Proof of a Conjectured 0-Rényi Entropy Inequality With Applications to Multipartite Entanglement

Zhiwei Song, Lin Chen, Yize Sun, and Mengyao Hu

***Abstract*—Characterizing the relations among the three bipartite reduced density operators ρ_{AB}, ρ_{AC} and ρ_{BC} of a tripartite mixed state ρ_{ABC} has been an open problem in quantum information. One of such relations has been reduced by [Cadney et al, LAA. 452, 153, 2014] to a conjectured inequality in terms of matrix rank, namely $r(\rho_{AB}) \cdot r(\rho_{AC}) \geq r(\rho_{BC})$ for any ρ_{ABC}. It is denoted as open problem 41 in the website "Open quantum problems-IQOQI Vienna". We prove the inequality, and thus establish a complete picture of the four-party linear inequalities in terms of the 0-entropy. Our proof is based on the construction of a novel canonical form of bipartite matrices under local equivalence. We extend our result to inequalities in multipartite systems, as well as the condition when the inequality is saturated.**

***Index Terms*—Entropy inequality, multipartite quantum state, Kronecker product, reduced density operator.**

I. INTRODUCTION

MULTIPARTITE Multipartite systems play a key role in quantum-information processing. For example, the inequality for the von Neumann entropy of reduced density operators of a multipartite has been proposed in [1] and [2], and the multipartite state conversion under many-copy cases has been shown under stochastic local operations and classical communications [3]. Further, the relation between the distillability of entanglement of three bipartite reduced density matrices from a tripartite pure state has been studied [4], [5], [6]. However, it is not easy to extend the relation to the tripartite mixed state, even we merely consider the rank of reduced density operators. It has been conjectured in [7] that the following inequality in terms of matrix rank may hold for any tripartite mixed state ρ_{ABC}, namely

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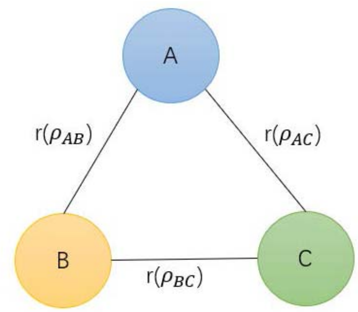

Fig. 1. The tripartite state ρ_{ABC} is conjectured to satisfy the inequality $r(\rho_{AB}) \cdot r(\rho_{BC}) \geq r(\rho_{AC})$ in terms of bipartite reduced density operators ρ_{AB}, ρ_{BC}, and ρ_{AC}, and $r(M)$ denotes the rank of matrix M. The inequality is known to be equivalent to the 0-entropy inequality $S_0(AB) + S_0(BC) \geq S_0(AC)$. We prove the inequality in this paper.

Conjecture 1:

$$r(\rho_{AB}) \cdot r(\rho_{AC}) \geq r(\rho_{BC}), \tag{1}$$

where $r(M)$ denotes the rank of matrix M, see Figure 1.

Note that Conjecture 1 has been listed as an open problem in [8]. It has been proven true when $r(\rho_{AB})$ is at most two and three in [7] and [9], respectively. In this paper we prove Conjecture 1 for any ρ_{ABC} in Theorem 2. The inequality together with the inequalities constructed in [7], establish basic inequalities for the tradeoff among the ranks of three bipartite reduced density operators, see (76)-(79). This is another point of view in contrast to the monogamy trade-off by Bell inequalities [10], [11], [12], [13], [14], [15], [16]. We thus manage to extend the results in [4], [5], and [6] from tripartite pure states to mixed states. Next, we extend the inequalities to multipartite systems in Lemma 13 - Theorem 17. We also discuss the condition when the inequality (1) is saturated in Lemma 18 and Proposition 20.

We review the meaning of Conjecture 1 in terms of the 0-entropy. Let $\alpha \in (0,1) \cup (1,\infty)$ and the logarithm has base two. The known α-Rényi entropy S_α of a quantum state ρ is defined as $S_\alpha(\rho) := \frac{1}{1-\alpha} \log \operatorname{Tr} \rho^\alpha$. One can verify that $\lim_{\alpha\to 1} S_\alpha(\rho)$ is exactly the von Neumann entropy $S(\rho) := -\operatorname{Tr}(\rho \log \rho)$. In contrast, authors in [7] has defined the 0-entropy $S_0(A) := \lim_{\alpha\to 0} S_\alpha(\rho_A) = \log r(\rho_A)$, the 0-entropy vector as the 7-dimensional vector $(r(\rho_A), r(\rho_B), r(\rho_C), r(\rho_D), r(\rho_{AB}), r(\rho_{AC}), r(\rho_{AD}))$ of a 4-partite pure state of system A, B, C, D, as well as the subadditivity $S_0(A) + S_0(B) \geq S_0(AB)$, $S_0(AB) + S_0(AC) \geq S_0(A)$ as well as more inequalities, just like the counterpart inequalities of von Neumann entropy. Further, the 0-entropy inequalities determine a cone with extremal rays characterized by eight 0-entropy vectors. In [7], focusing on the four-party case, authors have found six 0-entropy vectors corresponding to extremal rays, as well as the set of inequalities they correspond to. The set turns out to be the known 0-entropy inequalities and Conjecture 1. Hence, our proof of Conjecture 1 helps complete the aforementioned characterization and construct a complete picture of the four-party linear inequalities for the 0-entropy. We now present the main theorem.

Theorem 2: The inequality $r(\rho_{AB}) \cdot r(\rho_{AC}) \geq r(\rho_{BC})$ holds for any tripartite mixed state ρ_{ABC}.

Remark 1: It has been proved in [7] that Theorem 2 has an equivalent form in terms of the partial transpose and Schmidt rank, see Theorem 3. Hence in this paper, we prove Theorem 2 by proving Theorem 3.

The partial transpose is a positive map of extensive applications in quantum information. Firstly it is known that a separable state has a positive partial transpose (PPT), and it is the most efficient method of detecting entanglement so far [17]. Next, the two-qutrit PPT entangled states were constructed in 1997 [18], and such states of rank four have been characterized [19], [20]. The PPT entanglement represents quantum resources which cannot be distillable into pure entangled states under local operations and classical communications (LOCC). What's more, bipartite non-PPT states of rank at most four turn out to be distillable [21], [22], [23], and some non-PPT states are conjectured to be non distillable [24], [25], [26], [27], [28]. On the other hand, the Schmidt rank is a basic parameter of characterizing bipartite pure states, and has been extended to multipartite pure states as an entanglement monotone [3], [29]. Our result shows novel understanding of the aforementioned quantum-information applications in terms of partial transpose and Schmidt rank.

The rest of this paper is organized as follows. In Sec. II we introduce Theorem 3 and the preliminary facts in Lemmas 5 - 9 for the proof of Theorem 3. We show the proof of Theorem 3 supported by Theorems 11 and 12 in Sec. III. Then we apply and extend our result to Lemma 13 - Proposition 20 in Sec. IV. Finally we conclude in Sec. V.

II. Preliminaries

In this section we introduce the preliminary knowledge and facts of this paper. Let $\mathbb{M}_{m,n}$ be the set of $m \times n$ complex matrices, and $\mathbb{M}_n := \mathbb{M}_{n,n}$. Let I_n be the order-n identity matrix. We denote $det(M)$ as the determinant of matrix M, denote M^T as the transpose of matrix M. We denote the bipartite matrix $M \in \mathbb{M}_{m_1,n_1} \otimes \mathbb{M}_{m_2,n_2}$ as $M = \sum_{i=1}^{m_1}\sum_{j=1}^{n_1} |i\rangle\langle j| \otimes M_{i,j}$ with $M_{i,j} \in \mathbb{M}_{m_2,n_2}$. We denote the partial transposes of M w.r.t. system A and B as $M^{\Gamma_A} = \sum_{i=1}^{m_1}\sum_{j=1}^{n_1} |j\rangle\langle i| \otimes M_{i,j}$, and $M^{\Gamma_B} = \sum_{i=1}^{m_1}\sum_{j=1}^{n_1} |i\rangle\langle j| \otimes M_{i,j}^T$, respectively. Next, the number of linearly independent blocks $M_{i,j}$ is referred to the Schmidt rank $Sr := Sr(M)$ of M. One can derive that $Sr(M) = Sr(M^{\Gamma_A}) = Sr(M^{\Gamma_B}) = Sr(M^T)$. They work for the statement of following theorem.

Theorem 3: For any $M \in \mathbb{M}_{m_1,n_1} \otimes \mathbb{M}_{m_2,n_2}$,

$$r(M^{\Gamma_B}) \leq Sr(M) \cdot r(M). \tag{2}$$

Remark 2: The conjecture is equivalent to $r(M^{\Gamma_A}) \leq Sr(M) \cdot r(M)$. If we write $M = \sum_{i=1}^{K} R_i \otimes S_i$, where $R_1, \ldots, R_K \in \mathbb{M}_{m_1,n_1}$ are linearly independent, and $S_1, \ldots, S_K \in \mathbb{M}_{m_2,n_2}$ are linearly independent, then it is also equivalent to $r\left(\sum_{i=1}^{K} R_i \otimes S_i^T\right) \leq K \cdot r\left(\sum_{i=1}^{K} R_i \otimes S_i\right)$.

Theorems 2 and 3 are shown to be equivalent in [7]. Next it follows from Theorem 5 of [7] that Theorem 3 holds for $Sr(M) \leq 2$. It has been proved in Theorem 2 of [9] that Theorem 3 holds for $Sr(M) = 3$. Nevertheless, these proven cases do not contribute to our proof in the next section. To explain our proof, we present the following definition.

Definition 4: Two matrices M and N are said to be locally equivalent, $M \sim N$, if there exist invertible tensor product matrices $U \otimes V$ and $W \otimes X$ such that $(U \otimes V)M(W \otimes X) = N$.

Specifically, for $M = \sum_{i=1}^{m_1}\sum_{j=1}^{n_1} |i\rangle\langle j| \otimes M_{i,j}$, U and W correspond to block-row and block-column operations on M respectively, V and X correspond to row and column operations on each block $M_{i,j}$ respectively, which is $M_{i,j} \rightarrow VM_{i,j}X$. So M and N have the same rank and Schmidt rank, one can also show that M^{Γ_B} and N^{Γ_B} have the same rank and Schmidt rank. Hence proving M satisfies Theorem 3 is equivalent to proving N satisfies Theorem 3, and we shall frequently use this fact in the next section.

In the rest of this section, we present five preliminary lemmas used for the proof of the next section. The following two lemmas can be straightforwardly proven by using the basic matrix theory.

Lemma 5: The following inequalities

$$r(A_1) \leq r(\begin{bmatrix} A_1 A_2 \cdots A_n \end{bmatrix}) \leq r(A_1) + \cdots + r(A_n), \tag{3}$$

$$r(A) + r(C) \leq r(\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}) \tag{4}$$

hold for any block matrix.

Lemma 6: (i) Suppose $A_1, \cdots, A_n \in \mathbb{M}_{m_2,n_2}$ are linearly independent, $R \in \mathbb{M}_{n_2}$ is invertible. Then $A_1 \cdot R, \cdots, A_n \cdot R$ are linearly independent.

(ii) Suppose $A_1, \cdots, A_n, A_{n+1} \in \mathbb{M}_{m_2,n_2}$, and $R \in \mathbb{M}_{n_2}$ is invertible. If $A_{n+1} \in \text{span}\{A_1, \cdots, A_n\}$, then $A_{n+1} \cdot R \in \text{span}\{A_1 \cdot R, \cdots, A_n \cdot R\}$.

The following lemma is assertion (a) of Theorem 6 in [9]. The proof also involves basic matrix theory, see details in [9].

Lemma 7: Suppose one block-row or block-column of the Schmidt-rank-K block matrix M has K linearly independent blocks. Then M satisfies Theorem 3.

Further, given a bipartite matrix M, we hope to arrange its linearly independent blocks in a more regular position, i.e, the left-upper blocks of M. The following lemma achieves this by applying block-row and block-column operations on M under local equivalence of the A system.

Lemma 8: (i) Suppose

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t,1} & M_{t,2} & \cdots & M_{t,n_1} \\ 0 & M_{t+1,2} & \cdots & M_{t+1,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & M_{m_1,2} & \cdots & M_{m_1,n_1} \end{bmatrix},$$

where $M_{1,1},\cdots,M_{t,1}$ are linearly independent, further, there exists at least one block $M_{i,j}$ for $t+1\le i\le m_1, 2\le j\le n_1$, which is linearly independent with $M_{1,1},\cdots,M_{t,1}$. Then M is locally equivalent to $M' := [M'_{i,j}]$, where the first block-column of M' has at least $t+1$ linearly independent blocks.

(ii) Suppose

$$M=\begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} & \cdots & M_{1,n_1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ M_{t,1} & M_{t,2} & M_{t,3} & \cdots & M_{t,n_1}\\ M_{t+1,1} & M_{t+1,2} & M_{t+1,3} & \cdots & M_{t+1,n_1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ M_{s,1} & M_{s,2} & M_{s,3} & \cdots & M_{s,n_1}\end{bmatrix},$$

where $M_{1,1},\cdots,M_{s,1},M_{1,2},\cdots,M_{t,2}$ are linearly independent, $s>t$, and $M_{i,2}$ is spanned by $M_{1,1},\cdots,M_{s,1}$ for all $t+1\le i\le s$. At the same time, there exists at least one block $M_{i,j}$ for $t+1\le i\le s, 3\le j\le n_1$, which is linearly independent with $M_{1,1},\cdots,M_{s,1},M_{1,2},\cdots,M_{t,2}$. Then M is locally equivalent to $M'' := [M''_{i,j}]$, where $M''_{1,1},\cdots,M''_{s,1}$ are linearly independent, the second block-column of M'' has at least $t+1$ linearly independent blocks, and they are linearly independent with $M''_{1,1},\cdots,M''_{s,1}$.

(iii) Suppose

$$M=\begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,n} & \cdots & M_{1,n_1}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ M_{t_n,1} & M_{t_n,2} & \cdots & M_{t_n,n} & \cdots & M_{t_n,n_1}\\ M_{t_n+1,1} & M_{t_n+1,2} & \cdots & M_{t_n+1,n} & \cdots & M_{t_n+1,n_1}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ M_{t_2,1} & M_{t_2,2} & \cdots & M_{t_2,n} & \cdots & M_{t_2,n_1}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ M_{t_1,1} & M_{t_1,2} & \cdots & M_{t_1,n} & \cdots & M_{t_1,n_1}\end{bmatrix},$$

where $M_{1,1},\cdots,M_{t_1,1}$, $M_{1,2},\cdots,M_{t_2,2},\cdots$, $M_{1,n},\cdots,M_{t_n,n}$ are linearly independent, $2<n<n_1$, and $t_1\ge t_2\ge\cdots\ge t_{n-1}>t_n$. Any $M_{i,n}$ is spanned by $M_{1,1},\cdots,M_{t_1,1}$, $M_{1,2},\cdots,M_{t_2,2},\cdots$, $M_{1,n-1},\cdots,M_{t_{n-1},n-1}$, where $t_n+1\le i\le t_1$. At the same time, there exists at least one block $M_{i,j}$ for $t_n+1\le i\le t_1, n+1\le j\le n_1$, which is linearly independent with $M_{1,1},\cdots,M_{t_1,1}$, $M_{1,2},\cdots,M_{t_2,2},\cdots$, $M_{1,n},\cdots,M_{t_n,n}$. Then M is locally equivalent to $M^{(n)} := [M^{(n)}_{i,j}]$, where the n-th block-column of $M^{(n)}$ has at least t_n+1 linearly independent blocks and they are linearly independent with the linearly independent blocks $M^{(n)}_{1,1},\cdots,M^{(n)}_{t_1,1},M^{(n)}_{1,2},\cdots,M^{(n)}_{t_2,2},\cdots,M^{(n)}_{1,n-1},\cdots,M^{(n)}_{t_{n-1},n-1}$.

Proof: (i) Without loss of generality, assume that $M_{t+1,2}$ is linearly independent with $M_{1,1},\cdots,M_{t,1}$. Let

$$W=\begin{bmatrix}1 & 0\\ k & 1\end{bmatrix}\oplus I_{n_1-2} \tag{5}$$

be the $n_1\times n_1$ invertible matrix in Definition 4, and k an undetermined nonzero number ($W=\begin{bmatrix}1 & 0\\ k & 1\end{bmatrix}$ if $n_1=2$). Therefore

$$M' = M(W\otimes I_{n_2}) \tag{6}$$

is locally equivalent to M, and

$$\begin{bmatrix} M'_{1,1}\\ M'_{2,1}\\ \vdots\\ M'_{t,1}\\ M'_{t+1,1}\end{bmatrix}=\begin{bmatrix} M_{1,1}\\ M_{2,1}\\ \vdots\\ M_{t,1}\\ 0\end{bmatrix}+k\begin{bmatrix} M_{1,2}\\ M_{2,2}\\ \vdots\\ M_{t,2}\\ M_{t+1,2}\end{bmatrix}. \tag{7}$$

Our aim is to find suitable k such that $M'_{1,1},\cdots,M'_{t+1,1}$ are linearly independent. Since $M_{1,1},\cdots,M_{t,1},M_{t+1,2}$ are linearly independent, we focus on $M_{1,2},\cdots,M_{t,2}$, and have two cases, namely (A) and (B), as follows.

(A). Suppose $M_{1,2},\cdots,M_{t,2}$ are spanned by $M_{1,1},\cdots,M_{t,1},M_{t+1,2}$. Set

$$M_{i,2}=a_{1i}M_{1,1}+\cdots+a_{ti}M_{t,1}+b_iM_{t+1,2} \tag{8}$$

with $1\le i\le t$. From (7) we obtain that

$$\begin{aligned}\begin{bmatrix} M'_{1,1}\\ M'_{2,1}\\ \vdots\\ M'_{t,1}\\ M'_{t+1,1}\end{bmatrix} &=\begin{bmatrix}(ka_{11}+1)M_{1,1}+ka_{21}M_{2,1}+\cdots+ka_{t1}M_{t,1}+kb_1M_{t+1,2}\\ ka_{12}M_{1,1}+(ka_{22}+1)M_{2,1}+\cdots+ka_{t2}M_{t,1}+kb_2M_{t+1,2}\\ \vdots\\ ka_{1t}M_{1,1}+ka_{2t}M_{2,1}+\cdots+(ka_{tt}+1)M_{t,1}+kb_tM_{t+1,2}\\ kM_{t+1,2}\end{bmatrix}\\ &=\begin{bmatrix} ka_{11}+1 & ka_{21} & \cdots & ka_{t1} & kb_1\\ ka_{12} & ka_{22}+1 & \cdots & ka_{t2} & kb_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ ka_{1t} & ka_{2t} & \cdots & ka_{tt}+1 & kb_t\\ 0 & 0 & \cdots & 0 & k\end{bmatrix}\cdot\begin{bmatrix} M_{1,1}\\ M_{2,1}\\ \vdots\\ M_{t,1}\\ M_{t+1,2}\end{bmatrix}.\end{aligned} \tag{9}$$

Further, there must exist a k such that

$$\det\left(\begin{bmatrix} ka_{11}+1 & ka_{21} & \cdots & ka_{t1} & kb_1\\ ka_{12} & ka_{22}+1 & \cdots & ka_{t2} & kb_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ ka_{1t} & ka_{2t} & \cdots & ka_{tt}+1 & kb_t\\ 0 & 0 & \cdots & 0 & k\end{bmatrix}\right)\neq 0. \tag{10}$$

(9) and (10) imply that $M'_{1,1},\cdots,M'_{t+1,1}$ are linearly independent. Since $M'\sim M$ by (6), we have proved the case.

(B). Suppose there exists at least one $M_{i,2}$ that is linearly independent with $M_{1,1},\cdots,M_{t,1},M_{t+1,2}$, where $1\le i\le t$. Without loss of generality, assume that $M_{1,1},\cdots,M_{t,1},M_{t+1,2},M_{1,2},\cdots,M_{v,2}$ are linearly independent, $1\le v\le t$. At the same time, any $M_{i,2}$ is spanned by the $t+1+v$ matrices for $v+1\le i\le t$. From (7) we obtain that $M'_{1,1},\cdots,M'_{v,1}$ are linearly independent for any k. Similar to the Case (A), we can find suitable k such that $M'_{v+1,1},\cdots,M'_{t+1,1}$ are linearly independent, at the same time, they are linearly independent with $M'_{1,1},\cdots,M'_{v,1}$. Since $M'\sim M$, we have proved this case.

We obtain assertion (i).

(ii) Without loss of generality, assume that $M_{t+1,3}$ is linearly independent with $M_{1,1},\cdots,M_{s,1},M_{1,2},\cdots,M_{t,2}$. Further, based on the assumption, we set

$$M_{t+1,2}=g_1M_{1,1}+\cdots+g_sM_{s,1}. \tag{11}$$

Let

$$W_1=1\oplus\begin{bmatrix}10\\k1\end{bmatrix}\oplus I_{n_1-3} \tag{12}$$

be the $n_1\times n_1$ invertible matrix in Definition 4, and k an undetermined nonzero number ($W_1=1\oplus\begin{bmatrix}10\\k1\end{bmatrix}$ if $n_1=3$). Therefore

$$M''=M(W_1\otimes I_{n_2}) \tag{13}$$

is locally equivalent to M, and

$$\begin{bmatrix}M''_{1,1}\\ \vdots\\ M''_{s,1}\\ M''_{1,2}\\ \vdots\\ M''_{t,2}\\ M''_{t+1,2}\end{bmatrix}=\begin{bmatrix}M_{1,1}\\ \vdots\\ M_{s,1}\\ M_{1,2}+kM_{1,3}\\ \vdots\\ M_{t,2}+kM_{t,3}\\ M_{t+1,2}+kM_{t+1,3}\end{bmatrix}. \tag{14}$$

Our aim is to find suitable k such that $M''_{1,1},\cdots,M''_{s,1}$, $M''_{1,2},\cdots,M''_{t,2},M''_{t+1,2}$ are linearly independent, and similar to (i), we have two cases, namely (C) and (D), as follows.

(C). Suppose $M_{1,3},\cdots,M_{t,3}$ are spanned by $M_{1,1},\cdots,M_{s,1},M_{1,2},\cdots,M_{t,2},M_{t+1,3}$, where $1\le i\le t$. Set

$$\begin{aligned}M_{i,3}=&\, c_{1i}M_{1,1}+\cdots+c_{si}M_{s,1}\\ &+d_{1i}M_{1,2}+\cdots+d_{ti}M_{t,2}+f_iM_{t+1,3},\end{aligned} \tag{15}$$

with $1\le i\le t$. From (11) and (15), we obtain that in (16), shown at the bottom of the page. There must exist a k such that

$$\det(\begin{bmatrix} & & & 0 & \cdots & 0 & 0\\ & I_s & & \vdots & \ddots & \vdots & \vdots\\ & & & 0 & \cdots & 0 & 0\\ kc_{11} & \cdots & kc_{s1} & kd_{11}+1 & \cdots & kd_{t1} & kf_1\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ kc_{1t} & \cdots & kc_{st} & kd_{1t} & \cdots & kd_{tt}+1 & kf_t\\ g_1 & \cdots & g_s & 0 & \cdots & 0 & k\end{bmatrix})$$

$$=\det(\begin{bmatrix}kd_{11}+1 & \cdots & kd_{t1} & kf_1\\ \vdots & \ddots & \vdots & \vdots\\ kd_{1t} & \cdots & kd_{tt}+1 & kf_t\\ 0 & \cdots & 0 & k\end{bmatrix})\neq 0. \tag{17}$$

Since $M_{1,1},\cdots,M_{s,1}$, $M_{1,2},\cdots,M_{t,2},M_{t+1,3}$ are linearly independent, (16) and (17) imply that $M''_{1,1},\cdots,M''_{s,1}$, $M''_{1,2},\cdots,M''_{t,2},M''_{t+1,2}$ are linearly independent. We have proved the case.

(D). Suppose there exists at least one $M_{i,3}$ that is linearly independent with $M_{1,1},\cdots,M_{s,1},M_{1,2},\cdots,M_{t,1},M_{t+1,3}$, where $1\le i\le t$. Without loss of generality, assume that $M_{1,1},\cdots,M_{s,1},M_{1,2},\cdots,M_{t,2},M_{t+1,3},M_{1,3},\cdots,M_{v,3}$ are linearly independent, where $1\le v\le t$. At the same time, any $M_{i,3}$ is spanned by the $s+t+1+v$ matrices for $v+1\le i\le t$. From (14), we obtain that $M''_{1,1},\cdots,M''_{s,1},M''_{1,2},\cdots,M''_{v,2}$ are linearly independent for any k. Similar to the Case (C), we can find suitable k such that $M''_{v+1,2},\cdots,M''_{t+1,2}$ are linearly independent, at the same time, they are linearly independent with $M''_{1,1},\cdots,M''_{s,1},M''_{1,2},\cdots,M''_{v,2}$. Since $M''\sim M$, we have proved this case.

We obtain assertion (ii).

(iii) The proof is similar to that of (ii). ∎

$$\begin{aligned}\begin{bmatrix}M''_{1,1}\\ \vdots\\ M''_{s,1}\\ M''_{1,2}\\ \vdots\\ M''_{t,2}\\ M''_{t+1,2}\end{bmatrix}&=\begin{bmatrix}M_{1,1}\\ \vdots\\ M_{s,1}\\ kc_{11}M_{1,1}+\cdots+kc_{s1}M_{s,1}+(kd_{11}+1)M_{1,2}+\cdots+kd_{t1}M_{t,2}+kf_1M_{t+1,3}\\ \vdots\\ kc_{1t}M_{1,1}+\cdots+kc_{st}M_{s,1}+kd_{1t}M_{1,2}+\cdots+(kd_{tt}+1)M_{t,2}+kf_tM_{t+1,3}\\ g_1M_{1,1}+\cdots+g_sM_{s,1}+kM_{t+1,3}\end{bmatrix}\\ &=\begin{bmatrix} & & & 0 & \cdots & 0 & 0\\ & I_s & & \vdots & \ddots & \vdots & \vdots\\ & & & 0 & \cdots & 0 & 0\\ kc_{11} & \cdots & kc_{s1} & kd_{11}+1 & \cdots & kd_{t1} & kf_1\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ kc_{1t} & \cdots & kc_{st} & kd_{1t} & \cdots & kd_{tt}+1 & kf_t\\ g_1 & \cdots & g_s & 0 & \cdots & 0 & k\end{bmatrix}\cdot\begin{bmatrix}M_{1,1}\\ \vdots\\ M_{s,1}\\ M_{1,2}\\ \vdots\\ M_{t,2}\\ M_{t+1,3}\end{bmatrix}.\end{aligned} \tag{16}$$

Lemma 9: Suppose the $m \times 1$ block matrix $P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \end{bmatrix}$ has rank k, where $P_1, \cdots, P_m$ are $\ell \times n$ matrices. Then $Q = \sum_{i=1}^{m} c_i P_i$ has at most rank k for arbitrary numbers $c_1, \cdots, c_m$.

Proof: Consider the map π from the column vectors of length $m\ell$ to those of length ℓ defined by $\pi([x_1, x_2, \cdots, x_m]^T) = \sum_{i=1}^{m} c_i x_i$, where each x_i is a column vector of length ℓ. Since this is a linear map, any linear dependence between column vectors of P is mapped to the same linear relation between the image of the column vectors, which are the columns of Q. So, in particular, if $n-k$ columns of P are linear combinations of the remaining k, the same with hold for the columns of Q. ■

III. Proof of Theorem 3

For any block matrix $M \in \mathbb{M}_{m_1,n_1} \otimes \mathbb{M}_{m_2,n_2}$ with Schmidt rank $K \leq m_1 \cdot n_1$, we write

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} & \cdots & M_{1,n_1} \\ M_{2,1} & M_{2,2} & M_{2,3} & \cdots & M_{2,n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{m_1,1} & M_{m_1,2} & M_{m_1,3} & \cdots & M_{m_1,n_1} \end{bmatrix}, \tag{18}$$

where M has K linearly independent blocks.

Our proof of Theorem 3 is divided into two subsections. In subsection III-A, we first present N in (19) as a canonical form of M under local equivalence of the A system in Theorem 11. Further, we present N_p in (50) as a canoniacal form of N under local equivalence of the B system in Theorem 12. In subsection III-B, we prove that N_p in (50) satisfies the theorm and hence $M \sim N \sim N_p$ satisfies the theorem.

A. Two Canonical Forms of Bipartite Matrices

Definition 10: A bipartite matrix set $\mathbb{M} \subseteq \mathbb{M}_{m_1,n_1} \otimes \mathbb{M}_{m_2,n_2}$ is called $\mathbb{M}_{canonical}$, if any Schmidt-rank-K matrix $N \in \mathbb{M}_{canonical}$ is written as

$$N = \left[\begin{array}{c|c|c|c|c|c} N_{1,1} & N_{1,2} & N_{1,3} & \cdots & N_{1,p} & \\ \vdots & \vdots & \vdots & \ddots & \vdots & A_p \\ N_{k_p,1} & N_{k_p,2} & N_{k_p,3} & \cdots & N_{k_p,p} & \\ \cline{5-5} \vdots & \vdots & \vdots & \vdots & \multicolumn{2}{c}{A_{p-1}} \\ \cline{4-4} \vdots & \vdots & \vdots & \multicolumn{3}{c}{} \\ N_{k_3,1} & N_{k_3,2} & N_{k_3,3} & \multicolumn{3}{c}{\cdots} \\ \cline{3-3} \vdots & \vdots & \multicolumn{4}{c}{} \\ N_{k_2,1} & N_{k_2,2} & \multicolumn{4}{c}{} \\ \cline{2-2} \vdots & \multicolumn{5}{c}{A_2} \\ N_{k_1,1} & \multicolumn{5}{c}{} \\ \cline{1-1} 0 & \multicolumn{5}{c}{} \\ \vdots & \multicolumn{5}{c}{A_1} \\ 0 & \multicolumn{5}{c}{} \end{array}\right], \tag{19}$$

where $N_{1,1}, \cdots, N_{k_1,1}, N_{1,2}, \cdots, N_{k_2,2}, \cdots, N_{1,p}, \cdots, N_{k_p,p}$ are linearly independent, and

$$1 \leq p \leq n_1, \tag{20}$$

$$1 \leq k_p \leq \cdots \leq k_2 \leq k_1 \leq m_1, \tag{21}$$

$$k_1 + k_2 + \cdots + k_p = K. \tag{22}$$

At the same time, for any $1 \leq i \leq p$, each $m_2 \times n_2$ block in the L-shaped part A_i is spanned by $N_{1,1}, \cdots, N_{k_1,1}, N_{1,2}, \cdots, N_{k_2,2}, \cdots, N_{1,i}, \cdots, N_{k_i,i}$, we denote as

$$N_{A_i} \in \operatorname{span}\{N_{1,1}, \cdots, N_{k_1,1}, N_{1,2}, \cdots, N_{k_2,2}, N_{1,i}, \cdots, N_{k_i,i}\}, \tag{23}$$

and we shall use this notation from now on.

Note that if $m_1 = k_1$, then the zero blocks below $N_{k_1,1}$ disappear and A_1 becomes a $(k_1 - k_2) \times 1$ rectangular block matrix. If $k_1 = k_2$, then A_1 becomes a $(m_1 - k_1) \times (n_1 - 1)$ rectangular block matrix and A_2 becomes a $(k_2 - k_3) \times 1$ rectangular block matrix. If there exists i $(2 \leq i < p-1)$ such that $k_i = k_{i+1}$, then A_i becomes a $(k_{i-1} - k_i) \times (n_1 - i)$ rectangular block matrix, and A_{i+1} becomes a $(k_{i+1} - k_{i+2}) \times 1$ rectangular block matrix. If $k_{p-1} = k_p$, then A_{p-1} becomes a $(k_{p-2} - k_{p-1}) \times (n_1 - p + 1)$ rectangular block matrix, and A_p becomes a $k_p \times (n_1 - p)$ rectangular block matrix.

Theorem 11: For any bipartite matrix M in (18), there exists $N \in \mathbb{M}_{canonical}$ in (19), such that M is locally equivalent to N.

Proof: Our first aim is to obtain that

$$M \sim M' = \left[\begin{array}{c|ccc} M'_{1,1} & M'_{1,2} & \cdots & M'_{1,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ M'_{k_1,1} & M'_{k_1,2} & \cdots & M'_{k_1,n_1} \\ \hline 0 & & & \\ \vdots & & A'_1 & \\ 0 & & & \end{array}\right], \tag{24}$$

where $M'_{1,1}, \cdots, M'_{k_1,1}$ are linearly independent, $k_1 \leq m_1$, and every $m_2 \times n_2$ block in the rectangular block matrix A'_1 is spanned by $M'_{1,1}, \cdots, M'_{k_1,1}$, i.e.,

$$M_{A'_1} \in \operatorname{span}\{M'_{1,1}, \cdots, M'_{k_1,1}\}. \tag{25}$$

We apply the following three steps, namely **Steps** 1-3, to achieve this aim.

Step 1: Consider the first block-column of M in (18). Assume it has s_1 linearly independent blocks, $s_1 \leq m_1$. Up to block-row operations,

$$M \sim M_1 = \left[\begin{array}{c|ccc} M^{(1)}_{1,1} & M^{(1)}_{1,2} & \cdots & M^{(1)}_{1,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ M^{(1)}_{s_1,1} & M^{(1)}_{s_1,2} & \cdots & M^{(1)}_{s_1,n_1} \\ \hline 0 & & & \\ \vdots & & A^{(1)}_1 & \\ 0 & & & \end{array}\right], \tag{26}$$

where $M^{(1)}_{1,1}, \cdots, M^{(1)}_{s_1,1}$ are linearly independent. If $s_1 = m_1$ or $M_{A^{(1)}_1} \in \operatorname{span}\{M^{(1)}_{1,1}, \cdots, M^{(1)}_{s_1,1}\}$, then $M \sim M_1$ satisfies (24) and we have achieved this aim.

Step 2: Suppose $s_1 < m_1$ and there exists at least one block in $A_1^{(1)}$ that is linearly independent with $M^{(1)}_{1,1}, \cdots, M^{(1)}_{s_1,1}$. Using Lemma 8 (i), applying block-column and following block-row operations on M_1, we obtain that

$$M_1 \sim M_2 = \left[\begin{array}{c|ccc} M^{(2)}_{1,1} & M^{(2)}_{1,2} & \cdots & M^{(2)}_{1,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ M^{(2)}_{s_1+a,1} & M^{(2)}_{s_1+a,2} & \cdots & M^{(2)}_{s_1+a,n_1} \\ \hline 0 & & & \\ \vdots & & A_1^{(2)} & \\ 0 & & & \end{array}\right], \tag{27}$$

where $M^{(2)}_{1,1}, \cdots, M^{(2)}_{s_1+a,1}$ are linearly independent, $1 \leq a \leq m_1 - s_1$. If $s_1 + a = m_1$ or $M_{A_1^{(2)}} \in \mathrm{span}\{M^{(2)}_{1,1}, \cdots, M^{(2)}_{s_1+a,1}\}$, then $M \sim M_1 \sim M_2$ satisfies (24) and we have achieved this aim.

Step 3: If $s_1 + a < m_1$ and there exists at least one block in $A_1^{(2)}$ that is linearly independent with $M^{(1)}_{1,1}, \cdots, M^{(1)}_{s_1,1}$, then repeat **Step** 2.

Eventually we obtain that $M \sim M'$ in (24). Recall that M in (18) has Schmidt rank K, hence M' has Schmidt rank K. If $k_1 = K$ in M' in (24), then $M' \sim N$ in (19) with $p = 1$. Suppose

$$k_1 < K \tag{28}$$

in M'. Our next aim is to obtain that

$$M' \sim M'' = \left[\begin{array}{c|c|ccc} M''_{1,1} & M''_{1,2} & M''_{1,3} & \cdots & M''_{1,n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M''_{k_2,1} & M''_{k_2,2} & M''_{k_2,3} & \cdots & M''_{k_2,n_1} \\ \cline{2-5} \vdots & & & A_2'' & \\ M''_{k_1,1} & & & & \\ \cline{1-1}\cline{3-5} 0 & & & & \\ \vdots & & & A_1'' & \\ 0 & & & & \end{array}\right], \tag{29}$$

where $M''_{1,1}, \cdots, M''_{k_1,1}, M''_{1,2}, \cdots, M''_{k_2,2}$ are linearly independent, $1 \leq k_2 \leq k_1 \leq m_1$, and

$$M_{A_1''} \in \mathrm{span}\{M''_{1,1}, \cdots, M''_{k_1,1}\}, \tag{30}$$
$$M_{A_2''} \in \mathrm{span}\{M''_{1,1}, \cdots, M''_{k_1,1}, M''_{1,2}, \cdots, M''_{k_2,2}\}. \tag{31}$$

We next apply the following five steps, namely **Steps** 4-8, to achieve this aim.

Step 4: Consider the matrix M' in (24). (25) and (28) imply that there exists at least one block $M'_{i,j}$ that is linearly independent with $M'_{1,1}, \cdots, M'_{k_1,1}$ for $1 \leq i \leq k_1$ and $2 \leq j \leq n_1$. Up to block-row and block-column switches on M',

$$M' \sim M_3 = \left[\begin{array}{c|cccc} M^{(3)}_{1,1} & M^{(3)}_{1,2} & M^{(3)}_{1,3} & \cdots & M^{(3)}_{1,n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M^{(3)}_{s_2,1} & M^{(3)}_{s_2,2} & M^{(3)}_{s_2,3} & \cdots & M^{(3)}_{s_2,n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M^{(3)}_{k_1,1} & M^{(3)}_{k_1,2} & M^{(3)}_{k_1,3} & \cdots & M^{(3)}_{k_1,n_1} \\ \hline 0 & & & & \\ \vdots & & A_1^{(3)} & & \\ 0 & & & & \end{array}\right], \tag{32}$$

where $M^{(3)}_{1,1}, \cdots, M^{(3)}_{k_1,1}, M^{(3)}_{1,2}, \cdots, M^{(3)}_{s_2,2}$ are linearly independent, $1 \leq s_2 \leq k_1$, and

$$M^{(3)}_{i,2} \in \mathrm{span}\{M^{(3)}_{1,1}, \cdots, M^{(3)}_{k_1,1}, M^{(3)}_{1,2}, \cdots, M^{(3)}_{s_2,2}\} \tag{33}$$

holds for any $s_2 + 1 \leq i \leq k_1$ if $s_2 < k_1$.

Step 5: If $s_2 = k_1$, then $M' \sim M_3$ satisfies (29) and we have achieved this aim. If $s_2 < k_1$, by (33), applying block-row operations on M_3 in (32) such that

$$M_3 \sim M_4 = \left[\begin{array}{c|c|ccc} M^{(4)}_{1,1} & M^{(4)}_{1,2} & M^{(4)}_{1,3} & \cdots & M^{(4)}_{1,n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M^{(4)}_{s_2,1} & M^{(4)}_{s_2,2} & M^{(4)}_{s_2,3} & \cdots & M^{(4)}_{s_2,n_1} \\ \cline{2-5} \vdots & & & A_2^{(4)} & \\ M^{(4)}_{k_1,1} & & & & \\ \cline{1-1}\cline{3-5} 0 & & & & \\ \vdots & & & A_1^{(4)} & \\ 0 & & & & \end{array}\right], \tag{34}$$

where $M^{(4)}_{1,1}, \cdots, M^{(4)}_{k_1,1}, M^{(4)}_{1,2}, \cdots, M^{(4)}_{s_2,2}$ are linearly independent, $1 \leq s_2 \leq k_1$, and

$$M_{A_1^{(4)}} \in \mathrm{span}\{M^{(4)}_{1,1}, \cdots, M^{(4)}_{k_1,1}\}. \tag{35}$$

If $M_{A_2^{(4)}} \in \mathrm{span}\{M^{(4)}_{1,1}, \cdots, M^{(4)}_{k_1,1}, M^{(4)}_{1,2}, \cdots, M^{(4)}_{s_2,2}\}$, then $M' \sim M_4$ satisfies (29).

Step 6: If $s_2 < k_1$ and there exists at least one block in $A_2^{(4)}$ that is linearly independent with $M^{(4)}_{1,1}, \cdots, M^{(4)}_{k_1,1}$, $M^{(4)}_{1,2}, \cdots, M^{(4)}_{s_2,2}$ in (34), using Lemma 8 (ii) and repeat **Steps** 5, we obtain that

$$M_4 \sim M_5 = \left[\begin{array}{c|c|ccc} M^{(5)}_{1,1} & M^{(5)}_{1,2} & M^{(5)}_{1,3} & \cdots & M^{(5)}_{1,n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M^{(5)}_{s_2+a,1} & M^{(5)}_{s_2+a,2} & M^{(5)}_{s_2+a,3} & \cdots & M^{(5)}_{s_2+a,n_1} \\ \cline{2-5} \vdots & & & A_2^{(5)} & \\ M^{(5)}_{k_1,1} & & & & \\ \cline{1-1}\cline{3-5} 0 & & & & \\ \vdots & & & A_1^{(5)} & \\ 0 & & & & \end{array}\right], \tag{36}$$

where $M^{(5)}_{1,1}, \cdots, M^{(5)}_{k_1,1}$, $M^{(5)}_{1,2}, \cdots, M^{(5)}_{s_2+a,2}$ are linearly independent, $1 \leq a \leq k_1 - s_2$, and

$$M_{A_1^{(5)}} \in \mathrm{span}\{M^{(5)}_{1,1}, \cdots, M^{(5)}_{k_1,1}\}. \tag{37}$$

If $s_2 + a = k_1$ or $M_{A_2^{(5)}} \in \mathrm{span}\{M_{1,1}^{(5)}, \cdots, M_{k_1,1}^{(5)}, M_{1,2}^{(5)}, \cdots, M_{s_2+a,2}^{(5)}\}$, then $M' \sim M_5$ satisfies (29).

Step 7: If $s_2 + a < k_1$ and there exists at least one block in $M_{A_2^{(5)}}$ that is linearly independent with $M_{1,1}^{(5)}, \cdots, M_{k_1,1}^{(5)}, M_{1,2}^{(5)}, \cdots, M_{s_2+a,2}^{(5)}$, then repeat **Step** 6. Eventually we obtain that $M' \sim M''$ in (29), and M'' has Schmidt rank K. If $k_1 + k_2 = K$, then $M'' \sim N$ with $p = 2$ in (19).

Step 8: If $k_1 + k_2 < K$, then similar to **Steps** 4-7, consider the third block-column of M'' using Lemma 8 (iii). Continue this process until we obtain that $M'' \sim N$.

By achieving (24) and (29), we have shown that $M \sim M' \sim M'' \sim N$ in (19). We have finished the proof. ∎

Theorem 11 presents N in (19) as a canonical form of bipartite matrices under local equivalence of the A system. We further apply three steps on N, namely **Steps** 9-11, to present a canonical form of N under local equivalence of the B system. For this purpose, we denote $*_i$ as a matrix that contains exactly i columns, denote λ_i as a matrix that contains exactly i column vectors which are linearly independent, denote 0_i as a matrix that contains exactly i columns which are all zero column vectors.

Step 9: Consider $N_{1,1}, \cdots, N_{k_1,1}$ in (19), which form a matrix of n_2 columns. If the matrix has rank r_1, then

$$r_1 \le \min\{n_2, r(N)\}. \tag{38}$$

If $r_1 < n_2$, then there exists an $n_2 \times n_2$ invertible matrix R_1 such that

$$\begin{bmatrix} N_{1,1} \\ \vdots \\ N_{k_1,1} \end{bmatrix} \cdot R_1 = \begin{bmatrix} \lambda_{r_1} & 0_{n_2-r_1} \end{bmatrix}, \tag{39}$$

where the leftmost r_1 column vectors are linearly independent, and the rightmost $n_2 - r_1$ column vectors are zero vectors. Recall from (23), using Lemma 9 and (39), we have

$$N_{A_1} \cdot R_1 = \begin{bmatrix} *_{r_1} & 0_{n_2-r_1} \end{bmatrix}. \tag{40}$$

This implies that the rightmost $n_2 - r_1$ column vectors of each block in A_1 are zero vectors. Let

$$N_1 := [N_{i,j}^{(1)}] = N \cdot (I_{n_1} \otimes R_1), \tag{41}$$

from (19), (39) and (40) we obtain that

$$N \sim N_1 = \left[\begin{array}{c|c|c|c|c|c}
N^{(1)}_{1,1} & N^{(1)}_{1,2} & N^{(1)}_{1,3} & \cdots & N^{(1)}_{1,p} & \\
\vdots & \vdots & \vdots & \ddots & \vdots & A_p^{(1)} \\
N^{(1)}_{k_p,1} & N^{(1)}_{k_p,2} & N^{(1)}_{k_p,3} & \cdots & N^{(1)}_{k_p,p} & \\
\vdots & \vdots & \vdots & \vdots & A^{(1)}_{p-1} & \\
\vdots & \vdots & \vdots & & & \\
N^{(1)}_{k_3,1} & N^{(1)}_{k_3,2} & N^{(1)}_{k_3,3} & & \cdots & \\
\vdots & \vdots & & & & \\
N^{(1)}_{k_2,1} & N^{(1)}_{k_2,2} & & & & \\
\vdots & & & A_2^{(1)} & & \\
N^{(1)}_{k_1,1} & & & & & \\
0 & & & & & \\
\vdots & & A_1^{(1)} & & & \\
0 & & & & &
\end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c}
 & N^{(1)}_{1,2} & N^{(1)}_{1,3} & \cdots & N^{(1)}_{1,p} & \\
 & \vdots & \vdots & \ddots & \vdots & A_p^{(1)} \\
 & N^{(1)}_{k_p,2} & N^{(1)}_{k_p,3} & \cdots & N^{(1)}_{k_p,p} & \\
\lambda_{r_1}\ 0_{n_2-r_1} & \vdots & \vdots & \vdots & A_{p-1}^{(1)} & \\
 & \vdots & \vdots & & & \\
 & N^{(1)}_{k_3,2} & N^{(1)}_{k_3,3} & & \cdots & \\
 & \vdots & & & & \\
 & N^{(1)}_{k_2,2} & & & & \\
 & *_{r_1}\ 0_{n_2-r_1} & & A_2^{(1)} & & \\
0_{n_2} & *_{r_1}\ 0_{n_2-r_1} & *_{r_1}\ 0_{n_2-r_1} & \cdots & *_{r_1}\ 0_{n_2-r_1} & \cdots
\end{array}\right], \tag{42}$$

where $N_1 \in \mathbb{M}_{canonical}$ by Lemma 6. If $r_1 = n_2$, then the rightmost $n_2 - r_1$ zero vectors in (39) and (40) disappear and $N_1 = N$.

Step 10: Consider the second block-column of N_1 in (42), we write

$$\begin{bmatrix} N_{1,2}^{(1)} \\ \vdots \\ N_{k_2,2}^{(1)} \end{bmatrix} = \begin{bmatrix} *_{r_1} & *_{n_2-r_1} \end{bmatrix}. \tag{43}$$

For the rightmost $n_2 - r_1$ column vectors in (43). Suppose they form a matrix of rank r_2, then

$$r_2 \le \min\{n_2 - r_1, r(N_1)\}. \tag{44}$$

If $r_1 + r_2 < n_2$, then there exists an $(n_2 - r_1) \times (n_2 - r_1)$ invertible matrix R_2 such that

$$\begin{bmatrix} N_{1,2}^{(1)} \\ \vdots \\ N_{k_2,2}^{(1)} \end{bmatrix} \cdot (I_{r_1} \oplus R_2) = \begin{bmatrix} *_{r_1} & \lambda_{r_2} & 0_{n_2-r_1-r_2} \end{bmatrix}, \tag{45}$$

where the middle r_2 column vectors are linearly independent and the rightmost $n_2 - r_1 - r_2$ column vectors are zero vectors. Further, from (40), we have

$$\begin{aligned} & N_{A_1^{(1)}} \cdot (I_{r_1} \oplus R_2) \\ = & \begin{bmatrix} *_{r_1} & 0_{n_2-r_1} \end{bmatrix} \cdot (I_{r_1} \oplus R_2) = N_{A_1^{(1)}}. \end{aligned} \tag{46}$$

This implies each block in $N_{A_1^{(1)}}$ does not change after right-multiplying $(I_{r_1} \oplus R_2)$. Since $N_{A_2^{(1)}} \in \mathrm{span}\{N_{1,1}^{(1)}, \cdots, N_{k_1,1}^{(1)}, N_{1,2}^{(1)}, \cdots, N_{k_2,2}^{(1)}\}$, using Lemma 9 and (45), we have

$$N_{A_2^{(1)}} \cdot (I_{r_1} \oplus R_2) = \begin{bmatrix} *_{r_1+r_2} & 0_{n_2-r_1-r_2} \end{bmatrix}, \tag{47}$$

where the rightmost $n_2 - r_1 - r_2$ column vectors of each block in $A_2^{(1)}$ are zero vectors. Let

$$N_2 := [N_{i,j}^{(2)}] = N_1 \cdot [I_{n_1} \otimes (I_{r_1} \oplus R_2)]. \tag{48}$$

From (42), (45), (46) and (47), we have in (49), shown at the bottom of the next page, where $N_2 \in \mathbb{M}_{canonical}$ by

Lemma 6. Note that if $r_1 + r_2 = n_2$, then the rightmost $n_2 - r_1 - r_2$ column vectors in (45) and (47) disappear and $N_2 = N_1$.

Step 11: Consider $N^{(2)}_{1,3}, \cdots, N^{(2)}_{k_3,3}$ in (49) using the same way in **Steps** 9 and 10. Continuing this process, finally we obtain the following theorem.

Theorem 12: Any $N \in \mathbb{M}_{canonical}$ in (19) is locally equivalent to $N_p := [N^{(p)}_{i,j}] \in \mathbb{M}_{canonical}$. N_p is written in (50), shown at the bottom of the page, where $r_1 + \cdots + r_p \le n_2$, and the rightmost $n_2 - (r_1 + \cdots + r_p)$ column vectors of each block in $A^{(p)}_p$, if exist, are zero vectors.

Note that if $r_1 + \cdots + r_i = n_2$ for $1 \le i < p$, then $N \sim N_i = N_{i+1} = \cdots = N_p$. Hence we have presented another locally equivalent form of M in (18).

B. Proof of Theorem 3

The proof proceeds by induction on the Schmidt rank K of M. Denote $*^T_i$ as a matrix that contains exactly i rows, 0^T_i as a matrix that contains exactly i zero rows. Denote $0^{\Gamma_B}_i$ as the partial transpose of the B system of a bipartite matrix 0_i.

First, it is clear that Theorem 3 holds for any M of Schmidt rank one. Next, suppose Theorem 3 holds for any matrix of Schmidt rank at most $K-1$, with $K \ge 2$. We will prove that Theorem 3 holds for any M of Schmidt rank K. In subsections III-A, we have shown that $M \sim N \sim N_p$. Further, it has been proved by Lemma 7 that N in (19) satisfies Theorem 3 if $k_1 = K$. So from (21) and (22), we assume that

$$1 \le k_s < K \tag{51}$$

holds for any $1 \le s \le p$ in N in (19) and N_p in (50).

We next decompose N_p in (50) into the sum of p bipartite matrices. Firstly, let $N_{q_1} := [N^{(q_1)}_{i,j}] \in \mathbb{M}_{m_1,n_1} \otimes \mathbb{M}_{m_2,n_2}$, where the leftmost r_1 columns of $N^{(q_1)}_{i,j}$ in N_{q_1} are exactly the leftmost r_1 columns of $N^{(p)}_{i,j}$ in N_p in (50), the remaining $n_2 - r_1$ column vectors of $N^{(q_1)}_{i,j}$ in N_{q_1} are zero vectors. From (50), we obtain that

$$N_{q_1} = \left[\begin{array}{c|c|c|c} \lambda_{r_1}\ 0_{n_2-r_1} & *_{r_1}\ 0_{n_2-r_1} & \cdots & *_{r_1}\ 0_{n_2-r_1} \\ \hline 0_{n_2} & *_{r_1}\ 0_{n_2-r_1} & \cdots & *_{r_1}\ 0_{n_2-r_1} \end{array}\right]$$

$$N_1 \sim N_2 = \left[\begin{array}{c|c|c|c|c|c|c} \lambda_{r_1}\ 0_{n_2-r_1} & *_{r_1}\ \lambda_{r_2}\ 0_{n_2-r_1-r_2} & \begin{array}{c} N^{(2)}_{1,3} \\ \vdots \\ N^{(2)}_{k_p,3} \\ \vdots \\ \vdots \\ N^{(2)}_{k_4,3} \\ \vdots \\ N^{(2)}_{k_3,3} \end{array} & \begin{array}{c} N^{(2)}_{1,4} \\ \vdots \\ N^{(2)}_{k_p,4} \\ \vdots \\ \vdots \\ N^{(2)}_{k_4,4} \end{array} & \begin{array}{c} \cdots \\ \ddots \\ \cdots \\ \vdots \end{array} & \begin{array}{c} N^{(2)}_{1,p} \\ \vdots \\ N^{(2)}_{k_p,p} \\ \hline A^{(2)}_{p-1} \\ \hline \cdots \end{array} & A^{(2)}_p \\ & & *_{r_1+r_2}\ 0_{n_2-r_1-r_2} & \multicolumn{4}{c}{A^{(2)}_3} \\ \cline{2-7} & *_{r_1}\ 0_{n_2-r_1} & *_{r_1+r_2}\ 0_{n_2-r_1-r_2} & \cdots & \cdots & *_{r_1+r_2}\ 0_{n_2-r_1-r_2} & \cdots \\ \hline 0_{n_2} & *_{r_1}\ 0_{n_2-r_1} & *_{r_1}\ 0_{n_2-r_1} & \cdots & \cdots & *_{r_1}\ 0_{n_2-r_1} & \cdots \end{array}\right], \tag{49}$$

$$N_p = \left[\begin{array}{c|c|c|c|c|c} \lambda_{r_1}\ 0_{n_2-r_1} & *_{r_1}\ \lambda_{r_2}\ 0_{n_2-r_1-r_2} & *_{r_1+r_2}\ \lambda_{r_3}\ 0_{n_2-r_1-r_2-r_3} & \begin{array}{c} \cdots \\ \ddots \\ \cdots \\ \vdots \end{array} & \begin{array}{c} *_{r_1+\cdots+r_{p-1}} \lambda_{r_p} 0_{n_2-r_1-\cdots-r_p} \\ \hline *_{r_1+\cdots+r_{p-1}} 0_{n_2-r_1-\cdots r_{p-1}} \end{array} & A^{(p)}_p \\ & & & & *_{r_1+\cdots+r_{p-1}} 0_{n_2-r_1-\cdots r_{p-1}} & \cdots \\ & & *_{r_1+r_2}\ 0_{n_2-r_1-r_2} & \multicolumn{3}{c}{\cdots} \\ & *_{r_1}\ 0_{n_2-r_1} & *_{r_1+r_2}\ 0_{n_2-r_1-r_2} & \cdots & *_{r_1+r_2}\ 0_{n_2-r_1-r_2} & \cdots \\ \hline 0_{n_2} & *_{r_1}\ 0_{n_2-r_1} & *_{r_1}\ 0_{n_2-r_1} & \cdots & *_{r_1}\ 0_{n_2-r_1} & \cdots \end{array}\right], \tag{50}$$

$$:= \left[\begin{array}{c|ccc} N_{1,1}^{(q_1)} & N_{1,2}^{(q_1)} & \cdots & N_{1,n_1}^{(q_1)} \\ \vdots & \vdots & \ddots & \vdots \\ N_{k_1,1}^{(q_1)} & N_{k_1,2}^{(q_1)} & \cdots & N_{k_1,n_1}^{(q_1)} \\ \hline 0_{n_2} & & \omega_1^{(q_1)} & \end{array}\right], \tag{52}$$

where $\omega_1^{(q_1)}$ is an $(m_1 - k_1) \times (n_1 - 1)$ rectangular block matrix, and

$$N_{\omega_1^{(q_1)}} \in \operatorname{span}\{N_{1,1}^{(q_1)}, \cdots, N_{k_1,1}^{(q_1)}\}. \tag{53}$$

Thus by (51), we have

$$Sr(\omega_1^{(q_1)}) \leq k_1 < K, \tag{54}$$

i.e., the Schmidt rank of $\omega_1^{(q_1)}$ is strictly less than K. Further, from (52), recall the definition of λ_i, using Lemma 5, we have

$$r(\begin{bmatrix} N_{1,1}^{(q_1)} \\ \vdots \\ N_{k_1,1}^{(q_1)} \end{bmatrix}) + r(\omega_1^{(q_1)}) = r_1 + r(\omega_1^{(q_1)}) \leq r(N_{q_1}). \tag{55}$$

Secondly, let $N_{q_2} := [N_{i,j}^{(q_2)}] \in \mathbb{M}_{m_1,n_1} \otimes \mathbb{M}_{m_2,n_2}$, where the k-th column of $N_{i,j}^{(q_2)}$ in N_{q_2} is exactly the k-th column of $N_{i,j}^{(p)}$ in N_p in (50), $r_1 + 1 \leq k \leq r_1 + r_2$. At the same time, the remaining $n_2 - r_2$ column vectors of $N_{i,j}^{(q_2)}$ in N_{q_2} are zero vectors. From (50), we have

$$N_{q_2} = \left[\begin{array}{c|c|c|c|c} & 0_{r_1}\lambda_{r_2}0_{n_2-r_1-r_2} & 0_{r_1} *_{r_2} 0_{n_2-r_1-r_2} & \cdots & 0_{r_1} *_{r_2} 0_{n_2-r_1-r_2} \\ \hline 0_{n_2} & 0_{n_2} & 0_{r_1} *_{r_2} 0_{n_2-r_1-r_2} & \cdots & 0_{r_1} *_{r_2} 0_{n_2-r_1-r_2} \\ \hline & 0_{n_2} & 0_{n_2} & \cdots & 0_{n_2} \end{array}\right]$$

$$:= \left[\begin{array}{c|c|ccc} & N_{1,2}^{(q_2)} & N_{1,3}^{(q_2)} & \cdots & N_{1,n_1}^{(q_2)} \\ & \vdots & \vdots & \ddots & \vdots \\ 0_{n_2} & N_{k_2,2}^{(q_2)} & N_{k_2,3}^{(q_2)} & \cdots & N_{k_2,n_1}^{(q_2)} \\ \cline{2-5} & 0_{n_2} & & \omega_2^{(q_2)} & \\ \cline{2-5} & & & 0_{(n_1-1)n_2} & \end{array}\right], \tag{56}$$

where $\omega_2^{(q_2)}$ is a $(k_1 - k_2) \times (n_1 - 2)$ rectangular block matrix, and

$$N_{\omega_2^{(q_2)}} \in \operatorname{span}\{N_{1,1}^{(q_2)}, \cdots, N_{k_1,1}^{(q_2)}, N_{1,2}^{(q_2)}, \cdots, N_{k_2,2}^{(q_2)}\}. \tag{57}$$

Note that $N_{1,1}^{(q_2)}, \cdots, N_{k_1,1}^{(q_2)}$ are zero matrices, therefore

$$N_{\omega_2^{(q_2)}} \in \operatorname{span}\{N_{1,2}^{(q_2)}, \cdots, N_{k_2,2}^{(q_2)}\}, \tag{58}$$

and by (51), we have

$$Sr(\omega_2^{(q_2)}) \leq k_2 < K. \tag{59}$$

At the same time, using Lemma 5 and (56), we have

$$r(\begin{bmatrix} N_{1,2}^{(q_2)} \\ \vdots \\ N_{k_2,2}^{(q_2)} \end{bmatrix}) + r(\omega_2^{(q_2)}) = r_2 + r(\omega_2^{(q_2)}) \leq r(N_{q_2}). \tag{60}$$

Similarly, let $N_{q_s} := [N_{i,j}^{(q_s)}] \in \mathbb{M}_{m_1,n_1} \otimes \mathbb{M}_{m_2,n_2}$, where $3 \leq s \leq p$. For each N_{q_s}, the k-th column of $N_{i,j}^{(q_s)}$ is exactly the k-th column of $N_{i,j}^{(p)}$ in N_p in (50), $r_1 + \cdots + r_{s-1} + 1 \leq k \leq r_1 + \cdots + r_s$, and the remaining column vectors of $N_{i,j}^{(q_s)}$ are zero vectors. We obtain that in (61), shown at the bottom of the page, where $\omega_s^{(q_s)}$ is a $(k_{s-1} - k_s) \times (n_1 - s)$ rectangular block matrix, and

$$N_{\omega_s^{(q_s)}} \in \operatorname{span}\{N_{1,s}^{(q_s)}, \cdots, N_{k_s,s}^{(q_s)}\}, \tag{62}$$

$$r(\begin{bmatrix} N_{1,s}^{(q_s)} \\ \vdots \\ N_{k_s,s}^{(q_s)} \end{bmatrix}) + r(\omega_s^{(q_s)}) = r_s + r(\omega_s^{(q_s)}) \leq r(N_{q_s}), \tag{63}$$

$$Sr(\omega_s^{(q_s)}) \leq k_s < K, \tag{64}$$

hold for any $3 \leq s \leq p$. On the other hand, from the construction of each N_{q_s}, we have decomposed N_p in (50) into the sum of $N_{q_1}, \cdots, N_{q_p}$, i.e.,

$$N_p = N_{q_1} + N_{q_2} + \cdots + N_{q_p}. \tag{65}$$

Using Lemma 5, we have

$$r(N_{q_s}) \leq r(N_p) \tag{66}$$

holds for any $1 \leq s \leq p$. Note that if $r_1 + \cdots + r_i = n_2$ for $1 \leq i < p$ in N_p, then $N_{q_{i+1}}, \cdots, N_{q_p}$ disappear. We next consider the partial transpose of system B of each N_{q_s}. For

$$N_{q_s} = \left[\begin{array}{c|c|c|c|c} & 0_{\Sigma_{i=1}^{s-1} r_i}\lambda_{r_s}0_{n_2-\Sigma_{i=1}^{s} r_i} & 0_{\Sigma_{i=1}^{s-1} r_i} *_{r_s} 0_{n_2-\Sigma_{i=1}^{s} r_i} & \cdots & 0_{\Sigma_{i=1}^{s-1} r_i} *_{r_s} 0_{n_2-\Sigma_{i=1}^{s} r_i} \\ \hline 0_{(s-1)n_2} & 0_{n_2} & 0_{\Sigma_{i=1}^{s-1} r_i} *_{r_s} 0_{n_2-\Sigma_{i=1}^{s} r_i} & \cdots & 0_{\Sigma_{i=1}^{s-1} r_i} *_{r_s} 0_{n_2-\Sigma_{i=1}^{s} r_i} \\ \hline & 0_{n_2} & 0_{n_2} & \cdots & 0_{n_2} \end{array}\right] := \left[\begin{array}{c|c|ccc} & N_{1,s}^{(q_s)} & N_{1,s+1}^{(q_s)} & \cdots & N_{1,n_1}^{(q_s)} \\ & \vdots & \vdots & \vdots & \vdots \\ 0_{(s-1)n_2} & N_{k_s,s}^{(q_s)} & N_{k_s,s+1}^{(q_s)} & \cdots & N_{k_s,n_1}^{(q_s)} \\ \cline{2-5} & 0_{n_2} & & \omega_s^{(q_s)} & \\ \cline{2-5} & & & 0_{(n_1-s+1)n_2} & \end{array}\right], \tag{61}$$

$s=1$, from (52), we have

$$N_{q_1}^{\Gamma_B}=\left[\begin{array}{c|c} \begin{matrix} {N_{1,1}^{(q_1)}}^T \\ \vdots \\ {N_{k_1,1}^{(q_1)}}^T \end{matrix} & \begin{matrix} {N_{1,2}^{(q_1)}}^T & \cdots & {N_{1,n_1}^{(q_1)}}^T \\ \vdots & \ddots & \vdots \\ {N_{k_1,2}^{(q_1)}}^T & \cdots & {N_{k_1,n_1}^{(q_1)}}^T \end{matrix} \\ \hline 0_{n_2}^{\Gamma_B} & [\omega_1^{(q_1)}]^{\Gamma_B} \end{array}\right]$$

$$=\left[\begin{array}{c|c} \multicolumn{2}{c}{\begin{matrix} *_{r_1}^T & *_{r_1}^T & \cdots & *_{r_1}^T \\ 0_{n_2-r_1}^T & 0_{n_2-r_1}^T & & 0_{n_2-r_1}^T \end{matrix}} \\ \hline \multicolumn{2}{c}{\begin{matrix} \vdots & \vdots & \ddots & \vdots \end{matrix}} \\ \hline \multicolumn{2}{c}{\begin{matrix} *_{r_1}^T & *_{r_1}^T & \cdots & *_{r_1}^T \\ 0_{n_2-r_1}^T & 0_{n_2-r_1}^T & & 0_{n_2-r_1}^T \end{matrix}} \\ \hline 0_{n_2}^{\Gamma_B} & [\omega_1^{(q_1)}]^{\Gamma_B} \end{array}\right]. \tag{67}$$

Using Lemma 5 and (67), we have

$$r(N_{q_1}^{\Gamma_B})\leq k_1\cdot r_1+r(\left[\omega_1^{(q_1)}\right]^{\Gamma_B}). \tag{68}$$

Similar to N_{q_1}, from (61), for any $1\leq s\leq p$, one can show that

$$N_{q_s}^{\Gamma_B}=\left[\begin{array}{c|c} 0^{\Gamma_B}_{(s-1)n_2} & \begin{array}{c|c} \begin{matrix} {N_{1,s}^{(q_s)}}^T \\ \vdots \\ {N_{k_s,s}^{(q_s)}}^T \end{matrix} & \begin{matrix} {N_{1,s+1}^{(q_s)}}^T & \cdots & {N_{1,n_1}^{(q_s)}}^T \\ \vdots & \ddots & \vdots \\ {N_{k_s,s+1}^{(q_s)}}^T & \cdots & {N_{k_s,n_1}^{(q_s)}}^T \end{matrix} \\ \hline 0_{n_2}^{\Gamma_B} & [\omega_s^{(q_s)}]^{\Gamma_B} \\ \hline \multicolumn{2}{c}{0^{\Gamma_B}_{(n_1-s_1+1)n_2}} \end{array} \end{array}\right]$$

$$=\left[\begin{array}{c|c} 0_{n_2}^T & \begin{matrix} 0^T_{\sum_{i=1}^{s-1} r_i} & 0^T_{\sum_{i=1}^{s-1} r_i} & & 0^T_{\sum_{i=1}^{s-1} r_i} \\ *_{r_s}^T & *_{r_s}^T & \cdots & *_{r_s}^T \\ 0^T_{n_2-\sum_{i=1}^{s} r_i} & 0^T_{n_2-\sum_{i=1}^{s} r_i} & & 0^T_{n_2-\sum_{i=1}^{s} r_i} \end{matrix} \\ \hline \vdots & \begin{matrix} \vdots & \vdots & \ddots & \vdots \end{matrix} \\ \hline 0_{n_2}^T & \begin{matrix} 0^T_{\sum_{i=1}^{s-1} r_i} & 0^T_{\sum_{i=1}^{s-1} r_i} & & 0^T_{\sum_{i=1}^{s-1} r_i} \\ *_{r_s}^T & *_{r_s}^T & \cdots & *_{r_s}^T \\ 0^T_{n_2-\sum_{i=1}^{s} r_i} & 0^T_{n_2-\sum_{i=1}^{s} r_i} & & 0^T_{n_2-\sum_{i=1}^{s} r_i} \end{matrix} \\ \hline 0^{\Gamma_B}_{(s-1)n_2} & \begin{array}{c|c} 0_{n_2}^{\Gamma_B} & [\omega_s^{(q_s)}]^{\Gamma_B} \\ \hline \multicolumn{2}{c}{0^{\Gamma_B}_{(n_1-s_1+1)n_2}} \end{array} \end{array}\right], \tag{69}$$

and hence

$$r(N_{q_s}^{\Gamma_B})\leq k_s\cdot r_s+r([\omega_s^{(q_s)}]^{\Gamma_B}). \tag{70}$$

From (64), we obtain that the Schmidt rank of $\omega_s^{(q_s)}$ is at most $K-1$, where $\omega_s^{(q_s)}$ is from (61). Recalling the assumption of induction that Theorem 3 holds for any matrix of Schmidt rank at most $K-1$, hence for any $1\leq s\leq p$, we have

$$r([\omega_s^{(q_s)}]^{\Gamma_B})\leq Sr(\omega_s^{(q_s)})\cdot r(\omega_s^{(q_s)}). \tag{71}$$

Further, from (63), (64), (70) and (71) we obtain that

$$\begin{aligned} r(N_{q_s}^{\Gamma_B}) &\leq k_s\cdot r_s+Sr(\omega_s^{(q_s)})\cdot r(\omega_s^{(q_s)}) \\ &\leq k_s\cdot(r_s+r(\omega_s^{(q_s)})) \\ &\leq k_s\cdot r(N_{q_s}) \end{aligned} \tag{72}$$

holds for any $1\leq s\leq p$. On the other hand, from (65), we have

$$N_p^{\Gamma_B}=N_{q_1}^{\Gamma_B}+N_{q_2}^{\Gamma_B}+\cdots+N_{q_p}^{\Gamma_B}, \tag{73}$$

and by Lemma 5,

$$r(N_p^{\Gamma_B})\leq r(N_{q_1}^{\Gamma_B})+r(N_{q_2}^{\Gamma_B})+\cdots+r(N_{q_p}^{\Gamma_B}). \tag{74}$$

From (22), (66), (72) and (74), we have

$$\begin{aligned} r(N_p^{\Gamma_B}) &\leq k_1\cdot r(N_{q_1})+k_2\cdot r(N_{q_2})+\cdots+k_p\cdot r(N_{q_p}) \\ &\leq k_1\cdot r(N_p)+k_2\cdot r(N_p)+\cdots+k_p\cdot r(N_p) \\ &=(k_1+k_2+\cdots+k_p)\cdot r(N_p) \\ &=K\cdot r(N_p). \end{aligned} \tag{75}$$

This implies Theorem 3 holds for N_p in (50). Recall that M in (18) is locally equivalent to N_p, hence we prove that Theorem 3 also holds for M. This completes the proof.

IV. Application

In this section, we apply and extend Theorems 2 and 3. In subsection IV-A, we extend Theorem 2 and Theorem 3 to multipartite systems in Lemma 13 and Lemma 16 respectively. On the other hand, given a multipartite pure quantum state, the Schimidt measure, employing the tensor rank of the state, is a tool for quantifying entanglement [29]. In Theorem 17, we show a link between the Schmidt measure and the 0-entropies of some reduced density matrices of an even-partite pure state. In subsection IV-B, we partially discuss the condition when the inequality $r(\rho_{AB})\cdot r(\rho_{AC})\geq r(\rho_{BC})$ is saturated in Lemma 18. Specially, we give a general expression of tripatite PPT states that satisfy the condition in Proposition 20.

A. Applying and Extending Theorems 2 and 3 to Multipartite Cases

Firstly, we know that for a tripartite mixed state ρ_{ABC}, there are four inequalities in terms of the 0-entropy of ρ_A:

$$S_0(A)\geq 0, \tag{76}$$
$$S_0(A)+S_0(B)\geq S_0(AB), \tag{77}$$
$$S_0(AB)+S_0(AC)\geq S_0(BC), \tag{78}$$
$$S_0(AB)+S_0(AC)+S_0(BC)\geq 2S_0(A). \tag{79}$$

In particular, (77) is (8) of [7], (79) is Theorem 2 of [7], and (78) is Theorem 2. Note that the inequality

$$S_0(AB)+S_0(AC)\geq S_0(A) \tag{80}$$

obtained in [7] is a corollary of (78) and (79). So we have established a complete picture of the four-party linear inequalities in terms of the 0-entropy.

Next, we extend the inequalities (77)-(79) to multipartite quantum systems as follows.

Lemma 13: Given an n-partite mixed state of systems $A_1, \ldots, A_n$ with $n \geq 3$, we have

$$\sum_{j=1}^{k} S_0(A_j) \geq S_0(A_1 \ldots A_k), \tag{81}$$

$$\sum_{j=1}^{k-1} S_0(A_j A_{j+1}) \geq S_0(A_1 A_k) \tag{82}$$

hold for any $2 \leq k \leq n$, and

$$\sum_{j=1}^{k-1} S_0(A_j A_{j+1}) + S_0(A_k A_1) \geq 2S_0(A_1) \tag{83}$$

holds for any $3 \leq k \leq n$.

Proof: Firstly, (77) implies (81) for $k = 2$. Suppose (81) holds for $k-1$, then

$$\begin{aligned}\sum_{j=1}^{k} S_0(A_j) &= \sum_{j=1}^{k-1} S_0(A_j) + S_0(A_k) \\ &\geq S_0(A_1 \ldots A_{k-1}) + S_0(A_k) \\ &\geq S_0(A_1 \ldots A_k),\end{aligned} \tag{84}$$

where the last inequality follows from (77). Hence we have proved by induction that (81) holds.

Next, it is clear that (82) holds for $k = 2$. From (78), we obtain that

$$S_0(A_1 A_2) + S_0(A_2 A_3) \geq S_0(A_1 A_3) \tag{85}$$

holds by permuting the systems A_1 and A_2. Suppose (82) holds for $k-2$. Then

$$\begin{aligned}\sum_{j=1}^{k-1} S_0(A_j A_{j+1}) &= \sum_{j=1}^{k-2} S_0(A_j A_{j+1}) + S_0(A_{k-1} A_k) \\ &\geq S_0(A_1 A_{k-1}) + S_0(A_{k-1} A_k) \\ &\geq S_0(A_1 A_k),\end{aligned} \tag{86}$$

where the last inequality follows from (85). We have proved by induction that (82) holds.

Third, we have

$$\begin{aligned}&\sum_{j=1}^{k-1} S_0(A_j A_{j+1}) + S_0(A_k A_1) \\ =&\sum_{j=1}^{k-2} S_0(A_j A_{j+1}) + S_0(A_{k-1} A_k) + S_0(A_k A_1) \\ \geq& S_0(A_1 A_{k-1}) + S_0(A_{k-1} A_k) + S_0(A_k A_1) \\ \geq& 2S_0(A_1),\end{aligned} \tag{87}$$

where the first inequality follows from (82) and the last inequality follows from (79). Hence (83) holds. This completes the proof. ■

Now we apply the above results to obtain more inequalities.

Corollary 14: Given an n-partite mixed state of systems $A_1, \ldots, A_n$ with $n \geq 3$, we have

$$\sum_{j=1}^{k-1} S_0(A_j A_{j+1}) \geq S_0(A_i), \tag{88}$$

$$\sum_{j=1}^{k} S_0(A_1 \cdots A_{j-1} A_{j+1} \cdots A_k) \geq S_0(A_i), \tag{89}$$

$$\begin{aligned}&\sum_{j=1}^{k} S_0(A_1 \cdots A_{j-1} A_{j+1} \cdots A_k) \\ &\geq 2S_0(A_1 \cdots A_{i-1} A_{i+1} \cdots A_k)\end{aligned} \tag{90}$$

hold any for $3 \leq k \leq n$ and any $1 \leq i \leq k$.

Proof: By permuting the systems, we only need to prove (88)-(90) for $i = 1$. Firstly, (82) and (83) imply that (88) holds for $i = 1$. Next, one can show that

$$\begin{aligned}&2\sum_{j=1}^{k} S_0(A_1 \cdots A_{j-1} A_{j+1} \cdots A_k) \\ \geq&\sum_{j=1}^{k-1} S_0(A_j A_{j+1}) + S_0(A_k A_1) \geq 2S_0(A_1),\end{aligned} \tag{91}$$

where the first inequality follows from (78) and the second inequality follows from (83).

Third, we prove (90). If k is odd, then

$$\begin{aligned}&\sum_{j=2}^{k} S_0(A_1 \cdots A_{j-1} A_{j+1} \cdots A_k) \\ \geq& S_0(A_2 A_3) + \cdots + S_0(A_{k-1} A_k) \\ \geq& S_0(A_2 \cdots A_k),\end{aligned} \tag{92}$$

where the first inequality follows from (78) and the second inequality follows from (81). If $k = 4$, then

$$\begin{aligned}&S_0(A_1 A_3 A_4) + S_0(A_1 A_2 A_4) + S_0(A_1 A_2 A_3) \\ \geq& S_0(A_1 A_3 A_4) + S_0(A_1 A_2) \geq S_0(A_2 A_3 A_4).\end{aligned} \tag{93}$$

If $k > 4$ is even, then

$$\begin{aligned}&\sum_{j=2}^{k} S_0(A_1 \cdots A_{j-1} A_{j+1} \cdots A_k) \\ =&\sum_{j=2}^{4} S_0(A_1 \cdots A_{j-1} A_{j+1} \cdots A_k) \\ +&\sum_{j=5}^{k} S_0(A_1 \cdots A_{j-1} A_{j+1} \cdots A_k) \\ \geq& S_0(A_2 A_3 A_4) + S_0(A_5 A_6) + \cdots S_0(A_{k-1} A_k) \\ \geq& S_0(A_2 \cdots A_k).\end{aligned} \tag{94}$$

(92), (93) and (94) imply that (90) holds for $i = 1$. This completes the proof. ■

We have constructed a few novel inequalities every multipartite mixed state must satisfy. They shed new light to the marginal problem for multipartite systems, as well as the understanding of von Neumann entropy.

Third, we extend Theorem 3 to multipartite matrices. For this purpose, we refer to $\Gamma_{12\ldots m}$ as the partial transpose of systems $A_1, A_2, \ldots, A_m$. Next, we refer to the Schmidt rank $Sr(M)$ of an n-partite matrix M on the n-partite Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$ as the smallest integer d such that $M = \sum_{j=1}^{d} B_{j,1} \otimes B_{j,2} \otimes \ldots \otimes B_{j,n}$. Further, we refer to the Schmidt rank s of M over the system bipartition $A_{j_1}, \ldots, A_{j_m} : A_{j_{m+1}}, \ldots, A_{j_n}$, by rewriting

$$M = \sum_{k=1}^{s} C_k \otimes D_k, \tag{95}$$

where C_k's are linearly independent matrices on the systems $A_{j_1}, \ldots, A_{j_m}$, and D_k's are linearly independent matrices on the systems $A_{j_{m+1}}, \ldots, A_{j_n}$. Evidently $Sr(M) \geq s$. Now we propose the following observation.

Lemma 15: Suppose M is an n-partite matrix, and L is the largest Schmidt rank over all system bipartition of M. Then for any subset $\mathcal{S}$ of integers $j_1, j_2, \ldots, j_m \in \{1, \cdots, n\}$, we have

$$r(M^{\Gamma_{\mathcal{S}}}) \leq L \cdot r(M). \tag{96}$$

Proof: Using (95), we obtain that $L \geq s$. It follows from Theorem 3 that $r(M^{\Gamma_{\mathcal{S}}}) \leq s \cdot r(M) \leq L \cdot r(M)$. So the assertion holds. ∎

If $n = 2$ then Lemma 15 is exactly Theorem 3. Further, the lemma evidently holds when L is replaced by the Schmidt rank of M. That is

Lemma 16: Suppose M is an n-partite matrix. Then for any subset $\mathcal{S}$ of integers $j_1, j_2, \ldots, j_m \in \{1, \cdots, n\}$, we have

$$r(M^{\Gamma_{\mathcal{S}}}) \leq Sr(M) \cdot r(M). \tag{97}$$

Fourth, we apply Lemma 16 to multipartite quantum states. For an n-partite pure quantum state $|\psi\rangle$ on the Hilbert space $\mathcal{H}_{d_1} \otimes \cdots \otimes \mathcal{H}_{d_n}$, the tensor rank of $|\psi\rangle$ is the smallest integer d, such that

$$|\psi\rangle = \sum_{k=1}^{d} |\psi_{A_1}^{(k)}\rangle \otimes \cdots \otimes |\psi_{A_n}^{(k)}\rangle, \tag{98}$$

where $|\psi_{A_j}^{(k)}\rangle \in \mathcal{H}_{d_j}$, $j = 1, \cdots, n$. Next, in [29], the Schmidt measure is defined as $\mathcal{P}(|\psi\rangle\langle\psi|) = \log_2 d$, which can be used to quantify the degree of entanglement of $|\psi\rangle$. The following theorem shows that, given a $2n$-partite pure state, the Schmidt measure is also an upper bound of the difference between the 0-entropies of some reduced density matrices.

Theorem 17: Suppose $|\psi\rangle$ is a $2n$-partite pure state of systems $A_1 \cdots A_{2n}$ with dimensions $d_1, \cdots, d_{2n}$ respectively. Then

$$S_0(X_1 X_2 \cdots X_n) - S_0(Y_1 Y_2 \cdots Y_n) \leq \mathcal{P}(|\psi\rangle\langle\psi|), \tag{99}$$

where X_j (Y_j) is one of A_{2j-1} and A_{2j}, $1 \leq j \leq n$.

Proof: Our aim is to prove (99) holds for the situation that $Y_1 Y_2 \cdots Y_n = A_1 A_3 \cdots A_{2n-1}$, and X_j is choosen from A_{2j-1} and A_{2j} arbitrarily. If this is true, then (99) holds for any other situations by permuting the systems of $|\psi\rangle$.

From (98), we write

$$|\psi\rangle = \sum_{k=1}^{d} |\psi_{A_1}^{(k)}\rangle \otimes \cdots \otimes |\psi_{A_{2n}}^{(k)}\rangle \tag{100}$$

where d is the tensor rank of $|\psi\rangle$. Next, for any $j \in \{1, 3, \cdots, 2n-1\}$, the following identities

$$\begin{aligned} |\psi_{A_j}^{(k)}\rangle \otimes |\psi_{A_{j+1}}^{(k)}\rangle &= (I_{d_j} \otimes M_j^{(k)}) \cdot \sum_{i=1}^{d_j} |ii\rangle \\ &= \{[M_j^{(k)}]^T \otimes I_{d_{j+1}}\} \cdot \sum_{i=1}^{d_{j+1}} |ii\rangle \end{aligned} \tag{101}$$

hold, where $M_j^{(k)}$ are $d_{j+1} \times d_j$ matrices. Taking the first equality of (101) with all $j \in \{1, 3, \cdots, 2n-1\}$ to (100), we have

$$\begin{aligned} |\psi\rangle = \sum_{k=1}^{d} &(I_{d_1} \otimes M_1^{(k)} \otimes I_{d_3} \otimes M_3^{(k)} \otimes \cdots \otimes I_{d_{2n-1}} \\ &\otimes M_{2n-1}^{(k)}) \cdot (\sum_{i=1}^{d_1} |ii\rangle \otimes \sum_{i=1}^{d_3} |ii\rangle \otimes \cdots \otimes \sum_{i=1}^{d_{2n-1}} |ii\rangle). \end{aligned} \tag{102}$$

Let $M = \sum_{k=1}^{d} M_1^{(k)} \otimes M_3^{(k)} \otimes \cdots \otimes M_{2n-1}^{(k)}$, then M is an n-partite matrix with

$$Sr(M) \leq d. \tag{103}$$

Let $\rho = |\psi\rangle\langle\psi|$, from (102), it is clear that

$$r(\rho_{A_1 A_3 \cdots A_{2n-1}}) = r(M). \tag{104}$$

Further, given any reduced density matrix $\rho_{X_1 X_2 \cdots X_n}$ where X_j is one of A_{2j-1} and A_{2j}, $1 \leq j \leq n$, define the integer set $\mathcal{S} = \{s | X_s = A_{2s}, 1 \leq s \leq n\}$. For any $s \in \mathcal{S}$, take the second equality of (101) with $j = 2s - 1$ to (102), we obtain that

$$\begin{aligned} r(\rho_{X_1 X_2 \cdots X_n}) &= r(M^{\Gamma_{\mathcal{S}}}) \leq Sr(M) \cdot r(M) \\ &\leq d \cdot r(M) = d \cdot r(\rho_{A_1 A_3 \cdots A_{2n-1}}), \end{aligned} \tag{105}$$

where the first inequality follows from Lemma 16, the second inequality follows from (103) and the last equality follows from (104). By taking the logarithm of base two in (105), we have finished the proof. ∎

B. The Condition When the Inequality in Theorem 2 Is Saturated

For the inequality $r(\rho_{AB}) \cdot r(\rho_{AC}) \geq r(\rho_{BC})$, what is the condition by which $r(\rho_{AB}) \cdot r(\rho_{AC}) = r(\rho_{BC})$? We partially answer the problem as follows.

Lemma 18: (i) Suppose ρ_{ABC} is a tripartite pure state. Then the condition is $r(\rho_B) \cdot r(\rho_C) = r(\rho_A)$.

(ii) Suppose $\rho_{ABC} = \rho_A \otimes \rho_{BC}$ is a tripartite mixed state. Then the condition is $r(\rho_A) = 1$ and $r(\rho_{BC}) = r(\rho_B) \cdot r(\rho_C)$.

(iii) Suppose $\rho_{ABC} = \rho_B \otimes \rho_{AC}$ is a tripartite mixed state. Then the condition is $r(\rho_A) \cdot r(\rho_{AC}) = r(\rho_C)$.

(iv) Suppose $\rho_{ABC} = \rho_C \otimes \rho_{AB}$ is a tripartite mixed state. Then the condition is $r(\rho_A) \cdot r(\rho_{AB}) = r(\rho_B)$.

(v) Suppose ρ_{ABC} is a tripartite PPT state. Then the condition is $r(\rho_{AB}) = r(\rho_B)$, $r(\rho_{AC}) = r(\rho_C)$ and $r(\rho_B) \cdot r(\rho_C) = r(\rho_{BC})$.

Proof: (i) The assertion follows from the assumption of tripartite pure states.

(ii) We have $r(\rho_A)^2 \cdot r(\rho_B) \cdot r(\rho_C) = r(\rho_{AB}) \cdot r(\rho_{AC}) = r(\rho_{BC}) \le r(\rho_B) \cdot r(\rho_C)$. So assertion (ii) holds.

(iii) The condition $r(\rho_{AB}) \cdot r(\rho_{AC}) = r(\rho_{BC})$ is equivalent to $r(\rho_A \otimes \rho_B) \cdot r(\rho_{AC}) = r(\rho_B \otimes \rho_C)$, namely $r(\rho_A) \cdot r(\rho_{AC}) = r(\rho_C)$.

(iv) The condition $r(\rho_{AB}) \cdot r(\rho_{AC}) = r(\rho_{BC})$ is equivalent to $r(\rho_{AB}) \cdot r(\rho_A \otimes \rho_C) = r(\rho_B \otimes \rho_C)$, namely $r(\rho_A) \cdot r(\rho_{AB}) = r(\rho_B)$.

(v) It is known that the rank of a bipartite PPT state is lower bounded by that of any one of its reduced density operators. Hence (77) implies

$$r(\rho_{AB}) \cdot r(\rho_{AC}) \ge r(\rho_B) \cdot r(\rho_C) \ge r(\rho_{BC}). \tag{106}$$

If $r(\rho_{AB}) \cdot r(\rho_{AC}) = r(\rho_{BC})$ then we obtain assertion (v). ■

We point out that the states ρ_{ABC} satisfying the conditions of this lemma exist, as we show by the following examples. In (i) we assume $\rho_{ABC} = |0,0,0\rangle\langle 0,0,0|$. Actually the example applies to all of the five cases in Lemma 18, and we shall show more non-trivial examples. In (ii) we assume $\rho_{ABC} = |0\rangle\langle 0|_A \otimes |\beta\rangle\langle\beta|_B \otimes |\gamma\rangle\langle\gamma|_C$ where β and γ are arbitrary states. In (iii), we assume that ρ_{AC} is a pure state. In (iv) we assume that ρ_{AB} is a pure state. In (v), we assume that $\rho_{ABC} = |0\rangle\langle 0|_A \otimes |\beta\rangle\langle\beta|_B \otimes |\gamma\rangle\langle\gamma|_C$ where β and γ are arbitrary states. A more non-trivial example satisfying (v) is $\rho_{ABC} = |0\rangle\langle 0|_A \otimes \sigma_{BC}$ where σ_{BC} is a bipartite PPT state such that $r(\rho_B) \cdot r(\rho_C) = r(\rho_{BC})$. Actually, there is no other state satisfying (v), as we prove in the following Proposition 20. For this purpose, we refer to an $m \times n$ state ρ in the sense that $\operatorname{rank} \rho_A = m$ and $\operatorname{rank} \rho_B = n$, and we denote $\mathcal{R}(M)$ as the range of matrix M. We present the following preliminary lemma.

Lemma 19: (i) The $m \times n$ state of rank $\max\{m,n\}$ is separable if and only if it is PPT. In this case the state is the convex sum of $\max\{m,n\}$ pure product states.

(ii) The $m \times n$ state of rank r is entangled and NPT if $\max\{m,n\} > r$.

(iii) Suppose the separable $m \times n$ state of rank $n(\ge m)$ is written as $\rho = \sum_j |a_j, b_j\rangle\langle a_j, b_j|$. By removing proportional states, we can assume that the set $\{|a_j\rangle\}$ contains exactly k pairwise linearly independent states $|a_1\rangle, \dots, |a_k\rangle$. Then $\rho = \sum_{j=1}^k |a_j\rangle\langle a_j| \otimes \rho_j$, where $\mathcal{R}(\rho_i) \cap \mathcal{R}(\rho_j) = \{0\}$ for any $i \ne j$ and $\sum_{j=1}^k \mathcal{R}(\rho_j) = \mathcal{R}(\rho_B)$.

Proof: Assertion (i) and (ii) have been respectively proven in [30] and [31], and applied in [32], [33], [34], and [35]. We prove assertion (iii). It follows from assertion (i) that

$$\rho = \sum_{i=1}^n |c_i, d_i\rangle\langle c_i, d_i|, \tag{107}$$

where $|d_i\rangle$'s are linearly independent. By removing proportional states and up to the permutation of subscripts, we may assume that the set $\{|c_i\rangle\}$ contains exactly s pairwise linearly independent states $|c_1\rangle, \dots, |c_s\rangle$. Let $|c_1, d_{i_1}\rangle$ with $i_1 \in \mathcal{S}_1$,...,$|c_s, d_{i_s}\rangle$ with $i_s \in \mathcal{S}_s$, such that $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ and $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_s = \{1, 2, \dots, n\}$.

Next because $\rho = \sum_j |a_j, b_j\rangle\langle a_j, b_j|$, using (107) we have that $|a_j, b_j\rangle \in \mathcal{R}(\rho) = \operatorname{span}\{|c_i, d_i\rangle\}$. The above facts imply that $|a_j\rangle$ is proportional to one of $|c_1\rangle, \dots, |c_s\rangle$, say $|c_j\rangle$, and thus $|b_j\rangle \in \operatorname{span}\{|d_{i_j}\rangle\}$ for $i_j \in \mathcal{S}_j$. Hence $k = s$. We can write ρ as

$$\rho = \sum_{j=1}^k |a_j\rangle\langle a_j| \otimes \left(\sum_{i \in \mathcal{T}_j} |b_{\sigma(i)}\rangle\langle b_{\sigma(i)}| \right), \tag{108}$$

where $\mathcal{T}_m \cap \mathcal{T}_n = \emptyset$ for any $m \ne n$, $|b_{\sigma(i)}\rangle \in \operatorname{span} \mathcal{S}_j$ for $i \in \mathcal{T}_j$, and some integer permutation σ. By writing $\rho_j = \sum_{i \in \mathcal{T}_j} |b_{\sigma(i)}\rangle\langle b_{\sigma(i)}|$, one can show that $\mathcal{R}(\rho_i) \cap \mathcal{R}(\rho_j) = \{0\}$ for any $i \ne j$ and

$$\begin{aligned}\sum_{j=1}^k \mathcal{R}(\rho_j) =& \sum_{j=1}^k \operatorname{span}\{|d_{i_j}\rangle, i_j \in \mathcal{S}_j\} \\ =& \operatorname{span}\{|d_i\rangle\} = \mathcal{R}(\rho_B).\end{aligned} \tag{109}$$

We have proven the assertion. ■

Proposition 20: Every tripartite PPT state satisfying Lemma 18 (v) has the expression $\rho_{ABC} = |a\rangle\langle a|_A \otimes \sigma_{BC}$, where σ_{BC} is a bipartite PPT state such that $r(\rho_B) \cdot r(\rho_C) = r(\rho_{BC})$.

Proof: Using Lemma 18 (v), we may assume that ρ_{ABC} is a tripartite PPT state satisfying $r(\rho_{AB}) = r(\rho_B)$, $r(\rho_{AC}) = r(\rho_C)$ and $r(\rho_B) \cdot r(\rho_C) = r(\rho_{BC})$. Hence

$$\begin{aligned}r(\rho_{AB}) \cdot r(\rho_C) &\ge r(\rho_{ABC}) \ge r(\rho_{BC}) \\ &= r(\rho_B) \cdot r(\rho_C) = r(\rho_{AB}) \cdot r(\rho_C),\end{aligned} \tag{110}$$

where the second inequality follows from Lemma 19 (ii). So all inequalities are saturated, namely $r(\rho_{ABC}) = r(\rho_{BC})$. It follows from Lemma 19 (iii) that

$$\rho_{ABC} = \sum_{j=1}^s |a_j\rangle\langle a_j|_A \otimes (\rho_j)_{BC}, \tag{111}$$

where $|a_j\rangle$'s are pairwise linearly independent, and $\mathcal{R}(\rho_i)_{BC} \cap \mathcal{R}(\rho_j)_{BC} = \{0\}$ for any $i \ne j$. Using (111) we obtain that

$$\rho_{AB} = \sum_{j=1}^s |a_j\rangle\langle a_j|_A \otimes (\rho_j)_B, \tag{112}$$

$$\rho_{AC} = \sum_{j=1}^s |a_j\rangle\langle a_j|_A \otimes (\rho_j)_C. \tag{113}$$

Because $r(\rho_{AB}) = r(\rho_B)$ and $r(\rho_{AC}) = r(\rho_C)$, Lemma 19 (iii) implies that for any $i \ne j$,

$$\mathcal{R}((\rho_i)_B) \cap \mathcal{R}((\rho_j)_B) = \mathcal{R}((\rho_i)_C) \cap \mathcal{R}((\rho_j)_C) = \{0\}. \tag{114}$$

Hence, we obtain

$$\begin{aligned}r(\rho_{BC}) =& \operatorname{Dim} \mathcal{R}(\rho_{BC}) \\ =& \operatorname{Dim} \sum_{j=1}^s \mathcal{R}((\rho_j)_{BC}) \\ =& \sum_{j=1}^s \operatorname{Dim} \mathcal{R}((\rho_j)_{BC}) \\ =& \sum_{j=1}^s r((\rho_j)_{BC}) \\ \le& \sum_{j=1}^s r((\rho_j)_B) \cdot r((\rho_j)_C)\end{aligned}$$

$$\leq\left(\sum_{j=1}^{s} r((\rho_j)_B)\right)\cdot\left(\sum_{j=1}^{s} r((\rho_j)_C)\right)$$
$$=r(\rho_B)\cdot r(\rho_C), \tag{115}$$

where the last equality is from (114). The inequalities are saturated because of the condition $r(\rho_B)\cdot r(\rho_C)=r(\rho_{BC})$ in Lemma 18 (v). It holds only if $s=1$. Then the assertion follows from Eq. (111) and the fact that ρ_{ABC} is PPT. ■

We stress that, the conditions $r(\rho_{AB})=r(\rho_B)$, $r(\rho_{AC})=r(\rho_C)$ and $r(\rho_B)\cdot r(\rho_C)=r(\rho_{BC})$ are necessary. Otherwise Proposition 20 fails due to the following example. Let $\rho_{ABC}=\sum_j |jjj\rangle\langle jjj|$, which satisfies $r(\rho_{AB})=r(\rho_B)$ and $r(\rho_{AC})=r(\rho_C)$, though the third equality $r(\rho_B)\cdot r(\rho_C)=r(\rho_{BC})$ fails.

V. Conclusion

We have proven the inequality $r(\rho_{AB})\cdot r(\rho_{AC})\geq r(\rho_{BC})$ for any tripartite mixed state ρ_{ABC} by proving an equivalent theorem as well as the construction of a novel canonical form of bipartite matrices. So we have a complete picture of the four-party linear inequalities in terms of the 0-entropy. We also have extended the inequalities to the scenario of multipartite systems, sheding light on an inequality relation between the 0-entropy and the Schmidt measure, and discussed the condition when the equality $r(\rho_{AB})\cdot r(\rho_{AC})=r(\rho_{BC})$ holds.

We believe that the canonical form in Theorem 11 might be applied to more quantum-information problems concerning multipartite systems. Besides some open problems from this paper are as follows.

1) Can we obtain tighter lower bounds in the inequalities (81)-(83)?
2) Can we obtain more inequalities of multipartite systems, apart from Lemma 13 and Corollary 14 ?
3) Although we have provided some conditions by which the inequality in (1) is saturated in Lemma 18, a general condition is still missing. That is, can we characterize all states saturating Lemma 18 in a more non-trivial way?

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