follows with y_k^+ the largest solution to $(\lambda_0 y + \Omega_m)^2(1+y^2) - \frac{C^2}{\theta^2} = 0$.

LEMMA 6.4 If $C = \theta\Gamma\sqrt{1+\alpha^2} \leqslant 1$, then any SN bifurcation with $y_0 \notin I_y$ results in an 'annihilation' of two EFMs. Moreover, any other SN bifurcation satisfies $\Omega_m < \frac{C}{\theta}$ and $y_0 \in I_y$ or $\Omega_m > \frac{C}{\theta}$ and $y_0 \in \left[-\frac{\Omega_m}{\lambda_0}, y_k^+\right] \subset I_y$, and results in a 'creation' of two EFMs.

In the case $C > 1$, the results are more involved, see Lemma 6.10.

6.2 *Proofs of the main results*

Since the characterization of SN bifurcations is based on signs of certain derivatives of the function $f - g$, we start by deriving some properties of those derivatives.

LEMMA 6.5 In a bifurcation point (y_0, λ_0), the derivatives of $f - g$ satisfy the following:

(i) $(f-g)_\lambda(y_0;\lambda_0) < 0$ if $y_0 \in (-\infty, y_-) \cup (0, y_+)$ and $(f-g)_\lambda(y_0;\lambda_0) > 0$ if $y_0 \in (y_-, 0) \cup (y_+, \infty)$, where $y_\pm = -\frac{\Omega_m}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{\Omega_m^2 + 4\lambda/\theta}$,

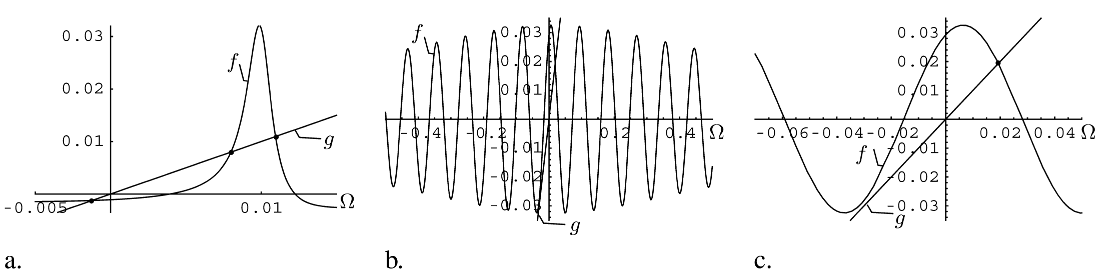

FIG. 7. The functions f and g as functions of Ω as λ is increased. (a) For $\lambda = 0.001$, there are three intersections of f and g, so there exist three EFMs. (b) $\lambda = 0.5$, with in (c) a blowup of the $\lambda = 0.5$ plot. In this case, there is only one intersection (EFM). The other parameters are fixed at $\Gamma = 0.0064$, $\theta = 70$, $\alpha = 5.0$, $\Omega_0 = 0.0414$ and $\Omega_m = 0.01$.

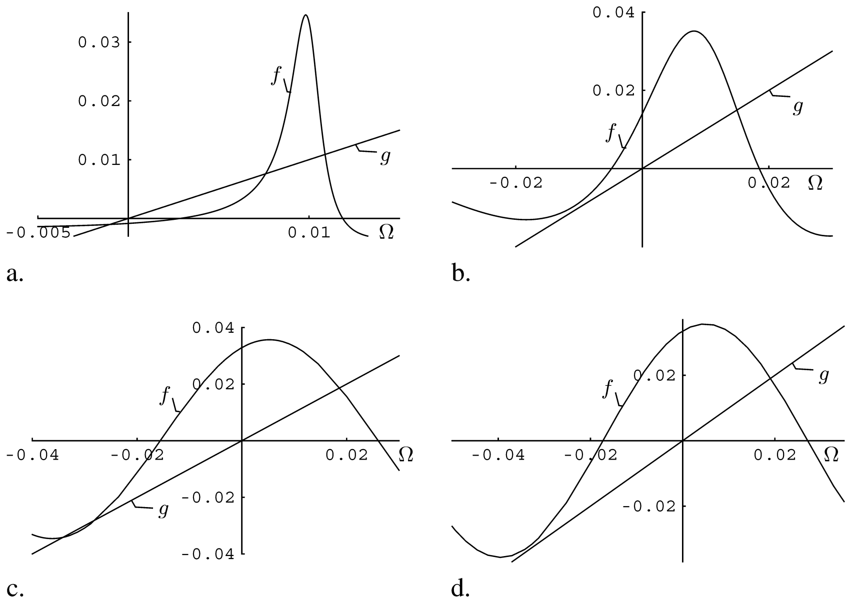

FIG. 8. The functions f and g as functions of Ω as λ is increased: (a) $\lambda = 0.001$, (b) $\lambda = 0.012$, (c) $\lambda = 0.18$ and (d) $\lambda = 1.3$. The number of intersections of f and g now changes from 3 to 1 to 3 to 1 again as λ is increased. The other parameters are fixed at $\Gamma = 0.007$, $\theta = 70$, $\alpha = 5.0$, $\Omega_0 = 0.04286$ and $\Omega_{\mathrm{m}} = 0.01$.

(ii) there exists a value $y^*_{\lambda_0} \in I_y$ such that $(f-g)_{yy}(y_0;\lambda_0) > 0$ if $y_0 < y^*_{\lambda_0}$ and $(f-g)_{yy}(y_0;\lambda_0) < 0$ if $y_0 > y^*_{\lambda_0}$,

(iii) if $\left(-\frac{\Omega_{\mathrm{m}}}{\lambda_0}, \lambda_0\right)$ is a bifurcation point, then $(f-g)_{yy}\left(-\frac{\Omega_{\mathrm{m}}}{\lambda_0}; \lambda_0\right) = \frac{2\lambda^2\Omega_{\mathrm{m}}(2\lambda+\lambda^2\theta+\Omega_{\mathrm{m}}^2\theta)}{(\lambda^2+\Omega_{\mathrm{m}}^2)(\lambda+\lambda^2\theta+\Omega_{\mathrm{m}}^2\theta)}$.

Proof. We compute polynomial expressions for $\frac{\partial f}{\partial \lambda}(y_0;\lambda_0)$ and $\frac{\partial^2 f}{\partial y^2}(y_0;\lambda_0)$ using expressions (3.8) and (3.9) with C and D as in (3.10). In a bifurcation point (y_0, λ_0), both $f(y_0;\lambda_0) = g(y_0;\lambda_0) = \lambda_0 y_0 + \Omega_{\mathrm{m}}$ and $f_y(y_0;\lambda_0) = g_y(y_0;\lambda_0) = \lambda_0$ hold. Substituting the first into the second yields

$$-\frac{y_0}{1+y_0^2}(\lambda_0 y_0 + \Omega_{\mathrm{m}}) - \frac{C}{\theta\sqrt{1+y_0^2}}\left[\lambda_0\theta + \frac{1}{1+y_0^2}\right]\cos(\lambda_0 y_0\theta + \arctan y_0 + D) = \lambda_0. \quad (6.5)$$

This leads to

$$\begin{aligned}\frac{\partial f}{\partial \lambda}(y_0;\lambda_0) &= -y_0\frac{C}{\sqrt{1+y_0^2}}\cos(\lambda_0 y_0\theta + \arctan y_0 + D)\\ &= y_0 + \frac{y_0^2\theta(\lambda_0 y_0 + \Omega_{\mathrm{m}}) - y_0}{\lambda_0\theta(y_0^2+1)+1}\end{aligned} \quad (6.6)$$

by substitution of (6.5). Hence,

$$\frac{\partial(f-g)}{\partial\lambda}(y_0;\lambda_0) = \frac{y_0^2\theta(\lambda_0 y_0 + \Omega_{\mathrm{m}}) - y_0}{\lambda_0\theta(y_0^2+1)+1}$$

in a bifurcation point. The denominator is positive since $\lambda_0\theta > 0$, and the zeroes of the nominator $y^2\theta(\lambda y + \Omega_{\mathrm{m}}) - y$ are $y = 0$ and $y_\pm = -\frac{\Omega_{\mathrm{m}}}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{\Omega_{\mathrm{m}}^2 + 4\lambda/\theta}$, where $y_- < 0 < y_+$. It is now easily checked that claim (i) of the lemma holds.

Similarly, using $f(y_0;\lambda_0) = g(y_0;\lambda_0)$ and (6.5), one derives

$$\frac{\partial^2 f}{\partial y^2}(y_0;\lambda_0) = -\frac{\Omega_{\mathrm{m}}(2 + 4\lambda_0\theta + 3\lambda_0^2\theta^2(1+y_0^2) + \lambda_0^3\theta^3(1+y_0^2)^2)}{(1+y_0^2)(1+\lambda_0\theta(1+y_0^2))} + \frac{\lambda_0 y_0(6 + 3\lambda_0^2\theta^2(1+y_0^2) + \lambda_0^3\theta^3(1+y_0^2)^2 + 2\lambda_0\theta(3+y_0^2))}{(1+y_0^2)(1+\lambda_0\theta(1+y_0^2))} \tag{6.7}$$

in a bifurcation point (y_0, λ_0). Again the denominator is positive, and the nominator is of the form $\Omega_{\mathrm{m}}A(y_0) + \lambda_0 y_0 B(y_0)$ with $B(y_0) > A(y_0) > 0$. Fixing λ_0 and considering y_0 as a variable in expression (6.7), one hence concludes that $\frac{\partial^2 f}{\partial y^2}(y_0;\lambda_0)$ has for each λ_0, a single zero $y^*_{\lambda_0} \in I_y$ with I_y as defined in (6.2). If for $\lambda = \lambda_0$ a bifurcation occurs in $y = y_0$, then $\frac{\partial^2 f}{\partial y^2}(y_0;\lambda_0) > 0$ if y_0 satisfies $y_0 < y^*_{\lambda_0}$ and $\frac{\partial^2 f}{\partial y^2}(y_0;\lambda_0) < 0$ if $y_0 > y^*_{\lambda_0}$. Direct computation in the points $y_0 = 0$ and $y_0 = \frac{-\Omega}{\lambda_0}$ confirms this:

$$\frac{\partial^2 f}{\partial y^2}(0;\lambda_0) = -\frac{\Omega_{\mathrm{m}}(2 + 4\lambda\theta + 3\lambda^2\theta^2 + \lambda^3\theta^3)}{1+\lambda\theta}$$

if $(y = 0, \lambda_0)$ is a bifurcation point and

$$\frac{\partial^2 f}{\partial y^2}\left(-\frac{\Omega_{\mathrm{m}}}{\lambda_0};\lambda_0\right) = \frac{2\lambda^2\Omega_{\mathrm{m}}(2\lambda + \lambda^2\theta + \Omega_{\mathrm{m}}^2\theta)}{(\lambda^2 + \Omega_{\mathrm{m}}^2)(\lambda + \lambda^2\theta + \Omega_{\mathrm{m}}^2\theta)}$$

if $\left(y = -\frac{\Omega_{\mathrm{m}}}{\lambda_0}, \lambda_0\right)$ is a bifurcation point. Since $\frac{\partial^2 g}{\partial y^2} \equiv 0$, the same results hold for the function $f - g$, so claims (ii) and (iii) are proved. □

Outside the interval I_y, the sign of $(f-g)_{yy}(f-g)_\lambda$ at an SN bifurcation in $(y_0;\lambda_0)$ is now known by the first two statements in Lemma 6.5. Therefore, using the first characterization of (6.1), we immediately conclude the following corollary.

COROLLARY 6.6 Let α, Γ, θ, Ω_{m} and Ω_0 be fixed and let $y_\pm = -\frac{\Omega_{\mathrm{m}}}{2\lambda_0} \pm \frac{1}{2\lambda_0}\sqrt{\Omega_{\mathrm{m}}^2 + 4\lambda_0/\theta}$. In a bifurcation point (y_0, λ_0) with $\lambda_0 > 0$, two EFMs are created when λ is increased from λ_0 if $y_0 \in (-\infty, y_-) \cup (y_+, \infty)$. Two EFMs disappear in a bifurcation point (y_0, λ_0) if $y_0 \in (y_-, y_+) \setminus \overline{I_y}$.

Combining Lemma 6.5(i) and (iii), it follows that if a bifurcation occurs in $y = -\frac{\Omega_{\mathrm{m}}}{\lambda} \neq 0$ (corresponding to $\Omega = 0$), then it is of annihilating type. Since $\lambda, \theta > 0$, $\frac{\partial^2(f-g)}{\partial y^2}(-\frac{\Omega_{\mathrm{m}}}{\lambda_0};\lambda_0) = 0$ iff $\Omega_{\mathrm{m}} = 0$, so in that case the bifurcation in $y_0 = -\frac{\Omega_{\mathrm{m}}}{\lambda_0} = 0$ is a supercritical pitchfork bifurcation. However, it is not known yet whether and when a SN bifurcation does take place in $y = -\frac{\Omega_{\mathrm{m}}}{\lambda}$. This is stated in the following corollary.

COROLLARY 6.7 Let α, Γ, θ, Ω_{m} and Ω_0 be fixed such that $\Omega_{\mathrm{m}} > 0$ and $\Omega_0\theta + \arctan\alpha \in (\frac{3\pi}{2}, 2\pi) + 2k\pi$, $k \in \mathbf{Z}$, or alternatively $\Omega_{\mathrm{m}} < 0$ and $\Omega_0\theta + \arctan\alpha \in (\pi, \frac{3\pi}{2}) + 2k\pi$, $k \in \mathbf{Z}$. Then, there exists a unique λ_{a}, satisfying $\arctan(-\frac{\Omega_{\mathrm{m}}}{\lambda_{\mathrm{a}}}) = -\Omega_0\theta - \arctan\alpha + \frac{3\pi}{2} + 2k\pi$, for which an annihilating bifurcation occurs in $y = -\frac{\Omega_{\mathrm{m}}}{\lambda} = -\frac{\Omega_{\mathrm{m}}}{\lambda_{\mathrm{a}}}$.

This means that if the parameters are chosen as in Corollary 6.7, at least one annihilating bifurcation (in $\Omega = 0$) occurs when λ is increased from 0 to ∞.

Proof. Necessary conditions for a SN bifurcation in $y = -\frac{\Omega_m}{\lambda}$ are $f(-\frac{\Omega_m}{\lambda}) = g(-\frac{\Omega_m}{\lambda}) = 0$ and $f_y(-\frac{\Omega_m}{\lambda}) = g_y(-\frac{\Omega_m}{\lambda}) = \lambda$. For this, it is necessary that, with D as in (3.10),

$$\sin\left(D - \Omega_m\theta + \arctan\left(-\frac{\Omega_m}{\lambda}\right)\right) = \sin\left(\Omega_0\theta + \arctan\alpha + \arctan\left(-\frac{\Omega_m}{\lambda}\right)\right) = 0$$

and

$$\operatorname{sgn} f_y\left(-\frac{\Omega_m}{\lambda}\right) = -\operatorname{sgn}\cos\left(\Omega_0\theta + \arctan\alpha + \arctan\left(-\frac{\Omega_m}{\lambda}\right)\right).$$

Combining these two results, one finds that if a SN bifurcation occurs in $y = -\frac{\Omega_m}{\lambda}$, then

$$\Omega_0\theta + \arctan\alpha + \arctan\left(-\frac{\Omega_m}{\lambda}\right) = \frac{3\pi}{2} + 2k\pi, \quad k \in \mathbf{Z}.$$

Since $\arctan(-\frac{\Omega_m}{\lambda}) \in (0, \frac{\pi}{2})$ if $\Omega_m < 0$ and $\arctan(-\frac{\Omega_m}{\lambda}) \in (-\frac{\pi}{2}, 0)$ if $\Omega_m > 0$, this can only occur for $\Omega_m < 0$ when $\Omega_0\theta + \arctan\alpha \in (\pi, \frac{3\pi}{2}) + 2k\pi$, $k \in \mathbf{Z}$, or for $\Omega_m > 0$ when $\Omega_0\theta + \arctan\alpha \in (\frac{3\pi}{2}, 2\pi) + 2k\pi$, $k \in \mathbf{Z}$. The observation that there is a unique $\lambda = \lambda_a$ such that

$$\Omega_0\theta + \arctan\alpha + \arctan\left(-\frac{\Omega_m}{\lambda}\right) = \frac{3\pi}{2} + 2k\pi$$

is easiest made in x-coordinates (recall $x = \lambda y$). It follows from an analysis of the zeros of

$$f(x;\lambda) = -\lambda\frac{C}{\theta\sqrt{\lambda^2 + x^2}}\sin\left(x\theta + \arctan\left(\frac{x}{\lambda}\right) + D\right)$$

and their position as λ varies from 0 to ∞. It turns out that exactly one of the zeros $x_n(\lambda)$ satisfies $x_n(\lambda_a) = -\Omega_m$ for some $\lambda_a \in (0, \infty)$. This ensures the uniqueness of λ_a. The geometry of the graphs of f and g confirms that the bifurcation is of annihilating type. □

As the zero $y^*_{\lambda_0}$ of $\frac{\partial^2 f}{\partial y^2}(y_0;\lambda_0)$ in Lemma 6.1(ii) cannot easily be determined directly, the question remains whether or not annihilating bifurcations can take place in the interval I_y, and if so, how large the annihilating region is. We use the second/third characterization in (6.1) to analyse this. We compute

$$\begin{aligned} f(y_0;\lambda_0+\delta) &= -\frac{C}{\theta\sqrt{1+y_0^2}}\sin(\lambda_0 y_0\theta + \delta y_0\theta + \arctan y_0 + D) \\ &= f(y_0;\lambda_0)\cos(\delta y_0\theta) + \frac{1}{y_0\theta}\frac{\partial f}{\partial\lambda}(y_0;\lambda_0)\sin(\delta y_0\theta). \end{aligned}$$

This yields, by using $f(y_0;\lambda_0) = g(y_0;\lambda_0)$, the exact expression

$$f(y_0;\lambda_0+\delta) - g(y_0;\lambda_0+\delta) = f(y_0;\lambda_0)[\cos(\delta y_0\theta) - 1] + \frac{\partial f}{\partial\lambda}(y_0;\lambda_0)\frac{\sin(\delta y_0\theta)}{y_0\theta} - \delta y_0 \qquad (6.8)$$

which determines whether EFMs are created or annihilated at (y_0, λ_0). It is easily estimated that the first term of (6.8) is of order $O(\delta^2)$ and the second term is of order $O(\delta)$ so that the right-hand side of (6.8) is of order O(δ). Moreover, the sign of $f(y_0; \lambda_0 + \delta) - g(y_0; \lambda_0 + \delta)$, which determines the creation or annihilation of the EFMs, depends on the relation between $\frac{\partial f}{\partial \lambda}(y_0; \lambda_0)$ and y_0 as is stated in the following lemma.

LEMMA 6.8 If there exists a constant $K > 0$ such that $\infty > \frac{\partial f}{\partial \lambda}(y_0; \lambda_0) \geqslant y_0 + K$, then $f(y_0; \lambda_0 + \delta) - g(y_0; \lambda_0 + \delta) > 0$ for $\delta > 0$ sufficiently small. Analogously, $f(y_0; \lambda_0 + \delta) - g(y_0; \lambda_0 + \delta) < 0$ for $\delta > 0$ sufficiently small if there exists a constant $K > 0$ such that $-\infty < \frac{\partial f}{\partial \lambda}(y_0; \lambda_0) < y_0 - K$.

To apply Lemma 6.8, we compare $\frac{\partial f}{\partial \lambda}(y_0; \lambda_0)$ with y_0. This comes down to determining the sign of $\frac{\partial (f-g)}{\partial \lambda}(y_0; \lambda_0)$ as in Lemma 6.5(i). Thus, the following lemma can be deduced.

LEMMA 6.9 In a bifurcation point (y_0, λ_0), the derivative $\frac{\partial f}{\partial \lambda}(y_0; \lambda_0)$ satisfies the following:

(i) $\forall\, y_0 \in (-\infty, y_-) \cup (0, y_+), \exists\, k > 0 : \frac{\partial f}{\partial \lambda}(y_0; \lambda_0) = y_0 - k$,

(ii) $\forall\, y_0 \in (y_-, 0) \cup (y_+, \infty), \exists\, k > 0 : \frac{\partial f}{\partial \lambda}(y_0; \lambda_0) = y_0 + k$,
with $y_\pm = -\frac{\Omega_\mathrm{m}}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{\Omega_\mathrm{m}^2 + 4\lambda/\theta}$.

Combining Lemmas 6.8 and 6.9 with the fact that $f(y_0; \lambda_0) = g(y_0; \lambda_0) > 0$ if $y > -\frac{\Omega_\mathrm{m}}{\lambda}$ and vice versa, we conclude that any bifurcation in (y_0, λ_0) with $y_0 \in I_y$, with I_y as defined in (6.2), is a creating bifurcation. Recalling the statements (6.1) and Lemma 6.5, y_0 must hence satisfy $y_0 \in (0, y^*_{\lambda_0})$ if $\Omega_\mathrm{m} < 0$ or $y_0 \in (y^*_{\lambda_0}, 0)$ if $\Omega_\mathrm{m} > 0$. These results are summarized in Theorem 6.1.

To analyse whether annihilating bifurcations may occur or not, it is useful to observe that the result in Theorem 6.1 is invariant under $\{\Omega_\mathrm{m} \to -\Omega_\mathrm{m}, y_0 \to -y_0\}$. Hence, we take, without loss of generality, $\Omega_\mathrm{m} > 0$ in the sequel.

In the following, we use the boundaries of the existence interval(s) of EFMs as a further restriction on the SN bifurcations. The envelope of f bounds the function f: $-\frac{C}{\theta\sqrt{1+y^2}} \leqslant f(y; \lambda) \leqslant \frac{C}{\theta\sqrt{1+y^2}}$ so that in a SN bifurcation point $(y_0; \lambda_0)$, the requirement

$$-\frac{C}{\theta\sqrt{1+y_0^2}} \leqslant \lambda_0 y_0 + \Omega_\mathrm{m} \leqslant \frac{C}{\theta\sqrt{1+y_0^2}} \tag{6.9}$$

holds. An equality at either side gives, after taking squares, $(\lambda_0 y_0 + \Omega_\mathrm{m})^2(1 + y_0^2) = \frac{C^2}{\theta^2}$. We therefore define the function

$$k(y_0) = (\lambda_0 y_0 + \Omega_\mathrm{m})^2(1 + y_0^2) - \frac{C^2}{\theta^2}, \tag{6.10}$$

and we analyse its sign depending on y_0 (since SN bifurcations can only occur for those y_0 with $k(y_0) \leqslant 0$). The function $k(y_0)$ satisfies

$$k'(y_0) = 0, \quad \text{for } y_0 = -\frac{\Omega_\mathrm{m}}{\lambda_0}, \quad \text{or} \quad y_0 = y_{p,m} := \frac{1}{4\lambda_0}[-\Omega_\mathrm{m} \pm \sqrt{\Omega_\mathrm{m}^2 - 8\lambda_0^2}],$$

where $y_{p,m}$ only exist if $\Omega_\mathrm{m}^2 \geqslant 8\lambda_0^2$ and $-\frac{\Omega_\mathrm{m}}{\lambda_0} < y_{p,m} < 0$ if $\Omega_\mathrm{m} > 0$. Moreover,

$$k(0) = \Omega_\mathrm{m}^2 - \frac{C^2}{\theta^2} \quad \text{and} \quad k\left(-\frac{\Omega_\mathrm{m}}{\lambda_0}\right) = -\frac{C^2}{\theta^2} < 0. \tag{6.11}$$

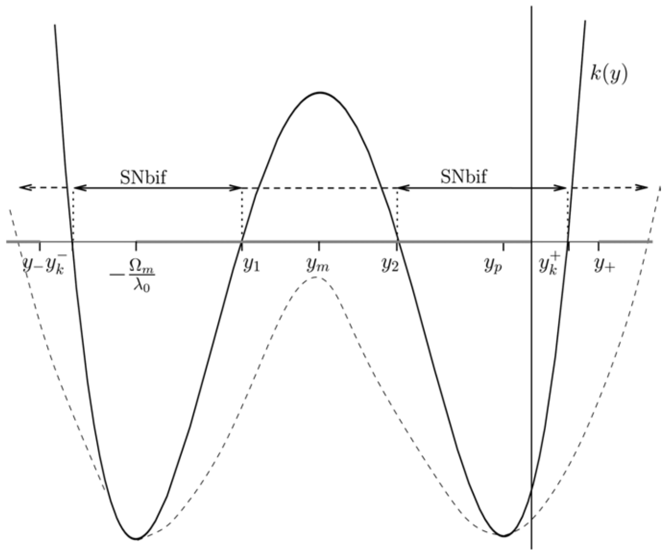

FIG. 9. Sketch of the function $k(y)$ with its zeroes and extrema. The dashed curve displays a different choice of the parameters, with a different form and number of zeroes of $k(y)$ as a result. The intervals where the SN bifurcation can take place are shown. Moreover, on the horizontal axis, the regions where a creating bifurcation occurs are given by the thicker red lines.

Hence, $k(y_0)$ always has a negative (local) minimum in $y_0 = -\frac{\Omega_\mathrm{m}}{\lambda_0}$. Moreover, k has at least two zeroes $y_k^\pm$, $y_k^- < y_k^+$. Thus, there is a bounded region $[y_k^-, y_k^+]$ outside which inequality (6.9) is never satisfied. Depending on the parameter values, this region can be split into two: when the local extrema y_m and y_p exist and satisfy $k(y_m) > 0$ and $k(y_p) < 0$, there are two more zeroes $y_{1,2}$ of k with $y_{1,2} \in I_y$, see Fig. 9. These latter zeroes need no further analysis since they give rise to possible SN bifurcations at $y_0 \in I_y$, that are by Theorem 6.1 clearly of creating nature as λ is increased.

The relative positions of zeroes of $k(y_0)$ and the points $y_0 = 0, -\frac{\Omega_\mathrm{m}}{\lambda_0}$ and $y_\pm$, that form the boundaries between creation and annihilation of EFMs in Theorem 6.1, determine whether bifurcations of creating or annihilating nature are possible or not. We compute

$$k(y_\pm) = \frac{1}{4}\left[\left(\Omega_\mathrm{m} \pm \sqrt{\Omega_\mathrm{m}^2 + \frac{4\lambda_0}{\theta}}\right)^2 + \frac{4}{\theta^2}\right] - \frac{C^2}{\theta^2},$$

from which it can be deduced that

$$k(y_+; \Omega_\mathrm{m}) = 0 \quad \text{iff} \;\; \Omega_\mathrm{m} = \Omega_\mathrm{m}^+ := \frac{1}{\sqrt{C^2-1}}\left[\frac{1}{\theta}(C^2-1) - \lambda_0\right], \tag{6.12}$$

$$k(y_-; \Omega_\mathrm{m}) = 0 \quad \text{iff} \;\; \Omega_\mathrm{m} = \Omega_\mathrm{m}^- := \frac{-1}{\sqrt{C^2-1}}\left[\frac{1}{\theta}(C^2-1) - \lambda_0\right]. \tag{6.13}$$

Therefore, $k(y_\pm) = 0$ is only possible if the remaining parameters satisfy

$$C = \theta\Gamma\sqrt{1+\alpha^2} > 1 \tag{6.14}$$

(recall that $C, \theta > 0$), i.e. while varying Ω_m, a zero of $k(y_0)$ can only pass through $y_\pm$ if (6.14) holds. We now have sufficient information to conclude Lemma 6.4 and its counterpart.

LEMMA 6.10 For $C = \theta\Gamma\sqrt{1+\alpha^2} > 1$, EFMs are ‘created’ in a SN bifurcation for

$$y_0 \in [y_k^-, y_-] \text{ when } 0 < \lambda_0 < \tfrac{1}{\theta}(C^2-1) \text{ and when } \lambda_0 > \tfrac{1}{\theta}(C^2-1) \text{ with } \Omega_m > \Omega_m^-,$$
$$y_0 \in I_y \text{ when } 0 < \Omega_m < \tfrac{C}{\theta},$$
$$y_0 \in \left[-\tfrac{\Omega_m}{\lambda_0}, y_k^+\right] \text{ when } \Omega_m > \tfrac{C}{\theta},$$
$$y_0 \in [y_+, y_k^+] \text{ when } 0 < \lambda_0 < \tfrac{1}{\theta}(C^2-1) \text{ with } 0 < \Omega_m < \Omega_m^+.$$

Annihilation of EFMs takes place in a SN bifurcation for

$$y_0 \in \left[y_-, -\tfrac{\Omega_m}{\lambda_0}\right] \text{ when } 0 < \lambda_0 < \tfrac{1}{\theta}(C^2-1) \text{ and when } \lambda_0 > \tfrac{1}{\theta}(C^2-1) \text{ with } \Omega_m > \Omega_m^-,$$
$$y_0 \in [0, y_k^+] \text{ when } 0 < \lambda_0 < \tfrac{1}{\theta}(C^2-1) \text{ with } \Omega_m^+ < \Omega_m < \tfrac{C}{\theta}$$
$$\text{and when } \lambda_0 > \tfrac{1}{\theta}(C^2-1) \text{ with } \Omega_m < \tfrac{C}{\theta},$$
$$y_0 \in [0, y^+] \text{ when } 0 < \lambda_0 < \tfrac{1}{\theta}(C^2-1) \text{ with } \Omega_m < \Omega_m^+,$$
$$y_0 \in \left[y_k^-, -\tfrac{\Omega_m}{\lambda_0}\right] \text{ when } \lambda_0 > \tfrac{1}{\theta}(C^2-1) \text{ with } 0 < \Omega_m < \Omega_m^-.$$

The proof of this lemma is by a careful analysis of the positions of the various zeroes of k.

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Appendix A. Rescaling (1.3) to (1.4)

In this appendix, we rescale (1.3) to (1.4) by introducing N and s and renaming E and F:

$$N = \frac{\xi n}{2\Gamma_0}, \quad s = \Gamma_0 t, \quad E \to \sqrt{\frac{\xi T_1}{2}}E, \quad F \to \sqrt{\frac{\xi T_1}{2}}F.$$

Thus, (1.3) is rewritten as

$$\begin{aligned} E' &= (1 + \mathrm{i}\alpha)NE + \Gamma F, \\ TN' &= P - N - (1 + 2N)|E|^2, \\ F' &= \lambda E(s - \theta)\mathrm{e}^{-\mathrm{i}\Omega_0\theta} + (\mathrm{i}\Omega_\mathrm{m} - \lambda)F. \end{aligned} \tag{A.1}$$

Here, prime means differentiation with respect to s. The dimensionless parameters T, P, θ, Γ, λ, Ω_0 and Ω_m are defined by

$$T \equiv \Gamma_0 T_1, \quad P \equiv \frac{\xi T_1}{2\Gamma_0}(J - J_\mathrm{thr}), \quad \theta \equiv \Gamma_0 \tau,$$

$$\Gamma \equiv \frac{\gamma}{\Gamma_0}, \quad \lambda \equiv \frac{\Lambda}{\Gamma_0}, \quad \Omega_0 \equiv \frac{\omega_0}{\Gamma_0} \quad \text{and} \quad \Omega_\mathrm{m} \equiv \frac{\omega_\mathrm{m}}{\Gamma_0}.$$

Furthermore, ω_m is in fact the detuning of the filter frequency with respect to the solitary laser frequency ω_0. Hence, the centre frequency of the filter is denoted by $\omega_\mathrm{f} \equiv \omega_\mathrm{m} + \omega_0$ or $\Omega_\mathrm{f} = \Omega_\mathrm{m} + \Omega_0$.

Typical values of the physical parameters are suggested in Yousefi *et al.* (2003) and lead to the values

$$\begin{aligned} T = 100, \quad P = 5.25, \quad \theta = 150, \quad \Gamma = 6.5 \times 10^{-3}, \\ \lambda = 6.7 \times 10^{-4}, \quad \Omega_\mathrm{f} = 17.51 \text{ rad}. \end{aligned} \tag{A.2}$$