Decompositions in algebra

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Stellingen

behorende bij het proefschrift
"Decompositions in algebra"

(i) Let $\alpha \in \overline{\mathbb{Z}}$ be of degree n and suppose all images of α in $\mathbb{C}$ have the same absolute value r. If α divides 2, then $r = 2^{e/n}$ for some integer $0 \leq e \leq n$. If also $3n \leq 4e$, then $\sqrt{2}$ divides α and in particular α has a non-trivial decomposition (see Definition 2.4.1).

(ii) Consider for $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ the norm on $\overline{\mathbb{Q}}$ given by

$$\|\alpha\|_p = \left(\frac{1}{[\mathbb{Q}(\alpha) : \mathbb{Q}]} \sum_{\sigma} |\sigma(\alpha)|^p \right)^{1/p},$$

where the sum ranges over all ring homomorphisms from $\mathbb{Q}(\alpha)$ to $\mathbb{C}$. Then for each $\alpha \in \overline{\mathbb{Q}}$ and $r < (\mathrm{e}/4)^{\frac{1}{2p}}$ there exist only finitely many $\beta \in \overline{\mathbb{Z}}$ such that $\|\alpha - \beta\|_p < r$.

(iii) Let p be a prime, k a commutative ring and R a connected reduced commutative k-algebra that is Noetherian as a k-module, such all $x, y \in R$ satisfy $px = p^2xy$ only if $x = 0$. We write $\mathcal{D}$ for the set of pairs (A, G) where $A \subseteq R$ is a k-subalgebra and $G \subseteq R^*$ is a p-subgroup such that the natural map $A[G] \to R$ is an isomorphism (see Definition 5.5.1). We equip $\mathcal{D}$ with the partial order where $(A, G) \leq (B, H)$ if $A \supseteq B$ and $G \subseteq H$. Then $\mathcal{D}$ has a maximal element. If $(A, G), (B, H) \in \mathcal{D}$ are maximal, then we have $(A, H), (B, G) \in \mathcal{D}$ and all four pairs are in the same orbit under $\mathrm{Aut}(R)$.

(iv) The theory of algorithms is improved when, whenever possible, the space of inputs and the space of outputs of each algorithm is equipped with a natural category structure and the algorithm computes a functor when restricting the class of morphisms to isomorphisms. This is also possible for some non-deterministic Las Vegas algorithms.

(v) If an abelian group is free as a module over its endomorphism ring, then its rank is 1 and its endomorphism ring is commutative. Moreover, there exists such a group of cardinality greater than that of the continuum.

(vi) For all rational $0 < r, s < 1$ there exists a Markov chain with a finite state space and with distinct vertices a, b, and c such that:
   (1) vertices b and c have no outgoing edges;
   (2) every other vertex has exactly two outgoing edges, one with probability r and another with $1 - r$; and
   (3) starting with probability 1 in a results in a limit distribution with probability s in b and $1 - s$ in c.

(vii) Suppose that R is a Noetherian commutative ring of Krull dimension at most 1, that its total ring of fractions K is Artinian, and that I is a fractional ideal of R (see Section 6.2). Then among all subrings $R \subseteq S \subseteq K$ for which SI is an invertible ideal of S there exists a unique minimum with respect to inclusion. It equals $I^n : I^n$ for some $n \in \mathbb{Z}_{\geq 1}$ and is Noetherian as an R-module.

(viii) Proof assistants, such as Lean and Coq, will never see use by a wide mathematical audience as long as the average user still has to worry about implementation details.

(ix) Baking is beneficial to the mental health of mathematics PhD students.

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Leiden, 5 maart 2024